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Dynamical behavior of a degenerate parabolic equation with memory on the whole space

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Abstract

This paper is concerned with the existence and uniqueness of global attractors for a class of degenerate parabolic equations with memory on $\mathbb{R}^n$. Since the corresponding equation includes the degenerate term $\operatorname{div}\{a(x)\nabla u\}$, it requires us to give appropriate assumptions about the weight function $a(x)$ for studying our problem. Based on this, we first obtain the existence of a bounded absorbing set, then verify the asymptotic compactness of a solution semigroup via the asymptotic contractive semigroup method. Finally, the existence and uniqueness of global attractors are proved. In particular, the nonlinearity f satisfies the polynomial growth of arbitrary order $p-1$ ($p \geq 2$) and the idea of uniform tail-estimates of solutions is employed to show the strong convergence of solutions.

Mathematics Subject Classification: 35K57; 35B40; 35B41

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1 Introduction

In this paper, we investigate the long-time behavior of solutions for the following degenerate parabolic equation with memory on $\mathbb{R}^n$ ($n \geq 2$):

$$u_t - \operatorname{div}\{a(x)\nabla u\} - \int_0^\infty k(s)\Delta u(t-s)\,ds + \lambda u + f(x,u) = g \quad \text{in } \mathbb{R}^n \times \mathbb{R}^+, \tag{1.1}$$

and initial value

$$u(x,t) = u_0(x) \quad \text{in } \mathbb{R}^n \times (-\infty, 0], \tag{1.2}$$

where the variable nonnegative weight coefficient $a(\cdot)$ denotes the diffusivity, the forcing term $g = g(x) \in L^2(\mathbb{R}^n)$ and the initial datum $u_0 \in L^2(\mathbb{R}^n)$ are given, λ is a positive constant, and $\mathbb{R}^+ = [0, \infty)$.

In order to study the equation (1.1) with initial condition (1.2), let us assume that the variable nonnegative diffusivity $a(\cdot)$, the nonlinearity f, and the memory $k(s)$ respectively satisfy the following conditions:

($\mathbf{H_1}$) The weight function $a(x)$ is a nonnegative function such that $a(x) \in L_{\text{loc}}(\mathbb{R}^n)$, and there exists some $0 < \alpha < 2$ such that, for each $z \in \mathbb{R}$,

$$\liminf_{x \to z} |x - z|^{-\alpha} a(x) > 0. \tag{1.3}$$

In addition, for $a(x)$ we also suppose that there exists $K > 0$ such that, for any $k \geq K$,

$$\sup_{k \leq |x| \leq \sqrt{2}k} a(x) < \infty. \tag{1.4}$$

($\mathbf{H_2}$) The memory kernel $k(s)$ is a nonnegative integrable function of total mass $\int_0^\infty k(s)\,ds = 1$. Let $\mu(s) = -k'(s)$, and we assume that

$$\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \qquad \mu(s) \geq 0, \qquad \mu'(s) \leq 0, \quad \forall s \in \mathbb{R}^+, \tag{1.5}$$

as well as there is a constant $\delta > 0$ such that

$$\mu'(s) + \delta \mu(s) \leq 0, \quad \forall s \in \mathbb{R}^+. \tag{1.6}$$

From (1.5) and (1.6), it is easy to infer that

$$\mu(\infty) = \lim_{s \to \infty} \mu(s) = 0. \tag{1.7}$$

To avoid the presence of unnecessary constants, we set

$$\int_0^\infty \mu(s)\,ds = 1. \tag{1.8}$$

($\mathbf{H_3}$) The nonlinearity $f \in C^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ fulfills $f(0) = 0$, along with the dissipation condition

$$f'(s) \geq -l, \tag{1.9}$$

and the arbitrary order polynomial growth restriction

$$\alpha_1 |s|^p - \beta_1 \varphi_1(x) \leq f(s)s \leq \alpha_2 |s|^p + \beta_2 \varphi_2(x), \quad p \geq 2, \tag{1.10}$$

where α_i, β_i $(i = 1, 2)$, and l are the positive constants, while $\varphi_1 \in L^1(\mathbb{R}^n)$, $\varphi_2 \in L^{\frac{p}{p-1}}(\mathbb{R}^n)$ are nonnegative functions.

In the light of the Dafermos' idea [1], we need to introduce a new variable η^t to characterize the past history of u, which is defined as follows:

$$\eta^t = \eta^t(x, s) := \int_0^s u(x, t - r)\,dr, \quad \forall s \in \mathbb{R}^+. \tag{1.11}$$

Denote $\eta_t^t = \frac{\partial}{\partial t}\eta^t$, $\eta_s^t = \frac{\partial}{\partial s}\eta^t$, then one easily gets

$$\eta_t^t = -\eta_s^t + u. \tag{1.12}$$

The historical variable $u_0(\cdot,-s)$ of u satisfies

$$\int_0^\infty e^{-\sigma s}\|u_0(-s)\|^2_{H^1(\mathbb{R}^n)}\,ds \le \Re, \tag{1.13}$$

where $\Re > 0$ and $\sigma \le \delta$ (δ is from (1.6)).

As a consequence, the problem (1.1)–(1.2) can be rewritten as follows:

$$\begin{cases} u_t - \operatorname{div}\{a(x)\nabla u\} - \int_0^\infty \mu(s)\Delta\eta^t(s)\,ds + \lambda u + f(x,u) = g, \\ \eta^t_t = -\eta^t_s + u, \end{cases} \tag{1.14}$$

with the initial data

$$u(x,0) = u_0(x), \qquad \eta^0(x,s) = \int_0^s u_0(x,-r)\,dr. \tag{1.15}$$

From (1.13), it is easy to obtain the following estimate:

$$\int_0^\infty \mu(s)\|\eta^0(s)\|^2_{H^1(\mathbb{R}^n)}\,ds \le \Re.$$

The integro-differential reaction–diffusion equation (1.1) with memory and $a(x) = 1$, as a model of heat diffusion with delay, depicts a reaction process that depends on the temperature itself, see, e.g., [2–7] and references therein. Such a model is also applied to some other physical phenomena, such as polymers and high viscosity liquids, etc.; see, e.g., [8–10]. However, equation (1.1), compared to the previous case of $a(x) = 1$, mainly describes a medium that is possibly somewhere a "perfect" insulator, see, e.g., [11].

The parabolic equations with degeneracy defined on bounded domain have been widely studied by some authors in recent years, including the well-posedness and long-time behavior of solutions (such as global and pullback attractors) for corresponding equation, see, e.g., [12–25] and references therein. In these published articles, the diffusivity $a(\cdot)$ is assumed to satisfy certain conditions, based on which the authors can ensure the compactness properties required for studying long-term dynamics. For the sake of simplicity, we will not go into much detail here. From the aforementioned works, we know that the global and pullback attractors on a bounded domain for the degenerate parabolic equation (1.16) given later have been thoroughly researched. However, for the unbounded case, it seems that few people thought about such questions.

To the best of our knowledge, the authors of [26, 27] have already studied the following degenerate parabolic equations on $\mathbb{R}^n$:

$$u_t - \operatorname{div}\{\sigma(x)\nabla u\} + \lambda u + f(x,u) = \text{"external force term"}. \tag{1.16}$$

In [26], for the autonomous semilinear degenerate parabolic equation, they obtained the existence of global attractors on $L^2(\mathbb{R}^n), L^p(\mathbb{R}^n)\cap\mathcal{H}^1(\mathbb{R}^n,\sigma)$. In [27], for the nonautonomous semilinear degenerate parabolic equation, the existence of pullback attractors on $L^p(\mathbb{R}^n)\cap\mathcal{H}^1(\mathbb{R}^n,\sigma)$ was proved under a new condition concerning a variable nonnegative diffusivity $\sigma(\cdot)$. Recently, the authors of [28] studied the well-posedness and the existence of global attractors on $L^2(\Omega)\times L^2_\mu(\mathbb{R};H^1_0(\Omega))$ of equation (1.1), which is an improvement compared with the studies of the semilinear degenerate parabolic equation (1.16). So, these works inspired us to consider the unbounded case of equation (1.1).

The problem (1.1)–(1.2) can be analyzed by following the Dafermos' idea of introducing an additional variable η^t, see, e.g., [3, 4, 29–34] and references therein. As is well known, if we want to study the existence of global attractors, the key is to obtain the asymptotic compactness of the solution semigroup in some sense. Note that the nonlinear term f of (1.1) satisfies the polynomial growth of arbitrary order $p-1$ ($p \geq 2$) and equation (1.1) contains the fading memory. As the problem (1.1)–(1.2) is considered on the whole space, this causes a series of difficulties.

(i) We cannot obtain higher regularity of solutions for equation (1.1) by utilizing the method of [3, 4, 29, 31, 35].

(ii) In the bounded case, the embedding $L^2_\mu(\mathbb{R}^+; H^2(\Omega) \cap H^1_0(\Omega)) \hookrightarrow L^2_\mu(\mathbb{R}^+; H^1_0(\Omega))$ is noncompact, let alone the embedding in the unbounded case. Thus the compact embedding method cannot be used to verify the asymptotic compactness of semigroup $\{S(t)\}_{t\geq 0}$.

In order to overcome these difficulties, the idea of the contractive function method is used to prove asymptotic compactness of the solution semigroup in [36–38]. Unfortunately, it is widely known that we cannot directly obtain that the semigroup $\{S(t)\}_{t\geq 0}$ is a contractive semigroup on $L^2(\mathbb{R}^n) \times L^2_\mu(\mathbb{R}; H^1_0(\mathbb{R}^n))$ since the phase space is an unbounded domain. Therefore, the asymptotic contractive semigroup method proposed in [39] and the uniform tail-estimates method proposed in [40] shall be applied to solve our problem. The main contribution of this paper is that we prove the existence and uniqueness of global attractors and the conclusions of the article extend some existing results in [18, 26–28] to whole space cases which have not been studied before.

The plan of this paper is as follows. In Sect. 2, we introduce some notations and recall some basic concepts on the asymptotic contractive function (semigroup) as well as some useful results later. In Sect. 3, we first sketch out the well-posedness of the problem (1.14)–(1.15), and then obtain the existence of a bounded absorbing set, as well as prove the asymptotic compactness of the semigroup corresponding to problem (1.14)–(1.15) by constructing an asymptotic contractive function. Finally, we obtain the existence and uniqueness of global attractors to problem (1.14)–(1.15) on the whole space $\mathbb{R}^n$.

2 Preliminaries

In this section, we introduce some notations and recall some of the existing abstract results, which shall be used to deal with our problem in the sequel.

2.1 Notation

For convenience, hereafter let C be an arbitrary positive constant and $Q(\cdot)$ be a strictly monotonically increasing positive function, which may be different from line to line, and even in the same line. Let

$$|\cdot|_p = \left(\int_{\mathbb{R}^n} \left| u(x) \right|^p dx \right)^{\frac{1}{p}}$$

be the norm of $L^p(\mathbb{R}^n)$ ($2 \leq p < \infty$), particularly, we denote the norms of $L^1(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ by $\|\cdot\|_{L^1}$ and $\|\cdot\|_{L^\infty}$, respectively. Furthermore, we consider $H := L^2(\mathbb{R}^n)$, equipped the following inner product:

$$(u,v)=\int_{\mathbb{R}^n} u(x)v(x)\,dx, \quad \forall u,v\in L^2(\mathbb{R}^n).$$

To describe our problem, we need to introduce the Hilbert space $\mathcal{H}^1(\mathbb{R}^n,a)$, equipped with the following norm:

$$\|u\|^2_{\mathcal{H}^1(\mathbb{R}^n,a)}:=\int_{\mathbb{R}^n}\big|u(x)\big|^2\,dx+\int_{\mathbb{R}^n}a(x)\big|\nabla u(x)\big|^2\,dx.$$

Denote the weighted spaces $\mathcal{V}_0=L^2_\mu(\mathbb{R}^+;L^2(\mathbb{R}^n))$ and $\mathcal{V}_1=L^2_\mu(\mathbb{R}^+;H^1(\mathbb{R}^n))$, as well as their inner products and norms as

$$\langle\psi,\eta\rangle_{\mathcal{V}_0}=\int_0^\infty \mu(s)(\psi,\eta)\,ds, \qquad \big\|\eta^t\big\|^2_{\mu,0}=\int_0^\infty \mu(s)\big|\eta^t(s)\big|_2^2\,ds,$$

and

$$\langle\psi,\eta\rangle_{\mathcal{V}_1}=\int_0^\infty \mu(s)\langle\psi,\eta\rangle_{H^1(\mathbb{R}^n)}\,ds, \qquad \big\|\eta^t\big\|^2_{\mu,1}=\int_0^\infty \mu(s)\big(\big|\eta^t(s)\big|_2^2+\big|\nabla\eta^t(s)\big|_2^2\big)\,ds,$$

respectively. According to the aforementioned notation, the phase space of the problem (1.14)–(1.15) can be represented as

$$\mathcal{L}_2:=H\times\mathcal{V}_1,$$

endowed the following norm:

$$\|\cdot\|^2_{\mathcal{L}_2}=|\cdot|_2^2+\|\cdot\|^2_{\mu,1}.$$

We denote the ball with radius R in $\mathcal{L}_2$ by

$$\mathscr{B}(R)=\big\{\phi\in\mathcal{L}_2:\|\phi\|_{\mathcal{L}_2}\le R\big\}.$$

2.2 Abstract results

In this subsection, we give some of the existing theoretical results, which shall be used to verify the asymptotic compactness of semigroup; for more detail, see [34, 36, 39].

Definition 2.1 Let X be a Banach space and B be a bounded subset of X. We call a function $\psi(\cdot,\cdot)$, defined on $X\times X$, an asymptotic contractive function if there exists a contractive function ϕ such that for any $\epsilon>0$ and any sequence $\{x_n\}_{n=1}^\infty\subset B$, there is a subsequence $\{x_{n_k}\}_{k=1}^\infty\subset\{x_n\}_{n=1}^\infty$ satisfying

$$\psi(x_{n_k},x_{n_l})\le\epsilon+\phi(x_{n_k},x_{n_l}),$$

where

$$\lim_{k\to\infty}\lim_{l\to\infty}\phi(x_{n_k},x_{n_l})=0.$$

We denote the set of all contractive functions on $B \times B$ by $\mathfrak{C}(B)$.

Lemma 2.2 *Let X be a Banach space and B be a bounded subset of X. Consider a semigroup $\{S(t)\}_{t\geq 0}$ with a bounded absorbing set B_0 on X. Moreover, assume that for all $\epsilon > 0$ there exist $T = T(B;\epsilon)$ and $\psi_T(\cdot,\cdot) \in \mathfrak{C}(B)$ such that*

$$\left\|S(T)x - S(T)y\right\|_X \leq \varepsilon + \psi_T(x,y), \quad \forall x,y \in B,$$

where ϕ_T depends on T. Then the semigroup $\{S(t)\}_{t\geq 0}$ is an asymptotic contractive semigroup on B.

Theorem 2.3 *Let $\{S(t)\}_{t\geq 0}$ be a continuous semigroup on X. Then $\{S(t)\}_{t\geq 0}$ has a global attractor in X provided that the following conditions hold true*:

(i) $\{S(t)\}_{t\geq 0}$ *has a bounded absorbing set B_0 in X;*

(ii) $\{S(t)\}_{t\geq 0}$ *is an asymptotic contractive semigroup on B_0.*

Lemma 2.4 ([41, 42]) *Let $X \subset\subset H \subset Y$ be Banach spaces, with X reflexive. Suppose that u_n is a sequence that is uniformly bounded in $L^2(0,T;X)$ and du_n/dt is uniformly bounded in $L^p(0,T;Y)$, for some $p > 1$. Then there is a subsequence of u_n that converges strongly in $L^2(0,T;H)$.*

3 Global attractors on $\mathcal{L}_2$

In this section, we shall consider the existence and uniqueness of the global attractors in $\mathcal{L}_2$. To this end, we first state the definition of a weak solution, and then give the well-posedness conclusion for the problem (1.14)–(1.15). Finally, we prove that the problem (1.14)–(1.15) possesses a bounded absorbing set in $\mathcal{L}_2$ and verify the asymptotic compactness of the corresponding solution process, which can ensure the existence and uniqueness of the global attractors in $\mathcal{L}_2$.

3.1 Well-posedness

The well-posedness of the problem (1.14)–(1.15) can be proved via the Faedo–Galerkin method (see, e.g., [26, 28, 43]). Of course, this needs to be based on the following definition of a weak solution.

Definition 3.1 Suppose that $g \in H$ and the initial value $z_0 = (u_0, \eta^0) \in \mathcal{L}_2$. Then for any $T > 0$, let $I = [0,T]$. The pair of functions $z(x,t) = (u(x,t), \eta^t(x,s))$ defined on $\mathbb{R}^n \times I$ is called a weak solution of the problem (1.14)–(1.15) if

$$\begin{aligned}
&u \in C(I;H) \cap L^2\big(0,T;\mathcal{H}^1\big(\mathbb{R}^n,a\big)\big) \cap L^p\big(0,T;L^p\big(\mathbb{R}^n\big)\big),\\
&u_t \in L^2\big(0,T;\mathcal{H}^{-1}\big(\mathbb{R}^n,a\big)\big), \qquad \eta^t \in C(0,T;\mathcal{V}_1),\\
&\eta_t^t + \eta_s^t \in L^\infty(0,T;\mathcal{V}_1) \cap L^2(0,T;\mathcal{V}_1),
\end{aligned}$$

where $\mathcal{H}^{-1}(\mathbb{R}^n,a)$ denotes the dual space of $\mathcal{H}^1(\mathbb{R}^n,a)$. Furthermore, the following identity:

$$\begin{cases}
(u_t,\omega) + (a(x)\nabla u, \nabla\omega) + \langle \eta^t, \omega\rangle_{\mathcal{V}_1} + \lambda(u,\omega) + \langle f(x,u),\omega\rangle = (g,\omega),\\
\langle \eta_t^t + \eta_s^t, \varphi\rangle_{\mathcal{V}_1} = \langle u,\varphi\rangle_{\mathcal{V}_1},
\end{cases} \tag{3.1}$$

holds true for any $(\omega,\varphi) \in C^\infty(\mathbb{R}^n) \times \mathcal{V}_1$ and a.e. $t \in I$.

Lemma 3.2 *For any $T > 0$ and $z_0 = (u_0, \eta^0) \in \mathcal{L}_2$, the problem (1.14)–(1.15) has a unique weak solution*

$$z(x,t) = \big(u(x,t), \eta^t(x,s)\big) \in C(I; H \times \mathcal{V}_1),$$

and there exists a positive constant κ, which is independent of t, such that the semigroup $S(t)$ is Lipschitz continuous:

$$\big\|S(t)z_0^1 - S(t)z_0^2\big\|_{\mathcal{L}_2} \le Ce^{\kappa T}\big\|z_0^1 - z_0^2\big\|_{\mathcal{L}_2}, \quad \forall t \in I, \tag{3.2}$$

where z_0^1 and z_0^2 denote the initial data of the problem (1.14)–(1.15).

Proof Let $z^1(t) = (u^1(t), \eta_1^t)$ and $z^2(t) = (u^2(t), \eta_2^t)$ be two solutions of the problem (1.14)–(1.15) with the initial data $z_0^1 = (u_0^1, \eta_1^0)$ and $z_0^2 = (u_0^2, \eta_2^0)$, respectively. Then we can obtain

$$\begin{aligned} &\frac{d}{dt}\big(\big|u^1(t) - u^2(t)\big|_2^2 + \big\|\nabla\eta_1^t - \nabla\eta_2^t\big\|_{\mu,0}^2\big) + \delta\big\|\nabla\eta_1^t - \nabla\eta_2^t\big\|_{\mu,0}^2 \\ &\quad + 2\int_{\mathbb{R}^n}\big(f\big(x,u^1(t)\big) - f\big(x,u^2(t)\big)\big)\big(u^1(t) - u^2(t)\big)\,dx + 2\lambda\big|u^1(t) - u^2(t)\big|_2^2 \le 0. \end{aligned} \tag{3.3}$$

In addition, we can also get from $(1.14)_2$ that

$$\frac{d}{dt}\big\|\eta_1^t - \eta_2^t\big\|_{\mu,0}^2 + \frac{\delta}{2}\big\|\eta_1^t - \eta_2^t\big\|_{\mu,0}^2 \le \frac{1}{2\delta}\big|u^1(t) - u^2(t)\big|_2^2. \tag{3.4}$$

By (1.9), it is easy to get that

$$\frac{d}{dt}\big(\big|u^1(t) - u^2(t)\big|_2^2 + \big\|\nabla\eta_1^t - \nabla\eta_2^t\big\|_{\mu,0}^2\big) \le \big\|\nabla\eta_1^t - \nabla\eta_2^t\big\|_{\mu,0}^2 + 2l\big|u^1(t) - u^2(t)\big|_2^2. \tag{3.5}$$

Combining with (3.4) and (3.5), we know that there exists $\kappa = \max\{1, 2l + \frac{1}{2\delta}\}$ such that for any $t \in [0,T]$,

$$\frac{d}{dt}\big(\big|u^1(t) - u^2(t)\big|_2^2 + \big\|\eta_1^t - \eta_2^t\big\|_{\mu,1}^2\big) \le \kappa\big(\big\|\eta_1^t - \eta_2^t\big\|_{\mu,1}^2 + \big|u^1(t) - u^2(t)\big|_2^2\big), \tag{3.6}$$

which implies (3.2). This proof is finished. □

By Lemma 3.2, it is easy to see that the solution semigroup $\{S(t)\}_{t\ge0}$ on $\mathcal{L}_2$ can be defined as

$$S(t): \mathcal{L}_2 \to \mathcal{L}_2, \qquad S(t)z_0 = z(t), \quad \forall t \ge 0. \tag{3.7}$$

Moreover, we know that the semigroup $\{S(t)\}_{t\ge0}$ is a strongly continuous semigroup on the phase space $\mathcal{L}_2$.

3.2 The existence of a bounded absorbing set in $\mathcal{L}_2$

Unless otherwise specified, we always assume that $g \in H$ and the conditions $(\mathbf{H_1})$–$(\mathbf{H_3})$ hold true throughout this article. Furthermore, we use $z(t) = (u(t), \eta^t)$ to denote the solution of the problem (1.14)–(1.15). In this subsection, we mainly address the dissipative feature of the semigroup $\{S(t)\}_{t\ge0}$. To this end, we give the following result.

Lemma 3.3 *Consider any $R > 0$ and $z_0 = (u_0, \eta^0) \in \mathscr{B}(R) \subset \mathcal{L}_2$. Then there exist two constants $c_1 > 0$ and $k_1 > 0$ such that for any $t \geq 0$, whenever $\|z_0\|_{\mathcal{L}_2} \leq R$, one has*

$$\|z(t)\|_{\mathcal{L}_2} \leq Q(R)e^{-c_1 t} + k_1, \tag{3.8}$$

where $k_1 = \frac{1}{c_1}(1 + \frac{4}{\delta^2})(\frac{1}{\lambda}|g|_2^2 + 2\beta_1\|\varphi_1\|_{L^1})$.

Proof Using u to multiply the first equation of (1.14) in H, we can obtain that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(|u|_2^2 + \|\nabla\eta^t\|_{\mu,0}^2\right) + \int_{\mathbb{R}^n} a(x)|\nabla u|^2\,dx + \frac{\delta}{2}\|\nabla\eta^t\|_{\mu,0}^2 \\ &\quad + \int_{\mathbb{R}^n} f(x,u)u\,dx + \frac{\lambda}{2}|u|_2^2 \leq \frac{1}{2\lambda}|g|_2^2, \end{aligned} \tag{3.9}$$

where we used

$$\langle\nabla\eta^t, \nabla\eta_s^t\rangle_{\mathcal{V}_0} \geq \frac{\delta}{2}\|\nabla\eta^t\|_{\mu,0}^2 \quad \text{and} \quad (g,u) \leq \frac{\lambda}{2}|u|_2^2 + \frac{1}{2\lambda}|g|_2^2.$$

From (1.10), we have

$$\int_{\mathbb{R}^n} f(x,u)u\,dx \geq \alpha_1|u|_p^p - \beta_1\|\varphi_1\|_{L^1}. \tag{3.10}$$

Combining with (3.9) and (3.10), it is easy to know that

$$\begin{aligned} &\frac{d}{dt}\left(|u|_2^2 + \|\nabla\eta^t\|_{\mu,0}^2\right) + \int_{\mathbb{R}^n} a(x)|\nabla u|^2\,dx + \delta\|\nabla\eta^t\|_{\mu,0}^2 \\ &\quad + \alpha_1|u|_p^p + \lambda|u|_2^2 \leq \frac{1}{\lambda}|g|_2^2 + 2\beta_1\|\varphi_1\|_{L^1}. \end{aligned} \tag{3.11}$$

Taking $c_1 = \{\lambda, \frac{\delta}{2}\}$ in (3.11), one has

$$\frac{d}{dt}\left(|u|_2^2 + \|\nabla\eta^t\|_{\mu,0}^2\right) + c_1\left(|u|_2^2 + \|\nabla\eta^t\|_{\mu,0}^2\right) \leq \frac{1}{\lambda}|g|_2^2 + 2\beta_1\|\varphi_1\|_{L^1}. \tag{3.12}$$

Applying Gronwall's inequality for (3.12), it follows that

$$|u|_2^2 + \|\nabla\eta^t\|_{\mu,0}^2 \leq Q(R)e^{-c_1 t} + \frac{1}{c_1}\left(\frac{1}{\lambda}|g|_2^2 + 2\beta_1\|\varphi_1\|_{L^1}\right). \tag{3.13}$$

Furthermore, we use η^t to multiply the second equation of (1.14) in $\mathcal{V}_0$ to get

$$\frac{1}{2}\frac{d}{dt}\|\eta^t\|_{\mu,0}^2 + \frac{\delta}{2}\|\eta^t\|_{\mu,0}^2 \leq \int_0^\infty \mu(s)\left(u,\eta^t\right)ds. \tag{3.14}$$

Due to Hölder's and Young's inequalities and (1.8), we have

$$\int_0^\infty \mu(s)\left(u,\eta^t\right)ds \leq \frac{1}{\delta}|u|_2^2 + \frac{\delta}{4}\left|\eta^t(s)\right|_{\mu,2}^2,$$

which, along with (3.14), yields

$$\frac{d}{dt}\left\|\eta^t\right\|_{\mu,0}^2+\frac{\delta}{2}\left\|\eta^t\right\|_{\mu,0}^2\le\frac{2}{\delta}|u|_2^2\le\frac{2}{\delta}\left[Q(R)e^{-c_1t}+\frac{1}{c_1}\left(\frac{1}{\lambda}|g|_2^2+2\beta_1\|\varphi_1\|_{L^1}\right)\right],\tag{3.15}$$

where (3.13) was used. Applying Gronwall's inequality in (3.15) again, we have

$$\left\|\eta^t\right\|_{\mu,0}^2\le e^{-\frac{\delta}{2}t}Q(R)+\frac{4}{\delta^2}\left[Q(R)e^{-c_1t}+\frac{1}{c_1}\left(\frac{1}{\lambda}|g|_2^2+2\beta_1\|\varphi_1\|_{L^1}\right)\right].\tag{3.16}$$

From (3.13) and (3.15), one gets

$$|u|_2^2+\left\|\eta^t\right\|_{\mu,1}^2\le e^{-\frac{\delta}{2}t}Q(R)+\left(1+\frac{4}{\delta^2}\right)\left[Q(R)e^{-c_1t}+\frac{1}{c_1}\left(\frac{1}{\lambda}|g|_2^2+2\beta_1\|\varphi_1\|_{L^1}\right)\right],$$

which implies (3.8). The proof is complete. □

Lemma 3.4 *For any given $R\in\mathbb{R}^+$, let $z_0=(u_0,\eta^0)\in\mathscr{B}(R)\subset\mathcal{L}_2$, then the semigroup $\{S(t)\}_{t\ge0}$ associated with the equation (1.14)–(1.15) admits an $(\mathcal{L}_2,\mathcal{L}_2)$-bounded absorbing set B_0, namely, there exists $t_0=t_0(R)<t$ such that, for any bounded set $B\subset\mathcal{L}_2$, one has*

$$S(t)B\subset B_0,\quad\forall t\ge t_0.$$

In fact, the uniformly bounded absorbing set B_0 can be given by

$$B_0=\left\{z\in\mathcal{L}_2:\|z\|_{\mathcal{L}_2}^2\le\rho_0\right\}.\tag{3.17}$$

Proof From Lemma (3.3), let $t_0:=t_0(R)=\frac{1}{c_1}\ln\frac{Q(R)}{k_1}$ and $\rho_0=2k_1$, then it is easy to see that the conclusion is true. □

Corollary 3.5 *For any $R>0$ and $z_0=(u_0,\eta^0)\in\mathscr{B}(R)\subset\mathcal{L}_2$, there exists $\rho_1=\rho_1(k_1)>0$ such that the relationship*

$$\int_t^{t+1}\left(\left|u(s)\right|_2^2+\int_{\mathbb{R}^n}a(x)\left|\nabla u(s)\right|^2dx+\left\|\eta^s\right\|_{\mu,1}^2+\left|u(s)\right|_p^p\right)ds\le Q(R)e^{-c_1t}+\rho_1$$

holds for any $t\ge0$.

Proof Integrating (3.11) with respect to t over $(t,t+1)$ and combining with Lemma 3.3, it is easy to obtain

$$\begin{aligned}&\int_t^{t+1}\left(\left|u(s)\right|_2^2+\int_{\mathbb{R}^n}a(x)\left|\nabla u(s)\right|^2dx+\left\|\nabla\eta^s\right\|_{\mu,0}^2+\left|u(s)\right|_p^p\right)ds\\&\quad\le\frac{1}{c_2}\left(\frac{1}{\lambda}|g|_2^2+2\beta_1\|\varphi_1\|_{L^1}+\left|u(t)\right|_2^2+\left\|\nabla\eta^t\right\|_{\mu,0}^2\right)\\&\quad\le\frac{1}{c_2}\left(\frac{1}{\lambda}|g|_2^2+2\beta_1\|\varphi_1\|_{L^1}+\left\|z(t)\right\|_{\mathcal{L}_2}^2\right)\\&\quad\le Q(R)e^{-c_1t}+\frac{1}{c_2}\left(\frac{1}{\lambda}|g|_2^2+2\beta_1\|\varphi_1\|_{L^1}\right)+\frac{k_1}{c_2},\end{aligned}\tag{3.18}$$

where $c_2=\min\{1,\delta,\alpha_1,\lambda\}$.

Similarly, we integrate (3.15) with respect to t over $(t,t+1)$ and obtain

$$\begin{aligned}\int_t^{t+1}\|\eta^s\|_{\mu,0}^2\,ds &\le \frac{2}{\delta}\|\eta^t\|_{\mu,0}^2+\frac{4}{\delta^2}\int_t^{t+1}|u(s)|_2^2\,ds\\&\le \frac{2}{\delta}\|z(t)\|_{\mathcal{L}_2}^2+\frac{4}{\delta^2}\left(Q(R)e^{-c_1t}+\frac{1}{c_2}\left(\frac{1}{\lambda}|g|_2^2+2\beta_1\|\varphi_1\|_{L^1}\right)+\frac{k_1}{c_2}\right)\\&\le Q(R)e^{-c_1t}+\frac{4}{c_2\delta^2}\left(\frac{1}{\lambda}|g|_2^2+2\beta_1\|\varphi_1\|_{L^1}\right)+\frac{4k_1}{c_2\delta^2}+\frac{2k_1}{\delta},\end{aligned}\tag{3.19}$$

where (3.8) and (3.18) were used.

Let

$$\rho_1=\frac{4+\delta^2}{c_2\delta^2}\left(\frac{1}{\lambda}|g|_2^2+2\beta_1\|\varphi_1\|_{L^1}\right)+\frac{4k_1}{c_2\delta^2}+\frac{2k_1}{\delta}+\frac{k_1}{c_2},$$

then from (3.18) and (3.19) we can get the desire conclusion. The proof is finished. □

To verify the asymptotic compactness of solution semigroup $\{S(t)\}_{t\ge0}$, the cut-off function technique shall be used to derive the following estimate.

Lemma 3.6 *Let B be any bounded subset $\mathcal{L}_2$ and $z_0\in B$. Then for any $\epsilon>0$, there exist the positive constants $K=K(\epsilon,B)$ and $T_2=T_2(\epsilon,B)$ such that for every $k\ge K$ and $t\ge T_2$,*

$$\int_{B_k^c}|u|^2\,dx+\int_0^\infty\mu(s)\int_{B_k^c}\left(\left|\nabla\eta^t(s)\right|^2+\left|\eta^t(s)\right|^2\right)dx\,ds\le C\epsilon,$$

where $B_k^c=\{x\in\mathbb{R}^n:|x|\ge k\}$.

Proof Let $\theta(\cdot):\mathbb{R}^+\to\mathbb{R}$ be a smooth function satisfying $0\le\theta(s)\le1$ for any $s\in\mathbb{R}^+$ and

$$\theta(s)=0,\quad \forall s\in[0,1);\qquad \theta(s)=1,\quad \forall s\in[2,\infty),$$

then it is easy to see that there exists a positive constant γ such that

$$0\le\theta'(s)\le\gamma,\quad \forall s\in\mathbb{R}^+.$$

Set $\theta_k=\theta(\frac{|x|^2}{k^2})$. Then we multiply the first equation of (1.14) by $\theta_k u$ in H to get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\theta_k|u|^2\,dx-\int_{\mathbb{R}^n}\theta_k u\,\mathrm{div}\{a(x)\nabla u\}\,dx+\lambda\int_{\mathbb{R}^n}\theta_k|u|^2\,dx\\&\quad-\int_0^\infty\mu(s)\int_{\mathbb{R}^n}\Delta\eta^t(s)\theta_k u\,dx\,ds+\int_{\mathbb{R}^n}\theta_k uf(x,u)\,dx=\int_{\mathbb{R}^n}\theta_k ug\,dx.\end{aligned}\tag{3.20}$$

We now deal with each term of the above formula (3.20). First of all, for the second term on the left-hand side of (3.20), one has

$$-\int_{\mathbb{R}^n}\theta_k u\,\mathrm{div}\{a(x)\nabla u\}\,dx=\int_{\mathbb{R}^n}\theta_k a(x)|\nabla u|^2\,dx+\frac{2}{k^2}\int_{\mathbb{R}^n}\theta_k'(x\cdot a(x)\nabla u)u\,dx.\tag{3.21}$$

For the fourth term on the left-hand side of (3.20), we get

$$
\begin{aligned}
&-\int_0^\infty \mu(s)\int_{\mathbb{R}^n} \Delta\eta^t(s)\theta_k u\,dx\,ds \\
&\quad= \int_0^\infty \mu(s)\int_{\mathbb{R}^n} \theta_k \nabla\eta^t(s)\cdot\nabla u\,dx\,ds + \frac{2}{k^2}\int_0^\infty \mu(s)\int_{\mathbb{R}^n} \theta_k' u\big(x\cdot\nabla\eta^t(s)\big)\,dx\,ds \\
&\quad\geq \frac{1}{2}\frac{d}{dt}\int_0^\infty \mu(s)\int_{\mathbb{R}^n} \theta_k\big|\nabla\eta^t(s)\big|^2\,dx\,ds + \frac{\delta}{2}\int_0^\infty \mu(s)\int_{\mathbb{R}^n} \theta_k\big|\nabla\eta^t(s)\big|^2\,dx\,ds \\
&\qquad + \frac{2}{k^2}\int_0^\infty \mu(s)\int_{\mathbb{R}^n} \theta_k' u\big(x\cdot\nabla\eta^t(s)\big)\,dx\,ds.
\end{aligned}
\tag{3.22}
$$

For the fifth term on the left-hand side of (3.20), by (1.10) we can obtain

$$
\int_{\mathbb{R}^n} \theta_k u f(x,u)\,dx \geq \alpha_1\int_{\mathbb{R}^n} \theta_k|u|^p\,dx - \beta_1\int_{\mathbb{R}^n} \theta_k\varphi_1(x)\,dx.
\tag{3.23}
$$

By Hölder's and Young's inequalities, we can handle the right-hand side of (3.20) and obtain

$$
\int_{\mathbb{R}^n} \theta_k u g\,dx \leq \frac{\lambda}{2}\int_{\mathbb{R}^n} \theta_k|u|^2\,dx + \frac{2}{\lambda}\int_{\mathbb{R}^n} \theta_k|g|^2\,dx.
\tag{3.24}
$$

By virtue of (3.20)–(3.24), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\int_{\mathbb{R}^n} \theta_k|u|^2\,dx + \int_0^\infty \mu(s)\int_{\mathbb{R}^n} \theta_k\big|\nabla\eta^t(s)\big|^2\,dx\,ds\right) + \int_{\mathbb{R}^n} \theta_k a(x)|\nabla u|^2\,dx \\
&\qquad + \frac{\delta}{2}\int_0^\infty \mu(s)\int_{\mathbb{R}^n} \theta_k\big|\nabla\eta^t(s)\big|^2\,dx\,ds + \frac{\lambda}{2}\int_{\mathbb{R}^n} \theta_k|u|^2\,dx + \alpha_1\int_{\mathbb{R}^n} \theta_k|u|^p\,dx \\
&\quad\leq -\frac{2}{k^2}\int_{\mathbb{R}^n} \theta_k'\big(x\cdot a(x)\nabla u\big)u\,dx - \frac{2}{k^2}\int_0^\infty \mu(s)\int_{\mathbb{R}^n} \theta_k' u\big(x\cdot\nabla\eta^t(s)\big)\,dx\,ds \\
&\qquad + \beta_1\int_{\mathbb{R}^n} \theta_k\varphi_1(x)\,dx + \frac{2}{\lambda}\int_{\mathbb{R}^n} \theta_k|g|^2\,dx.
\end{aligned}
\tag{3.25}
$$

According to the definition of θ_k, it is easy to find that

$$
\theta_k' = 0, \quad \text{when } |x| < k \text{ or } |x| > \sqrt{2}k.
$$

Therefore,

$$
\begin{aligned}
&-\frac{2}{k^2}\int_0^\infty \mu(s)\int_{\mathbb{R}^n} \theta_k' u\big(x\cdot\nabla\eta^t(s)\big)\,dx\,ds \\
&\quad\leq \frac{2\gamma}{k^2}\int_0^\infty \mu(s)\int_{k\leq|x|\leq\sqrt{2}k} |u||x|\big|\nabla\eta^t(s)\big|\,dx\,ds \\
&\quad\leq \frac{2\sqrt{2}\gamma}{k}\int_0^\infty \mu(s)\int_{k\leq|x|\leq\sqrt{2}k} |u|\big|\nabla\eta^t(s)\big|\,dx\,ds \\
&\quad\leq \frac{2\sqrt{2}\gamma}{k}\left(\frac{k\lambda}{8\sqrt{2}\gamma}\int_0^\infty \mu(s)\int_{k\leq|x|\leq\sqrt{2}k} |u|^2\,dx\,ds\right.
\end{aligned}
\tag{3.26}
$$

$$
\begin{aligned}
&\quad + \frac{2\sqrt{2}\gamma}{k\lambda} \int_0^\infty \mu(s) \int_{k\le|x|\le\sqrt{2}k} \left|\nabla\eta^t(s)\right|^2 dx\,ds\Big) \\
&\le \frac{\lambda}{4} \int_0^\infty \mu(s) \int_{k\le|x|\le\sqrt{2}k} |u|^2 dx\,ds + \frac{8\gamma^2}{k^2\lambda} \int_0^\infty \mu(s) \int_{k\le|x|\le\sqrt{2}k} \left|\nabla\eta^t(s)\right|^2 dx\,ds \\
&\le \frac{\lambda}{4}|u|_2^2 + \frac{8\gamma^2}{k^2\lambda}\left\|\nabla\eta^t\right\|_{\mu,0}^2.
\end{aligned}
$$

Similarly, by (1.4), we can obtain

$$
\begin{aligned}
&-\frac{2}{k^2}\int_{\mathbb{R}^n} \theta_k'\big(x\cdot a(x)\nabla u\big)u\,dx \\
&\quad\le \frac{2\sqrt{2}\gamma}{k}\int_{k\le|x|\le\sqrt{2}k} \left|a(x)\right||\nabla u||u|\,dx \\
&\quad\le \frac{\sqrt{2}\gamma}{k}\left(\frac{k}{4\sqrt{2}\gamma}\int_{k\le|x|\le\sqrt{2}k} a(x)|\nabla u|^2 dx + \frac{\sqrt{2}\gamma}{k}\int_{k\le|x|\le\sqrt{2}k} a(x)|u|^2 dx\right) \\
&\quad\le \frac{1}{4}\int_{k\le|x|\le\sqrt{2}k} a(x)|\nabla u|^2 dx + \frac{C_\gamma}{k^2}\int_{k\le|x|\le\sqrt{2}k} |u|^2 dx \\
&\quad\le \frac{1}{4}\int_{\mathbb{R}^n} \theta_k a(x)|\nabla u|^2 dx + \frac{C_\gamma}{k^2}|u|_2^2.
\end{aligned}
\tag{3.27}
$$

From (3.25)–(3.26), we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_{\mathbb{R}^n}\theta_k|u|^2 dx + \int_0^\infty \mu(s)\int_{\mathbb{R}^n}\theta_k\left|\nabla\eta^t(s)\right|^2 dx\,ds\right) + \frac{3}{2}\int_{\mathbb{R}^n}\theta_k a(x)|\nabla u|^2 dx \\
&\qquad + \delta\int_0^\infty \mu(s)\int_{\mathbb{R}^n}\theta_k\left|\nabla\eta^t(s)\right|^2 dx\,ds + \frac{\lambda}{2}\int_{\mathbb{R}^n}\theta_k|u|^2 dx + \alpha_1\int_{\mathbb{R}^n}\theta_k|u|^p dx \\
&\quad\le \frac{C_\gamma}{k^2}|u|_2^2 + \frac{8\gamma^2}{k^2\lambda}\left\|\nabla\eta^t\right\|_{\mu,0}^2 + \beta_1\|\varphi_1\|_{L^1} + \frac{2}{\lambda}\int_{\mathbb{R}^n}\theta_k|g|^2 dx \\
&\quad\le \frac{C}{k^2}\left(|u|_2^2 + \left\|\nabla\eta^t\right\|_{\mu,0}^2\right) + 2\beta_1\int_{\mathbb{R}^n}\theta_k\varphi_1(x)\,dx + \frac{4}{\lambda}\int_{\mathbb{R}^n}\theta_k|g|^2 dx.
\end{aligned}
\tag{3.28}
$$

Taking $c_3 = \min\{\delta, \frac{\lambda}{2}\}$, (3.28) implies that

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_{\mathbb{R}^n}\theta_k|u|^2 dx + \int_0^\infty \mu(s)\int_{\mathbb{R}^n}\theta_k\left|\nabla\eta^t(s)\right|^2 dx\,ds\right) \\
&\qquad + c_3\left(\int_{\mathbb{R}^n}\theta_k|u|^2 dx + \int_0^\infty \mu(s)\int_{\mathbb{R}^n}\theta_k\left|\nabla\eta^t(s)\right|^2 dx\,ds\right) \\
&\quad\le \frac{C}{k^2}\left(|u|_2^2 + \left\|\nabla\eta^t\right\|_{\mu,0}^2\right) + 2\beta_1\int_{\mathbb{R}^n}\theta_k\varphi_1(x)\,dx + \frac{4}{\lambda}\int_{\mathbb{R}^n}\theta_k|g|^2 dx.
\end{aligned}
$$

Due to $\varphi_1 \in L^1(\mathbb{R}^n)$, there exists $k_1 = k_1(\epsilon) > 0$ such that for all $k > k_1(\epsilon)$,

$$
\int_{\mathbb{R}^n}\theta_k\varphi_1(x)\,dx \le \frac{\epsilon}{2\beta_1}.
$$

Similarly, since $g \in H$, there exists $k_2 = k_2(\epsilon) > 0$ such that for all $k > k_2(\epsilon)$,

$$
\int_{\mathbb{R}^n}\theta_k|g|^2 dx \le \frac{\lambda\epsilon}{4}.
$$

Thus, by Lemma 3.4 and the aforementioned two estimates, we know that for all $t \geq t_0$,

$$\begin{aligned} &\frac{d}{dt}\left(\int_{\mathbb{R}^n} \theta_k |u|^2\,dx + \int_0^\infty \mu(s)\int_{\mathbb{R}^n} \theta_k \left|\nabla\eta^t(s)\right|^2\,dx\,ds\right) \\ &\quad + c_3\left(\int_{\mathbb{R}^n} \theta_k |u|^2\,dx + \int_0^\infty \mu(s)\int_{\mathbb{R}^n} \theta_k \left|\nabla\eta^t(s)\right|^2\,dx\,ds\right) \\ &\leq \frac{C\rho_0}{k^2} + 2\epsilon. \end{aligned} \tag{3.29}$$

Applying Gronwall's lemma, we get

$$\begin{aligned} &\int_{\mathbb{R}^n} \theta_k |u|^2\,dx + \int_0^\infty \mu(s)\int_{\mathbb{R}^n} \theta_k \left|\nabla\eta^t(s)\right|^2\,dx\,ds \\ &\quad \leq e^{-c_3 t}\left(\int_{\mathbb{R}^n} \theta_k |u_0|^2\,dx + \int_0^\infty \mu(s)\int_{\mathbb{R}^n} \theta_k \left|\nabla\eta^0(s)\right|^2\,dx\,ds\right) + \frac{C\rho_0}{k^2} + \frac{2}{c_3}\epsilon \\ &\quad \leq e^{-c_3 t}Q(R) + \frac{C\rho_0}{k^2} + \frac{2}{c_3}\epsilon. \end{aligned}$$

For the above given $\epsilon > 0$, let us take $K =: K(\epsilon) = \max\{k_1, k_2, \sqrt{\frac{C\rho_0}{\epsilon}}\}$, then there exists $T_0 =: T_0(\epsilon) = \max\{T_0, \frac{1}{c_3}\ln\frac{Q(R)}{\epsilon}\}$ such that, when $t \geq T_0$ and $k \geq K$, one has

$$\int_{\mathbb{R}^n} \theta_k |u|^2\,dx + \int_0^\infty \mu(s)\int_{\mathbb{R}^n} \theta_k \left|\nabla\eta^t(s)\right|^2\,dx\,ds \leq C\epsilon. \tag{3.30}$$

In addition, if we use $\theta(\frac{|x|^2}{k^2})\eta^t$ to take inner product with the second equation of (1.14) on $\mathcal{V}_0$, then by ($\mathbf{H_2}$), it yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \theta\left(\frac{|x|^2}{k^2}\right)\int_0^\infty \mu(s)\left|\eta^t(s)\right|^2\,ds\,dx \\ &\quad + \frac{\delta}{2}\int_{\mathbb{R}^n} \theta\left(\frac{|x|^2}{k^2}\right)\int_0^\infty \mu(s)\left|\eta^t(s)\right|^2\,ds\,dx \\ &\leq \int_{\mathbb{R}^n} \theta\left(\frac{|x|^2}{k^2}\right)\int_0^\infty \mu(s)\eta^t(s)u\,ds\,dx \\ &\leq \frac{1}{\delta}\int_{\mathbb{R}^n} \theta\left(\frac{|x|^2}{k^2}\right)|u|^2\,dx + \frac{\delta}{4}\int_{\mathbb{R}^n} \theta\left(\frac{|x|^2}{k^2}\right)\int_0^\infty \mu(s)\left|\eta^t(s)\right|^2\,ds\,dx. \end{aligned} \tag{3.31}$$

Combining with (3.30) and (3.31), we have

$$\frac{d}{dt}\int_{\mathbb{R}^N} \theta_k \int_0^\infty \mu(s)\left|\eta^t(s)\right|^2\,ds\,dx + \frac{\delta}{2}\int_{\mathbb{R}^N} \theta_k \int_0^\infty \mu(s)\left|\eta^t(s)\right|^2\,ds\,dx \leq \frac{C\epsilon}{\delta}. \tag{3.32}$$

Using Gronwall's lemma, we get

$$\begin{aligned} \int_0^\infty \mu(s)\int_{\mathbb{R}^n} \theta_k \left|\eta^t(s)\right|^2\,dx\,ds &\leq e^{-\frac{\delta}{2}t}\int_0^\infty \mu(s)\int_{\mathbb{R}^n} \theta_k \left|\eta^0(s)\right|^2\,dx\,ds + \frac{C\epsilon}{\delta^2} \\ &\leq Q(R)e^{-\frac{\delta}{2}t} + \frac{C\epsilon}{\delta^2}, \end{aligned} \tag{3.33}$$

and then there exists $T_1 =: T_1(\varepsilon) = \frac{2}{\delta} \ln \frac{Q(R)\delta^2}{C\epsilon}$ such that, when $t \geq T_1$,

$$\int_0^\infty \mu(s) \int_{\mathbb{R}^n} \theta_k \left|\eta^t(s)\right|^2 dx\, ds \leq C\epsilon. \tag{3.34}$$

By (3.30) and (3.34), we know that there exist K and $T_2 = \max\{T_0, T_1\}$ such that

$$\int_{\mathbb{R}^n} \theta_k |u|^2 dx + \int_0^\infty \mu(s) \int_{\mathbb{R}^n} \theta_k \left(\left|\nabla \eta^t(s)\right|^2 + \left|\eta^t(s)\right|^2\right) dx\, ds \leq C\epsilon$$

holds for all $k \geq K$ and $t \geq T_2$. So one has

$$\int_{B_k^c} |u|^2 dx + \int_0^\infty \mu(s) \int_{B_k^c} \left(\left|\nabla \eta^t(s)\right|^2 + \left|\eta^t(s)\right|^2\right) ds \leq C\epsilon.$$

The proof is complete. □

3.3 Asymptotic compactness

In this subsection, we will prove the existence of global attractors in $\mathcal{L}_2$ through the semigroup $S(t)$ defined by (3.7). In order to prove Theorem 3.8, first, we give the following lemma.

Lemma 3.7 *Assume that $z_n(t) = (u_n(t), \eta_n^t)$ $(n = 1, 2, \dots)$ are solutions of the problem (1.14)–(1.15). In addition, for any $k > 0$ and given $T > 0$, let $a \in \mathcal{C}(\bar{B}_k)$, where*

$$B_k = \left\{x \in \mathbb{R}^n : |x| < k\right\}.$$

Then there exists a subsequence of $\{u_n(t)\}$ such that it is convergent in $L^2(0, L^2(B_k))$.

Proof First of all, by Corollary 3.5, we can obtain that

$$\{u_n\}_{n=1}^\infty \text{ is uniformly bounded in } L^2\left(0, T; \mathcal{H}^1(B_k, a)\right), \tag{3.35}$$

as well as by further utilizing the assumption (1.10) in $(\mathbf{H_3})$, it is easy to show the following claim:

$$\left\{f(u_n)\right\}_{n=1}^\infty \text{ is uniformly bounded in } L^{\frac{p}{p-1}}\left(0, T; L^{\frac{p}{p-1}}(B_k)\right). \tag{3.36}$$

Next, for any $v \in \mathcal{C}([0, T]; \mathcal{C}_0^\infty(B_k))$, it is easy to get that

$$\begin{aligned}\int_0^T \int_{B_k} u_{nt} v\, dx\, dt = &\int_0^T \int_{B_k} \left(-a(x)\nabla u_n \cdot \nabla v - \left(\int_0^\infty \mu(s) \nabla \eta_n^t(s)\, ds\right) \nabla v\right) dx\, dt \\ &+ \int_0^T \int_{B_k} \left(-f(u_n)v - \lambda u_n v + g v\right) dx\, dt.\end{aligned} \tag{3.37}$$

By (3.37), we have

$$
\begin{aligned}
&\left|\int_0^T\int_{B_k} u_{nt} v \,dx\,dt\right| \\
&\quad\le \left(\int_0^T\int_{B_k} a(x)|\nabla u_n|^2\,dx\,dt\right)^{1/2}\left(\int_0^T\int_{B_k} a(x)|\nabla v|^2\,dx\,dt\right)^{1/2} \\
&\qquad + \left\|\nabla \eta_n^t\right\|_{L^2(0,T;L^2_\mu(\mathbb{R}^+;L^2(B_k)))}\|\nabla v\|_{L^2(0,T;L^2(B_k))} \\
&\qquad + \left\|f(u_n)\right\|_{L^{\frac{p}{p-1}}(0,T;L^{\frac{p}{p-1}}(B_k))}\|v\|_{L^p(0,T;L^p(B_k))} \\
&\qquad + \lambda\|u_n\|_{L^2(0,T;L^2(B_k))}\|v\|_{L^2(0,T;L^2(B_k))} \\
&\qquad + T^{1/2}|g|_2\|v\|_{L^2(0,T;L^2(B_k))} \\
&\quad\le C\big(\|a\|^{1/2}_{C(\bar{B}_k)},T,|g|_2\big)\left[\left(\int_0^T\int_{B_k} a(x)|\nabla u_n|^2\,dx\,dt\right)^{1/2} + \|u_n\|_{L^2(0,T;L^2(B_k))}\right. \\
&\qquad \left. + \left\|\nabla \eta_n^t\right\|_{L^2(0,T;L^2_\mu(\mathbb{R}^+;L^2(B_k)))} + \left\|f(u_n)\right\|_{L^{\frac{p}{p-1}}(0,T;L^{\frac{p}{p-1}}(B_k))} + 1\right] \\
&\qquad \times \left[\|v\|_{L^2(0,T;H^1(B_k))} + \|v\|_{L^p(0,T;L^p(B_k))}\right].
\end{aligned}
$$

Since $C([0,T];C_0^\infty(B_k))$ is dense in $L^2(0,T;H^1(B_k))\cap L^p(0,T;L^p(B_k))$, by Corollary 3.5 and (3.36) in $(\mathbf{H_3})$, we further know that

$$
\{u_{nt}\}_{n=1}^\infty \text{ is uniformly bounded in } L^2\big(0,T;H^{-1}(B_k)\big) + L^{\frac{p}{p-1}}\big(0,T;L^{\frac{p}{p-1}}(B_k)\big). \tag{3.38}
$$

Combining with (3.35), (3.38), and Lemma 2.4, one knows that there exists a subsequence of $\{u_n(t)\}$ (not relabeled) such that

$$
u_n \to u \quad \text{strongly in } L^2\big(0,T;L^2(B_k)\big).
$$

This proof is finished. □

Theorem 3.8 *Under the assumptions of Lemma 3.7, the semigroup $\{S(t)\}_{t\ge 0}$ generated by the solutions of the system (1.14)–(1.15) is an asymptotic contractive semigroup on $\mathcal{L}_2$.*

Proof For any $z_0^i \in B_0$ $(i=1,2)$, suppose that $z_i(t) = (u_i(t),\eta_i^t) = S(t)z_0^i$ $(i=1,2)$ are solutions of the following equation:

$$
\begin{cases}
u_t^i - \operatorname{div}\{a(x)\nabla u^i\} - \int_0^\infty \mu(s)\Delta\eta_i^t(s)\,ds + \lambda u^i + f(x,u^i) = g & \text{in } \mathbb{R}^n, \\
\partial_t \eta_i^t = u^i - \partial_s \eta_i^t,
\end{cases} \tag{3.39}
$$

with initial conditions

$$
\begin{cases}
u(x,0) = u_0^i(x), & x\in\mathbb{R}^n, \\
\eta^0(x,s) = \int_0^s u_0(x,-r)\,dr, & (x,s)\in\mathbb{R}^n\times\mathbb{R}^+.
\end{cases}
$$

Let $(\omega(t),\varsigma^t)=(u_1(t)-u_2(t),\eta_1^t-\eta_2^t)$, then it satisfies the following system:

$$\begin{cases}\omega_t-\operatorname{div}\{a(x)\nabla\omega\}-\int_0^\infty\mu(s)\Delta\varsigma^t(s)\,ds+\lambda\omega+f(x,u^1)-f(x,u^2)=0,\\ \varsigma_t^t=\omega-\varsigma_s^t,\\ \omega_0=u_0^1-u_0^2,\quad x\in\mathbb{R}^n\\ \varsigma^0=\eta_1^0-\eta_2^0,\quad (x,s)\in\mathbb{R}^n\times\mathbb{R}^+.\end{cases}\tag{3.40}$$

Multiplying the first equation of (3.40) by ω and integrating over $\mathbb{R}^n$, we get

$$\begin{aligned}&\frac{d}{dt}\big(|\omega|_2^2+\|\nabla\varsigma^t\|_{\mu,0}^2\big)+2\int_{\mathbb{R}^n}a(x)|\nabla\omega|^2\,dx+2\lambda|\omega|_2^2+\delta\|\nabla\varsigma^t\|_{\mu,0}^2\\&\quad+2\int_{\mathbb{R}^N}\big(f(x,u^1)-f(x,u^2)\big)\big(u^1-u^2\big)\,dx\le 0.\end{aligned}\tag{3.41}$$

Taking the inner product of the second equation of (3.40) and θ^t on $\mathcal{V}_0$ and using Young's inequality gives

$$\frac{d}{dt}\|\varsigma^t\|_{\mu,0}+\frac{\delta}{2}\|\varsigma^t\|_{\mu,0}\le\frac{2}{\delta}|\omega|_2^2.\tag{3.42}$$

Moreover, by (1.9), we have

$$\int_{\mathbb{R}^N}\big(f(x,u^1)-f(x,u^2)\big)\big(u^1-u^2\big)\,dx\ge -l|\omega|_2^2.\tag{3.43}$$

Together with (3.41)–(3.43), one gets

$$\frac{d}{dt}\big(|\omega|_2^2+\|\varsigma^t\|_{\mu,1}^2\big)+c_4\big(|\omega|_2^2+\|\varsigma^t\|_{\mu,1}^2\big)\le\left(l+\frac{2}{\delta}\right)|\omega|_2^2,$$

where $c_4=\min\{\lambda,\frac{\delta}{2}\}$.

So, from Gronwall's inequality, we get

$$|\omega|_2^2+\|\varsigma^t\|_{\mu,1}^2\le e^{-c_4t}\big(|\omega_0|_2^2+\|\varsigma^0\|_{\mu,1}^2\big)+\left(l+\frac{2}{\delta}\right)e^{-c_4t}\int_0^t e^{c_4s}\big|\omega(s)\big|_2^2\,ds.\tag{3.44}$$

For a fixed $T_2>0$ (from Lemma 3.6), one has

$$\begin{aligned}|\omega|_2^2+\|\varsigma^t\|_{\mu,1}^2&\le e^{-c_4t}\big(|\omega_0|_2^2+\|\varsigma^0\|_{\mu,1}^2\big)\\&\quad+\left(l+\frac{2}{\delta}\right)e^{-c_4t}\int_0^{T_2}e^{c_4s}\big|\omega(s)\big|_2^2\,ds\\&\quad+\left(l+\frac{2}{\delta}\right)e^{-c_4t}\int_{T_2}^t e^{c_4s}\big|\omega(s)\big|_2^2\,ds,\end{aligned}\tag{3.45}$$

Thus, there exists $\mathscr{T}=\mathscr{T}(\varepsilon)\ge T_2$ such that

$$|\omega|_2^2+\|\varsigma^t\|_{\mu,1}^2\le\epsilon+\left(l+\frac{2}{\delta}\right)e^{-c_4t}\int_{T_2}^t\big|\omega(s)\big|_2^2e^{\alpha_2s}\,ds\tag{3.46}$$

holds true for any $\epsilon>0$ and all $t\ge\mathscr{T}$.

Let $T \geq \mathscr{T}$ be given and consider

$$\begin{aligned}\psi_T(u^1,u^2) &= \left(l+\frac{2}{\delta}\right)e^{-c_4T}\int_{T_2}^{T}\left|\omega(s)\right|_2^2 e^{c_4s}\,ds\\ &= \left(l+\frac{2}{\delta}\right)e^{-c_4T}\int_{T_2}^{T}e^{c_4s}\left(\int_{B_k}\left|\omega(s)\right|^2dx+\int_{B_k^c}\left|\omega(s)\right|^2dx\right)ds\end{aligned}$$

and

$$\begin{aligned}\phi_T(u^1,u^2) &= \left(l+\frac{2}{\delta}\right)e^{-c_4T}\int_{T_2}^{T}e^{c_4s}\int_{B_k}\left|\omega(s)\right|^2dx\,ds\\ &\leq \left(l+\frac{2}{\delta}\right)\int_0^T\int_{B_k}\left|\omega(s)\right|^2dx\,ds = \left(l+\frac{2}{\delta}\right)\|\omega\|^2_{L^2(0,T;L^2(B_k))}.\end{aligned}$$

By Lemma 3.6, for any $\epsilon > 0$ and $k \geq K(\epsilon)$, one has

$$\int_{B_k^c}\left|\omega(s)\right|^2dx \leq \frac{\delta c_4\epsilon}{2\delta l+4}.$$

Thus, we get that

$$\psi_T(u^1,u^2) \leq \varepsilon + \phi_T(u^1,u^2)$$

and

$$\left\|S(T)u^1 - S(T)u^2\right\|^2_{\mathcal{L}_2} \leq \varepsilon + \psi_T(u^1,u^2).$$

Combining with Lemma 3.7 and Definition 2.1, we know that ϕ_T is a contractive function. Therefore, ψ_T is an asymptotically contractive function, which implies that the semigroup $\{S(t)\}_{t\geq 0}$ is an asymptotically contractive semigroup on $\mathcal{L}_2$ by Definition 2.2. This proof is complete. □

Now, we will present the main conclusion.

Theorem 3.9 *Under the assumptions of Lemma 3.7, assume that $\{S(t)\}_{t\geq 0}$ is the solution semigroup of equation (1.1) with initial value $z_0 \in \mathcal{L}_2$, then $\{S(t)\}_{t\geq 0}$ possesses a nonempty, invariable, compact global attractor in $\mathcal{L}_2$, which attracts any bounded set of $\mathcal{L}_2$.*

Proof Since we have proved Lemma 3.4 and Theorem 3.8, together with Theorem 2.3, we now easily get the existence of the global attractor $\tilde{\mathscr{A}}$ for the semigroup $S(t)$ defined by (3.7) in $\mathcal{L}_2$. □

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

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R.G. and X.L. wrote the main manuscript text and revised it. All authors reviewed the manuscript.

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