Hezenci et al. *Boundary Value Problems* (2024) 2024:12
https://doi.org/10.1186/s13661-023-01818-y

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

Open Access

Novel results of Milne-type inequalities involving tempered fractional integrals

Fatih Hezenci[1], Hüseyin Budak[1], Hasan Kara[1*] and Umut Baş[1]

[*]Correspondence:
hasan64kara@gmail.com
[1]Department of Mathematics, Faculty of Science and Arts, Duzce University, Duzce, Türkiye

Abstract

In this current research, we focus on the domain of tempered fractional integrals, establishing a novel identity that serves as the cornerstone of our study. This identity paves the way for the Milne-type inequalities, which are explored through the framework of differentiable convex mappings inclusive of tempered fractional integrals. The significance of these mappings in the realm of fractional calculus is underscored by their ability to extend classical concepts into more complex, fractional dimensions. In addition, by using the Hölder inequality and power-mean inequality, we acquire some new Milne-type inequalities. Moreover, the practicality and theoretical relevance of our findings are further demonstrated through the application of specific cases derived from the theorems.

Mathematics Subject Classification: 26D07; 26D10; 26D15; 65D32

Keywords: Milne-type inequality; Tempered fractional integrals; Convex functions

1 Introduction and preliminaries

Nowadays many scientists are focusing on fractional calculus and studying the theory of inequalities. The studies of fractional calculus have become the most popular field of research in physics, engineering, and mathematics. Most of the well-known mathematicians like Euler, Fourier, Laplace, Lacroix, Leibnitz, Abel, etc., were attracted to fractional calculus. Recently, tremendous research has been done on fractional calculus. The researchers who have contributed directly and indirectly to the development of fractional calculus are N. H. Abel (1823–1826), J. Liouville (1832–1873), G. F. B. Riemann (1847), H. Holmgren (1865–1867), A. K. Grunwald (1867–1872), A. V. Letnikov (1868–1872), H. Laurent (1884), P. A. Nekrassov (1888), A. Kurg (1889), J. Hadamard (1892), O. Heaviside (1892–1912), G. H. Hardy and J. E. Littlewood (1917–1928), H. Weyl (1917), Buss (1929), P. Levy (1923), A. Marchaud (1927), H.T. Davis (1924–1927), Goldman (1949), K. Oldham and J. Spanier (1974), L. Debnath (1992), K. S. Miller and B. Ross (1993), R. Gorenflo and F. Mainardi (2000), I. Podlubny (2003), and many more. While integer orders provide models that are suitable for nature in classical analysis, fractional computation in which arbitrary orders are examined enables us to obtain more realistic approaches. Moreover, the application of arithmetic carried out in classical analysis is very important in fractional analysis in terms of obtaining more realistic results in the solution of many problems. More

© The Author(s) 2024. **Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

general results are obtained with different approaches and operators in fractional calculus. Fractional integral operators have provided solutions to integrals in many studies. In the literature, many integral operators such as Riemann–Liouville, Hadamard, Katugompola, and tempered differential integral operators are considered. The most widely used of them is the Riemann–Liouville integral. There is a lot of research on these integrals. Tempered fractional integral operators generalizing Riemann–Liouville integrals are given in papers [12, 16]. In the literature, the theory of inequalities is an important area of mathematics. There are a lot of studies on the well-known Hermite–Hadamard inequality. Many researchers have studied the Hermite–Hadamard-type and related inequalities such as trapezoid and midpoint type. Also, a lot of researchers focused on Simpson-, Newton-, and Milne-type inequalities and contributed to science. Many researchers have contributed to the refinement and generalization of these integral inequalities. Sarikaya et al. [19] established Hermite–Hadamard-type inequalities via Riemann–Liouville fractional integrals. Sarikaya and Yildirim acquired some new inequalities of Hermite–Hadamard and midpoint type with the help of the Riemann–Liouville fractional integrals via differentiable convex functions in [18]. Khan et al. [1] established some new versions of Hermite–Hadamard-type inequalities for Riemann–Liouville and conformable fractional integrals. The authors of [15] proved several Simpson-type inequalities involving Riemann–Liouville fractional integrals and in the case of general convex functions. See references [5, 21], and the references therein, for further information concerning Simpson-type inequalities and several properties of Riemann–Liouville fractional integrals as well as those of various fractional integral operators. In the paper [9], Djenaoui and Meftah proved several new estimates of Milne's quadrature rule for functions whose first derivative is s-convex. Budak et al. [6] acquired fractional versions of Milne-type inequalities by utilizing differentiable convex functions. Ali et al. [2] gave error bounds via one of the open Newton–Cotes formulas, namely Milne's formula for differentiable convex functions in fractional and classical calculus. Alomari and Liu [3] established error estimates for Milne's rule for mappings of bounded variation and for absolutely continuous mappings.

Before giving Milne-type inequalities, let us give the following preliminary information. Simpson-type inequalities are inequalities that are created from Simpson's rules:

i. Simpson's quadrature formula (Simpson's 1/3 rule) is formulated as follows:

$$\int_{\eta}^{\mu} \mathfrak{F}(\kappa)\,d\kappa \approx \frac{\mu-\eta}{6}\left[\mathfrak{F}(\eta)+4\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+\mathfrak{F}(\mu)\right]. \tag{1.1}$$

ii. Simpson's second formula or Newton–Cotes quadrature formula (Simpson's 3/8 rule (cf. [8])) is formulated as follows:

$$\int_{\eta}^{\mu} \mathfrak{F}(\kappa)\,d\kappa \approx \frac{\mu-\eta}{8}\left[\mathfrak{F}(\eta)+3\mathfrak{F}\left(\frac{2\eta+\mu}{3}\right)+3\mathfrak{F}\left(\frac{\eta+2\mu}{3}\right)+\mathfrak{F}(\mu)\right]. \tag{1.2}$$

Formulae (1.1) and (1.2) are satisfied for any function $\mathfrak{F}$ with a continuous fourth derivative on $[\eta,\mu]$.

The most popular Newton–Cotes quadrature involving three points is Simpson-type inequality and formulated as follows:

Theorem 1 *Let $\mathfrak{F} : [\eta,\mu] \to \mathbb{R}$ be a four times continuously differentiable function on (η,μ), and let $\|\mathfrak{F}^{(4)}\|_\infty = \sup_{\kappa\in(\eta,\mu)} |\mathfrak{F}^{(4)}(\kappa)| < \infty$. Then, one has the following inequality*:

$$\left|\frac{1}{6}\left[\mathfrak{F}(\eta)+4\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+\mathfrak{F}(\mu)\right]-\frac{1}{\mu-\eta}\int_\eta^\mu \mathfrak{F}(\kappa)\,d\kappa\right| \le \frac{1}{2880}\left\|\mathfrak{F}^{(4)}\right\|_\infty(\mu-\eta)^4.$$

One of the classical closed-type quadrature rules is the Simpson 3/8 rule based on the Simpson 3/8 inequality as follows:

Theorem 2 ([8]) *If $\mathfrak{F} : [\eta,\mu] \to \mathbb{R}$ is a four times continuously differentiable function on (η,μ), and $\|\mathfrak{F}^{(4)}\|_\infty = \sup_{\kappa\in(\eta,\mu)} |\mathfrak{F}^{(4)}(\kappa)| < \infty$, then one has the inequality*

$$\begin{aligned}&\left|\frac{1}{8}\left[\mathfrak{F}(\eta)+3\mathfrak{F}\left(\frac{2\eta+\mu}{3}\right)+3\mathfrak{F}\left(\frac{\eta+2\mu}{3}\right)+\mathfrak{F}(\mu)\right]-\frac{1}{\mu-\eta}\int_\eta^\mu \mathfrak{F}(\kappa)\,d\kappa\right|\\&\quad\le \frac{1}{6480}\left\|\mathfrak{F}^{(4)}\right\|_\infty(\mu-\eta)^4.\end{aligned}$$

In terms of Newton–Cotes formulas, Milne's formula, which is of open type, is parallel to Simpson's formula, which is of closed type, since they hold under the same conditions.

Theorem 3 ([4]) *Suppose that $\mathfrak{F} : [\eta,\mu] \to \mathbb{R}$ is a four times continuously differentiable mapping on (η,μ), and let $\|\mathfrak{F}^{(4)}\|_\infty = \sup_{\kappa\in(\eta,\mu)} |\mathfrak{F}^{(4)}(\kappa)| < \infty$. Then, one has the inequality*

$$\left|\frac{1}{3}\left[2\mathfrak{F}(\eta)-\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+2\mathfrak{F}(\mu)\right]-\frac{1}{\mu-\eta}\int_\eta^\mu \mathfrak{F}(\kappa)\,d\kappa\right| \le \frac{7(\mu-\eta)^4}{23040}\left\|\mathfrak{F}^{(4)}\right\|_\infty. \tag{1.3}$$

In this research, we will establish a fractional version of the left-hand side of (1.3) and will consider several new bounds by using various mapping classes. Recall that the *gamma function, incomplete gamma function, λ-incomplete gamma function* are respectively defined by

$$\begin{aligned}\Gamma(\alpha) &:= \int_0^\infty \delta^{\alpha-1}e^{-\delta}\,d\delta,\\ \Upsilon(\alpha,\kappa) &:= \int_0^\kappa \delta^{\alpha-1}e^{-\delta}\,d\delta,\end{aligned}$$

and

$$\Upsilon_\lambda(\alpha,\kappa) := \int_0^\kappa \delta^{\alpha-1}e^{-\lambda\delta}\,d\delta.$$

Here, $0 < \alpha < \infty$ and $\lambda \ge 0$.

We list some properties of the λ-incomplete gamma function as follows:

Remark 1 ([14]) For the real numbers $\alpha > 0$, $\kappa,\lambda \ge 0$, and $\eta < \mu$, we readily have

i. $\Upsilon_{\lambda(\mu-\eta)}(\alpha,1) = \int_0^1 \delta^{\alpha-1}e^{-\lambda(\mu-\eta)\delta}\,d\delta = \frac{1}{(\mu-\eta)^\alpha}\Upsilon_\lambda(\alpha,1)$,
ii. $\int_0^1 \Upsilon_{\lambda(\mu-\eta)}(\alpha,\kappa)\,d\kappa = \frac{\Upsilon_\lambda(\alpha,\mu-\eta)}{(\mu-\eta)^\alpha} - \frac{\Upsilon_\lambda(\alpha+1,\mu-\eta)}{(\mu-\eta)^{\alpha+1}}$.

Recall also that the *Riemann–Liouville integrals* of order $\alpha > 0$ are given by

$$J^{\alpha}_{\eta+}\mathfrak{F}(\kappa) = \frac{1}{\Gamma(\alpha)}\int_{\eta}^{\kappa}(\kappa-\delta)^{\alpha-1}\mathfrak{F}(\delta)\,d\delta, \quad \kappa > \eta, \tag{1.4}$$

and

$$J^{\alpha}_{\mu-}\mathfrak{F}(\kappa) = \frac{1}{\Gamma(\alpha)}\int_{\kappa}^{\mu}(\delta-\kappa)^{\alpha-1}\mathfrak{F}(\delta)\,d\delta, \quad \kappa < \mu, \tag{1.5}$$

for $\mathfrak{F} \in L_1[\eta,\mu]$. See [10, 11] for details and unexplained subjects. Note that the Riemann–Liouville integrals become classical integrals when $\alpha = 1$.

We shall now present the fundamental definitions and new notations of tempered fractional operators.

Definition 1 ([12, 16]) The fractional tempered integral operators are given as follows:

$$\mathcal{J}^{(\alpha,\lambda)}_{\eta+}\mathfrak{F}(\kappa) = \frac{1}{\Gamma(\alpha)}\int_{\eta}^{\kappa}(\kappa-\delta)^{\alpha-1}e^{-\lambda(\kappa-\delta)}\mathfrak{F}(\delta)\,d\delta, \quad \kappa \in [\eta,\mu], \tag{1.6}$$

and

$$\mathcal{J}^{(\alpha,\lambda)}_{\mu-}\mathfrak{F}(\kappa) = \frac{1}{\Gamma(\alpha)}\int_{\kappa}^{\mu}(\delta-\kappa)^{\alpha-1}e^{-\lambda(\delta-\kappa)}\mathfrak{F}(\delta)\,d\delta, \quad \kappa \in [\eta,\mu]. \tag{1.7}$$

Here, $\mathfrak{F} \in L_1[\eta,\mu]$, $\alpha > 0$, and $\lambda \geq 0$.

If we choose $\lambda = 0$, then the fractional integrals in (1.6) and (1.7) coincide with the Riemann–Liouville fractional integrals in (1.4) and (1.5), respectively.

It can be said that the inequalities obtained with the help of tempered fractional integral operators generalize the inequalities established through the Riemann–Liouville integral operators. The descriptions of fractional integration with weak singular and exponential kernels were firstly reported in Buschman's earlier work [7]. For more research on the different cases of tempered fractional integrals, see the books [13, 17, 20]. In the paper [14], Mohammed et al. acquired several Hermite–Hadamard-type inequalities with the help of the tempered fractional integrals by employing convex functions, which cover the previously published results such as Riemann and Riemann–Liouville fractional integrals.

Inspired by the ongoing studies, we acquire the tempered fractional version of Milne's formula-type inequalities via differentiable convex mappings. The main advantage of these inequalities is that they can be converted into Riemann–Liouville fractional Milne-type inequalities for $\lambda = 0$. If we choose $\alpha = 1$ in the inequalities, the result is reduced to classical Milne-type inequalities. The basic definitions of fractional calculus and other relevant research in this discipline are given in the above-mentioned references. We will prove an integral equality in Sect. 2 that is critical in establishing the primary results of the present paper. Furthermore, it will prove some Milne-type inequalities for the case of differentiable convex mappings, including tempered fractional integrals. By using the special cases of the established results, we will then present several important inequalities. In Sect. 3, we will suggest several ideas about the inequalities of Milne via further directions of research.

2 Main results

In this section, we first obtain an identity by using tempered fractional integrals. Then, using this identity, we obtain new Milne-type inequalities with the help of tempered fractional integrals.

Lemma 1 *Let us consider that* $\mathfrak{F}:[\eta,\mu]\to\mathbb{R}$ *is an absolutely continuous function* (η,μ) *such that* $\mathfrak{F}'\in L_1[\eta,\mu]$. *Then, the following equality holds*:

$$\begin{aligned}&\frac{1}{3}\left[2\mathfrak{F}\left(\frac{\eta+3\mu}{4}\right)-\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+2\mathfrak{F}\left(\frac{3\eta+\mu}{4}\right)\right]\\&\quad-\frac{\Gamma(\alpha)}{2\curlyvee_{\lambda}(\alpha,\mu-\eta)}\left[\mathcal{J}_{\mu-}^{(\alpha,\lambda)}\mathfrak{F}(\eta)+\mathcal{J}_{\eta+}^{(\alpha,\lambda)}\mathfrak{F}(\mu)\right]\\&\quad=\frac{(\mu-\eta)^{\alpha+1}}{2\curlyvee_{\lambda}(\alpha,\mu-\eta)}\sum_{i=1}^{4}I_i.\end{aligned}\tag{2.1}$$

Here,

$$\begin{cases}I_1=\int_0^{\frac14}\curlyvee_{\lambda(\mu-\eta)}(\alpha,\delta)[\mathfrak{F}'(\delta\mu+(1-\delta)\eta)-\mathfrak{F}'(\delta\eta+(1-\delta)\mu)]\,d\delta,\\ I_2=\int_{\frac14}^{\frac12}\{\curlyvee_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac23\curlyvee_{\lambda(\mu-\eta)}(\alpha,1)\}[\mathfrak{F}'(\delta\mu+(1-\delta)\eta)-\mathfrak{F}'(\delta\eta+(1-\delta)\mu)]\,d\delta,\\ I_3=\int_{\frac12}^{\frac34}\{\curlyvee_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac13\curlyvee_{\lambda(\mu-\eta)}(\alpha,1)\}[\mathfrak{F}'(\delta\mu+(1-\delta)\eta)-\mathfrak{F}'(\delta\eta+(1-\delta)\mu)]\,d\delta,\\ I_4=\int_{\frac34}^{1}\{\curlyvee_{\lambda(\mu-\eta)}(\alpha,\delta)-\curlyvee_{\lambda(\mu-\eta)}(\alpha,1)\}[\mathfrak{F}'(\delta\mu+(1-\delta)\eta)-\mathfrak{F}'(\delta\eta+(1-\delta)\mu)]\,d\delta.\end{cases}$$

Proof Using integration by parts, we get

$$\begin{aligned}I_1&=\int_0^{\frac14}\curlyvee_{\lambda(\mu-\eta)}(\alpha,\delta)\left[\mathfrak{F}'\left(\delta\mu+(1-\delta)\eta\right)-\mathfrak{F}'\left(\delta\eta+(1-\delta)\mu\right)\right]d\delta\\&=\frac{1}{\mu-\eta}\curlyvee_{\lambda(\mu-\eta)}(\alpha,\delta)\left[\mathfrak{F}\left(\delta\mu+(1-\delta)\eta\right)+\mathfrak{F}\left(\delta\eta+(1-\delta)\mu\right)\right]\Big|_0^{\frac14}\\&\quad-\frac{1}{\mu-\eta}\int_0^{\frac14}\delta^{\alpha-1}e^{-\lambda(\mu-\eta)\delta}\left[\mathfrak{F}\left(\delta\mu+(1-\delta)\eta\right)+\mathfrak{F}\left(\delta\eta+(1-\delta)\mu\right)\right]d\delta\\&=\frac{1}{\mu-\eta}\curlyvee_{\lambda(\mu-\eta)}\left(\alpha,\frac14\right)\left[\mathfrak{F}\left(\frac{\eta+3\mu}{4}\right)+\mathfrak{F}\left(\frac{3\eta+\mu}{4}\right)\right]\\&\quad-\frac{1}{\mu-\eta}\int_0^{\frac14}\delta^{\alpha-1}e^{-\lambda(\mu-\eta)\delta}\left[\mathfrak{F}\left(\delta\mu+(1-\delta)\eta\right)+\mathfrak{F}\left(\delta\eta+(1-\delta)\mu\right)\right]d\delta.\end{aligned}\tag{2.2}$$

Then, arguing similarly as above, we readily obtain

$$\begin{aligned}I_2&=\frac{2}{\mu-\eta}\left\{\curlyvee_{\lambda(\mu-\eta)}\left(\alpha,\frac12\right)-\frac23\curlyvee_{\lambda(\mu-\eta)}(\alpha,1)\right\}\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)\\&\quad-\frac{1}{\mu-\eta}\left\{\curlyvee_{\lambda(\mu-\eta)}\left(\alpha,\frac14\right)-\frac23\curlyvee_{\lambda(\mu-\eta)}(\alpha,1)\right\}\left[\mathfrak{F}\left(\frac{\eta+3\mu}{4}\right)+\mathfrak{F}\left(\frac{3\eta+\mu}{4}\right)\right]\\&\quad-\frac{1}{\mu-\eta}\int_{\frac14}^{\frac12}\delta^{\alpha-1}e^{-\lambda(\mu-\eta)\delta}\left[\mathfrak{F}\left(\delta\mu+(1-\delta)\eta\right)+\mathfrak{F}\left(\delta\eta+(1-\delta)\mu\right)\right]d\delta,\end{aligned}\tag{2.3}$$

$$I_3=\frac{1}{\mu-\eta}\left\{\curlyvee_{\lambda(\mu-\eta)}\left(\alpha,\frac34\right)-\frac13\curlyvee_{\lambda(\mu-\eta)}(\alpha,1)\right\}\left[\mathfrak{F}\left(\frac{\eta+3\mu}{4}\right)+\mathfrak{F}\left(\frac{3\eta+\mu}{4}\right)\right]\tag{2.4}$$

$$
\begin{aligned}
&-\frac{2}{\mu-\eta}\left\{\Upsilon_{\lambda(\mu-\eta)}\left(\alpha,\frac{1}{2}\right)-\frac{1}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right\}\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)\\
&-\frac{1}{\mu-\eta}\int_{\frac{1}{2}}^{\frac{3}{4}}\delta^{\alpha-1}e^{-\lambda(\mu-\eta)\delta}\left[\mathfrak{F}\big(\delta\mu+(1-\delta)\eta\big)+\mathfrak{F}\big(\delta\eta+(1-\delta)\mu\big)\right]d\delta,
\end{aligned}
$$

and

$$
\begin{aligned}
I_4=&-\frac{1}{\mu-\eta}\left\{\Upsilon_{\lambda(\mu-\eta)}\left(\alpha,\frac{3}{4}\right)-\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right\}\left[\mathfrak{F}\left(\frac{\eta+3\mu}{4}\right)+\mathfrak{F}\left(\frac{3\eta+\mu}{4}\right)\right]\\
&-\frac{1}{\mu-\eta}\int_{\frac{3}{4}}^{1}\delta^{\alpha-1}e^{-\lambda(\mu-\eta)\delta}\left[\mathfrak{F}\big(\delta\mu+(1-\delta)\eta\big)+\mathfrak{F}\big(\delta\eta+(1-\delta)\mu\big)\right]d\delta.
\end{aligned}
\tag{2.5}
$$

If we add equations (2.2) to (2.5), then we have

$$
\begin{aligned}
\sum_{i=1}^{4}I_i=&\frac{2\Upsilon_{\lambda}(\alpha,\mu-\eta)}{3(\mu-\eta)^{\alpha+1}}\left[2\mathfrak{F}\left(\frac{\eta+3\mu}{4}\right)-\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+2\mathfrak{F}\left(\frac{3\eta+\mu}{4}\right)\right]\\
&-\frac{1}{\mu-\eta}\int_{0}^{1}\delta^{\alpha-1}e^{-\lambda(\mu-\eta)\delta}\left[\mathfrak{F}\big(\delta\mu+(1-\delta)\eta\big)+\mathfrak{F}\big(\delta\eta+(1-\delta)\mu\big)\right]d\delta.
\end{aligned}
\tag{2.6}
$$

With the help of the change of the variable $\kappa=\delta\mu+(1-\delta)\eta$ and $\kappa=\delta\eta+(1-\delta)\mu$ for $\delta\in[0,1]$, respectively, equality (2.6) can be rewritten as follows:

$$
\begin{aligned}
\sum_{i=1}^{4}I_i=&\frac{2\Upsilon_{\lambda}(\alpha,\mu-\eta)}{3(\mu-\eta)^{\alpha+1}}\left[2\mathfrak{F}\left(\frac{\eta+3\mu}{4}\right)-\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+2\mathfrak{F}\left(\frac{3\eta+\mu}{4}\right)\right]\\
&-\frac{\Gamma(\alpha)}{(\mu-\eta)^{\alpha+1}}\left[\mathcal{J}_{\mu-}^{(\alpha,\lambda)}\mathfrak{F}(\eta)+\mathcal{J}_{\eta+}^{(\alpha,\lambda)}\mathfrak{F}(\mu)\right].
\end{aligned}
\tag{2.7}
$$

Multiplying both sides of (2.7) by $\frac{(\mu-\eta)^{\alpha+1}}{2\Upsilon_{\lambda}(\alpha,\mu-\eta)}$, the equality (2.1) is obtained. □

Theorem 4 *If the assumptions of Lemma 1 hold and the function $|\mathfrak{F}'|$ is convex on $[\eta,\mu]$, then we have the following corrected Euler–Maclaurin-type inequality*:

$$
\begin{aligned}
&\left|\frac{1}{3}\left[2\mathfrak{F}\left(\frac{\eta+3\mu}{4}\right)-\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+2\mathfrak{F}\left(\frac{3\eta+\mu}{4}\right)\right]\right.\\
&\left.\quad-\frac{\Gamma(\alpha)}{2\Upsilon_{\lambda}(\alpha,\mu-\eta)}\left[\mathcal{J}_{\mu-}^{(\alpha,\lambda)}\mathfrak{F}(\eta)+\mathcal{J}_{\eta+}^{(\alpha,\lambda)}\mathfrak{F}(\mu)\right]\right|\\
&\quad\leq\frac{(\mu-\eta)^{\alpha+1}}{2\Upsilon_{\lambda}(\alpha,\mu-\eta)}\big(\Omega_1(\alpha,\lambda)+\Omega_2(\alpha,\lambda)+\Omega_3(\alpha,\lambda)+\Omega_4(\alpha,\lambda)\big)\left[\left|\mathfrak{F}'(\eta)\right|+\left|\mathfrak{F}'(\mu)\right|\right].
\end{aligned}
\tag{2.8}
$$

Here,

$$
\begin{cases}
\Omega_1(\alpha,\lambda)=\int_0^{\frac{1}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)\right|d\delta,\\
\Omega_2(\alpha,\lambda)=\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{2}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|d\delta,\\
\Omega_3(\alpha,\lambda)=\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{1}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|d\delta,\\
\Omega_4(\alpha,\lambda)=\int_{\frac{3}{4}}^{1}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|d\delta.
\end{cases}
$$

Proof Let us take the modulus in Lemma 1. Then, we reality have

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathfrak{F}\left(\frac{\eta+3\mu}{4}\right)-\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+2\mathfrak{F}\left(\frac{3\eta+\mu}{4}\right)\right]\right. \\
&\qquad\left.-\frac{\Gamma(\alpha)}{2\Upsilon_{\lambda}(\alpha,\mu-\eta)}\left[\mathcal{J}_{\mu-}^{(\alpha,\lambda)}\mathfrak{F}(\eta)+\mathcal{J}_{\eta+}^{(\alpha,\lambda)}\mathfrak{F}(\mu)\right]\right| \\
&\quad\leq\frac{(\mu-\eta)^{\alpha+1}}{2\Upsilon_{\lambda}(\alpha,\mu-\eta)}\left[\int_{0}^{\frac{1}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)\right|\left|\mathfrak{F}'\left(\delta\mu+(1-\delta)\eta\right)-\mathfrak{F}'\left(\delta\eta+(1-\delta)\mu\right)\right|d\delta\right. \\
&\qquad+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{2}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|\left|\mathfrak{F}'\left(\delta\mu+(1-\delta)\eta\right)-\mathfrak{F}'\left(\delta\eta+(1-\delta)\mu\right)\right|d\delta \\
&\qquad+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{1}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|\left|\mathfrak{F}'\left(\delta\mu+(1-\delta)\eta\right)-\mathfrak{F}'\left(\delta\eta+(1-\delta)\mu\right)\right|d\delta \\
&\qquad\left.+\int_{\frac{3}{4}}^{1}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|\left|\mathfrak{F}'\left(\delta\mu+(1-\delta)\eta\right)-\mathfrak{F}'\left(\delta\eta+(1-\delta)\mu\right)\right|d\delta\right].
\end{aligned} \tag{2.9}$$

It is assumed that $|\mathfrak{F}'|$ is convex. Thus,

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathfrak{F}\left(\frac{\eta+3\mu}{4}\right)-\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+2\mathfrak{F}\left(\frac{3\eta+\mu}{4}\right)\right]\right. \\
&\qquad\left.-\frac{\Gamma(\alpha)}{2\Upsilon_{\lambda}(\alpha,\mu-\eta)}\left[\mathcal{J}_{\mu-}^{(\alpha,\lambda)}\mathfrak{F}(\eta)+\mathcal{J}_{\eta+}^{(\alpha,\lambda)}\mathfrak{F}(\mu)\right]\right| \\
&\quad\leq\frac{(\mu-\eta)^{\alpha+1}}{2\Upsilon_{\lambda}(\alpha,\mu-\eta)}\left[\int_{0}^{\frac{1}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)\right|\left[\delta\left|\mathfrak{F}'(\mu)\right|+(1-\delta)\left|\mathfrak{F}'(\eta)\right|\right.\right. \\
&\qquad\left.+\delta\left|\mathfrak{F}'(\eta)\right|+(1-\delta)\left|\mathfrak{F}'(\mu)\right|\right]d\delta \\
&\qquad+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{2}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|\left[\delta\left|\mathfrak{F}'(\mu)\right|+(1-\delta)\left|\mathfrak{F}'(\eta)\right|\right. \\
&\qquad\left.+\delta\left|\mathfrak{F}'(\eta)\right|+(1-\delta)\left|\mathfrak{F}'(\mu)\right|\right]d\delta \\
&\qquad+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{1}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|\left[\delta\left|\mathfrak{F}'(\mu)\right|+(1-\delta)\left|\mathfrak{F}'(\eta)\right|\right. \\
&\qquad\left.+\delta\left|\mathfrak{F}'(\eta)\right|+(1-\delta)\left|\mathfrak{F}'(\mu)\right|\right]d\delta \\
&\qquad+\int_{\frac{3}{4}}^{1}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|\left[\delta\left|\mathfrak{F}'(\mu)\right|+(1-\delta)\left|\mathfrak{F}'(\eta)\right|\right. \\
&\qquad\left.\left.+\delta\left|\mathfrak{F}'(\eta)\right|+(1-\delta)\left|\mathfrak{F}'(\mu)\right|\right]d\delta\right] \\
&\quad=\frac{(\mu-\eta)^{\alpha+1}}{2\Upsilon_{\lambda}(\alpha,\mu-\eta)}\left(\Omega_1(\alpha)+\Omega_2(\alpha)+\Omega_3(\alpha)+\Omega_4(\alpha)\right)\left[\left|\mathfrak{F}'(\eta)\right|+\left|\mathfrak{F}'(\mu)\right|\right].
\end{aligned}$$

This finishes the proof of Theorem 4. □

Remark 2 Let us consider $\lambda = 0$ in Theorem 4. Then, the following Milne-type inequality holds:

$$\left|\frac{1}{3}\left[2\mathfrak{F}\left(\frac{\eta+3\mu}{4}\right)-\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+2\mathfrak{F}\left(\frac{3\eta+\mu}{4}\right)\right]-\frac{\Gamma(\alpha+1)}{2(\mu-\eta)^{\alpha}}\left[J_{\mu-}^{\alpha}\mathfrak{F}(\eta)+J_{\eta+}^{\alpha}\mathfrak{F}(\mu)\right]\right|$$
$$\leq \frac{\alpha(\mu-\eta)}{2}\left(\Omega_1(\alpha,0)+\Omega_2(\alpha,0)+\Omega_3(\alpha,0)+\Omega_4(\alpha,0)\right)\left[\left|\mathfrak{F}'(\eta)\right|+\left|\mathfrak{F}'(\mu)\right|\right],$$

which is given by Ali et al. [2, Theorem 1].

Remark 3 If we assign $\lambda = 0$ and $\alpha = 1$ in Theorem 4, then we get the Milne-type inequality

$$\left|\frac{1}{3}\left[2\mathfrak{F}\left(\frac{\eta+3\mu}{4}\right)-\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+2\mathfrak{F}\left(\frac{3\eta+\mu}{4}\right)\right]-\frac{1}{\mu-\eta}\int_{\eta}^{\mu}\mathfrak{F}(\delta)\,d\delta\right|$$
$$\leq \frac{5(\mu-\eta)}{48}\left[\left|\mathfrak{F}'(\eta)\right|+\left|\mathfrak{F}'(\mu)\right|\right],$$

which is given by Ali et al. [2, Corollary 1]. This inequality helps us find an error bound of Milne's rule.

Theorem 5 *Suppose that the assumptions of Lemma 1 hold and the function $|\mathfrak{F}'|^q$, $q > 1$ is convex on $[\eta,\mu]$. Then, the following Milne-type inequality holds*:

$$\left|\frac{1}{3}\left[2\mathfrak{F}\left(\frac{\eta+3\mu}{4}\right)-\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+2\mathfrak{F}\left(\frac{3\eta+\mu}{4}\right)\right]\right.$$
$$\left.-\frac{\Gamma(\alpha)}{2\Upsilon_{\lambda}(\alpha,\mu-\eta)}\left[\mathcal{J}_{\mu-}^{(\alpha,\lambda)}\mathfrak{F}(\eta)+\mathcal{J}_{\eta+}^{(\alpha,\lambda)}\mathfrak{F}(\mu)\right]\right|$$
$$\leq \frac{(\mu-\eta)^{\alpha+1}}{2\Upsilon_{\lambda}(\alpha,\mu-\eta)}\left\{\left(\varphi_1^p(\alpha,\lambda)+\varphi_4^p(\alpha,\lambda)\right)\left[\left(\frac{7|\mathfrak{F}'(\eta)|^q+|\mathfrak{F}'(\mu)|^q}{32}\right)^{\frac{1}{q}}\right.\right.$$
$$\left.+\left(\frac{|\mathfrak{F}'(\eta)|^q+7|\mathfrak{F}'(\mu)|^q}{32}\right)^{\frac{1}{q}}\right]$$
$$\left.+\left(\varphi_2^p(\alpha,\lambda)+\varphi_3^p(\alpha,\lambda)\right)\left[\left(\frac{3|\mathfrak{F}'(\eta)|^q+5|\mathfrak{F}'(\mu)|^q}{32}\right)^{\frac{1}{q}}+\left(\frac{5|\mathfrak{F}'(\eta)|^q+3|\mathfrak{F}'(\mu)|^q}{32}\right)^{\frac{1}{q}}\right]\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$\begin{cases}\varphi_1^p(\alpha,\lambda)=\left(\int_0^{\frac{1}{4}}|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)|^p\,d\delta\right)^{\frac{1}{p}},\\ \varphi_2^p(\alpha,\lambda)=\left(\int_{\frac{1}{4}}^{\frac{1}{2}}|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{2}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)|^p\,d\delta\right)^{\frac{1}{p}},\\ \varphi_3^p(\alpha,\lambda)=\left(\int_{\frac{1}{2}}^{\frac{3}{4}}|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{1}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)|^p\,d\delta\right)^{\frac{1}{p}},\\ \varphi_4^p(\alpha,\lambda)=\left(\int_{\frac{3}{4}}^{1}|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)|^p\,d\delta\right)^{\frac{1}{p}}.\end{cases}$$

Proof If we apply Hölder inequality in (2.9), then

$$\left|\frac{1}{3}\left[2\mathfrak{F}\left(\frac{\eta+3\mu}{4}\right)-\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+2\mathfrak{F}\left(\frac{3\eta+\mu}{4}\right)\right]\right.$$

$$
\begin{aligned}
&- \frac{\Gamma(\alpha)}{2\Upsilon_{\lambda}(\alpha,\mu-\eta)}\left[\mathcal{J}_{\mu-}^{(\alpha,\lambda)}\mathfrak{F}(\eta)+\mathcal{J}_{\eta+}^{(\alpha,\lambda)}\mathfrak{F}(\mu)\right]\Bigg| \\
&\leq \frac{(\mu-\eta)^{\alpha+1}}{2\Upsilon_{\lambda}(\alpha,\mu-\eta)}\Bigg\{\left(\int_0^{\frac{1}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)\right|^p d\delta\right)^{\frac{1}{p}}\Bigg[\left(\int_0^{\frac{1}{4}}\left|\mathfrak{F}'\big(\delta\mu+(1-\delta)\eta\big)\right|^q d\delta\right)^{\frac{1}{q}} \\
&+ \left(\int_0^{\frac{1}{4}}\left|\mathfrak{F}'\big(\delta\eta+(1-\delta)\mu\big)\right|^q d\delta\right)^{\frac{1}{q}}\Bigg] \\
&+ \left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{2}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|^p d\delta\right)^{\frac{1}{p}} \\
&\times \Bigg[\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\mathfrak{F}'\big(\delta\mu+(1-\delta)\eta\big)\right|^q d\delta\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\mathfrak{F}'\big(\delta\eta+(1-\delta)\mu\big)\right|^q d\delta\right)^{\frac{1}{q}}\Bigg] \\
&+ \left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{1}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|^p d\delta\right)^{\frac{1}{p}}\Bigg[\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\mathfrak{F}'\big(\delta\mu+(1-\delta)\eta\big)\right|^q d\delta\right)^{\frac{1}{q}} \\
&+ \left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\mathfrak{F}'\big(\delta\eta+(1-\delta)\mu\big)\right|^q d\delta\right)^{\frac{1}{q}}\Bigg]+\left(\int_{\frac{3}{4}}^{1}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|^p d\delta\right)^{\frac{1}{p}} \\
&\times \Bigg[\left(\int_{\frac{3}{4}}^{1}\left|\mathfrak{F}'\big(\delta\mu+(1-\delta)\eta\big)\right|^q d\delta\right)^{\frac{1}{q}}+\left(\int_{\frac{3}{4}}^{1}\left|\mathfrak{F}'\big(\delta\eta+(1-\delta)\mu\big)\right|^q d\delta\right)^{\frac{1}{q}}\Bigg]\Bigg\}.
\end{aligned}
$$

By using the convexity of $|\mathfrak{F}'|^q$, we readily get

$$
\begin{aligned}
&\left|\frac{1}{3}\left[2\mathfrak{F}\left(\frac{\eta+3\mu}{4}\right)-\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+2\mathfrak{F}\left(\frac{3\eta+\mu}{4}\right)\right]\right. \\
&- \frac{\Gamma(\alpha)}{2\Upsilon_{\lambda}(\alpha,\mu-\eta)}\left[\mathcal{J}_{\mu-}^{(\alpha,\lambda)}\mathfrak{F}(\eta)+\mathcal{J}_{\eta+}^{(\alpha,\lambda)}\mathfrak{F}(\mu)\right]\Bigg| \\
&\leq \frac{(\mu-\eta)^{\alpha+1}}{2\Upsilon_{\lambda}(\alpha,\mu-\eta)}\Bigg\{\left(\int_0^{\frac{1}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)\right|^p d\delta\right)^{\frac{1}{p}} \\
&\times \Bigg[\left(\int_0^{\frac{1}{4}}\delta\left|\mathfrak{F}'(\mu)\right|^q+(1-\delta)\left|\mathfrak{F}'(\eta)\right|^q d\delta\right)^{\frac{1}{q}} \\
&+ \left(\int_0^{\frac{1}{4}}\delta\left|\mathfrak{F}'(\eta)\right|^q+(1-\delta)\left|\mathfrak{F}'(\mu)\right|^q d\delta\right)^{\frac{1}{q}}\Bigg] \\
&+ \left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{2}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|^p d\delta\right)^{\frac{1}{p}} \\
&\times \Bigg[\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\delta\left|\mathfrak{F}'(\mu)\right|^q+(1-\delta)\left|\mathfrak{F}'(\eta)\right|^q d\delta\right)^{\frac{1}{q}} \\
&+ \left(\int_{\frac{1}{4}}^{\frac{1}{2}}\delta\left|\mathfrak{F}'(\eta)\right|^q+(1-\delta)\left|\mathfrak{F}'(\mu)\right|^q d\delta\right)^{\frac{1}{q}}\Bigg] \\
&+ \left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{1}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|^p d\delta\right)^{\frac{1}{p}} \\
&\times \Bigg[\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\delta\left|\mathfrak{F}'(\mu)\right|^q+(1-\delta)\left|\mathfrak{F}'(\eta)\right|^q d\delta\right)^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\delta\left|\mathfrak{F}'(\eta)\right|^q+(1-\delta)\left|\mathfrak{F}'(\mu)\right|^q d\delta\right)^{\frac{1}{q}}\Bigg]\\
&+\left(\int_{\frac{3}{4}}^{1}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|^p d\delta\right)^{\frac{1}{p}}\\
&\times\Bigg[\left(\int_{\frac{3}{4}}^{1}\delta\left|\mathfrak{F}'(\mu)\right|^q+(1-\delta)\left|\mathfrak{F}'(\eta)\right|^q d\delta\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{3}{4}}^{1}\delta\left|\mathfrak{F}'(\eta)\right|^q+(1-\delta)\left|\mathfrak{F}'(\mu)\right|^q d\delta\right)^{\frac{1}{q}}\Bigg]\Bigg\}\\
=&\frac{(\mu-\eta)^{\alpha+1}}{2\Upsilon_{\lambda}(\alpha,\mu-\eta)}\Bigg\{\Bigg(\left(\left(\int_0^{\frac{1}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)\right|^p d\delta\right)^{\frac{1}{p}}\right.\\
&\left.+\left(\int_{\frac{3}{4}}^{1}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|^p d\delta\right)^{\frac{1}{p}}\right)\\
&\times\left[\left(\frac{7|\mathfrak{F}'(\eta)|^q+|\mathfrak{F}'(\mu)|^q}{32}\right)^{\frac{1}{q}}+\left(\frac{|\mathfrak{F}'(\eta)|^q+7|\mathfrak{F}'(\mu)|^q}{32}\right)^{\frac{1}{q}}\right]\\
&+\left(\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{2}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|^p d\delta\right)^{\frac{1}{p}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{1}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|^p d\delta\right)^{\frac{1}{p}}\right)\\
&\times\left[\left(\frac{3|\mathfrak{F}'(\eta)|^q+5|\mathfrak{F}'(\mu)|^q}{32}\right)^{\frac{1}{q}}+\left(\frac{5|\mathfrak{F}'(\eta)|^q+3|\mathfrak{F}'(\mu)|^q}{32}\right)^{\frac{1}{q}}\right]\Bigg\},
\end{aligned}
$$

which completes the proof of Theorem 5. □

Remark 4 Let us consider $\lambda = 0$ in Theorem 4. Then, the following Milne-type inequality holds:

$$
\begin{aligned}
&\left|\frac{1}{3}\left[2\mathfrak{F}\left(\frac{\eta+3\mu}{4}\right)-\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+2\mathfrak{F}\left(\frac{3\eta+\mu}{4}\right)\right]-\frac{\Gamma(\alpha+1)}{2(\mu-\eta)^\alpha}\left[J_{\mu-}^{\alpha}\mathfrak{F}(\eta)+J_{\eta+}^{\alpha}\mathfrak{F}(\mu)\right]\right|\\
&\leq\frac{\alpha(\mu-\eta)}{2}\Bigg\{\left(\varphi_1^p(\alpha,0)+\varphi_4^p(\alpha,0)\right)\Bigg[\left(\frac{7|\mathfrak{F}'(\eta)|^q+|\mathfrak{F}'(\mu)|^q}{32}\right)^{\frac{1}{q}}\\
&+\left(\frac{|\mathfrak{F}'(\eta)|^q+7|\mathfrak{F}'(\mu)|^q}{32}\right)^{\frac{1}{q}}\Bigg]\\
&+\left(\varphi_2^p(\alpha,0)+\varphi_3^p(\alpha,0)\right)\left[\left(\frac{3|\mathfrak{F}'(\eta)|^q+5|\mathfrak{F}'(\mu)|^q}{32}\right)^{\frac{1}{q}}+\left(\frac{5|\mathfrak{F}'(\eta)|^q+3|\mathfrak{F}'(\mu)|^q}{32}\right)^{\frac{1}{q}}\right]\Bigg\},
\end{aligned}
$$

which is given by Ali et al. [2, Theorem 2].

Remark 5 If we choose $\lambda = 0$ and $\alpha = 1$ in Theorem 5, then we obtain the following Milne-type inequality:

$$
\left|\frac{1}{3}\left[2\mathfrak{F}\left(\frac{\eta+3\mu}{4}\right)-\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+2\mathfrak{F}\left(\frac{3\eta+\mu}{4}\right)\right]-\frac{1}{\mu-\eta}\int_\eta^\mu\mathfrak{F}(\delta)\,d\delta\right|
$$

$$
\begin{aligned}
&\leq (\mu-\eta)\Bigg[\left(\frac{1}{(p+1)4^{p+1}}\right)^{\frac{1}{p}}\left[\left(\frac{7|\mathfrak{F}'(\eta)|^q+|\mathfrak{F}'(\mu)|^q}{32}\right)^{\frac{1}{q}}+\left(\frac{|\mathfrak{F}'(\eta)|^q+7|\mathfrak{F}'(\mu)|^q}{32}\right)^{\frac{1}{q}}\right] \\
&\quad+\left(\frac{5^{p+1}}{12^{p+1}(p+1)}-\frac{1}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\Bigg[\left(\frac{3|\mathfrak{F}'(\eta)|^q+5|\mathfrak{F}'(\mu)|^q}{32}\right)^{\frac{1}{q}} \\
&\quad+\left(\frac{5|\mathfrak{F}'(\eta)|^q+3|\mathfrak{F}'(\mu)|^q}{32}\right)^{\frac{1}{q}}\Bigg]\Bigg],
\end{aligned}
$$

which is presented by Ali et al. [2, Corollary 2]. This inequality helps us find an error bound of Milne's rule.

Theorem 6 *Assume that the assumptions of Lemma 1 are valid. Assume also that the function* $|\mathfrak{F}'|^q$, $q \geq 1$ *is convex on* $[\eta,\mu]$. *Then, the following Milne-type inequality holds*:

$$
\begin{aligned}
&\bigg|\frac{1}{3}\left[2\mathfrak{F}\left(\frac{\eta+3\mu}{4}\right)-\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+2\mathfrak{F}\left(\frac{3\eta+\mu}{4}\right)\right] \\
&\quad-\frac{\Gamma(\alpha)}{2\Upsilon_{\lambda}(\alpha,\mu-\eta)}\left[\mathcal{J}_{\mu-}^{(\alpha,\lambda)}\mathfrak{F}(\eta)+\mathcal{J}_{\eta+}^{(\alpha,\lambda)}\mathfrak{F}(\mu)\right]\bigg| \\
&\leq \frac{(\mu-\eta)^{\alpha+1}}{2\Upsilon_{\lambda}(\alpha,\mu-\eta)}\Big\{\big(\Omega_1(\alpha,\lambda)\big)^{1-\frac{1}{q}}\Big[\Big(\Omega_5(\alpha,\lambda)\big|\mathfrak{F}'(\mu)\big|^q \\
&\quad+\big(\Omega_1(\alpha,\lambda)-\Omega_5(\alpha,\lambda)\big)\big|\mathfrak{F}'(\eta)\big|^q\Big)^{\frac{1}{q}} \\
&\quad+\Big(\Omega_5(\alpha,\lambda)\big|\mathfrak{F}'(\eta)\big|^q+\big(\Omega_1(\alpha,\lambda)-\Omega_5(\alpha,\lambda)\big)\big|\mathfrak{F}'(\mu)\big|^q\Big)^{\frac{1}{q}}\Big] \\
&\quad+\big(\Omega_2(\alpha,\lambda)\big)^{1-\frac{1}{q}}\Big[\Big(\Omega_6(\alpha,\lambda)\big|\mathfrak{F}'(\mu)\big|^q+\big(\Omega_2(\alpha,\lambda)-\Omega_6(\alpha,\lambda)\big)\big|\mathfrak{F}'(\eta)\big|^q\Big)^{\frac{1}{q}} \\
&\quad+\Big(\Omega_6(\alpha,\lambda)\big|\mathfrak{F}'(\eta)\big|^q+\big(\Omega_2(\alpha,\lambda)-\Omega_6(\alpha,\lambda)\big)\big|\mathfrak{F}'(\mu)\big|^q\Big)^{\frac{1}{q}}\Big] \\
&\quad+\big(\Omega_3(\alpha,\lambda)\big)^{1-\frac{1}{q}}\Big[\Big(\Omega_7(\alpha,\lambda)\big|\mathfrak{F}'(\mu)\big|^q+\big(\Omega_3(\alpha,\lambda)-\Omega_7(\alpha,\lambda)\big)\big|\mathfrak{F}'(\eta)\big|^q\Big)^{\frac{1}{q}} \\
&\quad+\Big(\Omega_7(\alpha,\lambda)\big|\mathfrak{F}'(\eta)\big|^q+\big(\Omega_3(\alpha,\lambda)-\Omega_7(\alpha,\lambda)\big)\big|\mathfrak{F}'(\mu)\big|^q\Big)^{\frac{1}{q}}\Big] \\
&\quad+\big(\Omega_4(\alpha,\lambda)\big)^{1-\frac{1}{q}}\Big[\Big(\Omega_8(\alpha,\lambda)\big|\mathfrak{F}'(\mu)\big|^q+\big(\Omega_4(\alpha,\lambda)-\Omega_8(\alpha,\lambda)\big)\big|\mathfrak{F}'(\eta)\big|^q\Big)^{\frac{1}{q}} \\
&\quad+\Big(\Omega_8(\alpha,\lambda)\big|\mathfrak{F}'(\eta)\big|^q+\big(\Omega_4(\alpha,\lambda)-\Omega_8(\alpha,\lambda)\big)\big|\mathfrak{F}'(\mu)\big|^q\Big)^{\frac{1}{q}}\Big]\Big\}.
\end{aligned}
$$

Here, $\Omega_1(\alpha,\lambda)$, $\Omega_2(\alpha,\lambda)$, $\Omega_3(\alpha,\lambda)$, *and* $\Omega_4(\alpha,\lambda)$ *are defined in Theorem 4 and*

$$
\begin{cases}
\Omega_5(\alpha,\lambda)=\int_0^{\frac{1}{4}}\delta|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)|\,d\delta, \\
\Omega_6(\alpha,\lambda)=\int_{\frac{1}{4}}^{\frac{1}{2}}\delta|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{2}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)|\,d\delta, \\
\Omega_7(\alpha,\lambda)=\int_{\frac{1}{2}}^{\frac{3}{4}}\delta|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{1}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)|\,d\delta, \\
\Omega_8(\alpha,\lambda)=\int_{\frac{3}{4}}^{1}\delta|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)|\,d\delta.
\end{cases}
$$

Proof Let us first apply the power-mean inequality in (2.9). Then, we get

$$
\begin{aligned}
&\bigg|\frac{1}{3}\left[2\mathfrak{F}\left(\frac{\eta+3\mu}{4}\right)-\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+2\mathfrak{F}\left(\frac{3\eta+\mu}{4}\right)\right] \\
&\quad-\frac{\Gamma(\alpha)}{2\Upsilon_{\lambda}(\alpha,\mu-\eta)}\left[\mathcal{J}_{\mu-}^{(\alpha,\lambda)}\mathfrak{F}(\eta)+\mathcal{J}_{\eta+}^{(\alpha,\lambda)}\mathfrak{F}(\mu)\right]\bigg|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{(\mu-\eta)^{\alpha+1}}{2\Upsilon_{\lambda}(\alpha,\mu-\eta)}\left\{\left(\int_0^{\frac{1}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)\right|d\delta\right)^{1-\frac{1}{q}}\right.\\
&\quad\times\left(\int_0^{\frac{1}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)\right|\left|\mathfrak{F}'\big(\delta\mu+(1-\delta)\eta\big)\right|^q d\delta\right)^{\frac{1}{q}}\\
&\quad+\left(\int_0^{\frac{1}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)\right|d\delta\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)\right|\left|\mathfrak{F}'\big(\delta\eta+(1-\delta)\mu\big)\right|^q d\delta\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{2}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|d\delta\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{2}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|\left|\mathfrak{F}'\big(\delta\mu+(1-\delta)\eta\big)\right|^q d\delta\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{2}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|d\delta\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{2}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|\left|\mathfrak{F}'\big(\delta\eta+(1-\delta)\mu\big)\right|^q d\delta\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{1}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|d\delta\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{1}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|\left|\mathfrak{F}'\big(\delta\mu+(1-\delta)\eta\big)\right|^q d\delta\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{1}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|d\delta\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{1}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|\left|\mathfrak{F}'\big(\delta\eta+(1-\delta)\mu\big)\right|^q d\delta\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{3}{4}}^{1}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|d\delta\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_{\frac{3}{4}}^{1}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|\left|\mathfrak{F}'\big(\delta\mu+(1-\delta)\eta\big)\right|^q d\delta\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{3}{4}}^{1}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|d\delta\right)^{1-\frac{1}{q}}\\
&\quad\left.\times\left(\int_{\frac{3}{4}}^{1}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|\left|\mathfrak{F}'\big(\delta\eta+(1-\delta)\mu\big)\right|^q d\delta\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Using the fact that $|\mathfrak{F}'|^q$ is convex, it follows that

$$
\begin{aligned}
&\left|\frac{1}{3}\left[2\mathfrak{F}\left(\frac{\eta+3\mu}{4}\right)-\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+2\mathfrak{F}\left(\frac{3\eta+\mu}{4}\right)\right]\right.\\
&\quad\left.-\frac{\Gamma(\alpha)}{2\Upsilon_{\lambda}(\alpha,\mu-\eta)}\left[\mathcal{J}_{\mu-}^{(\alpha,\lambda)}\mathfrak{F}(\eta)+\mathcal{J}_{\eta+}^{(\alpha,\lambda)}\mathfrak{F}(\mu)\right]\right|\\
&\leq \frac{(\mu-\eta)^{\alpha+1}}{2\Upsilon_{\lambda}(\alpha,\mu-\eta)}\left\{\left(\int_0^{\frac{1}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)\right|d\delta\right)^{1-\frac{1}{q}}\right.
\end{aligned}
$$

$$
\begin{aligned}
&\times\left[\left(\int_0^{\frac{1}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)\right|\left[\delta\left|\mathfrak{F}'(\mu)\right|^q+(1-\delta)\left|\mathfrak{F}'(\eta)\right|^q\right]d\delta\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^{\frac{1}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)\right|\left[\delta\left|\mathfrak{F}'(\eta)\right|^q+(1-\delta)\left|\mathfrak{F}'(\mu)\right|^q\right]d\delta\right)^{\frac{1}{q}}\right]\\
&+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{2}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|d\delta\right)^{1-\frac{1}{q}}\\
&\times\left[\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{2}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|\left[\delta\left|\mathfrak{F}'(\mu)\right|^q+(1-\delta)\left|\mathfrak{F}'(\eta)\right|^q\right]d\delta\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{2}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|\left[\delta\left|\mathfrak{F}'(\eta)\right|^q+(1-\delta)\left|\mathfrak{F}'(\mu)\right|^q\right]d\delta\right)^{\frac{1}{q}}\right]\\
&+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{1}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|d\delta\right)^{1-\frac{1}{q}}\\
&\times\left[\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{1}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|\left[\delta\left|\mathfrak{F}'(\mu)\right|^q+(1-\delta)\left|\mathfrak{F}'(\eta)\right|^q\right]d\delta\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\frac{1}{3}\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|\left[\delta\left|\mathfrak{F}'(\eta)\right|^q+(1-\delta)\left|\mathfrak{F}'(\mu)\right|^q\right]d\delta\right)^{\frac{1}{q}}\right]\\
&+\left(\int_{\frac{3}{4}}^{1}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|d\delta\right)^{1-\frac{1}{q}}\\
&\times\left[\left(\int_{\frac{3}{4}}^{1}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|\left[\delta\left|\mathfrak{F}'(\mu)\right|^q+(1-\delta)\left|\mathfrak{F}'(\eta)\right|^q\right]d\delta\right)^{\frac{1}{q}}\right.\\
&\left.\left.+\left(\int_{\frac{3}{4}}^{1}\left|\Upsilon_{\lambda(\mu-\eta)}(\alpha,\delta)-\Upsilon_{\lambda(\mu-\eta)}(\alpha,1)\right|\left[\delta\left|\mathfrak{F}'(\eta)\right|^q+(1-\delta)\left|\mathfrak{F}'(\mu)\right|^q\right]d\delta\right)^{\frac{1}{q}}\right]\right\}.
\end{aligned}
$$

Finally, we obtain the desired result of Theorem 6. □

Corollary 1 *Consider* $\lambda = 0$ *in Theorem* 6. *Then, the following Milne-type inequality holds*:

$$
\begin{aligned}
&\left|\frac{1}{3}\left[2\mathfrak{F}\left(\frac{\eta+3\mu}{4}\right)-\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+2\mathfrak{F}\left(\frac{3\eta+\mu}{4}\right)\right]-\frac{\Gamma(\alpha+1)}{2(\mu-\eta)^\alpha}\left[J_{\mu-}^{\alpha}\mathfrak{F}(\eta)+J_{\eta+}^{\alpha}\mathfrak{F}(\mu)\right]\right|\\
&\leq\frac{\alpha(\mu-\eta)}{2}\left\{\left(\Omega_1(\alpha,0)\right)^{1-\frac{1}{q}}\left[\left(\Omega_5(\alpha,0)\left|\mathfrak{F}'(\mu)\right|^q+\left(\Omega_1(\alpha,0)-\Omega_5(\alpha,0)\right)\left|\mathfrak{F}'(\eta)\right|^q\right)^{\frac{1}{q}}\right.\right.\\
&\left.+\left(\Omega_5(\alpha,0)\left|\mathfrak{F}'(\eta)\right|^q+\left(\Omega_1(\alpha,0)-\Omega_5(\alpha,0)\right)\left|\mathfrak{F}'(\mu)\right|^q\right)^{\frac{1}{q}}\right]\\
&+\left(\Omega_2(\alpha,0)\right)^{1-\frac{1}{q}}\left[\left(\Omega_6(\alpha,0)\left|\mathfrak{F}'(\mu)\right|^q+\left(\Omega_2(\alpha,0)-\Omega_6(\alpha,0)\right)\left|\mathfrak{F}'(\eta)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\Omega_6(\alpha,0)\left|\mathfrak{F}'(\eta)\right|^q+\left(\Omega_2(\alpha,0)-\Omega_6(\alpha,0)\right)\left|\mathfrak{F}'(\mu)\right|^q\right)^{\frac{1}{q}}\right]\\
&+\left(\Omega_3(\alpha,0)\right)^{1-\frac{1}{q}}\left[\left(\Omega_7(\alpha,0)\left|\mathfrak{F}'(\mu)\right|^q+\left(\Omega_3(\alpha,0)-\Omega_7(\alpha,0)\right)\left|\mathfrak{F}'(\eta)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\Omega_7(\alpha,0)\left|\mathfrak{F}'(\eta)\right|^q+\left(\Omega_3(\alpha,0)-\Omega_7(\alpha,0)\right)\left|\mathfrak{F}'(\mu)\right|^q\right)^{\frac{1}{q}}\right]\\
&+\left(\Omega_4(\alpha,0)\right)^{1-\frac{1}{q}}\left[\left(\Omega_8(\alpha,0)\left|\mathfrak{F}'(\mu)\right|^q+\left(\Omega_4(\alpha,0)-\Omega_8(\alpha,0)\right)\left|\mathfrak{F}'(\eta)\right|^q\right)^{\frac{1}{q}}\right.
\end{aligned}
$$

$$+\left(\Omega_8(\alpha,0)\left|\mathfrak{F}'(\eta)\right|^q+\left(\Omega_4(\alpha,0)-\Omega_8(\alpha,0)\right)\left|\mathfrak{F}'(\mu)\right|^q\right)^{\frac{1}{q}}\Big]\Big\}.$$

Corollary 2 *Let us consider* $\lambda = 0$ *and* $\alpha = 1$ *in Theorem 6. Then, the following Milne-type inequality holds*:

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathfrak{F}\left(\frac{\eta+3\mu}{4}\right)-\mathfrak{F}\left(\frac{\eta+\mu}{2}\right)+2\mathfrak{F}\left(\frac{3\eta+\mu}{4}\right)\right]-\frac{1}{\mu-\eta}\int_\eta^\mu\mathfrak{F}(\delta)\,d\delta\right|\\&\quad\le(\mu-\eta)\left\{\frac{1}{32}\left[\left(\frac{|\mathfrak{F}'(\mu)|^q+5|\mathfrak{F}'(\eta)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{|\mathfrak{F}'(\eta)|^q+5|\mathfrak{F}'(\mu)|^q}{6}\right)^{\frac{1}{q}}\right]\right.\\&\qquad\left.+\frac{7}{96}\left[\left(\frac{5|\mathfrak{F}'(\mu)|^q+9|\mathfrak{F}'(\eta)|^q}{14}\right)^{\frac{1}{q}}+\left(\frac{5|\mathfrak{F}'(\eta)|^q+9|\mathfrak{F}'(\mu)|^q}{14}\right)^{\frac{1}{q}}\right]\right\}.\end{aligned}$$

This inequality helps us find an error bound of Milne's rule.

3 Summary and concluding remarks

In the current investigation, several new versions of Milne-type inequalities were presented with the help of the differentiable convex mappings by utilizing tempered fractional integrals. What is more, Milne-type inequalities were established by taking advantage of the convexity, as well as Hölder and the power mean inequalities. Also, previous and new results were presented by using special cases of the obtained theorems.

In future work, the ideas and techniques via our results related to Milne-type inequalities by tempered fractional integrals may open new ways for researchers in this field. Moreover, readers can generalize our results by utilizing a different convex function classes or another type of fractional integral operator. Moreover, one can obtain Milne-type inequalities by tempered fractional integrals for convex functions by utilizing quantum integrals.

Funding
There is no funding.

Data Availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
The research contributions were as follows: F.H. and H.B. jointly contributed to the research conception and design, and F.H. was responsible for drafting the introduction and methodology sections, while H.B. focused on data collection and analysis. H.K. and U.B. collaborated on the literature review and results interpretation, with H.K. firstly responsible for the literature review and U.B. for results interpretation. H.K. and F.H. worked together on the discussion section, and U.B. and H.B. contributed valuable insights and revisions throughout the manuscript. All authors read and approved the final version of the manuscript.

Received: 29 November 2023 Accepted: 30 December 2023 Published online: 18 January 2024

References

1. Adil Khan, M., Iqbal, A., Suleman, M., Chu, Y.M.: Hermite–Hadamard type inequalities for fractional integrals via Green's function. J. Inequal. Appl. **2018**(1), Article ID 161 (2018)
2. Ali, M.A., Zhang, Z., Fečkan, M.: On some error bounds for Milne's formula in fractional calculus. Mathematics **11**(1), 146 (2023)
3. Alomari, M.: New error estimations for the Milne's quadrature formula in terms of at most first derivatives. Konuralp J. Math. **1**(1), 17–23 (2013)
4. Booth, A.D.: Numerical Methods, 3rd edn. Butterworths, California (1966)

5. Budak, H., Hezenci, F., Kara, H.: On generalized Ostrowski, Simpson and Trapezoidal type inequalities for co-ordinated convex functions via generalized fractional integrals. Adv. Differ. Equ. **2021**, Article ID 312 (2021)
6. Budak, H., Kösem, P., Kara, H.: On new Milne-type inequalities for fractional integrals. J. Inequal. Appl. **2023**, Article ID 10 (2023)
7. Buschman, R.G.: Decomposition of an integral operator by use of Mikusiński calculus. SIAM J. Math. Anal. **3**(1), 83–85 (1972)
8. Davis, P.J., Rabinowitz, P.: Methods of Numerical Integration. Academic Press, New York (2007)
9. Djenaoui, M., Meftah, B.: Milne type inequalities for differentiable s-convex functions. Honam Math. J. **44**(3), 325–338 (2022)
10. Gorenflo, R., Mainardi, F.: Fractional calculus. In: Fractals and Fractional Calculus in Continuum Mechanics, pp. 223–276. Springer, Vienna (1997)
11. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations, vol. 204. Elsevier, Amsterdam (2006)
12. Li, C., Deng, W., Zhao, L.: Well-posedness and numerical algorithm for the tempered fractional ordinary differential equations. Discrete Contin. Dyn. Syst., Ser. B **24**, 1989–2015 (2015)
13. Meerschaert, M.M., Sikorskii, A.: Stochastic models for fractional calculus. In: Stochastic Models for Fractional Calculus, vol. 43. de Gruyter, Berlin (2019)
14. Mohammed, P.O., Sarikaya, M.Z., Baleanu, D.: On the generalized Hermite–Hadamard inequalities via the tempered fractional integrals. Symmetry **12**(4), 595 (2020)
15. Peng, C., Zhou, C., Du, T.S.: Riemann–Liouville fractional Simpson's inequalities through generalized (m,h_1,h_2)-preinvexity. Ital. J. Pure Appl. Math. **38**, 345–367 (2017)
16. Sabzikar, F., Meerschaert, M.M., Chen, J.: Tempered fractional calculus. J. Comput. Phys. **293**, 14–28 (2015)
17. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives (Vol. 1). Gordon & Breach, Yverdon-les-Bains (1993)
18. Sarikaya, M.Z., Budak, H.: Some Hermite–Hadamard type integral inequalities for twice differentiable mappings via fractional integrals. Facta Univ., Ser. Math. Inform. **29**(4), 371–384 (2015)
19. Sarikaya, M.Z., Set, E., Yaldiz, H., Başak, N.: Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. Math. Comput. Model. **57**(9–10), 2403–2407 (2013)
20. Srivastava, H.M., Buschman, R.G.: Convolution Integral Equations, with Special Function Kernels. Wiley, New York (1977)
21. You, X., Hezenci, F., Budak, H., Kara, H.: New Simpson-type inequalities for twice differentiable functions via generalized fractional integrals. AIMS Math. **7**(3), 3959–3971 (2022)

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Submit your manuscript to a SpringerOpen journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com