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A necessary and sufficient condition for the existence of global solutions to reaction-diffusion equations on bounded domains

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Abstract

The purpose of this paper is to give a necessary and sufficient condition for the existence and non-existence of global solutions of the following semilinear parabolic equations

$$u_t = \Delta u + \psi(t)f(u), \quad \text{in } \Omega \times (0,\infty),$$

under the mixed boundary condition on a bounded domain Ω. In fact, this has remained an open problem for a few decades, even for the case $f(u) = u^p$. As a matter of fact, we prove:

> there is no global solution for any initial data if and only if
> $$\int_0^\infty \psi(t)\frac{f(\|S(t)u_0\|_\infty)}{\|S(t)u_0\|_\infty}\,dt = \infty$$
> for every nonnegative nontrivial initial data $u_0 \in C_0(\Omega)$.

Here, $(S(t))_{t\geq 0}$ is the heat semigroup with the mixed boundary condition.

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1 Introduction

In his seminal paper [1], Fujita first studied the reaction-diffusion equation

$$u_t = \Delta u + u^p, \quad \text{in } \mathbb{R}^N \times (0,\infty),$$

where $p > 1$, and obtained that

(i) if $1 < p < p^*$, then there is no global solution for any nonnegative and nontrivial initial data,
(ii) if $p > p^*$, then there exists a global solution whenever the nonnegative and nontrivial initial data are sufficiently small,

where $p^* = 1 + \frac{2}{N}$ is called the critical exponent. After his results, researchers obtained that there is no global solution for $p = p^*$ (see [2] for the case $N = 1$ or 2 and [3] for the case $N \geq 3$).

It is easy to see that the critical exponent p^* leads to a necessary and sufficient condition for the existence of the global solutions as above. Therefore, lots of researchers studied the critical exponent for various reaction-diffusion equations to find necessary and sufficient conditions for the existence of the global solutions (see the survey articles [4, 5]).

In this paper, we discuss the existence and nonexistence of the global solutions to the reaction-diffusion equation for a general source term $\psi(t)f(u)$, under mixed boundary condition:

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + \psi(t)f(u(x,t)), & (x,t) \in \Omega \times (0,\infty), \\ B[u](z,t) = 0, & (z,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, ψ is a nonnegative continuous function on $[0,\infty)$, and f is a locally Lipschitz continuous function satisfying that $f(0) = 0$ and $f(s) > 0$ for $s > 0$, and $B[u](z,t) = 0$ stands for the boundary condition

$$\mu(z)u(z,t) + \sigma(z)\frac{\partial u}{\partial n}(z,t) = 0, \quad (z,t) \in \partial\Omega \times (0,\infty).$$

Here, μ and σ are nonnegative continuous functions on $\partial\Omega$ with $\mu + \sigma > 0$ on $\partial\Omega$. Also, u_0 is a nonnegative and nontrivial $C_0(\Omega)$-function satisfying the compatibility condition $B[u_0] = 0$.

In his pioneering paper [6], Meier studied the global existence and nonexistence of the solutions to the reaction-diffusion equations (1) under the Dirichlet boundary condition and obtained the following result:

Theorem 1.1 ([6]) *Assume that $\psi \in C[0,\infty)$ and $f(u) = u^p$ for $p > 1$.*
(i) *If $\limsup_{t\to\infty} \|S(t)u_0\|_\infty^{p-1} \int_0^t \psi(\tau)\,d\tau = \infty$ for every $u_0 \in C_0(\Omega)$, then there is no global solution for any nonnegative and nontrivial initial data.*
(ii) *If $\int_0^\infty \psi(\tau)\|S(\tau)u_0\|_\infty^{p-1}\,d\tau < \infty$ for some $u_0 \in C_0(\Omega)$, then there exists global solution for sufficiently small initial data.*

Here, $(S(t))_{t\geq 0}$ is the heat semigroup with the Dirichlet boundary condition.

Meier gives a sufficient condition for the existence of the global solutions and a sufficient condition for the nonexistence of the global solutions, respectively. However, a necessary and sufficient condition for the existence of the global solutions to the equation (1) has been unknown, even for the case $f(u) = u^p$ and has remained as an open problem for a few decades. To our best knowledge, researches on necessary and sufficient conditions for the global existence of solutions for the reaction-diffusion equations in the current literature consider several specific source terms such as $t^\sigma u^p$, $e^{\beta t}u^p$, etc. (see [6–9]).

Recent researches of the equation (1) have adopted Meier's criterion and give several sufficient conditions for the blow-up solutions and global solutions (for example, see [10–12]). In conclusion, the open problem has faced methodological limitations and there has been no progress in research on necessary and sufficient conditions for the general source term $\psi(t)f(u)$.

From the above point of view, the purpose of this paper is twofold:

(i) to obtain the necessary and sufficient condition for the existence of the global solutions for more general source term $\psi(t)f(u)$.

(ii) to introduce a method, so-called a minorant method, to deal with $f(u)$ in the source term.

Finally, we obtained the following results to see 'completely' whether or not we have global solutions:

Theorem 1.2 *Let f be a convex and locally Liptchitz continuous function and ψ be a nonnegative continuous function. Suppose that f satisfies*

$$\int_1^\infty \frac{ds}{f_m(s)} < \infty, \tag{2}$$

where $f_m(u) := \inf_{0<\alpha<1} \frac{f(\alpha u)}{f(\alpha)}$ for $u \geq 0$. Then the following statements are equivalent.

(i) *there is no global solution u to the equation (1) for any nonnegative and nontrivial initial data u_0.*

(ii)

$$\int_0^\infty \psi(t) e^{\lambda_0 t} f\left(\epsilon e^{-\lambda_0 t}\right) dt = \infty$$

for every $\epsilon > 0$.

(iii)

$$\int_0^\infty \psi(t) \frac{f(\|S(t)u_0\|_\infty)}{\|S(t)u_0\|_\infty} dt = \infty$$

for every nonnegative nontrivial initial data $u_0 \in C_0(\Omega)$.

Here, $(S(t))_{t\geq 0}$ is the heat semigroup with the mixed boundary condition and λ_0 is the first eigenvalue of the Laplace operator Δ under the mixed boundary condition.

Theorem 1.2 is the form of a necessary and sufficient condition for global solutions of the equation (1). Therefore, the open problem mentioned above is solved with more general source term $\psi(t)f(u)$ and general boundary condition. Here, we note that the boundary condition $B[u] = 0$ includes the various boundary value problems such as the Dirichlet boundary problem, Neumann boundary problem, and Robin boundary problems. It is clear that these boundary conditions can express various natural phenomena. We note here that one of the meanings of our result is a unified approach.

In general, the case $p = p^*$ and $p < p^*$ are dealt in a different way and cannot be solved at the same time, when $f(u) = u^p$ (see [1–3, 13]). However, we prove the cases all at once.

As far as authors know, there is no paper which discuss the necessary and sufficient condition (or Fujita's blow-up solutions) on the source term $\psi(t)f(u)$ instead of $\psi(t)u^p$.

To deal with the general source term $\psi(t)f(u)$, we use the minorant method, which were introduced in [14] by authors.

We organized this paper as follows: In Sect. 2, we discuss Meier's criterion. We introduce the minorant method and discuss main results in Sect. 3.

2 Discussion on Meier's conditions

The purpose of this section is to discuss the necessary and sufficient condition for the existence of the global solutions, which was previously unknown and remained as an open problem. Let us deal with sufficient conditions for the blow-up solutions and global solutions to check that these conditions can be a necessary and sufficient condition.

From this point of view, let us discuss Meier's conditions. If the domain Ω is bounded, then it is well-known that $\|S(t)u_0\|_\infty \sim e^{-\lambda_0 t}$ for $t \geq 0$, for every nonnegative and nontrivial initial data $u_0 \in C_0(\Omega)$. Here, λ_0 is the first eigenvalue of the Laplace operator Δ under the mixed boundary condition discussed in [15, 16]:

Lemma 2.1 (See [15, 16]) *There exists $\lambda_0 \geq 0$ and $\phi_0 \in W^{1,2}(\Omega)$ with $\phi_0 > 0$ in Ω and $\partial\Omega \setminus \Gamma$ such that*

$$\begin{cases} -\Delta\phi_0(x) = \lambda_0\phi_0(x), & x \in \Omega, \\ \mu(z)\phi_0(z) + \sigma(z)\frac{\partial\phi_0}{\partial n}(z) = 0, & z \in \partial\Omega. \end{cases}$$

Moreover, λ_0 is given by

$$\lambda_0 := \inf_{w \in \mathcal{A}, w \not\equiv 0} \frac{\int_\Omega |\nabla w|^2\,dx + \int_\Gamma \frac{\mu(z)}{\sigma(z)}|w|^2\,dS}{\int_\Omega |w|^2\,dx},$$

where $\mathcal{A} := \{w \in W^{1,2}(\Omega) : w = 0 \text{ on } \Gamma\}$ with $\Gamma := \{z \in \partial\Omega : \sigma(z) = 0\}$.

Remark 2.2 It is known that the first eigenvalue $\lambda_0 = 0$ if and only if the boundary condition is the Neumann boundary condition. i.e., $\mu \equiv 0$.

Hence, under the Dirichlet boundary condition, Theorem 1.1 can be understood as follows:

- If

$$(C1): \quad \limsup_{t\to\infty} e^{-(p-1)\lambda_0 t}\int_0^t \psi(\tau)\,d\tau = \infty,$$

  then there is no global solution to the equation (1) for any nonnegative and nontrivial initial data.
- If

$$(C2): \quad \int_0^\infty \psi(t)e^{-(p-1)\lambda_0 t}\,dt < \infty$$

  then there exists a global solution to the equation (1) for sufficiently small initial data.

Let us consider the function ψ defined by $\psi(t) := (t+1)^{-\delta}e^{(p-1)\lambda_0 t}$ for $0 \le \delta \le 1$. Then it follows that

$$\begin{aligned}\limsup_{t\to\infty} e^{-(p-1)\lambda_0 t}\int_0^t \psi(\tau)\,d\tau &= \limsup_{t\to\infty} e^{-(p-1)\lambda_0 t}\int_0^t (\tau+1)^{-\delta}e^{(p-1)\lambda_0\tau}\,d\tau \\ &\le \limsup_{t\to\infty} e^{-(p-1)\lambda_0 t}\int_0^t e^{(p-1)\lambda_0\tau}\,d\tau \\ &= \frac{1}{(p-1)\lambda_0} < \infty\end{aligned}$$

and

$$\int_0^\infty \psi(t)e^{-(p-1)\lambda_0 t}\,dt = \int_0^\infty (t+1)^{-\delta}\,dt = \infty.$$

This implies that if the function $\psi(t) := (t+1)^{-\delta}e^{(p-1)\lambda_0 t}$ for $0 \le \delta \le 1$, then we do not know whether the solution exists globally or not.

Now, we are going to check whether the solution exists globally or not, by considering the simple example. Let's consider the functions ψ and f defined by $\psi(t) := (t+1)^{-\frac{1}{2}}e^{\lambda_0 t}$ and $f(u) := u^2$ in the equation (1). Then the equation (1) follows that

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + (t+1)^{-\frac{1}{2}}e^{\lambda_0 t}u^2, & (x,t)\in\Omega\times(0,\infty), \\ u(x,t) = 0, & (x,t)\in\partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x) \ge 0, & x\in\Omega. \end{cases} \tag{3}$$

Now, we consider the eigenfunction ϕ_0 to be $\int_\Omega \phi_0(x)\,dx = 1$, corresponding to the first Dirichlet eigenvalue λ_0. Suppose that the solution u to the equation (3) exists globally. Multiplying the first equation of (3) by ϕ_0 and integrating over Ω, we use Green's theorem and Jensen's inequality to obtain

$$\begin{aligned}&\int_\Omega u_t(x,t)\phi_0(x)\,dx \\ &\quad= \int_\Omega \phi_0(x)\Delta u(x,t)\,dx + (t+1)^{-\frac{1}{2}}e^{\lambda_0 t}\int_\Omega u^2\phi_0(x)\,dx \\ &\quad\ge -\lambda_0\int_\Omega u(x,t)\phi_0(x)\,dx + (t+1)^{-\frac{1}{2}}e^{\lambda_0 t}\left(\int_\Omega u(x,t)\phi_0(x)\,dx\right)^2,\end{aligned}$$

for all $t > 0$. Putting $y(t) := \int_\Omega u(x,t)\phi_0(x)\,dx$, for $t \ge 0$, then $y(t)$ exists for all time t and satisfies the following inequality

$$\begin{cases} y'(t) \ge -\lambda_0 y(t) + (t+1)^{-\frac{1}{2}}e^{\lambda_0 t}y^2(t), & t > 0, \\ y(0) = y_0 := \int_\Omega u_0(x)\phi_0(x)\,dx > 0. \end{cases} \tag{4}$$

Multiplying $e^{\lambda_0 t}$ by the inequality (4), we have

$$\left[e^{\lambda_0 t}y(t)\right]' \ge (t+1)^{-\frac{1}{2}}\left[e^{\lambda_0 t}y(t)\right]^2 \ge 0,$$

for all $t > 0$, which implies that

$$\frac{d}{dt}\left[e^{\lambda_0 t}y(t)\right]^{-1} \leq -(t+1)^{-\frac{1}{2}}$$

for all $t > 0$. Solving the differential inequality, we obtain that

$$y(t) \geq \frac{e^{-\lambda_0 t}}{y_0^{-1} - \int_0^t (\tau+1)^{-\frac{1}{2}}\, d\tau},$$

for all $t > 0$, which leads a contradiction. Hence, the solution u to the equation (3) blows up at finite time.

The above example implies that the condition $(C1)$ is no longer necessary condition for the nonexistence of global solution. In fact, the main part of this paper is focused on the condition $(C2)$ to see whether $(C2)$ is necessary and sufficient condition for the existence of the global solution.

On the other hand, if the function f in the equation (1) has no multiplicative property, we cannot apply Meier's results. For example, let us consider the function $f(u) = \frac{u^2+u}{2}$. Then it is easy to see that $u \leq f(u) \leq u^2$ for $u \geq 1$ and $u^2 \leq f(u) \leq u$ for $0 \leq u \leq 1$. Therefore, we cannot determine the parameter p in Theorem 1.1, in the case of $f(u) = \frac{u^2+u}{2}$. From this point of view, we have to consider a new method, so called the minorant method, to deal with a function f which is not multiplicative. In conclusion, we provide a formula $\frac{f(\|S(t)u_0\|_\infty)}{\|S(t)u_0\|_\infty}$ instead of $\|S(t)u_0\|_\infty^{p-1}$ to give a criterion of the existence of the global solution when the source term is $\psi(t)f(u)$.

3 Main results

In this section, we firstly introduce the minorant function and the majorant function. Next, we prove the main theorem by using the minorant function and majorant function.

First, we discuss multiplicative minorants and majorants of the function f, which will play an important role in this work.

Definition 3.1 For a function f, the minorant function $f_m : [0,\infty) \to [0,\infty)$ and the majorant function $f_M : [0,\infty) \to [0,\infty)$ are defined by

$$f_m(u) := \inf_{0<\alpha<1} \frac{f(\alpha u)}{f(\alpha)}, \quad u \geq 0,$$
$$f_M(u) := \sup_{0<\alpha<1} \frac{f(\alpha u)}{f(\alpha)}, \quad u \geq 0.$$

Then the following properties:

- $f(\alpha)f_m(u) \leq f(\alpha u) \leq f(\alpha)f_M(u)$, $0 < \alpha < 1$, $u > 0$.
- If g and h be functions satisfying that

$$f(\alpha)g(u) \leq f(\alpha u) \leq f(\alpha)h(u), \quad 0 < \alpha < 1, u > 0,$$

then it follows that $g(u) \leq f_m(u)$ and $f_M(u) \leq h(u)$, $u > 0$.

It follows that f_m and f_M are natural to call the multiplicative minorant and majorant of a function f respectively. In fact, the values of f_m and f_M depend strongly on the value of f near zero, since

$$f_m(u) \leq \inf_{0<\alpha<\frac{1}{u}} \frac{f(\alpha u)}{f(\alpha)} \quad \text{and} \quad f_M(u) \geq \sup_{0<\alpha<\frac{1}{u}} \frac{f(\alpha u)}{f(\alpha)},$$

for each $u \geq 1$. Also, if f is convex, then $\frac{f(u)}{u}$ is nondecreasing. Then it is easy to see that the function f, the minorant f_m, and the majorant f_M satisfy the following properties:

(i) $f_m(u) \leq \frac{f(u)}{f(1)} \leq f_M(u)$ for $u \geq 0$.
(ii) $f_m(\frac{1}{\alpha}) \leq \frac{f(1)}{f(\alpha)} \leq f_M(\frac{1}{\alpha})$ for $0 < \alpha \leq 1$.
(iii) $f_m(1) = f_M(1) = 1$.
(iv) $\frac{f_m(u)}{u}$ and $\frac{f_M(u)}{u}$ are nondecreasing in $(0, 1)$.
(v) $f_m(u) \leq u$ and $f_M(u) \leq u$ for $0 < u \leq 1$, since $\frac{f(\alpha u)}{f(\alpha)} = \frac{f(\alpha u)}{\alpha u} \frac{\alpha}{f(\alpha)} u \leq u$.
(vi) $\int_\eta^\infty \frac{ds}{f_M(s)} \leq f(1) \int_\eta^\infty \frac{ds}{f(s)} \leq \int_\eta^\infty \frac{ds}{f_m(s)}$ for $\eta \geq 1$.
(vii) $\int_0^1 \frac{ds}{f_m(s)} = \int_0^1 \frac{ds}{f(s)} = \int_0^1 \frac{ds}{f_M(s)} = \infty$.

We obtain from the property (vi) that $\int_\eta^\infty \frac{ds}{f(s)} < \infty$ implies $\int_\eta^\infty \frac{ds}{f_M(s)} < \infty$. However, the converse is not true, in general. The examples and the properties the minorant function f_m and the majorant function f_M were discussed in [14].

Now, we introduce the definition of the blow-up solutions and global solutions.

Definition 3.2 We say that a solution u blows up at finite time t^*, if there exists $0 < t^* < \infty$ such that $\|u(\cdot, t)\|_\infty \to \infty$ as $t \to t^*$. On the other hand, a solution u exists globally whenever $\|u(\cdot, t)\|_\infty$ is bounded for each time $t \geq 0$.

Now, we prove Theorem 1.2.

Proof (ii) $\Leftrightarrow$ (iii): It is well-known that $\|S(t)u_0\|_\infty \sim e^{-\lambda_0 t}$ for $t \geq 0$, for every nonnegative and nontrivial initial data $u_0 \in C_0(\Omega)$. Therefore, we easily see that for a nonnegative and nontrivial initial data $u_0 \in C_0(\Omega)$, there exist positive constants c_1 and c_2 such that

$$c_1 \psi(t) e^{\lambda_0 t} f\left(\epsilon e^{\lambda_0 t}\right) \leq \psi(t) \frac{f(\epsilon \|S(t)u_0\|_\infty)}{\|S(t)u_0\|_\infty} \leq c_2 \psi(t) e^{\lambda_0 t} f\left(\epsilon e^{\lambda_0 t}\right),$$

for each $\epsilon > 0$, since $\frac{f(u)}{u}$ is nondecreasing. Hence, by considering $w_0 := \epsilon u_0$, then we have

$$\frac{c_1}{\epsilon} \int_0^t \psi(\tau) e^{\lambda_0 \tau} f\left(\epsilon e^{\lambda_0 \tau}\right) d\tau \leq \int_0^t \psi(\tau) \frac{f(\|S(\tau)w_0\|_\infty)}{\|S(\tau)w_0\|_\infty} d\tau \leq \frac{c_2}{\epsilon} \int_0^t \psi(\tau) e^{\lambda_0 \tau} f\left(\epsilon e^{\lambda_0 \tau}\right) d\tau,$$

which completes the proof.

(ii) $\Rightarrow$ (i): Suppose that

$$\int_0^\infty \psi(t) e^{\lambda_0 t} f\left(\epsilon e^{-\lambda_0 t}\right) dt = \infty$$

for every $\epsilon > 0$. First of all, we choose eigenfunction ϕ_0 to satisfy $\int_\Omega \phi_0(x)\, dx = 1$, corresponding to the first eigenvalue λ_0 with the mixed boundary condition. Suppose that the solution u exists globally, on the contrary. Multiplying the equation (1) by ϕ_0 and integrating over Ω, we use Green's theorem and Jensen's inequality to obtain

$$\begin{aligned}\int_\Omega u_t(x,t)\phi_0(x)\,dx &= \int_\Omega \phi_0(x)\Delta u(x,t)\,dx + \psi(t)\int_\Omega f\big(u(x,t)\big)\phi_0(x)\,dx \\ &\geq -\lambda_0\int_\Omega u(x,t)\phi_0(x)\,dx + \psi(t)f\left(\int_\Omega u(x,t)\phi_0(x)\,dx\right),\end{aligned}$$

for all $t>0$. Putting $y(t) := \int_\Omega u(x,t)\phi_0(x)\,dx$, for $t\geq 0$, then $y(t)$ exists for all time t and satisfies the following inequality

$$\begin{cases} y'(t) \geq -\lambda_0 y(t) + \psi(t)f(y(t)), & t>0, \\ y(0) = y_0 := \int_\Omega u_0(x)\phi_0(x)\,dx > 0. \end{cases}$$

Then the inequality can be written as

$$\big[e^{\lambda_0 t}y(t)\big]' \geq \psi(t)e^{\lambda_0 t}f\big(y(t)\big) \geq 0, \tag{5}$$

for $t>0$ so that $e^{\lambda_0 t}y(t)$ is nondecreasing on $[0,\infty)$. On the other hand, by the properties of f_m, we can find $\nu_1 \in [0,1]$ such that $f_m = 0$ on $[0,\nu_1)$ and $f_m > 0$ on (ν_1,∞). Then there exists $\epsilon > 0$ such that $y_0 > \epsilon\nu_1$. *i.e.* $\nu_1 < \frac{y(0)}{\epsilon} \leq \frac{e^{\lambda_0 t}y(t)}{\epsilon}$ for $t\geq 0$. Combining all these arguments, it follows from (5) and the definition of f_m that

$$\frac{\frac{e^{\lambda_0 t}y(t)}{\epsilon}}{f_m(\frac{e^{\lambda_0 t}y(t)}{\epsilon})} \geq \frac{1}{\epsilon}\psi(t)e^{\lambda_0 t}f\big(\epsilon e^{-\lambda_0 t}\big),$$

for all $t>0$. Now, define a function $F_m : (\nu_1,\infty) \to (0,\nu_\infty)$ by

$$F_m(\nu) := \int_\nu^\infty \frac{dw}{f_m(w)}, \nu > \nu_1$$

where $\nu_\infty := \lim_{\nu\to\nu_1}\int_\nu^\infty \frac{dw}{f_m(w)}$. Then it is easy to see that F_m is well-defined continuous function, which is a strictly decreasing bijection with its inverse F_m^{-1} and $\lim_{\nu\to\infty}F_m(\nu) = 0$. Integrating the inequality (5) over $[0,t]$, we obtain

$$F_m\left(\frac{y(0)}{\epsilon}\right) - F_m\left(\frac{e^{\lambda_0 t}y(t)}{\epsilon}\right) \geq \frac{1}{\epsilon}\int_0^t \psi(\tau)e^{\lambda_0\tau}f\big(\epsilon e^{-\lambda_0\tau}\big)\,d\tau,$$

for all $t\geq 0$. Hence, we obtain

$$y(t) \geq \epsilon e^{-\lambda_0 t}F_m^{-1}\left[F_m\left(\frac{y(0)}{\epsilon}\right) - \frac{1}{\epsilon}\int_0^t \psi(\tau)e^{\lambda_0\tau}f\big(\epsilon e^{-\lambda_0\tau}\big)\,d\tau\right]$$

for all $t\geq 0$, which implies that $y(t)$ cannot be global. That is, the solution u doesn't exist globally.

(i) $\Rightarrow$ (ii): Suppose that

$$\int_0^\infty \psi(t)e^{\lambda_0 t}f\big(\epsilon e^{-\lambda_0 t}\big)\,dt < \infty$$

for some $\epsilon > 0$. We note that there exists a maximal interval $[0, m^*)$ on which f_M is finite. Therefore, consider a function $F_M : (0, m^*) \to (0, \infty)$ defined by

$$F_M(v) := \int_v^{m^*} \frac{dw}{f_M(w)}, \quad v \in (0, m^*).$$

Since f is convex, it is true that the value of $\int_v^{m^*} \frac{dw}{f_M(w)}$ is finite for each $v \in (0, m^*)$, $\lim_{v\to 0} \int_v^{m^*} \frac{dw}{f_M(w)} = \infty$, and $\lim_{v\to m^*} \int_v^{m^*} \frac{dw}{f_M(w)} = 0$. Hence, F_M is a well-defined continuous function, which is a strictly decreasing bijection with its inverse F_M^{-1}. Now, take a number z_0 such that

$$0 < z_0 < F_M^{-1}\left[\frac{1}{\epsilon}\int_0^\infty \psi(t)e^{\lambda_0 t} f\left(\epsilon e^{-\lambda_0 t}\right) dt\right]$$

and define a nondecreasing function $z : [0, \infty) \to [z_0, \infty)$ by

$$z(t) := F_M^{-1}\left[F_M(z_0) - \frac{1}{\epsilon}\int_0^t \psi(\tau)e^{\lambda_0 \tau} f\left(\epsilon e^{-\lambda_0 \tau}\right) d\tau\right], \quad t \geq 0.$$

Then $z(t)$ is a bounded solution of the following ODE problem:

$$\begin{cases} z'(t) = \frac{1}{\epsilon}\psi(t)e^{\lambda_0 t} f(\epsilon e^{-\lambda_0 t}) f_M(z(t)), & t > 0, \\ z(0) = z_0. \end{cases}$$

Now, we choose the eigenfunction ϕ_0 to satisfy $\sup_{x\in\Omega} \phi_0(x) = \epsilon$ and consider a function $v(x,t) := e^{-\lambda_0 t}\phi_0(x)$ on $\overline{\Omega} \times [0, \infty)$. Define $\overline{u}(x,t) := z(t)v(x,t)$ for $(x,t) \in \overline{\Omega} \times [0, \infty)$. Then $\overline{u}$ exists globally. Since f is convex, $\frac{f(u)}{u}$ is nondecreasing. Then it follows that

$$\begin{aligned} \overline{u}_t(x,t) &= \Delta\overline{u}(x,t) + \psi(t)v(x,t)\left[\frac{f(\epsilon e^{-\lambda_0 t})}{\epsilon e^{-\lambda_0 t}}\right] f_M\left(z(t)\right) \\ &\geq \Delta\overline{u}(x,t) + \psi(t)v(x,t)\left[\frac{f(v(x,t))}{v(x,t)}\right] f_M\left(z(t)\right) \\ &\geq \Delta\overline{u}(x,t) + \psi(t) f\left(\overline{u}(x,t)\right) \end{aligned}$$

for all $(x,t) \in \Omega \times (0, \infty)$. Also, $\overline{u} = 0$ on $\partial\Omega \times [0, \infty)$. Therefore, if we choose a nonnegative and nontrivial initial data u_0 sufficiently small that $u_0(x) \leq z_0\phi_0(x)$ for $x \in \Omega$, then $\overline{u}$ is the supersolution to the equation (1). This implies that there is a nonnegative and nontrivial initial data u_0 such that u exists globally. □

Remark 3.3 The proof of Theorem 1.2 works for all cases $\lambda_0 \geq 0$, including the case $\lambda_0 = 0$. In fact, if $\lambda_0 = 0$, then (ii) in Theorem 1.2 is equivalent to

$$\int_0^\infty \psi(t)\, dt = \infty.$$

Corollary 3.4 *Let the function* ψ *be a nonnegative continuous function and the function* f *be a nonnegative continuous and quasi-multiplitive function, i.e., there exist* $\gamma_2 \geq \gamma_1 > 0$

such that

$$\gamma_1 f(\alpha)f(u) \leq f(\alpha u) \leq \gamma_2 f(\alpha)f(u), \tag{6}$$

for $0 < \alpha < 1$ *and* $u > 0$. *Then the following statements are equivalent*:

(i) $\int_0^\infty \psi(t)e^{\lambda_0 t}f(e^{-\lambda_0 t})\,dt = \infty$.

(ii) $\int_0^\infty \psi(t)\frac{f(\|S(t)w_0\|_\infty)}{\|S(t)w_0\|_\infty}\,dt = \infty$ *for every nonnegative and nontrivial* $w_0 \in C_0(\Omega)$.

(iii) $\int_0^\infty \psi(t)\frac{dt}{F(e^{-\lambda_0 t})} = \infty$, *where* $F(v) := \int_v^\infty \frac{dw}{f(w)}$.

(iv) *There is no global solution to the equation* (1) *for any initial data*.

Proof The relation (6) implies that $f_m(u) = \gamma_1 f(u)$ and $f_M(u) = \gamma_2 f(u)$ for $u \geq 0$. Therefore, Theorem 1.2 says that (i), (ii), and (iv) are equivalent. Therefore, we now discuss (iii).

(i) $\Leftrightarrow$ (iii): Let $F(v) = \int_v^\infty \frac{dw}{f(w)}$. Then the assumption (6) follows that

$$\int_1^\infty \frac{z}{\gamma_2 f(z)f(s)}\,ds \leq \int_z^\infty \frac{dw}{f(w)} \leq \int_1^\infty \frac{z}{\gamma_1 f(z)f(s)}\,ds,$$

for $z > 0$. This implies that

$$\frac{F(1)}{\gamma_2}\frac{z}{f(z)} \leq F(z) \leq \frac{F(1)}{\gamma_1}\frac{z}{f(z)}.$$

i.e. $F(z) \sim \frac{z}{f(z)}$, $z > 0$. Therefore, the proof is complete. □

Remark 3.5 In 2014, Loayza and Paixão [10] studied the conditions for existence and nonexistence of the global solutions to the equation (1) under the general domain and obtained the following statements:

(i) for every $w_0 \in C_0(\Omega)$, there exist $\tau > 0$ such that

$$\int_{\|S(\tau)w_0\|_\infty}^\infty \frac{dw}{f(w)} \leq \int_0^\tau \psi(\sigma)\,d\sigma, \tag{7}$$

then there is no global solution u for every initial data,

(ii) the solution u exists globally for small initial data whenever

$$\int_0^\infty \psi(t)\frac{f(\|S(t)w_0\|_\infty)}{\|S(t)w_0\|_\infty}\,dt < 1, \tag{8}$$

for some $w_0 \in C_0(\Omega)$.

In fact, Corollary 3.4 imply that the conditions (7) and (8) have a strong relation, even though (7) and (8) have different formulas.

Also, by using Corollary 3.4, the example in Sect. 2 can be characterized completely as follows:

Example 3.6 Let the domain Ω be bounded in $\mathbb{R}^N$, $\psi(t) := (t+1)^{-\sigma}e^{kt}$, and $f(u) := u^p$ where $\sigma \in \mathbb{R}$, $k \in \mathbb{R}$, and $p > 1$. Then the following statements are true.

(i) If $k > (p-1)\lambda_0$, then there is no global solution u to the equation (1) for any nonnegative and nontrivial initial data $u_0 \in C_0(\Omega)$.

(ii) If $k < (p-1)\lambda_0$, then there exists a global solution to the equation (1) for sufficiently small initial data $u_0 \in C_0(\Omega)$.

(iii) If $k = (p-1)\lambda_0$ and $\sigma \leq 1$, then there is no global solution u to the equation (1) for any nonnegative and nontrivial initial data $u_0 \in C_0(\Omega)$.

(iv) If $k = (p-1)\lambda_0$ and $\sigma > 1$, then there exists a global solution to the equation (1) for sufficiently small initial data $u_0 \in C_0(\Omega)$.

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Data availability
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Declarations

Competing interests
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