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Three solutions for fractional elliptic systems involving ψ-Hilfer operator

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This work is in memory of the father of the first author: Mohamed Elhadi Guefaifia: 26 August1947 – 06 April 2022.

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Abstract
In this paper, using variational methods introduced in the previous study on fractional elliptic systems, we prove the existence of at least three weak solutions for an elliptic nonlinear system with a p-Laplacian ψ-Hilfer operator.

Mathematics Subject Classification: 35J60; 35B30; 35B40

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1 Introduction

Recently, fractional differential equation modeling has led to significant development in several fields due to the important results obtained, see [6, 13], as well as some basic theory of fractional differential equations have been given in [17] This is due to the fact that fractional differential equations have several applications in many models, for example in physics, engineering [11], mechanics, and medicine [14], which has led to great interest in these equations from a mathematical viewpoint, see for example [8, 9]. The authors in [13] introduced the ψ-Hilfer fractional operator with several examples. Also in reference [15], where the space $\mathbb{H}_p^{\alpha,\beta,\psi}([0,T],\mathbb{R})$ was constructed, which allows the study of many fractional differential equations involving the ψ-Hilfer fractional operator.

In [16] Sousa, J.V.C et .al, they discussed the existence and nonexistence of weak solutions to a nonlinear problem with a fractional p-Laplacian operator problem

$$\begin{cases} {}^{\mathbb{H}}\mathbb{D}_T^{\alpha,\beta,\psi}(|\mathbb{D}_{0^+}^{\alpha,\beta,\psi}\xi(t)|^{p-2}\mathbb{D}_{0^+}^{\alpha,\beta,\psi}\xi(t)) = \lambda|\xi(t)|^{p-2}\xi(t) + b(x)|\xi(t)|^{q-2}\xi(t), \\ I_{0^+}^{\beta(\beta-1),\psi}\xi(0) = I_T^{\beta(\beta-1),\psi}\xi(T) \end{cases} \tag{1.1}$$

where $\frac{1}{p} < \alpha < 1, 0 \le \beta \le 1, 1 < q < p-1 < \infty, b \in L^\infty([0,T])$, and $\lambda > 0$. by using the Nehari manifolds technique and combining with fiber maps. Also, Sousa, J.V.C in [12] attacked the bifurcation from infinity for problem (1.1).

In the reference [10], Ezati and Nyamoradi, using the genus properties of critical-point theory, studied the existence and multiplicity of solutions of the Kirchhoff equation ψ-Hilfer fractional operator p-Laplacian. Also, [3] A class of perturbed partial nonlinear systems is studied. With a Lipschitz condition of order $(p-1)$. The multiplicity of weak solutions is proved by variational method and three critical points theorems. An illustrative example was analyzed in order to highlight the result obtained.

In this research we are interested in studying the nonlinear system equipped with the ψ-Hilfer operator:

$$\begin{cases} {}^{\mathbb{H}}\mathbb{D}_T^{\alpha_i,\beta_i,\psi}(\Phi_p(\mathbb{D}_{0^+}^{\alpha_i,\beta_I,\psi}\xi_i(t))) \\ \quad = \varrho\chi_{\xi_i}(t,\xi_1(t),\xi_2(t),\dots,\xi_n(t)) \\ \qquad + \int_0^T k_{1,i}(t,\tau)g_{1,i}(\xi_i(\tau))\,d\tau + \int_0^t k_{2,i}(t,\tau)g_{2,i}(\xi_i(\tau))\,d\tau, \quad 1 \le i \le n, \\ \xi_i^{p-1}(t) = f_i(t)\xi_i^{p-1}(t) + \int_0^T k_{1,i}(t,\tau)g_{1,i}(\xi_i(\tau))\,d\tau \\ \qquad\quad + \int_0^t k_{2,i}(t,\tau)g_{2,i}(\xi_i(\tau))\,d\tau, \quad 1 \le i \le n, \text{a.e. } t \in J = [0,T], \\ \xi_i\backslash_{\partial J} = 0, \quad 1 \le i \le n, \end{cases} \tag{1.2}$$

where $\Phi_p(s) = |s|^{p-2}s$, $p > 1$, ϱ is positive parameter, $f_i : J \to \mathbb{R}$ is a continuous function with the maximum norm $\|f_i\|_\infty = \max_{t\in[0,T]} |f_i(t)| = M_i$, and $\chi : J \times \mathbb{R}^n \to \mathbb{R}$ is continuous and continuously differentiable according to ξ_i i.e,

$$\chi(t,\xi_1,\xi_2,\dots,\xi_n) \in C(J)$$

and

$$\chi(t,.,.,\dots,.) \in C^1(\mathbb{R}^n),$$

we assume

$$g_{1,i}, g_{2,i} : \mathbb{R} \to \mathbb{R}$$

are two continuous functions and satisfy the $(p-1)$ Lipschitz conditions, i.e,

$$\left|g_{1,i}(\zeta_1) - g_{1,i}(\zeta_2)\right| \le L_i'|\zeta_1 - \zeta_2|^{p-1} \tag{1.3}$$

and

$$\left|g_{2,i}(\zeta_1) - g_{2,i}(\zeta_2)\right| \le M_i'|\zeta_1 - \zeta_2|^{p-1}, \quad 1 \le i \le n, \tag{1.4}$$

for all $\zeta_1, \zeta_2 \in \mathbb{R}$, where L_i', $M_i' > 0$,

Moreover, the kernels $k_{1,i}$ and $k_{2,i}$, where

$$k_{1,i}(.,.), k_{2,i}(.,.) \in \bigl(C(J), J\bigr), \tag{1.5}$$

are bounded by the positive constants L_i and M_i, respectively. We know χ_s the partial derivative of χ with respect to s.

Motivated by the above works, applying the well-known three critical point theory of Bonanno and Marano [1]. We prove the existence of at least three different weak solutions of the nonlinear elliptic system (1.2).

Our paper is organized as follows: In Sect. 2, we present some definitions of fractional space and its properties. In the last section, we prove our results presented in Theorem 2.

2 Mathematical background

In this section, we present some preliminaries and lemmas that are useful for the proof of the main results.

Definition 2.1 [7] Let $\frac{1}{p} < \alpha_i \leq 1$, $0 \leq \beta_i \leq 1$ for $1 \leq i \leq n$, and $1 < p < \infty$. The ψ-fractional space $\mathbb{H}_p^{\alpha_i,\beta_i,\psi}$ is defined by the closure of $\overline{C_0^\infty(J,\mathbb{R})}^{\|\cdot\|_{\mathbb{H}_p^{\alpha_i,\beta_i,\psi}}}$, with respect to the following norm

$$\|\xi\|_{\mathbb{H}_p^{\alpha_i,\beta_i,\psi}} = \left(\|\xi\|_{L_\psi^p}^p + \left\|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha_i,\beta_i,\psi}\xi\right\|_{L_\psi^{p_i}}^p\right) \tag{2.1}$$

for all $\xi \in \mathbb{H}_p^{\alpha_i,\beta_i,\psi}$, $1 \leq i \leq n$.

Lemma 2.1 [7] *If $0 < \alpha_i \leq 1$, $0 \leq \beta_i \leq 1$ for $1 \leq i \leq n$, and $1 < p < \infty$. For all $\xi \in \mathbb{H}_p^{\alpha_i,\beta_i,\psi}(J,\mathbb{R})$, we have*

$$\|\xi\|_{L_\psi^p} \leq \frac{(\psi(T)-\psi(0))}{\Gamma(\alpha_i+1)}\left\|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha_i,\beta_i,\psi}\xi\right\|_{L_\psi^p}. \tag{2.2}$$

Moreover, if $\alpha_i > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|\xi\|_{\infty,\psi} \leq \frac{(\psi(T)-\psi(0))^{\alpha_i-\frac{1}{p}}}{\Gamma(\alpha_i)((\alpha_i-1)q+1)^{\frac{1}{q}}}\left\|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha_i,\beta_i,\psi}\xi\right\|_{L_\psi^p}, \tag{2.3}$$

where $\|\xi\|_{\infty,\psi} = \sup_{t\in J}|\xi(t)|$.

From the Inequality (2.3), we also have

$$\|\xi\|_{\infty,\psi} \leq \frac{(\psi(T)-\psi(0))^{\alpha_i-\frac{1}{p}}}{\Gamma(\alpha_i)((\alpha_i-1)q+1)^{\frac{1}{q}}}\left\|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha_i,\beta_i,\psi}\xi\right\|_{\mathbb{H}_p^{\alpha_i,\beta_i,\psi}}$$

Remark 1 The defined norm in (2.1) is equivalent to

$$\|\xi\|_{\alpha_{i,\beta_i}} = \|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha_i,\beta_i,\psi}\xi\|_{L_\psi^p}, \quad \text{for all } \xi \in \mathbb{H}_p^{\alpha_i,\beta_i,\psi}, 1 \leq i \leq n. \tag{2.4}$$

Proposition 2.2 [16] *Let $0 < \alpha_i \leq 1$, $0 \leq \beta_i \leq 1$ for $1 \leq i \leq n$, and $1 < p < \infty$. Assume that $\alpha_i > \frac{1}{p}$ and the sequence $\{\xi_k\}$ converges weakly to ξ in $\mathbb{H}_p^{\alpha_i,\beta_i,\psi}(J,\mathbb{R})$, i.e., $\xi_k \rightharpoonup \xi$ in $C(J,\mathbb{R})$, i.e., $\|\xi_k - \xi\|_\infty \to 0$ as $k \to \infty$.*

Proposition 2.3 [16] *The spaces $\mathbb{H}_p^{\alpha_i,\beta_i,\psi}$, $1 \leq i \leq n$ is compactly embedded in $C(J,\mathbb{R})$.*

Proposition 2.4 [16] *Let $0 < \alpha_i \leq 1$, $0 \leq \beta_i \leq 1$ for $1 \leq i \leq n$, and $1 < p < \infty$. The fractional space $\mathbb{H}_p^{\alpha_i,\beta_i,\psi}$, $1 \leq i \leq$ is a reflexive and separable Banach spaces.*

In this paper, we consider $E = \mathbb{H}_p^{\alpha_1,\beta_1,\psi}(J,\mathbb{R}) \times \cdots \times \mathbb{H}_p^{\alpha_n,\beta_n,\psi}(J,\mathbb{R})$ equipped with the norm

$$\|\xi\|_E = \sum_{i=1}^n \|\xi_i\|_{\alpha_{i,\beta_i}}, \quad \xi_i \in \mathbb{H}_p^{\alpha_i,\beta_i,\psi}, \xi = (\xi_1,\xi_2,\ldots,\xi_n) \in E, \tag{2.5}$$

Definition 2.2 We call $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in E$ a weak solution to the nonlinear system (1.2) if the following relationship holds

$$\begin{aligned}
&\int_0^T \sum_{i=1}^n \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha_i,\beta_i,\psi}\xi_i(t)\right|^{p-2}{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha_i,\beta_i,\psi}\xi_i(t).{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha_i,\beta_i,\psi}\nu_i(t)\,dt \\
&\quad - \sum_{i=1}^n \int_0^T \int_0^T k_{1,i}(t,\tau)g_{1,i}\big(\xi_i(\tau)\big)\nu_i(t)\,d\tau\,dt \\
&\quad - \sum_{i=1}^n \int_0^T \int_0^t k_{2,i}(t,\tau)g_{2,i}\big(\xi_i(\tau)\big)\nu_i(t)\,d\tau\,dt \\
&\quad = \varrho \int_0^T \sum_{i=1}^n \chi_{\xi_i}\big(t,\xi_1(t),\xi_2(t),\dots,\xi_n(t)\big)\nu_i(t)\,dt,
\end{aligned} \tag{2.6}$$

for all $\nu = (\nu_1, \nu_2, \dots, \nu_n) \in E$

Definition 2.3 Define the operator $\mathcal{G}_i : \mathbb{H}_p^{\alpha_i,\beta_i,\psi} \to \mathbb{H}_p^{\alpha_i,\beta_i,\psi}$ as

$$\begin{aligned}
\mathcal{G}_i(\xi_i) &= \frac{1}{p}\int_0^T k_{1,i}(t,\tau)g_{1,i}\big(\xi_i(\tau)\big)\xi_i(t)\,d\tau \\
&\quad + \frac{1}{p}\int_0^t k_{2,i}(t,\tau)g_{2,i}\big(\xi_i(\tau)\big)\xi_i(t)\,d\tau, \quad 1 \le i \le n, \text{and } t \in J.
\end{aligned} \tag{2.7}$$

On the other hand, from the System (1.2), it can be written

$$\begin{aligned}
\big(\xi_i(t) + \theta\nu_i(t)\big)^p &= f_i(t)\big(\xi_i(t) + \theta\nu_i(t)\big)^p \\
&\quad + \int_0^T k_{1,i}(t,\tau)g_{1,i}\big(\xi_i(\tau) + \theta\nu_i(\tau)\big)\big(\xi_i(t) + \theta\nu_i(t)\big)\,d\tau \\
&\quad + \int_0^T k_{2,i}(t,\tau)g_{2,i}\big(\xi_i(\tau) + \theta\nu_i(\tau)\big)\big(\xi_i(t) + \theta\nu_i(t)\big)\,d\tau.
\end{aligned} \tag{2.8}$$

By direct calculation of the derivative of $\mathcal{G}_i$, we obtain

$$\begin{aligned}
&\mathcal{G}_i'\big(\xi_i(t)\big)\big(\nu_i(t)\big) \\
&\quad = \frac{d}{d\theta}\bigg\{\frac{1}{p}\int_0^T k_{1,i}(t,\tau)g_{1,i}\big(\xi_i(\tau) + \theta\nu_i(\tau)\big)\big(\xi_i(t) + \theta\nu_i(t)\big)\,d\tau \\
&\quad\quad + \frac{1}{p}\int_0^T k_{2,i}(t,\tau)g_{2,i}\big(\xi_i(\tau) + \theta\nu_i(\tau)\big)\big(\xi_i(t) + \theta\nu_i(t)\big)\,d\tau\bigg\}_{\theta=0} \\
&\quad = \frac{1}{p}\frac{d}{d\theta}\big\{\big(\xi_i(t) + \theta\nu_i(t)\big)^p - f_i(t)\big(\xi_i(t) + \theta\nu_i(t)\big)^p\big\}_{\theta=0} \\
&\quad = \xi_i^{p-1}(t)\nu_i(t) - f_i(t)\xi_i^{p-1}(t)\nu_i(t) \\
&\quad = \int_0^T k_{1,i}(t,\tau)g_{1,i}\big(\xi_i(\tau)\big)\nu_i(t)\,d\tau + \int_0^t k_{2,i}(t,\tau)g_{2,i}\big(\xi_i(\tau)\big)\nu_i(t)\,d\tau.
\end{aligned} \tag{2.9}$$

The following theorem, taken from [1], is the basic principle to prove our results

Theorem 2.5 ([1], *Theorem* 3.6) *Let E be a reflexive real Banach space; $\mathcal{J}: E \to \mathbb{R}$ be a coercive, continuously Gateaux differentiable and sequentially weakly lower semicontinuous functional whose Gateaux derivative admits a continuous inverse on E^*. Moreover, suppose that $\mathcal{E}: E \to \mathbb{R}$ be a sequentially weakly upper semicontinuous and continuously Gateaux differentiable functional in which its Gateaux derivative is compact such that*

$$\mathcal{J}(0) = \mathcal{E}(0) = 0.$$

We suppose that there exist $r \in \mathbb{R}$ and $\xi^* \in E$ with $0 < r < \mathcal{J}(\xi^*)$, which fulfills

(1) $\sup_{\xi \in \mathcal{J}^{-1}(]-\infty, r])} \mathcal{E}(\xi) < r \frac{\mathcal{E}(\xi^*)}{\mathcal{J}(\xi^*)}$;

(2) *For each $\varrho \in \Lambda_\varrho = (\frac{\mathcal{J}(\xi^*)}{\mathcal{E}(\xi^*)}, \frac{r}{\sup_{\xi \in \mathcal{J}^{-1}(]-\infty, r])} \mathcal{E}(\xi)})$, the functional $\mathcal{J} - \varrho\mathcal{E}$ is coercive.*

Then, for any $\varrho \in \Lambda_\varrho$, the functional $\mathcal{J} - \varrho\mathcal{E}$ admits at least three different critical points in E.

To prove the existence of at least three solutions for the nonlinear system (1.2), we assume the following

$$\theta_i = \max\left\{L_i L_i', M_i M_i'\right\},$$

$$\sigma = \min_{1 \le i \le n} \{\sigma_i\}, \quad \sigma_i = 1 - \frac{2\theta_i T^2 [\psi(T) - \psi(0)]^{p\alpha_i - 1}}{(\Gamma(\alpha_i))^p((\alpha_i - 1)q + 1)^{\frac{p}{q}}},$$

$$S = \max_{1 \le i \le n} \left\{ \frac{(\psi(T) - \psi(0))^{p\alpha_i - 1}}{(\Gamma(\alpha_i))^p((\alpha_i - 1)q + 1)^{\frac{p}{q}} - 2\theta_i T^2(\psi(T) - \psi(0))^{p\alpha_i - 1}} \right\}, \tag{2.10}$$

$$\Omega(c) = \left\{ \eta = (\eta_1, \eta_2 \dots, \eta_n) \in \mathbb{R}^n : \frac{1}{p} \sum_{i=1}^n \eta_i^p \le c \right\}, \quad c > 0 \tag{2.11}$$

and

$$k = \max_{1 \le i \le n} \left\{ \frac{(\psi(T) - \psi(0))^{p\alpha_i}}{\sigma(\Gamma(\alpha_i + 1))^p} \right\}. \tag{2.12}$$

3 Main result

We now present the main results

Theorem 3.1 *We consider $\chi: J \times \mathbb{R}^n \to \mathbb{R}$ to be a function that satisfies*

$$\chi(., \xi_1, \xi_2, \dots, \xi_n) \in C(J), \qquad \chi(t, ., \dots, .) \in C^1\left(\mathbb{R}^n\right)$$

and

$$\chi(t, 0, \dots, 0) = 0, \quad \textit{for all } t \in J.$$

Fix

$$\varrho_1 = \frac{\sum_{i=1}^n (\|z_i\|_{\alpha_i}^p - p \int_0^T \mathcal{G}_i(z_i(t))\, dt)}{p \int_0^T \chi(t, z_1(t), \dots, z_n(t))\, dt}$$

and

$$\varrho_2 = \frac{r}{\int_0^T \sup_{(\eta_1,\eta_2,\dots,\eta_n)\in\Omega(Sr)} \chi(t,\eta_1,\eta_2,\dots,\eta_n)\,dt}.$$

If there exist a positive constant r and a function $z(t) = (z_1(t),\dots,z_n(t))$ such that the following conditions are satisfied

$(H0)$ $\frac{1}{p} < \alpha_i \leq 1$;

$(H1)$ $\frac{2\theta_i T^2[\psi(T)-\psi(0)]^{p\alpha_i-1}}{(\Gamma(\alpha_i))^p((\alpha_i-1)q+1)^{\frac{p}{q}}} < 1$;

$(H2)$ $\sum_{i=1}^n \|z_i\|_{\alpha_i}^p \geq pr + p\sum_{i=1}^n \int_0^T \mathcal{G}_i(z_i(t))\,dt$;

$(H3)$ $\varrho_1 < \varrho_2$;

$(H4)$ $\liminf_{|\eta_i|\to+\infty} \frac{\chi(t,(\eta_1,\eta_2,\dots,\eta_n))}{\sum_{i=1}^n \eta_i^p} < \frac{1}{pk\varrho_2}$.

Then, for any $\varrho \in (\varrho_1,\varrho_2)$, nonlinear system (1.2) admits at least three different weak solutions in E.

Proof We consider that the space $E = \prod_{i=1}^n \mathbb{H}_p^{\alpha_i,\beta_i,\psi}(J,\mathbb{R})$ equipped with the norm $\|\xi\|_E$ defined by (2.5). For any

$$\xi = (\xi_1,\dots,\xi_n) \in E.$$

We define the functionals $\mathcal{J}$ and $\mathcal{E}$: $E \to \mathbb{R}$ by

$$\mathcal{J}(\xi) = \frac{1}{p}\sum_{i=1}^n \|\xi_i\|_{\alpha_i,\beta_i}^p - \sum_{i=1}^n \int_0^T \mathcal{G}_i\bigl(\xi_i(t)\bigr)\,dt, \tag{3.1}$$

and

$$\mathcal{E}(\xi) = \int_0^T \chi\bigl(t,\xi_1(t),\dots,\xi_n(t)\bigr)\,dt. \tag{3.2}$$

These functionals are well-defined Gateaux differentiable:

$$\begin{aligned}
\mathcal{J}'(\xi)(v) = {} & \int_0^T \sum_{i=1}^n \bigl|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha_i,\beta_i,\psi}\xi_i(t)\bigr|^{p-2}\,{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha_i,\beta_i,\psi}\xi_i(t).{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha_i,\beta_i,\psi}v_i(t)\,dt \\
& - \sum_{i=1}^n \int_0^T \int_0^T k_{1,i}(t,\tau)g_{1,i}\bigl(\xi_i(\tau)\bigr)v_i(t)\,d\tau\,dt \\
& - \sum_{i=1}^n \int_0^T \int_0^t k_{2,i}(t,\tau)g_{2,i}\bigl(\xi_i(\tau)\bigr)v_i(t)\,d\tau\,dt,
\end{aligned}$$

and

$$\mathcal{E}'(\xi)(v) = \int_0^T \sum_{i=1}^n \chi_{\xi_i}\bigl(t,\xi_1(t),\dots,\xi_n(t)\bigr)v_i(t)\,dt.$$

for all $v = (v_1,v_2,\dots,v_n) \in E$, where $\mathcal{J}'(\xi)$ and $\mathcal{E}'(\xi) \in E^*$, such that E^* is dual space of E.

Here we prove the conditions imposed on functional $\mathcal{J}$ in Theorem 1.

Since

$$\left|k_{1,i}(t,\tau)\right| \le L_i$$

and

$$\left|k_{2,i}(t,\tau)\right| \le M_i$$

from (1.3), (1.4), and (1.5), we get

$$\begin{aligned}
\mathcal{G}_i(\xi_i) &= \frac{1}{p}\int_0^T k_{1,i}(t,\tau)g_{1,i}\big(\xi_i(\tau)\big)\xi_i(t)\,d\tau \\
&\quad + \frac{1}{p}\int_0^t k_{2,i}(t,\tau)g_{2,i}\big(\xi_i(\tau)\big)\xi_i(t)\,d\tau \\
&\le \frac{1}{p}|\xi_i|^{p-1}TL_iL_i'\|\xi_i\|_\infty + \frac{1}{p}|\xi_i|^{p-1}tM_iM_i'\|\xi_i\|_\infty \\
&\le \frac{1}{p}TL_iL_i'\|\xi_i\|_\infty^p + \frac{1}{p}TM_iM_i'\|\xi_i\|_\infty^p \\
&\le \frac{2}{p}\theta_i T\|\xi_i\|_\infty^p.
\end{aligned} \tag{3.3}$$

Equations (2.3), (2.4), and (3.1) imply

$$\begin{aligned}
\mathcal{J}(\xi) &= \frac{1}{p}\sum_{i=1}^n \|\xi_i\|_{\alpha_i,\beta_i}^p - \sum_{i=1}^n \int_0^T \mathcal{G}_i\big(\xi_i(t)\big)\,dt \\
&\ge \frac{1}{p}\sum_{i=1}^n \|\xi_i\|_{\alpha_i,\beta_i}^p - \frac{2}{p}\sum_{i=1}^n \int_0^T \theta_i T\|\xi_i\|_{\infty,\psi}^p\,dt \\
&\ge \frac{1}{p}\sum_{i=1}^n \|\xi_i\|_{\alpha_i,\beta_i}^p - \frac{2}{p}\sum_{i=1}^n \theta_i T^2\|\xi_i\|_\infty^p \\
&\ge \frac{1}{p}\sum_{i=1}^n \|\xi_i\|_{\alpha_i,\beta_i}^p - \frac{2}{p}\sum_{i=1}^n \theta_i T^2\frac{(\psi(T)-\psi(0))^{p\alpha_i-1}}{(\Gamma(\alpha_i))^p((\alpha_i-1)q+1)^{\frac{p}{q}}}\left\|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha_i,\beta_i,\psi}\xi_i\right\|_{\mathbb{H}_p^{\alpha_i,\beta_i,\psi}}^p \\
&\ge \frac{1}{p}\sum_{i=1}^n \left(1-\frac{2\theta_i T^2[\psi(T)-\psi(0)]^{p\alpha_i-1}}{(\Gamma(\alpha_i))^p((\alpha_i-1)q+1)^{\frac{p}{q}}}\right)\left\|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha_i,\beta_i,\psi}\xi_i\right\|_{\mathbb{H}_p^{\alpha_i,\beta_i,\psi}}^p \\
&\ge \frac{1}{p}\sum_{i=1}^n \sigma_i\left\|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha_i,\beta_i,\psi}\xi_i\right\|_{\mathbb{H}_p^{\alpha_i,\beta_i,\psi}}^p \\
&\ge \frac{\sigma}{p}\|\xi\|_E.
\end{aligned}$$

Since σ is positive, under assumption $(H1)$, then $\lim_{\|\xi\|_X\to+\infty}\mathcal{J}(\xi) = +\infty$, i.e., it is coercive.

Here we prove the conditions imposed on functional $\mathcal{E}$ in Theorem 1.

Since

$$\mathcal{E}' : E \to E^*$$

is a compact operator.

If $\lim_{m\to+\infty}\xi_m \rightharpoonup \xi$ in E, where

$$\xi_m = (\xi_{m,1},\dots,\xi_{m,n}),$$

which ensures the convergence (converges uniformly) of ξ_m to ξ on the interval J. Therefore,

$$\begin{aligned}\lim_{m\to+\infty}\sup \mathcal{E}(\xi_m) &\le \int_0^T \lim_{m\to+\infty}\sup \chi(t,\xi_{m,1},\dots,\xi_{m,n})\,dt\\ &= \int_0^T \chi(t,\xi_1,\dots,\xi_n)\,dt\\ &= \mathcal{E}(\xi).\end{aligned}$$

Hence $\mathcal{E}$ is sequentially weakly upper semi-continuous.

Moreover, $\chi(t,.,\dots,.)\in C^1(\mathbb{R}^n)$, i.e.,

$$\lim_{m\to+\infty}\chi(t,\xi_{m,1},\dots,\xi_{m,n}) = \chi(t,\xi_1,\dots,\xi_n), \quad t\in J$$

According to Lebesgue dominant convergence theorem, $\mathcal{E}'(\xi_m)\to\mathcal{E}'(\xi)$ strongly, so $\mathcal{E}'$ is strongly continuous on E. Then, $\mathcal{E}':E\to E^*$ is a compact operator.

Suppose that $\xi_0(t)=(0,\dots,0)$ and $\xi^*(t)=z(t)$, then

$$\mathcal{J}\big(\xi_0(t)\big) = \mathcal{E}\big(\xi_0(t)\big) = 0$$

From hypothesis $(H2)$ it follows

$$0 < r \le \frac{1}{p}\sum_{i=1}^n \|z_i\|^p_{\alpha_i,\beta_i} - \sum_{i=1}^n\int_0^T \mathcal{G}_i\big(z_i(t)\big)\,dt = \mathcal{J}\big(\xi^*\big)$$

Problems (2.5), (2.6), (2.10), and (2.11) give

$$\begin{aligned}&\mathcal{J}^{-1}(]-\infty,r])\\ &\quad = \big\{\xi\in E:\mathcal{J}(\xi)\le r\big\}\\ &\quad = \left\{\xi\in E:\frac{1}{p}\sum_{i=1}^n\|\xi_i\|^p_{\alpha_i,\beta_i} - \sum_{i=1}^n\int_0^T\mathcal{G}_i\big(\xi_i(t)\big)\,dt\le r\right\}\\ &\quad \subseteq \left\{\xi\in E:\frac{1}{p}\sum_{i=1}^n\|\xi_i\|^p_{\alpha_i,\beta_i} - \frac{2}{p}\sum_{i=1}^n\int_0^T\theta_i T\|\xi_i\|^p_\infty\,dt\right\}\\ &\quad = \left\{\xi\in E:\sum_{i=1}^n\frac{(\Gamma(\alpha_i))^p((\alpha_i-1)q+1)^{\frac{p}{q}}}{p(\psi(T)-\psi(0))^{p\alpha_i-1}}\|\xi_i\|^p_{\infty,\psi} - \frac{2}{p}\sum_{i=1}^n\theta_i T^2\|\xi_i\|^p_{\infty,\psi}\le r\right\}\\ &\quad = \left\{\xi\in E:\sum_{i=1}^n\frac{(\Gamma(\alpha_i))^p((\alpha_i-1)q+1)^{\frac{p}{q}} - 2\theta_i T^2(\psi(T)-\psi(0))^{p\alpha_i-1}}{p(\psi(T)-\psi(0))^{p\alpha_i-1}}\|\xi_i\|^p_{\infty,\psi}\le r\right\}\\ &\quad \subseteq \left\{\xi\in E:\frac{1}{pS}\sum_{i=1}^n\|\xi_i\|^p_{\infty,\psi}\le r\right\}\end{aligned}$$

$$\subseteq \left\{ \xi \in E : \frac{1}{p} \sum_{i=1}^{n} |\xi_i|_{\infty,\psi}^{p} \leq rS, \text{ for all } t \in [0,T] \right\}$$
$$\subseteq \Omega(Sr),$$

which leads to

$$\begin{aligned}
\sup_{\xi \in \mathcal{J}^{-1}(]-\infty,r])} \mathcal{E}(\xi) &= \sup_{\xi \in \mathcal{J}^{-1}(]-\infty,r])} \int_0^T \chi(t,\xi_1,\ldots,\xi_n)\,dt \\
&\leq \sup_{\eta \in \Omega(Sr)} \int_0^T \chi(t,\eta_1,\ldots,\eta_n)\,dt \\
&= \int_0^T \sup_{\eta \in \Omega(Sr)} \chi(t,\eta_1,\ldots,\eta_n)\,dt.
\end{aligned}$$

By $(H3)$, we have

$$\begin{aligned}
\frac{\sup_{\xi \in \mathcal{J}^{-1}(]-\infty,r])} \mathcal{E}(\xi)}{r} &= \frac{\sup_{\xi \in \mathcal{J}^{-1}(]-\infty,r])} \int_0^T \chi(t,\xi_1,\ldots,\xi_n)\,dt}{r} \\
&\leq \frac{\int_0^T \sup_{\eta \in \Omega(Sr)} \chi(t,\eta_1,\ldots,\eta_n)\,dt}{r} \\
&< \frac{p \int_0^T \chi(t,z_1,\ldots,z_n)\,dt}{\sum_{i=1}^{n} (\|z_i\|_{\alpha_i,\beta_I}^{p} - p \int_0^T \mathcal{G}_i(z_i(t))\,dt)} \qquad (3.4) \\
&= \frac{\mathcal{E}(z(t))}{\mathcal{J}(z(t))} \\
&= \frac{\mathcal{E}(\xi^*)}{\mathcal{J}(\xi^*)},
\end{aligned}$$

thus,

$$\sup_{\xi \in \mathcal{J}^{-1}(]-\infty,r])} \mathcal{E}(\xi) < r \frac{\mathcal{E}(\xi^*)}{\mathcal{J}(\xi^*)}.$$

Hence, hypothesis (1) of Theorem 1 is fulfilled.

From assumption $(h4)$, there are constants μ and $\varepsilon \in \mathbb{R}$ that satisfy the following

$$\frac{\mu}{\sigma} < \frac{\int_0^T \sup_{\eta \in \Omega(Sr)} \chi(t,\eta_1,\ldots,\eta_n)\,dt}{r}. \qquad (3.5)$$

Also

$$\forall \eta \in \mathbb{R}^n : \chi(t,\eta_1,\ldots,\eta_n) \leq \frac{\mu}{pk\sigma} \sum_{i=1}^{n} |\eta_i|^p + \varepsilon,$$

for $t \in J$ and a fixed vector

$$\xi = (\xi_1,\xi_2,\ldots,\xi_n) \in E,$$

we have

$$\chi\big(\xi_1(t),\dots,\xi_n(t)\big) \le \frac{\mu}{pk\sigma}\sum_{i=1}^{n}|\xi_i|^p + \varepsilon \tag{3.6}$$

for all $t \in J$. Finally, it remains to check that the functional $\mathcal{J}(\xi) - \varrho\mathcal{E}(\xi)$ is coercive. Assume $\varrho \in \Lambda$, thus fetching into accounts (2.4), (2.12), (3.5), and (3.6), we have

$$\begin{aligned}
\mathcal{J}(\xi) - \varrho\mathcal{E}(\xi) &= \frac{1}{p}\sum_{i=1}^{n}\|\xi_i\|^p_{\alpha_{i,\beta_i}} \\
&\quad - \sum_{i=1}^{n}\int_0^T \mathcal{G}_i\big(\xi_i(t)\big)\,dt - \varrho\int_0^T \chi(t,\xi_1,\xi_2,\dots,\xi_n) \\
&\ge \frac{1}{p}\sum_{i=1}^{n}\|\xi_i\|^p_{\alpha_{i,\beta_i}} - \frac{2}{p}\sum_{i=1}^{n}\theta_i T^2\|\xi_i\|^p_{\alpha_{i,\beta_i}} \\
&\quad - \varrho\int_0^T \chi(t,\xi_1,\xi_2,\dots,\xi_n)\,dt \\
&\ge \frac{1}{p}\sum_{i=1}^{n}\|\xi_i\|^p_{\alpha_{i,\beta_i}} - \frac{2}{p}\sum_{i=1}^{n}\theta_i T^2\|\xi_i\|^p_{\alpha_{i,\beta_i}} - \frac{\varrho\mu}{pk\sigma}\int_0^T\left(\sum_{i=1}^{n}|\xi_i|^p\right)dt - \varrho\varepsilon T \\
&\ge \frac{1}{p}\sum_{i=1}^{n}\|\xi_i\|^p_{\alpha_{i,\beta_i}} - \frac{2}{p}\sum_{i=1}^{n}\theta_i T^2\frac{(\psi(T)-\psi(0))^{p\alpha_i-1}}{(\Gamma(\alpha_i))^p((\alpha_i-1)q+1)^{\frac{p}{q}}}\|\xi_i\|^p_{\alpha_{i,\beta_i}} \\
&\quad - \frac{\varrho\mu}{pk\sigma}\sum_{i=1}^{n}\frac{(\psi(T)-\psi(0))^{p\alpha_i}}{(\Gamma(\alpha_i+1))^p}\|\xi_i\|^p_{\alpha_{i,\beta_i}} - \varrho\varepsilon T \\
&\ge \frac{1}{p}\sum_{i=1}^{n}\sigma_i\|\xi_i\|^p_{\alpha_{i,\beta_i}} - \frac{\varrho\mu}{pk\sigma}\sum_{i=1}^{n}\frac{(\psi(T)-\psi(0))^{p\alpha_i}}{(\Gamma(\alpha_i+1))^p}\|\xi_i\|^p_{\alpha_{i,\beta_i}} - \varrho\varepsilon T \\
&\ge \frac{1}{p}\sum_{i=1}^{n}\sigma\|\xi_i\|^p_{\alpha_{i,\beta_i}} - \frac{\varrho\mu}{p}\sum_{i=1}^{n}\|\xi_i\|^p_{\alpha_{i,\beta_i}} - \varrho\varepsilon T \\
&\ge \frac{1}{p}\left(\sigma - \frac{\mu r}{\int_0^T \sup_{\eta\in\Omega(Sr)}\chi(t,\eta_1,\dots,\eta_n)\,dt}\right)\sum_{i=1}^{n}\|\xi_i\|^p_{\alpha_{i,\beta_i}} - \varrho\varepsilon T,
\end{aligned}$$

from (3.5) the term

$$\left(\sigma - \frac{\mu r}{\int_0^T \sup_{\eta\in\Omega(Sr)}\chi(t,\eta_1,\dots,\eta_n)\,dt}\right)$$

is clearly positive, thus

$$\lim_{\|\xi\|_E\to+\infty}\mathcal{J}(\xi) - \varrho\mathcal{E}(\xi) = +\infty. \tag{3.7}$$

This means, $\mathcal{J} - \varrho\mathcal{E}$ is coercive and thus the hypothesis (2) of Theorem 1 is also established.

Applying Theorem 1, the weak solutions of the nonlinear system (1.2) are exactly the critical points of the equation

$$\mathcal{J}'(\xi) - \varrho\mathcal{E}'(\xi) = 0.$$

Thus, the nonlinear system (1.2) accepts at least three critical points, which are weak solutions in E for $\varrho \in \Lambda_\varrho$, and the proof ends. □

4 Conclusion

In this work, by using variational methods introduced in the previous study on fractional elliptic systems, we prove the existence of at least three weak solutions for an elliptic nonlinear system with a p-Laplacian ψ-Hilfer operator, where we have based on some published works that extend the well-known three critical point theory of Bonanno and Marano [1]. In the next work, we will apply the same methods to the same problem with variable exponent.

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
There is no applicable.

Competing interests
The authors declare no competing interests.

Author contributions
RG, TB: writing original draft, Methodology, Resources, formal analysis, Conceptualization; SB: Corresponding author, review and editing, SB: Writing review and editing; RG: and SB: funding. All authors read and approved the final manuscript.

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