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On a composite obtained by a mixture of a dipolar solid with a Moore–Gibson–Thompson media

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Abstract

Our study is dedicated to a mixture composed of a dipolar elastic medium and a viscous Moore–Gibson–Thompson (MGT) material. The mixed problem with initial and boundary data, considered in this context, is approached from the perspective of the existence of a solution to this problem as well as the uniqueness of the solution. Considering that the mixed problem is very complex, both from the point of view of the basic equations and that of the initial conditions and the boundary data, the classical methods become difficult. That is why we preferred to transform it into a problem of Cauchy type on a conveniently constructed Hilbert space. In this way, we immediately proved both the existence and uniqueness of the solution, with techniques from the theory of semigroups of linear operators.

Keywords: Dipolar bodies; Lagrange identities; Uniqueness; Instability; Moore–Gibson–Thompson theory

1 Introduction

In recent years, the number of studies dedicated to systems of differential equations of MGT type has increased considerably. The researchers consider that the MGT theory appeared, like many other nonclassical theories, in order to avoid violating the causality principle, which happens in the classical theory. Many works on the MGT theory highlight its practical applicability, see [1–5], while other studies address different theoretical aspects of this theory, see [6–9]. To highlight the importance of media with a dipolar structure, it is enough to mention the works [10–12]. In other previous works, many different aspects are addressed regarding the bodies with nonclassical structure, see [13–23].

We must specify that in our paper we are dealing with the MGT model in the context of the Kelvin–Voigt viscoelasticity theory. It is known that the behavior of waves, from a mechanical point of view, in the Kelvin–Voigt viscoelasticity also violates the causality principle. That is why the theory of the viscoelastic media based on the MGT model can be considered more appropriate than the known Kelvin–Voigt viscoelastic theory because the aforementioned paradox can be avoided. There is a big number of specialists who consider that this theory of the viscoelastic media based on the MGT model is among the theories which allow thermal waves to propagate at a finite speed. Also, the viscoelastic

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effects modeled by the partial differential equations of MGT type can be an alternative for the known theories dedicated to heat conduction.

Another aspect that we take into account in our study is that of a mixture between two or more types of materials which interact, with the aim of obtaining another material with superior properties compared to the separate component materials.

The proposed mixture model has concrete applicability to real, modern materials, such as graphite, polymers, and other granular media having large molecules. Also, it applies for animal and human bones. Considering the great number of published works dedicated to the mixtures of different media, it can be concluded that these are very suitable for modeling a great number of media in continuum mechanics [24–29]. Other similar applicable results can be found in [30–33].

The plan of our study is as follows. First, in Sect. 2, we systematize the main equations, initial and boundary conditions, which describe the evolution of the proposed mixture. So, we obtain a mixed initial-boundary value problem. Also, we specify the assumptions, conditions, and restrictions imposed on the functions we use, in order to obtain the results we proposed. In Sect. 3, the formulated mixed problem is transformed into a new problem of the Cauchy type defined on a Hilbert space, which we build in advance using the data of the mixed problem. Our main results, regarding the existence and uniqueness of the solution to the Cauchy problem, are formulated and proved in Sect. 4. It should be stated that these results are obtained based on tools provided by the theory of operator semigroups.

2 Preliminaries

In all what follows we will use a domain D in the three-dimensional Euclidean space R^3 whose boundary, denoted by ∂D, is assumed to have at least the regularity that allows the application of the divergence theorem. Assume that D is occupied by a mixture composed of a dipolar elastic medium and a viscous Moore–Gibson–Thompson (MGT) material. The closure of the domain D is denoted by $\bar{D}$ and $\bar{D} = D \cup \partial D$. By convention, a vector field $\mathbf{v}$ has three components v_i, $i = 1, 2, 3$, and a tensor field $\mathbf{w}$ has nine components w_{ij}, $i, j = 1, 2, 3$. The derivative of a function $f(t, x)$ with respect to the time variable t is denoted by $\dot{f}$. The derivative of the function f with respect to its space variable x_i is designated by $f_{,i}$. In the case of repeating subscripts, Einstein's summation rule is used.

If there is no possibility of confusion, specifying the dependence of a function on its spatial or temporal arguments can be avoided. The evolution of our body is referred to in the fixed system of Cartesian axes Ox_i, $i = 1, 2, 3$.

For the considered mixture, a typical point has, at a moment t, the displacements of components denoted by $\mathbf{v} = (v_m)$ and $\mathbf{w} = (v_m)$, $m = 1, 2, 3$, respectively.

For the microrotations, we will use the notation $\boldsymbol{\phi} = (\phi_m)$, $m = 1, 2, 3$. These three vectors are functions which depend on time and the material point. At the initial moment, it is supposed that the particles are in the same position.

With the help of internal variables (v_m, w_m, ϕ_m), the kinematic characteristics of the media, that is, the tensors of strain can be defined through the following kinematic relations:

$$
\begin{aligned}
e_{mn} &= \frac{1}{2}(w_{n,m} + w_{m,n}), \\
\eta_{mn} &= v_{n,m} + \varepsilon_{knm}\phi_k, \\
\mu_{mn} &= \phi_{n,m}.
\end{aligned}
\tag{1}
$$

The components of the tensor of stress are denoted by τ_{mn}, the notation σ_{mn} is used for the components of the couple stress tensor, defined on Ω.

In the case of a thermoelastic body which is homogeneous and possesses in its reference state a point of symmetry, while the rest is considered nonisotropic, the tensors of stress can be introduced by using the following constitutive relations [29]:

$$\begin{aligned}
\tau_{mn} &= \int_{-\infty}^{t} \big[A_{mnkl}(t-\tau)\dot{e}_{kl}(\tau) + B_{mnkl}(t-\tau)\dot{\eta}_{kl}(\tau)\big]\,d\tau,\\
\sigma_{mn} &= \int_{-\infty}^{t} \big[B_{klmn}(t-\tau)\dot{e}_{kl}(\tau) + C_{mnkl}(t-\tau)\dot{\eta}_{kl}(\tau)\big]\,d\tau,\\
\gamma_{mn} &= \int_{-\infty}^{t} \big[D_{klmn}(t-\tau)\dot{\eta}_{kl}(\tau) + E_{mnkl}(t-\tau)\dot{\mu}_{kl}(\tau)\big]\,d\tau,
\end{aligned}\tag{2}$$

and for the internal body force and the internal body couple, we have:

$$\begin{aligned}
F_{m} &= \int_{-\infty}^{t} a_{ml}(t-\tau)\big[\dot{v}_{l}(\tau) - \dot{\phi}_{l}(\tau))\big]\tau,\\
G_{m} &= \int_{-\infty}^{t} b_{ml}(t-\tau)\big[\dot{w}_{l}(\tau) - \dot{\phi}_{l}(\tau)\big]\,d\tau,\\
p_{m} &= \int_{-\infty}^{t} c_{ml}(t-\tau)\big[\dot{v}_{l}(\tau) - \dot{w}_{l}(\tau)\big]\,d\tau.
\end{aligned}\tag{3}$$

The above constitutive tensors satisfy the symmetries:

$$A_{mnkl} = A_{klmn}, \qquad C_{mnkl} = C_{klmn}, \qquad E_{mnkl} = E_{klmn}.$$

Also, we suppose the following assumptions are satisfied:

$$\begin{aligned}
&A_{mnkl}(\tau,x) = A_{mnkl}(x) + \left[\frac{1}{p}A^{*}_{mnkl}(x) - A_{mnkl}(x)\right]e^{-\frac{s}{p}},\\
&B_{mnkl}(\tau,x) = B_{mnkl}(x), \qquad C_{mnkl}(\tau,x) = C_{mnkl}(x),\\
&D_{mnkl}(\tau,x) = D_{mnkl}(x), \qquad E_{mnkl}(\tau,x) = E_{mnkl}(x),\\
&a_{ml}(\tau,x) = a_{ml}(x), \qquad b_{ml}(\tau,x) = b_{ml}(x), \qquad c_{ml}(\tau,x) = c_{ml}(x),
\end{aligned}\tag{4}$$

in which the parameter p is a positive constant.

For our mixture, we have the following equations of evolution:

$$\begin{aligned}
&\tau_{mn,n} + F_{m} = \varrho_{1}\ddot{v}_{m},\\
&\sigma_{mn,n} + p_{m} = \varrho_{2}\ddot{w}_{m},\\
&\gamma_{mn,n} + \epsilon_{mnk}\tau_{nk} + G_{m} = I_{mn}\ddot{\phi}_{n},
\end{aligned}\tag{5}$$

where ϱ_1 and ϱ_2 are the mass densities of the two media of the mixture.

We will suppose that as $t \to -\infty$ the deformations vanish.

Considering relations (3) and (4) and using a notation of the form $\hat{u}_m = u_m + p\dot{u}_m$, we can rewrite the evolution equations from (5) in the following form:

$$\begin{aligned}
&\left(A_{mnkl}e_{k,l} + A^*_{mnkl}\dot{e}_{k,l} + B_{mnkl}\hat{\eta}_{kl}\right)_{,n} - a_{ml}(v_l + p\dot{v}_l - \hat{w}_l) = \varrho_1(p\dddot{v}_m + \ddot{v}_m),\\
&\left[B_{mnkl}(e_{k,l} + p\dot{e}_{k,l}) + C_{mnkl}\hat{\eta}_{kl}\right]_{,n} + a_{ml}(w_l + p\dot{w}_l - \hat{w}_l) = \varrho_2\ddot{\hat{w}}_m,\\
&\left[D_{mnkl}(\eta_{k,l} + p\dot{e}_{k,l}) + C_{mnkl}\hat{\mu}_{kl}\right]_{,n} - b_{ml}(\phi_l + p\dot{\phi}_l - \hat{w}_l)\\
&\qquad + \epsilon_{mnk}\left(A_{mnkl}e_{k,l} + A^*_{mnkl}\dot{e}_{k,l} + B_{mnkl}\hat{\eta}_{kl}\right)\\
&\quad = I_{mn}(p\dddot{\phi}_m + \ddot{\phi}_m),
\end{aligned} \tag{6}$$

where the tensors a_{mn}, b_{mn}, and c_{mn} satisfy the following symmetry relations:

$$a_{mn} = a_{nm}, \qquad b_{mn} = b_{nm}, \qquad c_{mn} = c_{nm}.$$

In what follows, in order not to complicate the writing, we will give up the hat in the notation.

To complete the mixed problem with initial and boundary values, in our context, we need the initial data, which we take in their general form:

$$\begin{aligned}
&v_m(0,x) = v^0_m(x), \qquad \dot{v}_m(0,x) = \tilde{v}^0_m(x), \qquad \ddot{v}_m(0,x) = \check{v}^0_m(x),\\
&\phi_m(0,x) = \phi^0_m(x), \qquad \dot{\phi}_m(0,x) = \tilde{\phi}^0_m(x), \qquad \ddot{\phi}_m(0,x) = \check{\phi}^0_m(x),\\
&w_m(0,x) = w^0_m(x), \qquad \dot{w}_m(0,x) = \tilde{w}^0_m(x), \qquad \forall x \in D.
\end{aligned} \tag{7}$$

Also, we must add the boundary conditions, which we will consider in their homogeneous form:

$$v_m(t,x) = w_m(t,x) = \phi_m(t,x) = 0, \quad \forall (t,x) \in (-\infty, 0] \times \partial D. \tag{8}$$

Now we have to systematize the restrictions that we have to impose on the functions we work with so that we can obtain the results we proposed. These are:

(H_1) The densities ϱ_1 and ϱ_2, and the inertia tensor I_{mn}, are assumed to be strictly positive.

(H_2) A positive constant c_1 can be determined so that

$$A_{mnkl}x_{mn}x_{kl} + 2B_{mnkl}x_{mn}y_{kl} + C_{mnkl}y_{mn}y_{kl} \geq c_1(x_{mn}x_{mn} + y_{mn}y_{mn}), \quad \forall x_{mn}, y_{mn}.$$

(H_3) Positive constants c_2, c_3, and c_4 can be determined so that

$$a_{mn}x_m x_n \geq c_2 x_m x_n, \qquad b_{mn}x_m x_n \geq c_3 x_m x_n, \qquad c_{mn}x_m x_n \geq c_4 x_m x_n, \quad \forall x_m.$$

(H_4) Two positive constants c_5, c_6, can be determined so that

$$\begin{aligned}
&A_{mnkl}x_{mn}x_{kl} \geq pc_5 x_{mn}x_{kl}, \quad \forall x_{mn},\\
&C_{mnkl}x_{mn}x_{kl} \geq pc_6 x_{mn}x_{kl}, \quad \forall x_{mn}.
\end{aligned}$$

Let us denote by $\mathcal{P}$ the mixed problem consisting of equations (6), the initial conditions (7), the boundary conditions (8), and we suppose that the assumptions (H_1), (H_2), (H_3), and (H_4) are satisfied.

In the next step, we will transform the problem $\mathcal{P}$ into an abstract Cauchy-type problem, considered on a Hilbert space that we will define in advance.

By using the known Sobolev spaces $W_0^{1,2}(D)$ and $L^2(D)$ (see [34]), we define the space

$$H = \mathbf{W}_0^{1,2}(D) \times \mathbf{W}_0^{1,2}(D) \times \mathbf{L}^2(D) \times \mathbf{W}_0^{1,2}(D) \times \mathbf{W}_0^{1,2}(D) \times \mathbf{L}^2(D) \times \mathbf{W}_0^{1,2}(D) \times \mathbf{L}^2(D).$$

Here, the vector notations have the following meaning:

$$\mathbf{W}_0^{1,2}(D) = \left[W_0^{1,2}(D)\right]^3, \qquad \mathbf{L}^2(D) = \left[L^2(D)\right]^3.$$

For an arbitrary element

$$W = (v_m, \tilde{v}_m, \breve{v}_m, w_m, \tilde{w}_m, \phi_m, \tilde{\phi}_m, \breve{\phi}_m)$$

from the space H, inspired by equations (6), we can define the operators:

$$\begin{aligned} &A = (A_m), \qquad \tilde{A} = (\tilde{A}_m), \qquad \breve{A} = (\breve{A}_m), \qquad B = (B_m), \qquad \tilde{B} = (\tilde{B}_m),\\ &C = (C_m), \qquad \tilde{C} = (\tilde{C}_m), \qquad \breve{C} = (\breve{C}_m), \quad m = 1, 2, 3, \end{aligned}$$

by

$$\begin{aligned} A_m v &= \frac{1}{p\varrho_1}\left[(A_{mnkl}e_{kl})_{,n} - a_{ml}v_l\right],\\ \tilde{A}_m \tilde{v} &= \frac{1}{p\varrho_1}\left[\left(A^*_{mnkl}\tilde{e}_{kl}\right)_{,n} - pa_{ml}\tilde{v}_l\right],\\ \breve{A}_m \breve{v} &= \frac{1}{p}\breve{v}_m,\\ B_m w &= \frac{1}{p\varrho_2}\left[(B_{mnkl}e_{kl})_{,n} + a_{ml}v_l\right],\\ \tilde{B}_m \tilde{w} &= \frac{1}{\varrho_2}\left[(pB_{mnkl}\tilde{\eta}_{kl})_{,n} - pa_{ml}\tilde{v}_l\right],\\ C_m \phi &= I_{ij}^{-1}\left[(D_{mnkl}\eta_{kl})_{,n} - b_{ml}\phi_l\right],\\ \tilde{C}_m \tilde{\phi} &= I_{ij}^{-1}\left[(D_{mnkl}\tilde{\eta}_{kl})_{,n} - pb_{ml}\tilde{\phi}_l\right],\\ \breve{C}_m \breve{\phi} &= I_{ij}^{-1}\left[(pE_{mnkl}\breve{\mu}_{kl})_{,n} + pb_{ml}\breve{\phi}_l\right]. \end{aligned} \tag{9}$$

Now, we can introduce on the Hilbert space H the Cauchy problem

$$\begin{aligned} &\frac{dW}{dt} = \Gamma W,\\ &W(0) = W^0, \end{aligned} \tag{10}$$

in which Γ is a matrix operator constructed with help of the operators defined in (9).

Also, in the notation above,

$$W^0 = \left(v_m^0, \tilde{v}_m^0, \breve{v}_m^0, w_m^0, \tilde{w}_m^0, \phi_m^0, \tilde{\phi}_m^0, \breve{\phi}_m^0\right),$$

that is, the initial values defined in (7).

A scalar product in the Hilbert space H can be defined as follows:

$$\begin{aligned}
&\big\langle (v_m, \tilde{v}_m, \breve{v}_m, w_m, \tilde{w}_m, \phi_m, \tilde{\phi}_m, \breve{\phi}_m), \left(v_m^*, \tilde{v}_m^*, \breve{v}_m^*, w_m^*, \tilde{w}_m^*, \phi_m^*, \tilde{\phi}_m^*, \breve{\phi}_m^*\right)\big\rangle \\
&\quad = \frac{1}{2}\int_D \Big\{\varrho_1 (p\tilde{v}_m + \breve{v}_m)\overline{\left(p\tilde{v}_m^* + \breve{v}_m^*\right)} + I_{mn}(p\tilde{\phi}_n + \breve{\phi}_n)\overline{\left(p\tilde{\phi}_n^* + \breve{\phi}_n^*\right)} + \varrho_2 w_m \overline{w_m^*} \\
&\qquad + p\overline{A^*}_{mnrs}\tilde{v}_{m,n}\overline{\tilde{v}_{r,s}^*} + A_{mnrs}(v_{m,n} + p\tilde{v}_{m,n})\overline{\left(v_{r,s}^* + p\tilde{v}_{r,s}^*\right)} + C_{mnrs}\phi_{m,n}\overline{\phi_{r,s}^*} \\
&\qquad + B_{mnrs}\big[(v_{m,n} + p\tilde{v}_{m,n})\overline{\phi_{r,s}^*} + (\overline{v_{m,n}} + p\overline{\tilde{v}_{m,n}})\phi_{r,s}^*\big] \\
&\qquad + a_{mn}(v_m + p\tilde{v}_m - w_m)\overline{\left(v_n^* + p\tilde{v}_n^* - w_n^*\right)} + D_{mnrs}(w_{m,n} + p\tilde{w}_{m,n})\overline{\left(w_{r,s}^* + p\tilde{w}_{r,s}^*\right)} \\
&\qquad + E_{mnrs}\big[(w_{m,n} + p\tilde{w}_{m,n})\overline{\phi_{r,s}^*} + (\overline{v_{m,n}} + p\overline{\tilde{w}_{m,n}})\phi_{r,s}^*\big] \\
&\qquad + b_{mn}(\phi_m + p\tilde{\phi}_m - w_m)\overline{\left(\phi_n^* + p\tilde{\phi}_n^* - w_n^*\right)}\Big\},
\end{aligned} \tag{11}$$

where a bar on a variable is used to designate the complex conjugate of the respective variable.

The inner form (11) induces a norm on the Hilbert space H of the following form:

$$\begin{aligned}
&\big\| (v_m, \tilde{v}_m, \breve{v}_m, w_m, \tilde{w}_m, \phi_m, \tilde{\phi}_m, \breve{\phi}_m)\big\|^2 \\
&\quad = \frac{1}{2}\int_D \Big\{\varrho_1 (p\tilde{v}_m + \breve{v}_m)\overline{(p\tilde{v}_m + \breve{v}_m)} + I_{mn}(p\tilde{\phi}_n + \breve{\phi}_n)\overline{(p\tilde{\phi}_n + \breve{\phi}_n)} + \varrho_2 w_m \overline{w_m} \\
&\qquad + p\overline{A^*}_{mnrs}\tilde{v}_{m,n}\overline{\tilde{v}_{r,s}} + A_{mnrs}(v_{m,n} + p\tilde{v}_{m,n})\overline{(v_{r,s} + p\tilde{v}_{r,s})} + C_{mnrs}\phi_{m,n}\overline{\phi_{r,s}} \\
&\qquad + B_{mnrs}\big[(v_{m,n} + p\tilde{v}_{m,n})\overline{\phi_{r,s}} + (\overline{v_{m,n}} + p\overline{\tilde{v}_{m,n}})\phi_{r,s}\big] \\
&\qquad + a_{mn}(v_m + p\tilde{v}_m - w_m)\overline{(v_n + p\tilde{v}_n - w_n)} + D_{mnrs}(w_{m,n} + p\tilde{w}_{m,n})\overline{(w_{r,s} + p\tilde{w}_{r,s})} \\
&\qquad + E_{mnrs}\big[(w_{m,n} + p\tilde{w}_{m,n})\overline{\phi_{r,s}} + (\overline{v_{m,n}} + p\overline{\tilde{w}_{m,n}})\phi_{r,s}\big] \\
&\qquad + b_{mn}(\phi_m + p\tilde{\phi}_m - w_m)\overline{(\phi_n + p\tilde{\phi}_n - w_n)}\Big\},
\end{aligned} \tag{12}$$

and it can be shown, in a usual way, that this norm is equivalent to the original one on the Hilbert space H.

3 Main results

Both basic results of our paper, that is, those concerning the existence and uniqueness of a solution for the Cauchy problem (10), will be obtained using the theory of semigroups of operators and are based on the well-known Hille–Yosida theorem.

To begin with, we must specify that the domain of the operator Γ consists of all the elements $(v, \tilde{v}, \breve{v}, w, \tilde{w}, \phi, \tilde{\phi}, \breve{\phi})$ for which

$$\begin{aligned}
&(v, \breve{v}, \phi, \breve{\phi}, \tilde{w}) \in W_0^{1,2}, \\
&Av + \tilde{A}\tilde{v} + C\phi + \tilde{C}\tilde{\phi} + Bw \in L^2(D),
\end{aligned}$$

$$\tilde{A}v + \breve{A}\tilde{v} + \tilde{C}\phi + \breve{C}\tilde{\phi} + \tilde{B}w \in L^2(D),$$
$$Bv + \tilde{B}\tilde{v} + C\phi + \tilde{C}\tilde{\phi} + \breve{C}w \in L^2(D).$$

We will denote by $\mathcal{D}(\Gamma)$ the domain of the operator Γ. In the known way, it can be shown that the domain $\mathcal{D}$ is dense in the Hilbert space H.

Considering an arbitrary element $W = (v, \tilde{v}, \breve{v}, w, \tilde{w}, \phi, \tilde{\phi}, \breve{\phi})$ from the domain $\mathcal{D}$ of the operator Γ, it can be shown that

$$(\Gamma W, W) = -\frac{1}{2}\int_D (A_{mnkl}\dot{v}_{m,n}\dot{v}_{k,l} + C_{mnkl}\dot{\phi}_{m,n}\dot{\phi}_{k,l})\,dV \le 0, \tag{13}$$

that is, the operator Γ is dissipative, as required in the Lummer–Phillips corollary of Hille–Yosida theorem (see, for instance, [34]).

In accordance with this corollary, in order to ensure that the problem (15) has a solution and that this solution is unique, it must be demonstrated that the operator Γ satisfies the condition of range. For this, some element $W^* = (v^*, \tilde{v}^*, \breve{v}^*, w^*, \tilde{w}^*, \phi^*, \tilde{\phi}^*, \breve{\phi}^*)$ is fixed in the above defined Hilbert space H. Then it is said that the operator Γ satisfies the range condition if the equation $\Gamma W = W^*$ admits a solution $W \in \mathcal{D}(\Gamma)$. With another equivalent formulation, we must show that the resolvent of the operator Γ contains the null element.

This means that, considering an arbitrary element $(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8) \in H$, we must find a solution to the following system:

$$\begin{aligned}
&\tilde{v} = u_1, \qquad \breve{v} = u_2, \qquad \tilde{\phi} = u_3, \qquad \breve{\phi} = u_4, \qquad \tilde{w} = u_5,\\
&Av + \tilde{A}\tilde{v} + C\phi + \tilde{C}\tilde{\phi} + Bw = u_6,\\
&\tilde{A}v + \breve{A}\tilde{v} + \tilde{C}\phi + \breve{C}\tilde{\phi} + \tilde{B}w = u_7,\\
&Bv + \tilde{B}\tilde{v} + C\phi + \tilde{C}\tilde{\phi} + \breve{C}w = u_8.
\end{aligned} \tag{14}$$

Considering that we can easily find the variables $\tilde{v}$, $\breve{v}$, $\tilde{\phi}$, $\breve{\phi}$, and $\tilde{w}$, for the other variables we obtain the following system:

$$\begin{aligned}
&Av + C\phi + Bw = u_6 - \tilde{A}u_1 - \tilde{C}u_3,\\
&\tilde{A}v + \tilde{C}\phi + \tilde{B}w = u_7 - \breve{A}u_1 - \breve{C}u_3,\\
&Bv + C\phi + \breve{C}w = u_8 - \tilde{B}u_1 - \tilde{C}u_3.
\end{aligned} \tag{15}$$

It is not difficult to see that the terms on the right-hand side of the system (15) are elements of the space $W^{-1,2}(D) \times W^{-1,2}(D) \times W^{-1,2}(D)$.

Considering this and taking into account the hypotheses H_1–H_4, Lax–Milgram lemma can be used, and it ensures the existence of a solution $(v, \phi, w) \in W_0^{1,2}(D) \times W_0^{1,2}(D) \times W_0^{1,2}(D)$ of system (15).

Also, the operator Γ generates a semigroup of contractions, the infinitesimal generator of this semigroup is unique, and it is the solution of Cauchy problem (10), considered for V_0 as an arbitrary element in the domain of the operator Γ. It is also the unique solution of the problem $\mathcal{P}$.

4 Conclusions

We proved both the existence and uniqueness of the solution to problem (10). If we take into account how this problem was constructed, we deduce the existence and uniqueness of the solution to the initially formulated problem $\mathcal{P}$.

It is necessary to specify for the variables above,

$$(v, \tilde{v}, \breve{v}, w, \tilde{w}, \phi, \tilde{\phi}, \breve{\phi}) \quad \text{and} \quad (u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8),$$

the existence of a positive constant K for which we have

$$\left\| (v, \tilde{v}, \breve{v}, w, \tilde{w}, \phi, \tilde{\phi}, \breve{\phi}) \right\| \leq K \left\| (u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8) \right\|.$$

Also, using arguments similar to those above related to the theory of operator semigroups, it is possible to demonstrate the continuous dependence of the solutions of the problem $\mathcal{P}$ with respect to the initial data and to the supply terms in the problem $\mathcal{P}$. Moreover, the exponential decay of the solutions of the problem $\mathcal{P}$ can be shown.

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