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# Existence and optimal controls of non-autonomous for impulsive evolution equation without Lipschitz assumption

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**Abstract**

In this paper, we investigate the existence of mild solutions as well as optimal controls for non-autonomous impulsive evolution equations with nonlocal conditions. Using the Schauder's fixed-point theorem as well as the theory of evolution family, we prove the existence of mild solutions for the concerned problem. Furthermore, without the Lipschitz continuity of the nonlinear term, the optimal control result is derived by setting up minimizing sequences twice. An example is given of the application of the results.

**Keywords:** Impulsive evolution equation; Optimal controls; Nonlocal conditions; Resolvent operator; Schauder's fixed point theorem

## 1 Introduction

In this paper, we consider the existence of mild solutions as well as optimal control of the following nonautonomous Volterra-type impulsive evolution equation with nonlocal conditions

$$\begin{cases} u'(t) - A(t)u(t) = Bv(t) + f(t, u(h(t)), Fu(t)), & t \in J := [0, a], t \neq t_k, \\ \Delta u(t_k) = u(t_k^+) - u(t_k^-) = I_k(u(t_k)), & k = 1, 2, \dots, m, \\ u(0) = u_0 + g(u), \end{cases} \tag{1}$$

in Banach space $E$, where $a > 0$ is a constant. $A(t) : D(A) \subseteq E \to E$ is a family of densely defined and closed linear operators generating an evolution system $\{H(t,s) : 0 \leq s \leq t \leq a\}$ on $E$, $D(A)$ is independent of $t$. $B : U \to E$ is a bounded linear operator, and the control function $v(t)$ is given in Banach space $L^2(J; U)$ of admissible control functions, $U$ is also a Banach space, $h(t) \in C(J, J)$, $f : J \times E \times E \to E$ is a continuous nonlinear mapping, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = a$, $I_k, k = 1, 2, \dots, m$ are impulsive functions, $m \in \mathbb{N}$, and $Fu(t) = \int_0^t K(t,s)u(s)\,ds$, $K \in C(D, R^+)$ is a Volterra integral operator with integral kernel $K$, $D = \{(t,s) : 0 \leq s \leq t \leq a\}$. Denote $K^* = \sup_{(t,s)\in D} \int_0^t K(t,s)u(s)\,ds$.

More recently, evolution equations have been used to describe states or processes that change over time in physics, mechanics, or other natural sciences. It is well known that the nonlocal problems are more widely used in application than the classical ones. Byszewski [1] was the first to investigate the nonlocal problems. He obtained the existence and uniqueness of mild solutions for nonlocal differential equations without impulsive conditions. Impulsive differential equations have emerged in many evolutionary processes such as population dynamics, electromagnetic waves, mathematical epidemiology, fed-batch culture in fermentative production, etc. The theory of impulsive differential systems was developed as a result of the presence of sudden changes in the state of the system. These changes of state are caused by transient forces (perturbations). There are state mutations in these applications, and pulses can better express these mutations. Therefore, this application process is more suitable to be represented by impulse differential equations, see for instance [2–6].

Deng [7] pointed out that the nonlocal initial condition can be applied in physics with better effect than the classical initial condition $u(0) = u_0$, and used the nonlocal conditions $u(0) = \sum_{k=1}^{m} c_k u(t_k)$ to describe the diffusion phenomenon of a small amount of gas in a transparent tube. The above findings have encouraged more authors to focus on differential equations with non-local conditions. Differential equations and integral differential equations are often applied to models of processes subject to abrupt changes in a given time. They have a wide range of applications in areas such as control, mechanical, electrical engineering fields and so on. In 2011, Tai [8] studied the exact controllability of fractional impulsive neutral functional integro-differential systems with nonlocal conditions by using the fractional power of operators and Banach contraction mapping theorem. Consequently, the nonlocal condition can be more useful for describing some physical phenomena than the standard initial condition $u(0) = u_0$. The importance of nonlocal conditions was also discussed in [8–12].

As we all know, the problem of controllability plays an important role in the analysis and design of control systems, engineering, deterministic and stochastic control theories. Controllability is a fundamental concept in the modern mathematical control theory. There are various studies on the approximate controllability of systems represented by differential equations, integral differential equations, differential inclusion, neutral-type generalized differential equations, and integer-order impulsive differential equations in Banach spaces. In 2020, Chen et al. [13] discussed that approximate controllability of non-autonomous evolution system with nonlocal conditions and introduced a new Green's function to prove the existence of mild solutions. Arora et al. [14] considered the non-autonomous semi-linear impulsive differential equations with state-dependent delay in 2021. The approximate controllability results of the first-order systems were obtained in a separable reflexive Banach space, which has a uniformly convex dual. They have used the theory of linear evolution systems, properties of the resolvent operator and Schauder's fixed-point theorem to establish sufficient conditions for the approximate controllability of such a system.

Optimal control, which plays a key role in control systems, is one of the fundamental problems in the field of mathematical control theory. Many researchers have shown an increased interest in the solvability and optimal control of systems governed by nonlocal differential equations(see [9, 11, 12, 15–27]).

In 2017, Kumar [28] considered fractional optimal control of a semi-linear system with fixed delay

$$\begin{cases} {}^C D_t^{\alpha} u(t) = Au(t) + B(t)v(t) + f(t, u(t-h)), & t \in ]0, \tau], \\ u(t) = \varphi(t), & t \in [-h, 0], \end{cases}$$

in a reflexive Banach space. $u(t)$ takes its values in a reflexive Banach space $V$; the control function $v(t)$ takes its values in another separable reflexive Banach space $\hat{V}; A : D(A) \subseteq V \to V$ is the infinitesimal generator of a compact $C_0$-semigroup $T(t), t \geq 0$ on $V$; $\{B(t) : t \geq 0\}$ is a family of linear operators from $\hat{V}$ to $V$; the function $f$: $[0, \tau] \times V \to V$ is nonlinear and $\varphi \in C([-h, 0]; V)$. Here, $C([-h, 0]; V)$ be the Banach space of all continuous functions from an interval $[-h, 0]$ to $V$ with the usual supremum norm. He used Weissinger's fixed-point theorem to obtain the existence and uniqueness of a mild solution, and introduced optimal control for the system governed by fractional-order semi-linear equation with fixed delay in the state. However, most of the above problems have been discussed in equations without impulses. It is necessary to discuss the existence of solutions and optimal control for evolution equations with impulses.

In 2017, Liu et al. [11] studied optimal control problems for system-governed semi-linear fractional differential equations with noninstantaneous impulses

$$\begin{cases} {}_0^c D_t^{\alpha} u(t) = Au(t) + f(t, u(t)) + B(t)v(t), & t \in \bigcup_{j=0}^{N} [s_i, t_{i+1}], \\ u(t) = g_i(t, u(t_i^-)), & t \in ]t_i, s_i[, i = 1, 2, \dots, N, \\ u(s_i^+) = u(s_i^-), & i = 1, 2, \dots, N, \\ u(0) = u_0 \in X, \end{cases}$$

where $\alpha \in ]0, 1]$, ${}_0^c D_t^{\alpha}$ denotes Caputo fractional of order $\alpha$ with the lower limit 0, the unbounded linear operator $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a $C_0$-semigroup $\{T(t), t \geq 0\}$ on $X$. $f : [0, T] \times X \times X \to X$ and $g_i : [t_i, s_i] \times X \to X$ are given continuous functions. The impulsive point $t_i$ and connection point $s_i$ are satisfied with the relationship

$$0 = s_0 < t_1 < s_1 < \cdots < s_{N-1} < t_N < s_N < t_{N+1} = T.$$

The symbols $u(s_i^+) := \lim_{\epsilon \to 0^+} u(s_i + \epsilon)$ and $u(s_i^-) := \lim_{\epsilon \to 0^-} u(s_i + \epsilon)$ represent the right and left limits of $u(t)$ at $t = s_i$, respectively. In addition, $B(t)$ is a linear operator from a separable reflexive Banach space $Y$ into $X$. Of course, the control function $v$ is chosen from a suitable control set $V_{ad} \subseteq Y$. They utilized fractional calculus, semigroup theory and fixed-point approach to present the solvability of the corresponding control system using the newly introduced concept of mild solutions. They then provided the result of optimal controls for Lagrange problem under the suitable conditions.

Inspired by all the above, in this article we discuss the existence and optimal controls of non-autonomous for impulsive evolution equation without Lipschitz assumption. We should point out that we do not require Lipschitz assumption for the $g$ term in this paper. Furthermore, we investigate the existence of the evolution equation using Schauder's fixed-point theorem and optimal controls of nonautonomous for impulsive evolution equation by setting up minimizing sequences twice.

## 2 Preliminaries

Let $E$ and $U$ be two real Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|_U$ .We denote by $C(J,E)$ the Banach space of all continuous functions from interval $J$ into $E$ equipped with the supremum norm

$$\|u\|_C = \sup_{t\in J} \|u(t)\|, \quad u \in C(J,E).$$

And by $L^p(E)$ the Banach space of all $E$-valued $p$-order Bochner integrable functions on $J$ equipped with the norm

$$\|f\|_{L^p} = (\int_0^a \|f(t)\|^p \, dt)^{\frac{1}{p}} \quad \text{for } p \geq 1.$$

We put $J_0 = [0,t_1], J_k = (t_k,t_{k+1}], k = 1,\dots,m$. Let $PC(J,E) := \{u : J \to E : u$ is continuous on $J_k$, and the right limit $u(t_k^+)$ exists, $k = 1,2,\dots,m\}$. It is easy to check that $PC(J,E)$ is a Banach space endowed with the norm $\|u\|_{PC} = \sup\{\|u(t)\|, t \in J\}$ and $C(J,E) \subseteq PC(J,E) \subseteq L^1(J,E)$.

Suppose that a family of linear operators $\{A(t) : 0 \leq t \leq a\}$ satisfies the following assumptions:

($A_1$) The family $\{A(t) : 0 \leq t \leq a\}$ is a closed linear operator;

($A_2$) For each $t \in [0,a]$, the resolvent $R(\lambda,A(t)) = (\lambda - A(t))^{-1}$ of linear operator $A(t)$ exists for all $\lambda$ such that $\mathrm{Re}\,\lambda \leq 0$, and there also exists $K > 0$ such that $\|R(\lambda,A(t))\| \leq K/(|\lambda| + 1)$;

($A_3$) There exist $0 < \delta \leq 1$ and $K > 0$ such that $\|(A(t) - A(s))A^{-1}(\tau)\| \leq K|t - s|^\delta$ for all $t$, $s$ and $\tau \in [0,a]$;

($A_4$) For each $t \in [0,a]$ and some $\lambda \in \rho(A(t))$, the resolvent set $R(\lambda,A(t))$ of linear operator $A(t)$ is compact.

Because of these conditions, the family $\{A(t) : 0 \leq t \leq T\}$ generates a unique linear evolution system, called linear evolution family $\{H(t,s) : 0 \leq s \leq t \leq T\}$, and there exists a family of bounded linear operators $\{\Psi(t,\tau) \mid 0 \leq \tau \leq t \leq T\}$ with norm $\|\Psi(t,\tau)\| \leq C|t - \tau|^{\delta-1}$ such that $H(t,s)$ can be represented as

$$H(t,s) = e^{-(t-s)A(t)} + \int_s^t e^{-(t-\tau)A(\tau)}\Psi(\tau,s)\,d\tau, \tag{2}$$

where $e^{-\tau A(t)}$ denotes the analytic semigroup with infinitesimal generator $(-A(t))$.

**Lemma 1** [13] *The family of linear operators* $\{H(t,s) : 0 \leq s \leq t \leq T\}$ *satisfies the following conditions*:

(i) *The mapping* $(t,s) \to H(t,s)x$ *is continuous, for each* $x \in E$, $H(t,s) \in L(E)$ *and* $0 \leq s \leq t \leq T$;

(ii) $H(t,s)H(s,\tau) = H(t,\tau)$ *for* $0 \leq \tau \leq s \leq t \leq T$, *and* $H(t,t) = I$;

(iii) $H(t,s)$ *is a compact operator whenever* $t - s > 0$;

(iv) *There holds, if* $0 < h < 1$, $0 < \gamma < 1$, *and* $t - \tau > h$,

$$\left\|H(t+h,\tau) - H(t,\tau)\right\| \leq \frac{Kh^\gamma}{|t-\tau|^\gamma}.$$

*Condition* ($A_4$) *ensures the generated evolution operator satisfies* (*iii*). *Hence, there exists a constant* $M \geq 1$ *such that*

$$\|H(t,s)\| \leq M \quad \text{for all } 0 \leq s \leq t \leq T. \tag{3}$$

**Definition 1** [13] The evolution family $\{H(t,s) : 0 \leq s \leq t \leq T\}$ is continuous and maps bounded subsets of $E$ into pre-compact subsets of $E$.

**Lemma 2** [18] *For each* $t \in [0,a]$ *and some* $\lambda \in \rho(A(t))$, *if the resolvent* $R(\lambda, A(t))$ *is a compact operator, then* $H(t,s)$ *is a compact operator whenever* $0 \leq s < t \leq a$.

**Lemma 3** [18] *Let* $\{H(t,s), 0 \leq s \leq t \leq a\}$ *be a compact evolution system in* $E$. *Then, for each* $s \in [0,a]$, $t \mapsto H(t,s)$ *is continuous by operator norm for* $t \in (s,a]$.

Let $Y$ be another separable reflexive Banach space, whose norm is also denoted by $\|\cdot\|$, in which the control function $v(t)$ takes its values, $U$ is a bounded subset of $Y$. Denoted by $P_c(Y)$ a class of nonempty closed and convex subsets of $Y$. We suppose that the multivalued map $\omega : J \to P_c(Y)$ is graph measurable, $\omega(\cdot) \subset U$. The admissible control set $V_{ad}$ is defined by

$$V_{ad} = \{v \in L^p(J,U) : v(t) \in \omega(t), \text{ a.e. } t \in J\}, \quad p > 1.$$

Obviously, $V_{ad} \neq \emptyset$ (see [29]) and $V_{ad} \subset L^p(J,Y)(p > 1)$ is bounded, closed and convex.

For any $r > 0$, let $\Omega_r = \{u \in PC(J,E) : \|u(t)\|_{PC} \leq r\}$. We denote by $S(v) := \{u^v \in \Omega_r : u^v$ is the mild solution of the system (1) corresponding to the control $v \in V_{ad}\}$ and $A_{ad} := \{(u^v, v) : v \in V_{ad}, u^v \in S(v)\}$. Hence, $A_{ad}$ is the set of all admissible state-control pair $(u^v, v)$.

**Definition 2** A function $u \in C(J,E)$ is said to be a mild solution of nonlocal problem (1) if for any $v \in L^2(J,U)$, $u(t)$ satisfies the integral equation

$$\begin{aligned} u(t) &= H(t,0)\big(u_0 + g(u)\big) + \int_0^t H(t,s)\big[f\big(s, u\big(h(s)\big), Fu(s)\big) + Bv(s)\big]\,ds \\ &\quad + \sum_{0<t_k<t} H(t,t_k) I_k\big(u(t_k)\big), \quad t \in J. \end{aligned}$$

*Remark* 1 A pair $(u^v, v)$ is said to be feasible for the system (1) if and only if $(u^v, v) \in V_{ad}$.

Our optimal control problem can be transformed into the limited Lagrange problem: the following integral cost functional $J(u^v, v) = \int_0^a L(t, u^v(h(t)), v(t))\,dt$.

Find an admissible state-control pair $(u^0, v^0) \in A_{ad}$ such that for all $v \in V_{ad}$,

$$J\big(u^0, v^0\big) := \inf\big\{J\big(u^v, v\big) : \big(u^v, v\big) \in A_{ad}\big\}, \tag{4}$$

where $u^0 \in \Omega_r$ denotes the mild solution of system (1), corresponding to the control $v^0 \in V_{ad}$. Then a pair $(u^0, v^0) \in A_{ad}$ satisfying (4) is called the optimal state-control pair.

**Lemma 4** [18] *If* $\Omega$ *is a compact subset of a Banach space* $E$, *its convex closure is compact*.

**Lemma 5** [18] *The closure and weak closure of a convex subset in a normed space are the same.*

## 3 Proof of the main results

In this section, using Schauder's fixed-point theorem and the theory of evolution system, we first consider the existence of mild solutions of the non-autonomous impulsive integro-differential evolution equation (1). To this end, we assume the following conditions:

- $(H_1)$ There exists a function $\psi \in L(J,\mathbb{R}^+)$ such that $\|Bv(t)\| \leq \psi(t)$ for all $v \in L^2(J,U)$ and $t \in J$.
- $(H_2)$ The function $f : J \times E \times E \to E$ satisfies:
  (i) for every $t \in J$, the function $f(t,\cdot,\cdot) : E \times E \to E$ is continuous and for each $(u,v) \in E \times E$, the function $f(\cdot,u,v) : J \to E$ is strongly measurable;
  (ii) for any $r > 0$, there exists a function $\varphi \in L^1(J,\mathbb{R}^+)$ such that $\sup\{\|f(t,u,x)\| : \|u\| \leq r, \|x\| \leq K^*r\} \leq \varphi(t)$, for all $u,x \in \Omega_r$, $t \in J$.
- $(H_3)$ The function $g : PC(J,E) \to E$ is supposed to be $g(0) = 0$ and $g(u)$ is continuous, for any $r > 0$, there exists a function $\omega \in L^1(J,\mathbb{R}^+)$ such that $\|g(u)\| \leq \omega(t)$ for all $u \in E$.
- $(H_4)$ The impulses $I_k : E \to E$, $k \in \{1,2,\dots,m\}$ are continuous and satisfy $\|I_k(u)\| \leq d_k$, for all $u \in E$, $k = 1,\dots,m$.

**Theorem 1** *Assume that the evolution family $\{H(t,s) : 0 \leq s \leq t \leq T\}$ generated by $\{A(t) : 0 \leq t \leq a\}$ is compact. If the assumptions $(H_1)$ – $(H_4)$ are satisfied, then the nonlocal problem (1) has at least one mild solution on $J$.*

*Proof* Define operator $Q : C(J,E) \to C(J,E)$, defined by

$$(Qu)(t) = H(t,0)\big(u_0 + g(u)\big) + \int_0^t H(t,s)\big[Bv(s) + f\big(s,u\big(h(s)\big),Fu(s)\big)\big]\,ds + \sum_{0<t_k<t} H(t,t_k)I_k\big(u(t_k)\big), \quad t \in J. \tag{5}$$

By direct calculation, we can see that the operator $Q$ is well defined on $C(J,E)$. From Definition 2, it is easy to see that the mild solution of control system (1) is equivalent to the fixed point of operator $Q$ defined by (5). In the following, we will prove that the operator $Q$ admits a fixed point by applying Schauder's fixed-point theorem. To make our later analysis more transparent, we discuss the proof in four steps.

**Step 1**. We prove that the operator $Q : C(J,E) \to C(J,E)$ is continuous. Let $\{u_n\}_{n=1}^{\infty} \subset C(J,E)$ be a sequence such that $\lim_{n\to+\infty} u_n = u$ in $C(J,E)$. By the continuity of the nonlinear term $f$, we have

$$\lim_{n\to+\infty} \big\|f\big(s,u_n\big(h(s)\big),Fu_n(s)\big) - f\big(s,u\big(h(s)\big),Fu(s)\big)\big\| = 0, \quad \forall s \in J, \tag{6}$$

$$\lim_{n\to+\infty} \big\|g(u_n) - g(u)\big\| = 0. \tag{7}$$

In addition, since

$$\big\|f\big(s,u_n(s),u_n\big(h(s)\big)\big) - f\big(s,u(s),u\big(h(s)\big)\big)\big\| \leq 2\varphi(s), \tag{8}$$

$$\big\|g(u_n) - g(u)\big\| \leq 2\omega(s), \tag{9}$$

and the Lebesgue's dominated convergence theorem follows that

$$
\begin{aligned}
&\|(Qu_n)(t)-(Qu)(t)\| \\
&\quad\le \|H(t,0)(g(u_n)-g(u))\| \\
&\qquad+\int_0^t \|H(t,s)\|\,\|f(s,u_n(h(s)),Fu_n(s))-f(s,u(h(s)),Fu(s))\|\,ds \\
&\qquad+\left\|\sum_{k=1}^{m} H(t,t_k)\big(I_k(u_n(t_k))-I_k(u(t_k))\big)\right\| \\
&\quad\le M\|g(u_n)-g(u)\|+M\int_0^t \|f(s,u_n(h(s)),Fu_n(s))-f(s,u(h(s)),Fu(s))\|\,ds \\
&\quad+M\sum_{k=1}^{m}\|\big(I_k(u_n(t_k))-I_k(u(t_k))\big)\| \\
&\quad\to 0 \quad \text{as } n\to\infty,
\end{aligned}
$$

which means that

$$
\|(Qu_n)-(Qu)\|_C=\sup_{t\in J}\|(Qu_n)(t)-(Qu)(t)\|\to 0 \quad \text{as } n\to\infty. \tag{10}
$$

Therefore, by (10), we know that $Q:C(J,E)\to C(J,E)$ is a continuous operator.

**Step 2**. We will prove that there exists a positive constant $r$ big enough such that $Q$ maps $\Omega_r:=\{u\in C(J,E):\|u(t)\|\le r, t\in J\}$ into itself. In fact, if we choose

$$
r\ge M\left(\|u_0+g(u)\|+N+\sum_{k=1}^{m} d_k\right), \tag{11}
$$

where

$$
N=\int_0^t \big[\varphi(s)+\psi(s)\big]\,ds. \tag{12}
$$

Then for each $u\in\Omega_r$, from (3), (11), and (12) and the conditions $(H_1)-(H_4)$, we have

$$
\begin{aligned}
\|(Qu)t\| &\le \|H(t,0)(u_0+g(u))\|+\int_0^t \|H(t,s)[f(s,u(h(s)),Fu(s))]+Bv(s)\|\,ds \\
&\quad+\left\|\sum_{0<t_k<t} H(t,t_k)I_k(u(t_k))\right\| \\
&\le M\|(u_0+g(u))\|+M\int_0^t \|[f(s,u(h(s)),Fu(s))]+Bv(s)\|\,ds+M\sum_{k=1}^{m} d_k \\
&\le M\|(u_0+g(u))\|+M\int_0^t \big[\varphi(s)+\psi(s)\big]\,ds+M\sum_{k=1}^{m} d_k \\
&= M\left(\|(u_0+g(u)\|+N+\sum_{k=1}^{m} d_k\right)\le r.
\end{aligned}
$$

Therefore, we know $Q:\Omega_r \to \Omega_r$ is a continuous operator.

**Step 3**. We show that $Q:(\Omega_r)$ is an equicontinuous operator of functions in $C(J,E)$. For every $u \in \Omega_r$ and $0 \le t_1 \le t_2 \le a$, by means of (3), (6), and the conditions $(H_1)-(H_4)$, one gets that

$$\begin{aligned}
&\big\|(Qu)(t_2)-(Qu)(t_1)\big\| \\
&\quad = \bigg\| \big[H(t_2,0)-H(t_1,0)\big]\big(u_0+g(u)\big) \\
&\qquad + \int_0^{t_2} H(t_2,s)\big[f\big(s,u\big(h(s)\big),Fu(s)\big)+Bv(s)\big]\,ds \\
&\qquad - \int_0^{t_1} H(t_1,s)\big[f\big(s,u\big(h(s)\big),Fu(s)\big)+Bv(s)\big]\,ds \\
&\qquad + \sum_{0<t_k<t_1}\big[H(t_2,t_k)-H(t_1,t_k)\big]I_k\big(u(t_k)\big) + \sum_{t_1\le t_k\le t_2} H(t_2,t_k)I_k\big(u(t_k)\big)\bigg\| \\
&\quad \le \big\|\big[H(t_2,0)-H(t_1,0)\big]\big(u_0+g(u)\big)\big\| \\
&\qquad + \bigg\|\int_0^{t_1}\big[H(t_2,s)-H(t_1,s)\big]\big[f\big(s,u\big(h(s)\big),Fu(s)\big)+Bv(s)\big]\,ds\bigg\| \\
&\qquad + \bigg\|\int_{t_1}^{t_2} H(t_2,s)\big[f\big(s,u\big(h(s)\big),Fu(s)\big)+Bv(s)\big]\,ds\bigg\| \\
&\qquad + \bigg\|\sum_{0<t_k<t_1}\big[H(t_2,t_k)-H(t_1,t_k)\big]I_k\big(u(t_k)\big)\bigg\| + \bigg\|\sum_{t_1\le t_k\le t_2} H(t_2,t_k)I_k\big(u(t_k)\big)\bigg\| \\
&\quad \le \big\|\big[H(t_2,0)-H(t_1,0)\big]\big\|_{L(E)}\big\|\big(u_0+g(u)\big)\big\| \\
&\qquad + \int_0^{t_1}\big\|\big[H(t_2,s)-H(t_1,s)\big]\big\|_{L(E)}\big\|f\big(s,u\big(h(s)\big),Fu(s)\big)+Bv(s)\big\|\,ds \\
&\qquad + M\int_{t_1}^{t_2}\big\|Bv(s)+f\big(s,u\big(h(s)\big),Fu(s)\big)\big\|\,ds \\
&\qquad + \sum_{0<t_k<t_1}\big\|H(t_2,t_k)-H(t_1,t_k)\big\|_{L(E)}d_k + M\sum_{t_1\le t_k\le t_2} d_k \\
&\quad := I_1+I_2+I_3+I_4+I_5,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \big\|\big[H(t_2,0)-H(t_1,0)\big]\big\|_{L(E)}\big\|\big(u_0+g(u)\big)\big\|; \\
I_2 &= \int_0^{t_1}\big\|\big[H(t_2,s)-H(t_1,s)\big]\big\|_{L(E)}\big\|f\big(s,u\big(h(s)\big),Fu(s)\big)+Bv(s)\big\|\,ds; \\
I_3 &= M\int_{t_1}^{t_2}\big\|Bv(s)+f\big(s,u\big(h(s)\big),Fu(s)\big)\big\|\,ds; \\
I_4 &= \sum_{0<t_k<t_1}\big\|H(t_2,t_k)-H(t_1,t_k)\big\|_{L(E)}d_k; \\
I_5 &= M\sum_{t_1\le t_k\le t_2} d_k.
\end{aligned}$$

Therefore, we only need to check if $I_i$ tend to 0, independently of $u \in \Omega_r$, when $t_2 \to t_1$, $i = 1,2,3,4,5$. Obviously, $I_1 = \|[H(t_2,0) - H(t_1,0)]\|_{L(E)}\|(u_0 + g(u))\| \to 0$, when $t_2 \to t_1$.

For $t_1 \equiv 0$, $0 < t_2 \leq a$, it is easy to see that $I_2 = 0$. If $t_1 = 0$, we choose $\delta \in (0,t_1)$ small enough, by the conditions $(H_1)$ and $(H_2)$, and we have

$$
\begin{aligned}
I_2 \leq{} & \int_0^{t_1-\delta} \bigl\|H(t_2,s) - H(t_1,s)\bigr\| \bigl\|f\bigl(s,u\bigl(h(s)\bigr),Fu(s)\bigr) + Bv(s)\bigr\|\,ds \\
& + \int_{t_1-\delta}^{t_1} \bigl\|H(t_2,s) - H(t_1,s)\bigr\| \bigl\|f\bigl(s,u\bigl(h(s)\bigr),Fu(s)\bigr) + Bv(s)\bigr\|\,ds \\
\leq{} & \int_0^{t_1-\delta} \bigl\|H(t_2,s) - H(t_1,s)\bigr\| \bigl[\varphi(s) + \psi(s)\bigr]\,ds \\
& + \int_{t_1-\delta}^{t_1} \bigl\|H(t_2,s) - H(t_1,s)\bigr\| \bigl[\varphi(s) + \psi(s)\bigr]\,ds \\
\leq{} & \sup_{s\in[0,t_1-\delta]} \bigl\|H(t_2,s) - H(t_1,s)\bigr\|_{L(E)} \int_0^{t_1-\delta} \bigl[\varphi(s) + \psi(s)\bigr]\,ds \\
& + M\int_{t_1-\delta}^{t_1} \varphi(s)\,ds + M\int_{t_1-\delta}^{t_1} \psi(s)\,ds \to 0 \quad \text{as } t_2 - t_1 \to 0 \text{ and } \delta \to 0.
\end{aligned}
$$

For $I_3$, by (3) and the conditions $(H_1)$ and $(H_2)$, we get that

$$
\begin{aligned}
& I_3 \leq M\int_{t_1}^{t_2} \varphi(s)\,ds + M\int_{t_1}^{t_2} \psi(s)\,ds \to 0 \quad \text{as } t_2 - t_1 \to 0, \\
& I_4 = \sum_{0<t_k<t_1} \bigl\|H(t_2,t_k) - H(t_1,t_k)\bigr\|_{L(E)} d_k \to 0 \quad \text{as } t_2 - t_1 \to 0, \\
& I_5 = M\sum_{t_1\leq t_k\leq t_2} d_k \to 0 \quad \text{as } t_2 - t_1 \to 0.
\end{aligned}
$$

Therefore, $\|(Qu)(t_2) - (Qu)(t_1)\| \to 0$ as $t_2 \to t_1$, which means that the operator $Q$ is equicontinuous in $\Omega_r$.

**Step 4**. We demonstrate that the operator $Q : \Omega_r \to \Omega_r$ is compact. To prove this, we first show that $G(t) = \{(Qu)(t) : u \in \Omega_r\}$ is relatively compact in $E$ for every $t \in J$.

For $t = 0$, it is easy to verify that the set $G(t)$ is relatively compact in $E$. Let $t \in (0,a]$ for any $v \in V_{ad}$, $u \in \Omega_r$, and $\varepsilon \in (0,t-s)$, then we define an operator $Q^\varepsilon$ by

$$
\begin{aligned}
\bigl(Q^\varepsilon u\bigr)(t) :={} & H(t,0)\bigl(u_0 + g(u)\bigr) + \int_0^{t-\varepsilon} H(t,s)\bigl[Bv(s) + f\bigl(s,u\bigl(h(s)\bigr),Fu(s)\bigr)\bigr]\,ds \\
& + \sum_{k=1}^{m} H(t,t_k)I_k\bigl(u(t_k)\bigr).
\end{aligned}
$$

It follows from the boundedness of $V_{ad}$ and $(H_1)$ that the set $X_\varepsilon = \{H(t,s)[Bv(s) + f(s,u(h(s)),Fu(s))] : 0 \leq s < t - \varepsilon\}$ is relatively compact owing to the compactness of $H(t,s)(t-s) > 0$. Then $\overline{co}(X_\varepsilon)$ is a compact set depend on Lemma 4. By the mean value theorem of Bochner integrals, we get $(Q^\varepsilon u)(t) \in (t-\varepsilon)\overline{co}(X_\varepsilon)$ for all $t \in J$. Thus, the set $G_\varepsilon(t) = \{(Q^\varepsilon u)(t) : u \in \Omega_r\}$ is relatively compact in $E$ for every $t \in (0,a]$. Moreover, by $(H_1)$

and $(H_2)$, we have

$$\begin{aligned}
&\big\|(Qu)(t)-\big(Q^{\varepsilon}u\big)(t)\big\| \\
&\quad=\bigg\|\int_0^t H(t,s)\big[Bv(s)+f\big(s,u\big(h(s)\big),Fu(s)\big)\big]\,ds \\
&\qquad-\int_0^{t-\varepsilon} H(t,s)\big[Bv(s)+f\big(s,u\big(h(s)\big),Fu(s)\big)\big]\,ds\bigg\| \\
&\quad\le\int_{t-\varepsilon}^t \big\|H(t,s)\big[Bv(s)+f\big(s,u\big(h(s)\big),Fu(s)\big)\big]\big\|\,ds \\
&\quad\le M\int_{t-\varepsilon}^t \big[\varphi(s)+\psi(s)\big]\,ds\to 0 \quad \text{as } \varepsilon\to 0.
\end{aligned}$$

So we have proved that there is a family of relatively compact sets $G_{\varepsilon}(t)$ arbitrarily close to the set $G(t)$. Thus, the set $G(t)$ is relatively compact in $E$ for every $t\in[0,a]$.

Consequently, by the the Ascoli–Arzela theorem, one gets that the operator $Q:\Omega_r\to\Omega_r$ is compact and continuous in $\Omega_r$. Therefore, by Schauder's fixed-point theorem we obtain that the operator $Q$ has at least one fixed point in $\Omega_r$, which is the mild solution of the control system (1) on $J$. This completes the proof of Theorem 1. □

## 4 Existence of optimal controls

In this section, without the Lipschitz continuity of the nonlinear term $f$, this is different from [28]. We use the method of setting up minimizing sequences twice to investigate the existence of optimal state-control pair of the limited Lagrange problem(L) governed by the control system (1). The main idea comes from [12] and [19].

On the cost integrand $L:J\times E\times Y\to\mathbb{R}\cup\{\infty\}$, we suppose that:

(HL) (i) The integrand $L:J\times E\times Y\to\mathbb{R}\cup\{\infty\}$ is Borel measurable;

(ii) The integrand $L(t,\cdot,\cdot)$ is sequentially lower semi-continuous on $E\times Y$ for almost all $t\in J$ and each $u\in E$;

(iii) There exist constants $c\ge 0$, $d>0$, and function $\omega\in L^1(J,R^+)$ such that

$$L(t,u,v)\ge\omega(t)+c\|u\|+d\|v\|^p,\quad \forall u\in E,v\in Y.$$

**Theorem 2** *Assume that the conditions* $(H_1)$–$(H_4)$ *and* (HL) *hold. Then the limited Lagrange problem*$(L)$*, governed by* (1)*, admits at least one optimal state-control pair, that is, there exists an admissible state-control pair* $(u^0,v^0)\in A_{ad}$ *such that* $J(u^0,v^0)=\int_0^a L(t,u^0(t),v^0(t))\le J(u^v,v)\in A_{ad}$.

*Proof* For fixed $v\in V_{ad}$ we define

$$J(v):=\inf_{u^v\in S(v)}J(u^v,v).$$

**Step 1**. We first show that there exists $\widetilde{u}^v\in S(v)$ such that $J(\widetilde{u}^v,v)=J(v)$.

If $S(v)$ contains only finitely many elements the proof is obvious without loss of generality, if $S(v)$ contains infinitely many elements, we can suppose that $J(v)<+\infty$, since it is trivial for the case of $J(v)=+\infty$. Using (HL), we obtain $J(v)>-\infty$. By the definition of infimum, there exists a minimizing sequence $\{u_n^v\}_{n=1}^{\infty}\in S(v)$ satisfying $\lim_{n\to\infty}J(u_n^v,v)=J(v)$.

Next, we prove that the set $\{u_n^\nu\}_{n=1}^\infty$ is relatively compact in $PC(J,E)$ for each $\nu \in V_{ad}$. Similar to the proof of Step 4 in Theorem 1, we can infer that $\{u_n^\nu\}_{n=1}^\infty$ is relatively compact set in $PC(J,E)$. Hence, we can suppose that there is a subsequence of set in $\{u_n^\nu\}_{n=1}^\infty$, notrelabled, and $\widetilde{u}^\nu \in PC(J,E)$ such that $\lim_{n\to\infty} u_n^\nu \to \widetilde{u}^\nu$. We know that

$$\bigl\|H(t,s)\bigl[f\bigl(s,u_n^\nu\bigl(h(s)\bigr),Fu_n^\nu(s)\bigr)+B\nu(s)\bigr]\bigr\| \le M\bigl[\varphi(s)+\psi(s)\bigr] \in L^1(J,E).$$

Therefore, taking $n \to \infty$ and according to the continuity of $f$, $g$, $I_k$ and the Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}\widetilde{u}^\nu(t) = {} & H(t,0)\bigl(u_0+g\bigl(\widetilde{u}^\nu\bigr)\bigr)+\int_0^t H(t,s)\bigl[f\bigl(s,\widetilde{u}^\nu\bigl(h(s)\bigr),F\widetilde{u}^\nu(s)\bigr)+B\nu(s)\bigr]\,ds \\ & + \sum_{0<t_k<t} H(t,t_k)I_k\bigl(\widetilde{u}^\nu(t_k)\bigr), \quad t \in J,\end{aligned}$$

that is, $\widetilde{u}^\nu \in S(\nu)$. Thus, due to $(\mathrm{H_L})$ and Theorem 2.1 of [30], we get

$$\begin{aligned} J(\nu) &= \lim_{n\to\infty}\int_0^a L\bigl(t,\widetilde{u}^\nu\bigl(h(t)\bigr),\nu(t)\bigr)\,dt \\ &\ge \int_0^a L\bigl(t,\widetilde{u}^\nu\bigl(h(t)\bigr),\nu(t)\bigr)\,dt = J\bigl(\widetilde{u}^\nu,\nu\bigr) \\ &\ge J(\nu),\end{aligned}$$

which yields that $J(\widetilde{u}^\nu,\nu) = J(\nu) = \inf_{u\in S(\nu)} J(u,\nu)$.

**Step 2**. We shall seek $\nu^0 \in V_{ad}$ such that $J(\nu^0) = \inf_{\nu\in V_{ad}} J(\nu)$.

It follows from $\{\nu_n\}_{n=1}^\infty \subseteq V_{ad}$ is bounded in $L^p(J,Y)$, $p>1$. Thus, we can extract a subsequence from $\{\nu_n\}_{n=1}^\infty$, not relabeled, weakly converging to some $\nu^0 \in L^p(J,Y)$ as $n \to \infty$. By utilizing the closedness and convexity of $V_{ad}$ due to Lemma 5.

On the other hand, in view of Step 1, for any $n \ge 1$, we can find $\widetilde{u}^{\nu_n} \in S(\nu_n)$ satisfying $J(\widetilde{u}^{\nu_n},\nu_n) = J(\nu_n)$. Since $\widetilde{u}^{\nu_n} \in S(\nu_n)$,

$$\begin{aligned}\widetilde{u}^{\nu_n}(t) = {} & H(t,0)\bigl(u_0+g\bigl(\widetilde{u}^{\nu_n}\bigr)\bigr)+\int_0^t H(t,s)\bigl[f\bigl(s,\widetilde{u}^{\nu_n}\bigl(h(s)\bigr),F\widetilde{u}^{\nu_n}(s)\bigr)+B\nu(s)\bigr]\,ds \\ & + \sum_{0<t_k<t} H(t,t_k)I_k\bigl(\widetilde{u}^{\nu_n}(t_k)\bigr), \quad t \in J. \end{aligned} \tag{13}$$

Hence, similar to Step 1, gives rise to relatively compact of $\{\widetilde{u}^{\nu_n}\}_{n=1}^\infty$ in $PC(J,E)$. We can get $\lim_{n\to\infty} \widetilde{u}^{\nu_n} = u^0$.

Taking the limit in both sides of the equation of (13) as $n \to \infty$, since the continuity of $f$, $g$, $I_k$ and Lebesgue's dominated convergence theorem yield that

$$\begin{aligned}u^0(t) = {} & H(t,0)\bigl(u_0+g\bigl(u^0\bigr)\bigr)+\int_0^t H(t,s)\bigl[f\bigl(s,u^0\bigl(h(s)\bigr),Fu^0(s)\bigr)+B\nu^0(s)\bigr]\,ds \\ & + \sum_{0<t_k<t} H(t,t_k)I_k\bigl(u^0(t_k)\bigr), \quad t \in J,\end{aligned}$$

that is, $(u^0,\nu^0) \in A_{ad}$ is an admissible state-control pair.

Thus, exploiting Theorem 2.1 of [30] yields

$$\begin{aligned}\inf_{v\in V_{ad}} J(v) &= \lim_{n\to\infty} J(v_n) = \lim_{n\to\infty}\int_0^a L\bigl(t,\widetilde{u}^{v_n}\bigl(h(t)\bigr),v_n(t)\bigr)\,dt \\ &\geq \int_0^a L\bigl(t,u^0\bigl(h(t)\bigr),v^0(t)\bigr)\,dt = J\bigl(u^0,v^0\bigr) \\ &\geq \inf_{v\in V_{ad}} J(v).\end{aligned}$$

Therefore, $(u^0,v^0)$ is an optimal state-control pair. This completes the proof of Theorem 2. □

## 5 Example

In this section, we provide an example to illustrate our abstract results.

*Example* 1 Consider the following non-autonomous partial differential equation with nonlocal problem:

$$\begin{cases}\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial^2 x}u(x,t) + b(t)u(x,t) \\ \qquad + \frac{\sqrt{u(x,\sin(t))}\cdot\sin(\int_0^t K(t,s)u(x,s)\,ds)}{1+t^2} + 3v(x,t), & x\in[0,1], t\in J\backslash\{\frac{1}{2}\}, \\ u(0,t) = u(1,t) = 0, & t\in J\backslash\{\frac{1}{2}\}, \\ u(x,0) = u_0(x) + \sin[u(x,t)], & x\in[0,1], \\ \Delta u(x,\frac{1}{2}) = \frac{|u(x,\frac{1}{2})|}{2+|u(x,\frac{1}{2})|},\end{cases} \tag{14}$$

where $J := [0,a]$, $a > \frac{1}{2}$, $b(t) : J \to \mathbb{R}$ is a continuously differentiable function and satisfies

$$b_{\min} := \min_{t\in[0,1]} b(t) < 1, \tag{15}$$

and $v \in L^2(J,L^2(0,1;\mathbb{R}))$. Let $E = L^2(0,1;\mathbb{R})$ with the norm $\|\cdot\|_2$ and inner product $\langle\cdot,\cdot\rangle$. Consider the operator $A$ on $X$ defined by

$$Au := \frac{\partial^2}{\partial x^2}u, \quad u\in D(A),$$

where

$$D(A) := \bigl\{u\in L^2(0,1;\mathbb{R}), u''\in L^2(0,1;\mathbb{R}), u(0) = u(1) = 0\bigr\}.$$

Then it is easy to check that $A(t)$ generates an evolution system $H(t,s) : 0\leq s\leq t\leq a$ in E and there exists a constant $M > 0$ such that $\|H(t,s)\|\leq M$. Then the assumption $(H_1)$ is satisfied. Let

$$\mathcal{J}(u,v) = \int_0^1\int_0^a \bigl|u(x,t)\bigr|^2\,dx\,dt + \int_0^1\int_0^a \bigl|v(x,t)\bigr|^2\,dx\,dt.$$

The family $\{A(t) : 0 \leq t \leq a\}$ generates a strongly continuous evolution family $\{H(t,s) : 0 \leq s \leq t \leq a\}$ defined by

$$H(t,s)u = \sum_{n=1}^{\infty} e^{-(\int_s^t a(\tau)\,d\tau + n^2(t-s))} \langle u, v_n \rangle v_n, \quad 0 \leq s \leq t \leq 1, u \in E. \tag{16}$$

A direct calculation gives

$$\|H(t,s)\|_{L(E)} \leq e^{-(1+b_{\min})(t-s)}, \quad 0 \leq s \leq t \leq 1.$$

(15) and (16) mean that

$$M := \sup_{0 \leq s \leq t \leq a} \|H(t,s)\|_{L(E)} = 1$$

(see [13]).

The cost function is

$$\mathcal{J}(u^v, v) = \int_0^a \left( \|u^v(t)\|_E^2 + \|v(t)\|_Y^2 \right) dt,$$

where $Y := E = L^2[0,1]$.

For any $t \in [0,a]$, we define

$$\begin{aligned}
&u(t)(x) = u(x,t);\\
&f\big(t, u\big(h(t), Fu(t)\big) = \frac{\sqrt{u(x,\sin(t))} \cdot \sin(\int_0^t K(t,s)u(x,s)\,ds)}{1+t^2};\\
&g\big(u(t)\big)(x) = \sin\big[u(x,t)\big];\\
&Bv(t)(x) = 3v(x,t);\\
&I_k u\left(x, \frac{1}{2}\right) = \frac{|u(x,\frac{1}{2})|}{2 + |u(x,\frac{1}{2})|}.
\end{aligned}$$

For any $r > 0$, let $\Omega_r := \{u \in PC(J,X) : \|u(t)\|_{PC} \leq r, t \in J\}$. For any $u \in \Omega_r$ and $t \in J$, we have

$$\begin{aligned}
&\big\|f\big(t, u\big(h(t)\big), Fu(t)\big)\big\| = \left\| \frac{\sqrt{u(x,\sin(t))} \cdot \sin(\int_0^t K(t,s)u(x,s)\,ds)}{1+t^2} \right\| \leq \sqrt{r};\\
&\left\| I_k u\left(x, \frac{1}{2}\right) \right\| \leq \frac{r(2-r)}{4-r^2};\\
&\big\|g\big(u(t)\big)(x)\big\| = \big\|\sin\big[u(x,t)\big]\big\| \leq 1.
\end{aligned}$$

From the definition of nonlinear term $f$ and bounded linear operator $A$ combined with the above discussion, we can easily verify that the assumptions $(H_1)$ – $(H_4)$ are satisfied with $\psi(t) = 3v(t)$.

Therefore, the non-autonomous partial differential equation (14) is equivalent to the evolution equation (1). According to Theorem 1, we know that (14) has at least one mild solution $u \in [C(0,1) \times (0,a)]$. By Theorem 2, if the condition (HL) is satisfied and its corresponding limited Lagrange problem admits at least one optimal state-control pair.

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**Data Availability**
No datasets were generated or analysed during the current study.

## Declarations

**Competing interests**
The authors declare no competing interests.

**Author contributions**
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