# Infinite system of nonlinear tempered fractional order BVPs in tempered sequence spaces 

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Abstract
This paper deals with the existence results of the infinite system of tempered fractional BVPs

$$
\begin{aligned}
& { }_{0}^{\mathrm{R}} D_{r}^{Q, \lambda} z_{j}(\mathrm{r})+\psi_{j}(\mathrm{r}, \mathrm{z}(\mathrm{r}))=0, \quad 0<\mathrm{r}<1, \\
& z_{j}(0)=0, \quad{ }_{0}^{\mathrm{R}} \mathrm{D}_{\mathrm{r}}^{\mathrm{m}, \lambda} z_{j}(0)=0, \\
& \mathrm{~b}_{1} z_{j}(1)+\mathrm{b}_{2}{ }_{0}^{\mathrm{R}} \mathrm{D}_{\mathrm{r}}^{\mathrm{m}, \lambda} z_{j}(1)=0,
\end{aligned}
$$

where $j \in \mathbb{N}, 2<\varrho \leq 3,1<m \leq 2$, by utilizing the Hausdorff measure of noncompactness and Meir-Keeler fixed point theorem in a tempered sequence space.

Keywords: Tempered fractional derivative; Measure of noncompactness; Meir-Keeler fixed point theorem; Tempered sequence spaces

## 1 Introduction

Fractional calculus is an important branch of mathematics that deals with the description of the possibility of computation of unknown functions via suitable derivative and integral operators of real order and studies the relationships between them [17, 19, 21]. The concept of fractional calculus has been widely used to model physical problems engineering systems, and their applications which significantly leads to a set of linear or nonlinear fractional differential equations [8, 35, 39, 42, 43]. The development of this discipline has inspired scholars to look into its existence and uniqueness [14, 28, 32-34, 38].

The tempered fractional derivative is obtained by multiplying the fractional derivative by an exponential factor. The traditional Riemann-Liouville (RL) and Caputo fractional derivatives are obtained under special circumstances for $\lambda=0$, and this new fractional operator depends on the parameter $\lambda$. Due to its use in physics, groundwater hydrology, poroelasticity, geophysical flow, and finance [7,12, 13, 22, 23, 36], the tempered fractional derivative has recently gained popularity as a subject of study. In [44], Zaky studied the

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well-posedness of the solution to the following two-point nonlinear tempered fractional boundary value problem (TFBVP)
\[

$$
\begin{aligned}
& { }_{0}^{c} D_{r}^{\alpha, \lambda} z(r)=g(r, z(r)) \quad r \in[0, T], \\
& \alpha z(0)+\beta e^{\lambda T} z(T)=\gamma,
\end{aligned}
$$
\]

where $g \in C([0, T] \times \mathbb{R}, \mathbb{R}), \alpha, \beta, \gamma$ are real constants with $\alpha+\beta \neq 0$, and ${ }_{0}^{c} D_{r}^{\alpha, \lambda}$ is the Caputo tempered fractional derivative of rational order $\alpha \in(0,1)$. Pandey et al. [29] by applying fractional variational approach, studied the properties of eigenvalues for the TFBVP

$$
\begin{aligned}
& { }_{\mathrm{r}}^{\mathrm{c}} D_{b}^{\vartheta, \lambda}\left[\phi(\mathrm{r})_{a}^{\mathrm{c}} D_{\mathrm{r}}^{\vartheta, \lambda} \mathrm{z}(\mathrm{r})\right]+\phi(\mathrm{r}) \mathrm{z}(\mathrm{r})=\mu \psi_{\mu}(\mathrm{r}) z(\mathrm{r}), \quad \mathrm{r} \in[\alpha, \beta], \vartheta \in(0,1), \\
& \mathrm{z}(\alpha)=\mathrm{z}(\beta)=0,
\end{aligned}
$$

where $\phi, \phi$ and $\psi_{\mu}$ are real-valued continuous functions defined on $[\alpha, \beta]$. Recently, Khuddush and Prasad [15] studied the thermistor problem with two-point boundary conditions

$$
\begin{aligned}
& { }_{0}^{{ }_{0}^{\mathrm{C}}} \mathrm{D}_{\mathrm{r}}^{2 \varrho, \lambda} \mathrm{z}(\mathrm{r})=\frac{\mu \mathrm{g}(\mathrm{z}(\mathrm{r}))}{\left[\int_{0}^{\mathrm{T}} \mathrm{~g}(\mathrm{z}(\tau)) \mathrm{d} \tau\right]^{2}}, \quad 0<\varrho<\frac{1}{2}, 0<\mathrm{r}<\mathrm{T}, \\
& a \mathrm{z}(0)+b e^{\lambda \mathrm{T}} \mathrm{z}(\mathrm{~T})=c,
\end{aligned}
$$

where $\lambda \geq 0, \mu>0,{ }^{c} D_{0^{+}}^{2 \rho}$ denotes the tempered Caputo fractional derivative of order $2 \varrho$, $g \in C([0, T] \times \mathbb{R}, \mathbb{R})$ and derived sufficient conditions for the existence, uniqueness and Hyers-Ulam stability of solutions.
The measure of noncompactness (MNC) plays a dominant role in functional analysis, as introduced by Kuratowski [18]. In [41], Srivastava et al. applied the MNC on $\mathrm{C}([0, a] \times[0, a])$ to study the two variable functional integral equations. Many scholars have also used the notion of an MNC for the existence of solutions for results of infinite systems of differential and integral equations $[2,6,10,11,25,26,30,31,40]$. In the following, we mention a few recent works on MNC. By utilizing the Hausdorff measure of noncompactness (HMNC) in tempered sequence spaces, Das et al. [9] established the existence of solutions to the infinite system of TFBVP

$$
\begin{gathered}
{ }_{0}^{\mathrm{R}} \mathrm{D}_{0^{+}}^{\beta}\left(\mathrm{z}_{\mathrm{k}}(\mathrm{r})\right)+\mathrm{h}_{\mathrm{k}}\left(\mathrm{r}, \mathrm{z}_{\mathrm{k}}(\mathrm{r})\right), \quad 0<\mathrm{r}<\mathrm{T}, \\
\mathrm{z}_{\mathrm{k}}(0)=\mathrm{z}_{\mathrm{k}}(\mathrm{~T})=0, \quad \mathrm{k}=1,2,3, \ldots,
\end{gathered}
$$

where $h_{k} \in C([0, T], \mathbb{R})$, and ${ }_{0}^{\mathrm{R}} \mathrm{D}_{0^{+}}^{\beta}$ is a Riemann-Liouville fractional derivative of order $\beta \in(1,2)$. Recently, Khuddush et al., [16] established the existence results by applying the concept of a family of measures of noncompactness in the space of functions $\mathbb{C}^{3, \alpha}\left(\mathbb{R}^{+}\right)$to
the following $\infty$-point fractional BVP

$$
\begin{aligned}
& { }_{0}^{R} D_{0^{+}}^{\alpha} z(r)=h(r, z(r)), \quad m<\alpha<m+1, r \in \mathbb{R}^{+}, \\
& \lim _{r \rightarrow \infty}{ }_{0}^{R} D_{0^{+}}^{\delta-1} z(r)+\sum_{j=1}^{\infty} C_{j} z\left(\varphi\left(\tau_{j}\right)\right)=0, \\
& z(0)=0, \quad z^{\prime \prime}(0)=0, \ldots, z^{(m)}(0)=0,
\end{aligned}
$$

where $m$ is a fixed nonnegative integer, $c_{j} \in \mathbb{R}^{+}, \varphi \in C\left(\mathbb{R}^{+}, \mathbb{R}\right), h \in C\left(\mathbb{R}^{+} \times \mathbb{R}, \mathbb{R}\right)$. Inspired by the above-mentioned works, in this paper, we derive sufficient conditions for the existence of solutions via the HMNC in tempered sequence spaces to the following infinite system of TFBVP

$$
\begin{align*}
& { }_{0}^{\mathrm{R}} \mathrm{D}_{\mathrm{r}}^{\varrho, \lambda} \mathrm{z}_{\mathrm{j}}(\mathrm{r})+\psi_{j}(\mathrm{r}, \mathrm{z}(\mathrm{r}))=0, \quad 0<\mathrm{r}<1,  \tag{1}\\
& \left\{\begin{array}{l}
\mathrm{z}_{\mathrm{j}}(0)=0, \quad{ }_{0}^{\mathrm{R}} \mathrm{D}_{\mathrm{r}}^{\mathrm{m}, \lambda} z_{j}(0)=0, \\
\mathrm{~b}_{1} z_{j}(1)+\mathrm{b}_{2}{ }_{0}^{\mathrm{R}} D_{r}^{\mathrm{m}, \lambda} z_{j}(1)=0,
\end{array}\right. \tag{2}
\end{align*}
$$

where $j \in \mathbb{N}, 2<\varrho \leq 3,1<m \leq 2, \lambda>0,{ }_{0}^{R} D_{r}^{\varrho, \lambda}$ denotes the RL-tempered fractional derivative of order $\varrho, \mathrm{z}(\mathrm{r})=\left(\mathrm{z}_{\mathrm{j}}(\mathrm{r})\right)_{j=1}^{\infty}$, and $\psi_{j}:(0,1) \rightarrow(0,1)$ is continuous. We also provide an example to illustrate the theoretical results.

## 2 Preliminaries

Essential results are stated here prior to proceeding to the main results in the subsequent sections.

- Denote by $\mathcal{A C}[b, c]$ the space of real-valued absolutely continuous functions $z(r)$ on [b, c].
- Denote by $\mathcal{A C}^{k}[\mathrm{~b}, \mathrm{c}]$ the space of real-valued functions $\mathrm{z}(\mathrm{r})$, which have continuous derivatives of order $k-1$ on $[\mathrm{b}, \mathrm{c}]$ such that $\frac{\mathrm{d}^{k-1} \mathrm{z}(\mathrm{r})}{\mathrm{rr}^{k-1}} \in \mathcal{A C}[\mathrm{~b}, \mathrm{c}]$.
- Denote by $\mathcal{L}([b, c])$ the family of all Lebesgue measurable functions on $[b, c]$.

Definition $2.1([20,37])$ Let $z(r) \in \mathcal{L}([b, c]), \lambda \geq 0$ and $\eta>0$. The RL-tempered fractional integral of order $\eta$ is defined as

$$
{ }_{b}^{\mathrm{R}} \mathrm{I}_{\mathrm{r}}^{\eta, \lambda} \mathrm{z}(\mathrm{r})=e^{-\lambda \mathrm{r} \mathrm{R}} \mathrm{I}_{\mathrm{r}}^{\eta}\left(e^{\lambda \mathrm{r}} \mathrm{z}(\mathrm{r})\right)=\frac{1}{\Gamma(\eta)} \int_{\mathrm{b}}^{\mathrm{r}} e^{-\lambda(\mathrm{r}-\xi)}(\mathrm{r}-\xi)^{\eta-1} \mathrm{z}(\xi) d \xi,
$$

where ${ }_{b} I_{r}^{\eta}$ is the classical RL-fractional integral [17]

$$
{ }_{\mathrm{b}}^{\mathrm{R}} \mathrm{I}_{\mathrm{r}}^{\eta} \mathrm{z}(\mathrm{r})=\frac{1}{\Gamma(\eta)} \int_{\mathrm{b}}^{\mathrm{r}}(\mathrm{r}-\xi)^{\eta-1} \mathrm{z}(\xi) d \xi
$$

Definition 2.2 ([20,37]) Let $k-1<\eta<k, k \in \mathbb{N}^{+}$and $\lambda \geq 0$. The RL-tempered fractional derivative of order $\eta$ is defined as

$$
{ }_{b}^{\mathrm{R}} \mathrm{D}_{\mathrm{r}}^{\eta, \lambda} \mathrm{z}(\mathrm{r})=e^{-\lambda \mathrm{rR}} \mathrm{D}_{\mathrm{r}}^{\eta}\left(e^{\lambda \mathrm{r}} \mathrm{z}(\mathrm{r})\right)=\frac{e^{-\lambda \mathrm{r}}}{\Gamma(k-\eta)} \frac{d^{k}}{d \xi^{k}} \int_{\mathrm{b}}^{\mathrm{r}} \frac{e^{\lambda \xi} \mathrm{z}(\xi)}{(\mathrm{r}-\xi)^{\mathrm{\eta}-k+1}} d \xi,
$$

where ${ }_{b}^{R} D_{r}^{\eta}$ is the classical RL-fractional derivative [17]

$$
{ }_{b}^{\mathrm{R}} D_{\mathrm{r}}^{\eta} z(\mathrm{r})=\frac{1}{\Gamma(k-\eta)} \frac{d^{k}}{d \xi^{k}} \int_{\mathrm{b}}^{\mathrm{r}} \frac{z(\xi)}{(\mathrm{r}-\xi)^{\eta-k+1}} d \xi .
$$

Lemma 2.3 (Composite property [20]) Let $\mathrm{z}(\mathrm{r}) \in \mathcal{A C}^{k}[\mathrm{~b}, \mathrm{c}]$ and $k-1<\eta<k$. Then, the composite property between RL-tempered fractional derivative and RL-tempered fractional integral is given by

$$
\begin{equation*}
{ }_{b}^{\mathrm{R}} \mathrm{I}_{\mathrm{r}}^{\eta}, \lambda\left[{ }_{b}^{\mathrm{R}} D_{\mathrm{r}}^{\eta, \lambda} \mathrm{z}(\mathrm{r})\right]=\mathrm{z}(\mathrm{r})-\sum_{k=0}^{n-1} \frac{e^{-\lambda \mathrm{r}}(\mathrm{r}-\mathrm{b})^{\eta-k-1}}{\Gamma(\eta-k)}\left[\left.{ }_{b}^{\mathrm{R}} D_{\mathrm{r}}^{\eta-k-1}\left(e^{\lambda \mathrm{r}} \mathrm{z}(\mathrm{r})\right)\right|_{\mathrm{r}=\mathrm{b}}\right] \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
{ }_{b}^{\mathrm{R}} \mathrm{D}_{\mathrm{r}}^{\eta, \lambda}\left[{ }_{b}^{\mathrm{R}} \mathrm{r}_{\mathrm{r}}^{\eta, \lambda} z(\mathrm{r})\right]=\mathrm{z}(\mathrm{r}) . \tag{4}
\end{equation*}
$$

To study the boundary value problem (1)-(2), first we solve the following linear fractional differential equation

$$
\begin{equation*}
{ }_{0}^{\mathrm{R}} \mathrm{D}_{\mathrm{r}}^{\varrho, \lambda} \mathrm{z}(\mathrm{r})+\mathrm{V}(\mathrm{r})=0, \quad 2<\varrho \leq 3,0<\mathrm{r}<1, \tag{5}
\end{equation*}
$$

satisfying the boundary conditions (2).

Lemma 2.4 Let z be a solution of (5) and (2) if and only if z solves the integral equation

$$
\mathrm{z}(\mathrm{r})=\int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) \mathrm{V}(\mathrm{p}) e^{-\lambda(\mathrm{r}-\mathrm{p})} d \mathrm{p}
$$

where

$$
\begin{aligned}
& \mho(\mathrm{r}, \mathrm{p})= \begin{cases}\mho_{1}(\mathrm{r}, \mathrm{p}), & 0 \leq \mathrm{r} \leq \mathrm{p} \leq 1 \\
\mho_{2}(\mathrm{r}, \mathrm{p}), & 0 \leq \mathrm{p} \leq \mathrm{r} \leq 1,\end{cases} \\
& \mho_{1}(\mathrm{r}, \mathrm{p})={\mathbb{k} \mathrm{b}_{1} \frac{\mathrm{r}^{\varrho-1}(1-\mathrm{p})^{\varrho-1}}{\Gamma(\varrho)}+\mathbb{k} \mathrm{b}_{2} \frac{\mathrm{r}^{\varrho-1}(1-\mathrm{p})^{\varrho-m-1}}{\Gamma(\varrho-m)},}_{\Gamma}^{\Gamma(\varrho)},
\end{aligned}
$$

and $\mathbb{k}=\left[\mathrm{b}_{1}+\mathrm{b}_{2} \frac{\Gamma(\varrho)}{\Gamma(\varrho-\mathrm{m})}\right]^{-1}>1$.
Proof Assume that $\mathrm{z} \in C^{[e]+1}[0,1]$ is a solution of (5). According to Lemma 2.3, we obtain

$$
z(r)=A e^{-\lambda r} r^{\varrho-1}+B e^{-\lambda r} r^{\varrho-2}+C e^{-\lambda r} r^{\varrho-3}-I^{\varrho, \lambda} V(r)
$$

where $A, B$, and $C$ are constants. Using the boundary condition $z(0)=0$, we get $C=0$, and hence

$$
\begin{equation*}
\mathrm{z}(\mathrm{r})=\mathrm{A} e^{-\lambda \mathrm{r}} \mathrm{r}^{\varrho-1}+\mathrm{B} e^{-\lambda \mathrm{r}} \mathrm{r}^{\varrho-2}-\mathrm{I}^{\varrho, \lambda} \mathrm{V}(\mathrm{r}) . \tag{6}
\end{equation*}
$$

Applying the tempered fractional order derivative operator ${ }_{0}^{R} D_{r}^{m, \lambda}$ on both sides of (6), we get

$$
\begin{equation*}
{ }_{0}^{\mathrm{R}} \mathrm{D}_{\mathrm{r}}^{\mathrm{m}, \lambda} \mathrm{z}(\mathrm{r})=\mathrm{A} \frac{\Gamma(\varrho)}{\Gamma(\varrho-\mathrm{m})} e^{-\lambda \mathrm{r}} \mathrm{r}^{\varrho-\mathrm{m}-1}-\mathrm{B} \frac{\Gamma(\varrho-1)}{\Gamma(\varrho-\mathrm{m}-1)} e^{-\lambda \mathrm{r}} \mathrm{r}^{\varrho-\mathrm{m}-2}-\mathrm{I}^{\varrho-\mathrm{m}, \lambda} \mathrm{~V}(\mathrm{r}) \tag{7}
\end{equation*}
$$

Using condition ${ }_{0}^{R} D_{r}^{m, \lambda} z(0)=0$, we obtain $B=0$. So, equation (6) reduced to

$$
\begin{equation*}
\mathrm{z}(\mathrm{r})=\mathrm{A} e^{-\lambda \mathrm{r}} \mathrm{r}^{\varrho-1}-\mathrm{I}^{\mathrm{\rho}, \lambda} \mathrm{~V}(\mathrm{r}) \tag{8}
\end{equation*}
$$

Taking $r=1$ into (7), (8) and using condition $b_{1} z(1)+b_{2}{ }_{0}^{R} D_{r}^{m, \lambda} z(1)=0$, we get

$$
\mathrm{A}=\int_{0}^{1} \mathbb{k} \mathrm{~b}_{1} \frac{(1-\mathrm{p})^{\varrho-1}}{\Gamma(\varrho)} e^{\lambda \mathrm{p}} \mathrm{~V}(\mathrm{p}) d \mathrm{p}+\int_{0}^{1} \mathbb{k} \mathrm{~b}_{2} \frac{(1-\mathrm{p})^{\varrho-m-1}}{\Gamma(\varrho-\mathrm{m})} e^{\lambda \mathrm{p}} \mathrm{~V}(\mathrm{p}) d \mathrm{p}
$$

Plugging A value into (8), we get

$$
\begin{aligned}
\mathrm{z}(\mathrm{r})= & \int_{0}^{1} \mathbb{k} \mathrm{~b}_{1} \frac{(1-\mathrm{p})^{\varrho-1}}{\Gamma(\varrho)} \mathrm{r}^{\varrho-1} e^{-\lambda(\mathrm{r}-\mathrm{p})} \mathrm{V}(\mathrm{p}) d \mathrm{p}+\int_{0}^{1} \mathbb{k} \mathrm{~b}_{2} \frac{(1-\mathrm{p})^{\varrho-\mathrm{m}-1}}{\Gamma(\varrho-\mathrm{m})} \mathrm{r}^{\varrho-1} e^{-\lambda(\mathrm{r}-\mathrm{p})} \mathrm{V}(\mathrm{p}) d \mathrm{p} \\
& -\int_{0}^{\mathrm{r}} \frac{(\mathrm{r}-\mathrm{p})^{\varrho-1}}{\Gamma(\varrho)} e^{-\lambda(\mathrm{r}-\mathrm{p})} \mathrm{V}(\mathrm{p}) d \mathrm{p} \\
= & \int_{0}^{\mathrm{r}}\left[\frac{\mathbb{k} \mathrm{~b}_{1} \mathrm{r}^{\varrho-1}(1-\mathrm{p})^{\varrho-1}-(\mathrm{r}-\mathrm{p})^{\varrho-1}}{\Gamma(\varrho)}+\mathbb{k} \mathrm{b}_{2} \frac{(1-\mathrm{p})^{\varrho-\mathrm{m}-1}}{\Gamma(\varrho-\mathrm{m})} \mathrm{r}^{\varrho-1}\right] e^{-\lambda(\mathrm{r}-\mathrm{p})} \mathrm{V}(\mathrm{p}) d \mathrm{p} \\
& +\int_{\mathrm{r}}^{1}\left[\mathbb{k} \mathrm{r}^{\varrho-1} \frac{(1-\mathrm{p})^{\varrho-1}}{\Gamma(\varrho)}+\mathbb{k} \mathrm{b}_{2} \frac{(1-\mathrm{p})^{\varrho-\mathrm{m}-1}}{\Gamma(\varrho-\mathrm{m})} \mathrm{r}^{\varrho-1}\right] e^{-\lambda(\mathrm{r}-\mathrm{p})} \mathrm{V}(\mathrm{p}) d \mathrm{p} \\
= & \int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) e^{-\lambda(\mathrm{r}-\mathrm{p})} \mathrm{V}(\mathrm{p}) d \mathrm{p} .
\end{aligned}
$$

It is clear from Lemma 2.4 that $z_{j}$ is a solution of (1)-(2) iff $z_{j}$ solves the following integral equation

$$
\begin{equation*}
\mathrm{z}_{\mathrm{j}}=\int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) \psi_{\mathrm{j}}\left(\mathrm{p}, \mathrm{z}_{\mathrm{j}}(\mathrm{p})\right) e^{-\lambda(\mathrm{r}-\mathrm{p})} d \mathrm{p} \tag{9}
\end{equation*}
$$

Lemma 2.5 Suppose that $\mathbb{k}>0$, then for all $\mathrm{r}, \mathrm{p} \in[0,1]$, the kernel $\mho(\mathrm{r}, \mathrm{p})$ satisfies the following
(i) $\mho(\mathrm{r}, \mathrm{p}) \geq 0$ and continuous on $[0,1] \times[0,1]$.
(ii) $\mho(\mathrm{r}, \mathrm{p}) \leq \mathbb{k} \mathrm{b}_{1} \frac{(1-\mathrm{p})^{\varrho-1}}{\Gamma(\varrho)}+\mathbb{k} \mathrm{b}_{2} \frac{(1-\mathrm{p}) \varrho^{\varrho-\mathrm{m}-1}}{\Gamma(\varrho-\mathrm{m})}=: \mho^{\star}(\mathrm{p})$.
(iii) $\max _{\mathrm{r} \in[0,1]} \int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) d \mathrm{p}=\frac{\mathrm{kb}_{1}}{\Gamma(\varrho+1)}+\frac{\mathrm{kb}_{2}}{\Gamma(\varrho-\mathrm{m}+1)}=: \mathrm{C}^{\star}$.

Proof From the definition of $\mho(\mathrm{r}, \mathrm{p})$, it is clear that $\mho(\mathrm{r}, \mathrm{p})$ is continuous on $[0,1] \times[0,1]$. For $r, p \in[0,1]$, we have

$$
\begin{aligned}
\mho_{2}(\mathrm{r}, \mathrm{p})= & \mathbb{k} \mathrm{b}_{1} \frac{\mathrm{r}^{\varrho-1}(1-\mathrm{p})^{\varrho-1}}{\Gamma(\varrho)}+\mathbb{k} \mathrm{b}_{2} \frac{\mathrm{r}^{\varrho-1}(1-\mathrm{p})^{\varrho-m-1}}{\Gamma(\varrho-\mathrm{m})}-\frac{(\mathrm{r}-\mathrm{p})^{\varrho-1}}{\Gamma(\varrho)} \\
= & \frac{1}{\Gamma(\varrho)}\left[\mathbb{k} \mathrm{b}_{1} \mathrm{r}^{\varrho-1}(1-\mathrm{p})^{\varrho-1}-(\mathrm{r}-\mathrm{p})^{\varrho-1}\right]+\mathbb{k} \mathrm{b}_{2} \frac{\mathrm{r}^{\varrho-1}(1-\mathrm{p})^{\varrho-m-1}}{\Gamma(\varrho-\mathrm{m})} \\
= & \frac{1}{\Gamma(\varrho)}\left[\mathbb{k} \mathrm{b}_{1} \mathrm{r}^{\varrho-1}(1-\mathrm{p})^{\varrho-1}-\mathrm{r}^{\varrho-1}\left(1-\frac{\mathrm{p}}{\mathrm{r}}\right)^{\varrho-1}\right]+\mathbb{k} \mathrm{b}_{2} \frac{\mathrm{r}^{\varrho-1}(1-\mathrm{p})^{\varrho-m-1}}{\Gamma(\varrho-\mathrm{m})} \\
\geq & \frac{1}{\Gamma(\varrho)}\left[\mathbb{k} \cdot \mathrm{br}^{\varrho-1}(1-\mathrm{p})^{\varrho-1}-\mathrm{r}^{\varrho-1}(1-\mathrm{p})^{\varrho-1}\right]+\mathbb{k} \mathrm{b}_{2} \frac{\mathrm{r}^{\varrho-1}(1-\mathrm{p})^{\varrho-m-1}}{\Gamma(\varrho-\mathrm{m})} \\
= & \frac{1}{\Gamma(\varrho)}\left[\mathbb{k} \mathrm{b}_{1} \mathrm{r}^{\varrho-1}(1-\mathrm{p})^{\varrho-1}-\mathbb{k}\left(\mathrm{b}+\mathrm{b}_{2} \frac{\Gamma(\varrho)}{\Gamma(\varrho-\mathrm{m})}\right) \mathrm{r}^{\varrho-1}(1-\mathrm{p})^{\varrho-1}\right] \\
& +\mathbb{k} \mathrm{b}_{2} \frac{\mathrm{r}^{\varrho-1}(1-\mathrm{p})^{\varrho-m-1}}{\Gamma(\varrho-m)} \\
= & \frac{\mathbb{k} \mathrm{b}_{2}}{\Gamma(\varrho-\mathrm{m})}\left[(1-\mathrm{p})^{-m}-1\right] \mathrm{r}^{\varrho-1}(1-\mathrm{p})^{\varrho-1} \geq 0 .
\end{aligned}
$$

This proves (i). For $r, p \in[0,1]$, we have

$$
\begin{aligned}
\mho_{2}(\mathrm{r}, \mathrm{p}) & =\mathbb{k} \mathrm{b}_{1} \frac{\mathrm{r}^{\varrho-1}(1-\mathrm{p})^{\varrho-1}}{\Gamma(\varrho)}+\mathbb{k} \mathrm{b}_{2} \frac{\mathrm{r}^{\varrho-1}(1-\mathrm{p})^{\varrho-m-1}}{\Gamma(\varrho-\mathrm{m})}-\frac{(\mathrm{r}-\mathrm{p})^{\varrho-1}}{\Gamma(\varrho)} \\
& \leq \mathbb{k} \mathrm{b}_{1} \frac{\mathrm{r}^{\varrho-1}(1-\mathrm{p})^{\varrho-1}}{\Gamma(\varrho)}+\mathbb{k} \mathrm{b}_{2} \frac{\mathrm{r}^{\varrho-1}(1-\mathrm{p})^{\varrho-m-1}}{\Gamma(\varrho-m)} \\
& =\left[\mathbb{k}_{\mathrm{b}_{1}} \frac{(1-\mathrm{p})^{\varrho-1}}{\Gamma(\varrho)}+\mathbb{k} \mathrm{b}_{2} \frac{(1-\mathrm{p})^{\varrho-m-1}}{\Gamma(\varrho-m)}\right] \mathrm{r}^{\varrho-1} \\
& \leq \mathbb{k} \mathrm{b}_{1} \frac{(1-\mathrm{p})^{\varrho-1}}{\Gamma(\varrho)}+\mathbb{k} \mathrm{b}_{2} \frac{(1-\mathrm{p})^{\varrho-m-1}}{\Gamma(\varrho-m)} .
\end{aligned}
$$

This proves (ii). Finally,

$$
\begin{aligned}
\max _{\mathrm{r} \in[0,1]} \int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) d \mathrm{p} & =\frac{\mathbb{k b _ { 1 }}}{\Gamma(\varrho)} \int_{0}^{1}(1-\mathrm{p})^{\varrho-1} \mathrm{dp}+\frac{\mathbb{k b}_{2}}{\Gamma(\varrho-\mathrm{m})} \int_{0}^{1}(1-\mathrm{p})^{\varrho-m-1} \mathrm{dp} \\
& =\frac{\mathbb{k} \mathrm{b}_{1}}{\Gamma(\varrho+1)}+\frac{\mathbb{k b _ { 2 }}}{\Gamma(\varrho-\mathrm{m}+1)}
\end{aligned}
$$

Definition 2.6 ([18]) The Kurtowski MNC of F, where F is a subset of a metric space E, is given by

$$
\operatorname{Kur}(\mathrm{F})=\inf \left\{\eta>0: F \subset \bigcup_{\mathrm{k}=1}^{\mathrm{p}} \mathrm{G}_{\mathrm{k}}, \mathrm{G}_{\mathrm{k}} \subset \mathrm{E}, \operatorname{diam}\left(\mathrm{G}_{\mathrm{k}}\right)<\eta, \mathrm{p}=1,2, \ldots\right\} .
$$

From above definition, we have

$$
\operatorname{Kur}(F) \leq \operatorname{diam}(F) \quad \text { for all } F \subset E .
$$

Let $B(\zeta, r)=\{\xi \in \mathfrak{Z}:\|\xi-\zeta\| \leq r\}$, where $\mathfrak{Z}$ is a Banach space equipped with the norm $\|\cdot\|$.

Denote by $\mathfrak{M}_{\mathfrak{Z}}$ the class of all non-empty and bounded subsets of $\mathfrak{Z}$ and $\mathfrak{N}_{\mathfrak{Z}}$ its subclass containing all relatively compact sets.

Definition 2.7 ([4]) A real-valued function $\mu$ from $\mathcal{M}_{\mathcal{3}}$ to $[0,1)$ is said to be an MNC if
(i) $\operatorname{Ker}(\mu)=\left\{\mathrm{H} \in \mathfrak{M}_{\mathcal{Z}}: \mu(\mathrm{H})=0\right\}$ is non-empty and $\operatorname{Ker}(\mu) \subset \mathcal{N}_{\mathcal{Z}}$.
(ii) $\mu(\mathrm{H}) \leq \mu(\mathrm{G})$, if $\mathrm{H} \subset \mathrm{G}$.
(iii) $\mu(\mathfrak{c o n v}(\mathrm{H}))=\mu(\mathrm{H})$, where $\mathfrak{c o n v}(\mathrm{H})$ is the convex closure of H .
(iv) $\mu(\overline{\mathrm{H}})=\mu(\mathrm{H})$.
(v) $\mu(\xi \mathrm{G}+(1-\xi) \mathrm{H}) \leq \xi \mu(\mathrm{G})+(1-\xi) \mu(\mathrm{H})$ for all $0 \leq \xi \leq 1$.
(vi) if $\mathrm{G}_{\mathrm{k}} \in \mathcal{M}_{\mathfrak{Z}}, \mathrm{G}_{\mathrm{k}}=\overline{\mathrm{G}}_{\mathrm{k}}, \mathrm{G}_{\mathrm{k}+1} \subset \mathrm{G}_{\mathrm{k}}$ for $\mathrm{k} \in \mathbb{N}$ and $\lim _{\mathrm{k} \rightarrow+\infty} \mu\left(\mathrm{G}_{\mathrm{k}}\right)=0$ then $\bigcap_{\mathrm{k}=1}^{\infty} \mathrm{G}_{\mathrm{k}} \neq \emptyset$.

Definition 2.8 ([6]) The HMNC is defined as

$$
\chi(H)=\inf \left\{\eta>0: G \subset \bigcup_{k=1}^{p} B\left(\zeta_{k}, r_{k}\right), \zeta_{k} \in G, r_{k}<\eta, p=1,2, \ldots\right\},
$$

where H is a bounded subset of a metric space G .

Next, we define some Banach spaces as

$$
\begin{aligned}
& \mathrm{c}_{0}=\left\{\varpi \in \omega: \lim _{\mathrm{p} \rightarrow+\infty} \varpi_{\mathrm{p}}=0,\|\varpi\|_{c_{0}}=\sup _{\mathrm{p}}\left|\varpi_{\mathrm{p}}\right|\right\}, \\
& \mathrm{c}=\left\{\varpi \in \omega: \lim _{\mathrm{p} \rightarrow+\infty} \varpi_{\mathrm{p}}=\mathfrak{z}, \mathfrak{z} \in \mathbb{C},\|\varpi\|_{\mathrm{C}}=\sup _{\mathrm{p}}\left|\varpi_{\mathrm{p}}\right|\right\} .
\end{aligned}
$$

We also define

$$
\chi(J)=\lim _{\mathrm{k} \rightarrow \infty}\left\{\sup _{\sigma(\mathrm{z}) \in \mathrm{J}}\left[\max _{\mathrm{p} \geq \mathrm{k}}\left|\varpi_{\mathrm{p}}\right|\right]\right\}, \quad \mathrm{J} \in \mathfrak{M}_{\mathrm{c}_{0}}
$$

which is called the HMNC on the Banach space ( $\mathrm{c}_{0},\|\cdot\|_{c_{0}}$ ); for more details, see [6].

Definition 2.9 ([27]) The MNC $\mu$ on the Banach space (c, $\|\cdot\|_{c}$ ) is defined by

$$
\begin{equation*}
\mu(\mathrm{J})=\lim _{\mathrm{k} \rightarrow \infty}\left\{\sup _{\varpi(\mathrm{z}) \in \mathrm{J}}\left[\sup _{\mathrm{p} \geq \mathrm{k}}\left|\varpi_{\mathrm{p}}-\lim _{\mathrm{q} \rightarrow \infty} \varpi_{\mathrm{q}}\right|\right]\right\}, \quad \mathrm{J} \in \mathfrak{M}_{\mathrm{C}} . \tag{10}
\end{equation*}
$$

Definition 2.10 ([5]) If $\ell=\left(\ell_{j}\right)$, then $\ell$ is called a tempering sequence, if $\ell_{j}>0$ for all $j$ and $\ell$ is nonincreasing. Define

$$
M=\left\{\varpi=\left(\varpi_{j}\right)_{j=1}^{\infty}: \lim _{j \rightarrow \infty} \ell_{j} \varpi_{j}=0\right\}
$$

Then, $M$ is a linear space over $\mathbb{R}$. We denote the space by $c_{0}^{\ell}$ and $c_{0}^{\ell}$ that is a Banach space with the norm $\|\varpi\|_{C_{0}^{\ell}}=\sup \left\{\ell_{\mathrm{k}}\left|\varpi_{\mathrm{k}}\right|\right\}$. Next, let

$$
\mathrm{N}=\left\{\varpi=\left(\varpi_{j}\right)_{j=1}^{\infty}: \lim _{j \rightarrow \infty} \ell_{j} \varpi_{j}=\text { finite }\right\} .
$$

Then, $M$ is a linear space over $\mathbb{R}$. We denote the space by $c^{\ell}$ and $c^{\ell}$ that is a Banach space with the norm $\|\varpi\|_{c^{\ell}}=\sup \left\{\ell_{\mathrm{k}}\left|\varpi_{\mathrm{k}}\right|\right\}$.

Here, we note that there is a isometry between the spaces $c_{0}^{\ell}$ and $c_{0}$ and the spaces $c^{\ell}$ and c: In [5], the HMNC on $\mathfrak{M}_{\mathrm{c}_{0}^{\ell}}$ is defined as

$$
\chi(J)=\lim _{k \rightarrow+\infty}\left\{\sup _{w \in \mathrm{~J}}\left[\sup _{\mathrm{p} \geq \mathrm{k}}\left(\ell_{\mathrm{p}}\left|\mathrm{w}_{\mathrm{p}}\right|\right)\right]\right\}, \quad \mathrm{J} \in \mathfrak{M}_{\mathrm{c}_{0}^{\ell}} .
$$

Also, the HMNC on $c^{\ell}$ is defined as

$$
\mu_{c^{\ell}}(\mathrm{J})=\lim _{\mathrm{p} \rightarrow \infty}\left\{\sup _{\mathrm{w} \in \mathrm{~J}}\left[\sup _{\mathrm{k} \geq \mathrm{p}}\left|\ell_{\mathrm{k}} \mathrm{w}_{\mathrm{k}}-\lim _{\mathrm{r} \rightarrow \infty}\left(\ell_{\mathrm{r}} \mathrm{w}_{\mathrm{r}}\right)\right|\right]\right\}, \quad \mathrm{J} \in \mathfrak{M}_{\mathrm{c}^{\ell}} .
$$

Let $Q=(0,1)$. Denote by $C\left(Q, C_{0}^{\ell}\right)$ the space of all continuous functions on $Q$ with values in $\mathrm{C}_{0}^{\ell}$, which is a Banach space with the norm

$$
\|\varpi\|_{\mathrm{C}\left(Q, c_{0}^{\ell}\right)}=\max \left\{\|\varpi(\mathrm{w})\|_{\mathrm{C}_{0}^{\ell}}: \mathrm{w} \in \mathrm{Q}\right\}, \quad \varpi \in \complement\left(\mathrm{Q}, \mathrm{C}_{0}^{\ell}\right) .
$$

Denote by $\complement\left(Q, c^{\ell}\right)$ the space of all continuous functions on $Q$ with values in $c^{\ell}$, which is a Banach space with the norm

$$
\|\varpi\|_{C_{\left(Q, c^{\ell}\right)}}=\max \left\{\|\varpi(\mathrm{w})\|_{\mathrm{c}^{\ell}}: \mathrm{w} \in \mathrm{Q}\right\}, \quad \varpi \in \complement\left(\mathrm{Q}, \mathrm{c}^{\ell}\right),
$$

for more details, see [9].
Let $G \neq \emptyset$ be a bounded, closed and convex subset of $\complement\left(Q, c^{\ell}\right)$ and $z \in Q$ or $\complement\left(Q, c_{0}^{\ell}\right)$, Then,

$$
\chi_{\mathrm{C}\left(\mathrm{Q}, \mathrm{C}_{0}^{\ell}\right)}(\mathrm{G})=\sup \left\{\chi_{\mathrm{C}_{0}^{\ell}}(\mathrm{G}(\mathrm{w})): \mathrm{w} \in \mathrm{Q}\right\}
$$

satisfy all the axioms of MNC on $\complement\left(Q, \mathrm{c}_{0}^{\ell}\right)$ and

$$
\mu_{\mathrm{C}\left(\mathrm{Q}, \mathrm{c}^{\ell}\right)}(\mathrm{G})=\sup \left\{\mu_{\mathrm{c}^{\ell}}(\mathrm{G}(\mathrm{w})): \mathrm{w} \in \mathrm{Q}\right\}
$$

satisfy all the axioms of MNC on $\complement\left(\mathrm{Q}, \mathrm{c}^{\ell}\right)$, which can be found in [9].

Definition 2.11 ([24]) Let G be a metric space with metric $d$. The mapping $\mathrm{F}: \mathrm{G} \rightarrow \mathrm{G}$ is called a Meir-Keeler contraction if for any $\delta>0$, there exists $\eta>0$ such that

$$
\delta \leq d(z, w)<\delta+\eta \quad \Longrightarrow \quad d(F z, F w)<\delta, \quad \text { for all } z, w \in G .
$$

Theorem 2.12 ([24]) Let G be a complete metric space. If $\mathrm{F}: \mathrm{G} \rightarrow \mathrm{G}$ is a Meir-Keeler contraction, then F has a unique fixed point.

Definition 2.13 ([1]) The mapping $F$ on a non-empty subset $U$ of a Banach space $G$ is said to be a Meir-Keeler condensing operator if for any $\eta>0$, there exists $\delta>0$ such that

$$
\eta \leq \mu(\mathrm{H})<\eta+\delta \quad \Longrightarrow \quad \mu(\mathrm{F}(\mathrm{H}))<\eta, \quad \mathrm{H} \subset \mathrm{U} .
$$

Theorem 2.14 In addition to Definition 2.13, if U is a closed, bounded and convex subset of G, then F has a fixed point.

## 3 Solvability of the Boundary Value Problem (1) in $\complement\left(\Omega, c_{0}^{\ell}\right)$

This portion deals with the existence of solutions for BVP (1) in $\complement\left(Q, c_{0}^{\ell}\right)$.
We assume that the following conditions are met in this section:
(A) Let $\psi_{j}: Q \times \mathbb{R}^{\infty} \rightarrow \mathbb{R}$ and define an operator $\psi$ from $Q \times C_{0}^{\ell}$ to $C_{0}^{\ell}$ as $(\mathrm{r}, \mathrm{z}(\mathrm{r})) \rightarrow(\psi \mathrm{z})(\mathrm{r})=\left(\psi_{j}(\mathrm{r}, \mathrm{z}(\mathrm{r}))_{j=1}^{\infty}\right.$, which is the family of all functions $((\psi \mathrm{z})(\mathrm{r}))_{\mathrm{r} \in \mathrm{e}}$ equicontinuous on $\mathrm{C}_{0}^{\ell}$.
(B) $\zeta_{j}(\mathrm{r}), \zeta_{j}(\mathrm{r}): Q \rightarrow \mathbb{R}$ are continuous functions, $\ell_{j} \zeta_{j}(\mathrm{r}) \xrightarrow{\text { uniformly }} 0$ converges on $Q$, and the sequence $\left(\zeta_{j}(r)\right)$ is equibounded on $Q$. Let $\zeta(r)=\sup \left\{\zeta_{j}(r): j \in \mathbb{N}\right\}$, $\zeta^{\star}=\sup \{\zeta(\mathrm{r}): \mathrm{r} \in \mathrm{Q}\}, \ell^{\star}=\sup \left\{\ell_{j} \xi_{j}(\mathrm{r}): j \in \mathbb{N}, \mathrm{r} \in \mathrm{Q}\right\}$ and

$$
\left|\psi_{j}(\mathrm{r}, \mathrm{z}(\mathrm{r}))\right| \leq \xi_{j}(\mathrm{r})+\zeta_{j}(\mathrm{r})\left|z_{j}(\mathrm{r})\right|, \quad z_{j} \in C_{0}^{\ell}, \mathrm{r} \in \mathrm{Q}, \mathrm{j} \in \mathbb{N} .
$$

Theorem 3.1 Let $\widehat{\mho}:=\sup _{\mathrm{p} \in \mathrm{Q}} \mho^{\star}(\mathrm{p})$, and suppose $e^{\lambda} \widehat{\mho}<1$ and $(\mathrm{A})-(\mathrm{B})$ hold, then $B V P(1)$ has at least one solution $\mathrm{z}(\mathrm{r})=\left(\mathrm{z}_{\mathrm{j}}(\mathrm{r})\right)_{j=1}^{\infty}$ in $\mathrm{C}\left(\mathrm{Q}, \mathrm{c}_{0}^{\ell}\right)$.

Proof Since $\sup \left\{\ell_{j}\left|z_{j}(\mathrm{r})\right|\right\}<+\infty$ for all $\mathrm{z}(\mathrm{r})=\left(\mathrm{z}_{\mathrm{j}}(\mathrm{r})\right)_{j=1}^{\infty} \in \mathrm{C}\left(\mathrm{Q}, \mathrm{c}_{0}^{\ell}\right)$ and $\mathrm{r} \in \mathrm{Q} \exists \kappa>0 \ni$ $\sup \left\{\ell_{j}\left|z_{j}(\mathrm{r})\right|\right\}<\kappa$. From (B) and (9), we get

$$
\begin{aligned}
\|\mathrm{z}(\mathrm{r})\|_{C_{0}^{\ell}} & =\sup _{j \in \mathbb{N}}\left\{\ell_{\mathrm{j}}\left|\int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) e^{-\lambda(\mathrm{r}-\mathrm{p})} \psi_{j}(\mathrm{r}, \mathrm{z}(\mathrm{r})) d \mathrm{p}\right|\right\} \\
& \leq e^{\lambda} \sup _{j \in \mathbb{N}}\left\{\ell_{j} \int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) \psi_{j}(\mathrm{r}, \mathrm{z}(\mathrm{r})) d \mathrm{p}\right\} \\
& \leq e^{\lambda} \sup _{j \in \mathbb{N}}\left\{\int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) \ell_{j} \psi_{j}(\mathrm{r}, \mathrm{z}(\mathrm{r})) d \mathrm{p}\right\} \\
& \leq e^{\lambda} \sup _{j \in \mathbb{N}}\left\{\int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) \ell_{j}\left[\xi_{j}(\mathrm{p})+\zeta_{j}(\mathrm{p})\left|\mathrm{z}_{j}(\mathrm{p})\right|\right] d \mathrm{p}\right\} \\
& \leq e^{\lambda} \sup _{j \in \mathbb{N}}\left\{\int_{0}^{1} \mho(\mathrm{r}, \mathrm{p})\left[\ell^{\star}+\zeta^{\star} \kappa\right] d \mathrm{p}\right\} \\
& \leq\left(\ell^{\star}+\zeta^{\star} \kappa\right) e^{\lambda} \int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) d \mathrm{p} \\
& \leq \mathrm{C}^{\star}\left(\ell^{\star}+\zeta^{\star} \kappa\right) e^{\lambda}:=a_{1} .
\end{aligned}
$$

Therefore,

$$
\max _{\mathrm{r} \in \mathrm{Q}}\|z(\mathrm{r})\|_{c_{0}^{\ell}} \leq a_{1} \text {, i.e., }\|\mathrm{z}(\mathrm{r})\|_{\mathrm{C}_{\left((,), \varepsilon_{0}^{\ell}\right)}} \leq a_{1} .
$$

Let $\mathcal{E}=\mathcal{E}\left(z^{0}(r), a_{1}\right)$ be a closed ball centered at $z^{0}(r)=\left(z^{0}(r)\right)_{j=1}^{\infty}$, for all $r \in Q$, and radius $a_{1}$. So, $\mathcal{E}$ is a non-empty closed, bounded and convex subset of $\mathrm{C}\left(\mathrm{Q}, \mathrm{c}_{0}^{\ell}\right)$.
For fixed $\mathrm{r} \in \mathrm{Q}$, define an operator $\wp=\left(\wp_{j}\right)_{j=1}^{\infty}:\left\lceil\left(Q, \mathrm{c}_{0}^{\ell}\right) \rightarrow\left\lceil\left(Q, \mathrm{c}_{0}^{\ell}\right)\right.\right.$ as

$$
(\wp \mathrm{z})(\mathrm{r})=\left\{\left(\wp_{j} \mathrm{z}\right)(\mathrm{r})\right\}_{j=1}^{\infty}=\left\{\int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) e^{-\lambda(\mathrm{r}-\mathrm{p})} \psi_{j}(\mathrm{r}, \mathrm{z}(\mathrm{r})) d \mathrm{p}\right\}_{\mathrm{j}=1}^{\infty} .
$$

Since $\left(\psi_{j}(r, z(r))\right)_{j=1}^{\infty} \in C_{0}^{\ell}$, for $r \in Q$, it follows that

$$
\begin{aligned}
\lim _{j \rightarrow \infty}\left\{\ell_{j}\left(\wp_{j} \mathrm{z}\right)(\mathrm{r})\right\} & =\lim _{j \rightarrow \infty}\left\{\ell_{j} \int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) e^{-\lambda(\mathrm{r}-\mathrm{p})} \psi_{j}(\mathrm{r}, \mathrm{z}(\mathrm{r})) d \mathrm{p}\right\} \\
& \leq e^{\lambda} \int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) \lim _{j \rightarrow \infty}\left[\ell_{j} \psi_{j}(\mathrm{r}, \mathrm{z}(\mathrm{r}))\right] d \mathrm{p} \\
& =0 .
\end{aligned}
$$

Thus, $(\wp z)(r) \in \complement\left(Q, c_{0}^{\ell}\right)$. It is easy to see that $\left(\wp_{j} z\right)(r)$ satisfies

$$
\begin{aligned}
& \left(\wp_{\mathrm{j}} z\right)(0)=0, \quad{ }_{0}^{\mathrm{R}} D_{\mathrm{r}}^{\mathrm{m}, \lambda}\left(\wp_{\mathrm{j}} z(0)\right)=0, \\
& \mathrm{~b}_{1}\left(\wp_{\mathrm{j}} z(1)\right)+\mathrm{b}_{2}^{\mathrm{R}} D_{\mathrm{r}}^{\mathrm{m}, \lambda}\left(\wp_{\mathrm{j}} z(1)\right)=0 .
\end{aligned}
$$

For fixed $r \in Q$ and $z(r) \in \mathcal{E}$, we get

$$
\begin{aligned}
\left\|(\wp \mathrm{z})(\mathrm{r})-\mathrm{z}^{0}(\mathrm{r})\right\|_{\mathrm{C}_{0}^{\ell}} \leq a_{1} & \Longrightarrow \max _{\mathrm{r} \in Q}\left\|(\wp \mathrm{z})(\mathrm{r})-\mathrm{z}^{0}(\mathrm{r})\right\|_{\mathrm{C}_{0}^{\ell}} \leq a_{1} \\
& \Longrightarrow\left\|(\wp \mathrm{z})(\mathrm{r})-\mathrm{z}^{0}(\mathrm{r})\right\|_{\mathrm{C}\left(Q, \mathrm{c}_{0}^{\ell}\right)} \leq a_{1} .
\end{aligned}
$$

Thus, $\wp$ is a self-mapping on $\mathcal{E}$. From (A), for any $z(r)=\left(z_{j}(r)\right)_{j=1}^{\infty} \in \mathcal{E}$ and for any $\eta>$ 0 , there exists $\xi>0$ such that $\|(\psi z)(r)-(\psi w)(r)\|_{c_{0}^{\ell}}<\frac{\eta}{e^{\lambda} \overparen{\mho}}$ for each $z(r) \in \mathcal{E}$, whenever $|z(r)-w(r)| \leq \xi$, where $r \in Q$. So, for $r \in Q$, we have

$$
\begin{aligned}
\|(\wp \mathrm{z})(\mathrm{r})-(\wp \mathrm{w})(\mathrm{r})\|_{c_{0}^{\ell}} & =\sup _{j \in \mathbb{N}}\left\{\left|\ell_{j} \int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) e^{-\lambda(\mathrm{r}-\mathrm{p})}\left[\psi_{j}(\mathrm{p}, \mathrm{z}(\mathrm{p}))-\psi_{j}(\mathrm{p}, \mathrm{w}(\mathrm{p}))\right] d \mathrm{p}\right|\right\} \\
& \leq e^{\lambda} \sup _{j \in \mathbb{N}}\left\{\int_{0}^{1} \mho^{\star}(\mathrm{p}) \ell_{\mathrm{j}}\left|\psi_{j}(\mathrm{p}, \mathrm{z}(\mathrm{p}))-\psi_{j}(\mathrm{p}, \mathrm{w}(\mathrm{p}))\right| d \mathrm{p}\right\} \\
& \leq e^{\lambda} \widehat{\delta} \frac{\eta}{e^{\lambda} \widehat{\delta}}<\eta .
\end{aligned}
$$

Thus, $\wp$ is continuous on $\mathcal{E} \forall r \in Q$.
Now, we have

$$
\begin{aligned}
\chi(\wp \mathcal{E}) & =\lim _{j \rightarrow \infty}\left\{\sup _{z(\mathrm{r}) \in \mathcal{E}} \sup _{m \geq j}\left[\ell_{m}\left|\int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) e^{-\lambda(\mathrm{r}-\mathrm{p})} \psi_{m}(\mathrm{r}, \mathrm{z}(\mathrm{r})) d \mathrm{p}\right|\right]\right\} \\
& \leq e^{\lambda} \widehat{\mho} \lim _{j \rightarrow \infty}\left\{\sup _{z(\mathrm{r}) \in \mathcal{E}} \sup _{m \geq j}\left[\int_{0}^{1}\left(\ell_{m} \xi_{m}(\mathrm{p})+\ell_{m} \zeta_{m}(\mathrm{p})\left|\mathrm{z}_{m}(\mathrm{p})\right|\right) d \mathrm{p}\right]\right\} \\
& \leq e^{\lambda} \widehat{\mho} \chi(\mathcal{E}) .
\end{aligned}
$$

Thus,

$$
\sup _{\mathrm{r} \in \mathrm{Q}} \chi(\wp \mathcal{E}) \leq e^{\lambda} \widehat{\mho} \sup _{\mathrm{r} \in \mathrm{Q}} \chi(\mathcal{E})
$$

It follows that

$$
\chi_{\mathrm{C}\left(Q, c_{0}^{\ell}\right)}(\wp \mathcal{E}) \leq e^{\lambda} \widehat{\mho}_{\chi_{\left(\left(Q, c_{0}^{\ell}\right)\right.}}(\mathcal{E})<\eta .
$$

That is

$$
\chi_{\mathrm{C}\left(\mathrm{Q}, \mathrm{c}_{0}^{\ell}\right)}(\mathcal{E})<\frac{\eta}{e^{\lambda} \widehat{\widetilde{\mathcal{U}}}} .
$$

Setting $\xi=\frac{\eta}{e^{\lambda} \widehat{\mho}}\left[1-e^{\lambda} \widehat{\mho}\right]$, we get $\eta \leq \chi_{C\left(Q, c_{0}^{\ell}\right)}(\mathcal{E})<\eta+\xi$.
Therefore, $\wp$ is a Meir-Keeler condensing operator on $\mathcal{E}$. Further, $\wp$ satisfies all the conditions of Theorem 2.14, i.e., $\wp$ has a fixed point in $\mathcal{E}$. Hence, BVP (1) has a solution in $\complement\left(Q, c_{0}^{\ell}\right)$.

## 4 Solvability of the BVP (1) in $\subset\left(Q, c^{\ell}\right)$

In this section, we study the solvability of BVP (1) in C( $\left.Q, C^{\ell}\right)$.
We assume the following conditions hold throughout this section:
(C) Let $\psi_{j}: Q \times \mathbb{R}^{\infty} \rightarrow \mathbb{R}$ and define an operator $\psi$ from $Q \times C^{\ell}$ to $C^{\ell}$ as $(r, z(r)) \rightarrow(\psi z)(r)=\left(\psi_{j}(r, z(r))\right)_{j=1}^{\infty}$, which is the family of all functions $((\psi z)(r))_{r \in Q}$ equicontinuous on $c^{\ell}$.
(D) $\eta_{j}(r), \mathfrak{z}_{j}(r): Q \rightarrow \mathbb{R}$ are continuous functions such that the sequence $\ell_{j} \eta_{j}(r) \xrightarrow{\text { uniformly }} 0$ on $Q$ and the sequence $\left(\mathfrak{z}_{j}(r)\right)$ is convergence on $Q$, so we take $\mathfrak{z}(r)=\sup \left\{\mathfrak{z}_{j}(r): j \in \mathbb{N}\right\}, \mathfrak{z}^{\star}=\sup \{\mathfrak{z}(r): r \in Q\}, \ell^{\star}=\sup \left\{\ell_{j} \eta_{j}(r): j \in \mathbb{N}, r \in Q\right\}$ and

$$
\psi_{j}(r, z(r)) \leq \eta_{j}(r)+\mathfrak{z}_{j}(r) z_{j}(r), \quad z_{j} \in c^{\ell}, r \in Q, j=1,2,3, \ldots
$$

Theorem 4.1 Let $\widehat{\mathcal{V}}:=\sup _{\mathrm{p} \in Q} \mho^{\star}(\mathrm{p})$, suppose $e^{\lambda} \widehat{\mathcal{J}}_{\mathfrak{z}^{\star}}<1$ and $(\mathscr{C})-(\mathscr{D})$ hold, then BVP (1) has at least one solution $\mathrm{z}(\mathrm{r})=\left(\mathrm{z}_{\mathrm{j}}(\mathrm{r})\right)$ in $\complement\left(\mathrm{Q}, \mathrm{c}^{\ell}\right)$.

Proof Since $\sup \left\{\ell_{j}\left|z_{j}(r)\right|\right\}<+\infty$ for all $z(r)=\left(z_{j}(r)\right)_{j=1}^{\infty} \in C\left(Q, c^{\ell}\right)$ and $r \in Q$, there exists $\rho>0$ such that $\sup \left\{\ell_{j}\left|z_{j}(r)\right|\right\}<\rho$. From (D) and (9), we get

$$
\begin{aligned}
\|z(\mathrm{r})\|_{c^{\ell}} & =\sup _{j \in \mathbb{N}}\left\{\ell_{j}\left|\int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) e^{-\lambda(\mathrm{r}-\mathrm{p})} \psi_{j}(\mathrm{r}, \mathrm{z}(\mathrm{r})) d \mathrm{p}\right|\right\} \\
& \leq \sup _{j \in \mathbb{N}}\left\{\ell_{j} e^{\lambda} \int_{0}^{1}|\mho(\mathrm{r}, \mathrm{p})|\left|\psi_{j}(\mathrm{r}, \mathrm{z}(\mathrm{r}))\right| d \mathrm{p}\right\} \\
& \leq e^{\lambda} \sup _{j \in \mathbb{N}}\left\{\int_{0}^{1}|\mho(\mathrm{r}, \mathrm{p})| \ell_{j}\left|\psi_{j}(\mathrm{r}, \mathrm{z}(\mathrm{r}))\right| d \mathrm{p}\right\} \\
& \leq e^{\lambda} \sup _{j \in \mathbb{N}}\left\{\int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) \ell_{j}\left[\eta_{j}(\mathrm{p})+\mathfrak{z}_{j}(\mathrm{p})\left|\mathrm{z}_{\mathrm{j}}(\mathrm{p})\right|\right] d \mathrm{p}\right\} \\
& \leq e^{\lambda}\left\{\int_{0}^{1} \mho(\mathrm{r}, \mathrm{p})\left[\ell^{\star}+\mathfrak{z}^{\star} \rho\right] d \mathrm{p}\right\} \\
& \leq C^{\star} e^{\lambda}\left(\ell^{\star}+\mathfrak{z}^{\star} \rho\right):=b .
\end{aligned}
$$

Thus,

$$
\max _{\mathrm{r} \in \mathrm{Q}}\left\|_{\mathrm{z}(\mathrm{r})}\right\|_{\mathrm{c}^{\ell}} \leq b \text {, i.e., }\|\mathrm{z}(\mathrm{r})\|_{\mathrm{C}\left(\ell, c^{\ell}\right)} \leq b .
$$

Let $\mathcal{S}=\mathcal{S}\left(z^{0}(r), r_{1}\right)$ be closed with center $z^{0}(r)=\left(z^{0}(r)\right)_{j=1}^{\infty}$ for all $r \in Q$ and radius $b$. So, $\mathcal{S}$ is a non-empty bounded, closed convex subset of $\complement\left(\mathrm{Q}, \mathrm{c}^{\ell}\right)$.

For fixed $\mathrm{r} \in \mathrm{Q}$, define an operator $\$=\left(\$_{j}\right)_{j=1}^{\infty}: \complement\left(\mathrm{Q}, \mathrm{c}_{0}^{\ell}\right) \rightarrow \complement\left(\mathrm{Q}, \mathrm{c}_{0}^{\ell}\right)$ as

$$
(\$ \mathrm{z})(\mathrm{r})=\left\{\left(\$_{\mathrm{j}} \mathrm{z}\right)(\mathrm{r})\right\}_{j=1}^{\infty}=\left\{\int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) e^{-\lambda(\mathrm{r}-\mathrm{p})} \psi_{\mathrm{j}}(\mathrm{r}, \mathrm{z}(\mathrm{r})) d \mathrm{p}\right\}_{\mathrm{j}=1}^{\infty}
$$

Now, let $i \in \mathbb{N}$ and

$$
\begin{aligned}
&\left|\ell_{j}\left(\$_{j} \mathrm{z}\right)(\mathrm{r})-\ell_{\mathrm{i}}\left(\$_{\mathrm{i}} \mathrm{z}\right)(\mathrm{r})\right| \\
&=\left|\ell_{\mathrm{j}} \int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) e^{-\lambda(\mathrm{r}-\mathrm{p})} \psi_{\mathrm{j}}(\mathrm{r}, \mathrm{z}(\mathrm{r})) d \mathrm{p}-\ell_{\mathrm{i}} \int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) e^{-\lambda(\mathrm{r}-\mathrm{p})} \psi_{\mathrm{i}}(\mathrm{r}, \mathrm{z}(\mathrm{r})) d \mathrm{p}\right| \\
& \leq \mid \ell_{\mathrm{j}} \int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) e^{-\lambda(\mathrm{r}-\mathrm{p})}\left(\eta_{j}(\mathrm{p})+\mathfrak{z}_{j}(\mathrm{p}) \mathrm{z}_{\mathrm{j}}(\mathrm{p})\right) d \mathrm{p} \\
&-\ell_{\mathrm{i}} \int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) e^{-\lambda(\mathrm{r}-\mathrm{p})}\left(\eta_{\mathrm{i}}(\mathrm{p})+\mathfrak{z}_{\mathrm{i}}(\mathrm{p}) \mathrm{z}_{\mathrm{i}}(\mathrm{p})\right) d \mathrm{p} \mid \\
& \leq\left|e^{-\lambda(\mathrm{r}-\mathrm{p})}\right|\left\{\int_{0}^{1}|\mho(\mathrm{r}, \mathrm{p})| \ell_{j} \eta_{j}(\mathrm{p})-\ell_{i} \eta_{\mathrm{i}}(\mathrm{p}) \mid d \mathrm{p}\right. \\
&\left.+\int_{0}^{1}|\mho(\mathrm{r}, \mathrm{p})| \ell_{j} \mathfrak{z}_{j}(\mathrm{p}) \mathrm{z}_{j}(\mathrm{p})-\ell_{i} \mathfrak{z}_{\mathrm{j}}(\mathrm{p}) \mathrm{z}_{\mathrm{i}}(\mathrm{p}) \mid d \mathrm{p}\right\} \\
& \leq e^{\lambda}\left\{\int_{0}^{1}|\mho(\mathrm{r}, \mathrm{p})| \ell_{j} \eta_{j}(\mathrm{p})-\ell_{i} \eta_{\mathrm{i}}(\mathrm{p}) \mid d \mathrm{p}\right. \\
&\left.+\int_{0}^{1}|\mho(\mathrm{r}, \mathrm{p})| \ell_{j} \mathfrak{z}_{j}(\mathrm{p}) \mathrm{z}_{j}(\mathrm{p})-\ell_{\mathrm{i}} \mathfrak{z}_{\mathrm{j}}(\mathrm{p}) \mathrm{z}_{\mathrm{i}}(\mathrm{p}) \mid d \mathrm{p}\right\} .
\end{aligned}
$$

Observe that

$$
\left|\ell_{j} \mathfrak{z}_{j}(\mathrm{p}) \mathrm{z}_{\mathrm{j}}(\mathrm{p})-\ell_{\mathrm{i}} \mathfrak{z}_{\mathrm{i}}(\mathrm{p}) \mathrm{z}_{\mathrm{i}}(\mathrm{p})\right| \leq \ell_{j}\left|\mathrm{z}_{j}(\mathrm{p})\right|\left|\mathfrak{z}_{j}(\mathrm{p})-\mathfrak{z}_{i}(\mathrm{p})\right|+\left|\mathfrak{z}_{\mathrm{i}}(\mathrm{p})\right|\left|\ell_{\mathrm{j}} \mathrm{z}_{\mathrm{j}}(\mathrm{p})-\ell_{\mathrm{i}} z_{\mathrm{i}}(\mathrm{p})\right| .
$$

As ${ }_{j}, i \rightarrow \infty$, we obtain $\left|\mathfrak{z}_{j}(p)-\mathfrak{z}_{i}(p)\right| \rightarrow 0,\left|\ell_{j} z_{j}(p)-\ell_{i} z_{i}(p)\right| \rightarrow 0$ and $\mid \ell_{j} \eta_{j}(p)-$ $\ell_{i} \eta_{i}(p) \mid \rightarrow 0$. Since $\left(\mathfrak{z}_{j}\right),\left(\ell_{j} \eta_{j}\right)$ are convergent on $Q$ and $z_{j}(p) \in C\left(Q, c^{\ell}\right)$, it follows that

$$
\left|\ell_{j}\left(\$_{j} z\right)(r)-\ell_{i}\left(\$_{i} z\right)(r)\right| \rightarrow 0 \quad \text { as } j, i \rightarrow \infty
$$

Hence, $(\$ z)(r) \in C\left(Q, c^{\ell}\right)$. We also note that $\left(\$_{j} z\right)(r)$ satisfies

$$
\begin{aligned}
& \left(\$_{j} z\right)(0)=0, \quad{ }_{0}^{\mathrm{R}} D_{r}^{\mathrm{m}, \lambda}\left(\$_{j} z(0)\right)=0, \\
& b_{1}\left(\$_{j} z(1)\right)+b_{2}{ }_{0}^{\mathrm{R}} D_{r}^{\mathrm{m}, \lambda}\left(\$_{j} z(1)\right)=0 .
\end{aligned}
$$

For fixed $r \in Q$ and $z(r) \in \mathcal{S}$, we get

$$
\begin{aligned}
\left\|(\$ \mathrm{z})(\mathrm{r})-\mathrm{z}^{0}(\mathrm{r})\right\|_{\mathrm{c}^{\ell}} \leq b & \Longrightarrow \max _{\mathrm{r} \in \mathrm{Q}}\left\|(\$ \mathrm{z})(\mathrm{r})-\mathrm{z}^{0}(\mathrm{r})\right\|_{\mathrm{c}^{\ell}} \leq b \\
& \Longrightarrow\left\|(\$ \mathrm{z})(\mathrm{r})-\mathrm{z}^{0}(\mathrm{r})\right\|_{\left(\mathrm{C}, \mathrm{c}^{\ell}\right)} \leq b,
\end{aligned}
$$

which proves that $\$$ is a self-mapping on $\mathcal{S}$. From (C), for any $w(r)=\left(w_{j}(r)\right)_{j=1}^{\infty} \in \mathcal{S}$ and for any $\eta>0$, there exists $\xi>0$ such that $\|(\psi z)(r)-(\psi w)(r)\|_{c^{\ell}}<\frac{\eta}{e^{\lambda} \widehat{\jmath}}$ for each $z(r) \in \mathcal{S}$,
whenever $|z(r)-w(r)| \leq \xi$, where $r \in Q$. So, for $r \in Q$, we have

$$
\begin{aligned}
\|(\$ \mathrm{z})(\mathrm{r})-(\$ \mathrm{w})(\mathrm{r})\|_{\mathrm{c}^{\ell}} & =\sup _{j \in \mathbb{N}}\left\{\left|\ell_{\mathrm{j}} \int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) e^{-\lambda(\mathrm{r}-\mathrm{p})}\left[\psi_{j}(\mathrm{p}, \mathrm{z}(\mathrm{p}))-\psi_{\mathrm{j}}(\mathrm{p}, \mathrm{w}(\mathrm{p}))\right] d \mathrm{p}\right|\right\} \\
& \leq e^{\lambda} \sup _{j \in \mathbb{N}}\left\{\int_{0}^{1} \mho^{\star}(\mathrm{p}) \ell_{j}\left|\psi_{j}(\mathrm{p}, \mathrm{z}(\mathrm{p}))-\psi_{j}(\mathrm{p}, \mathrm{w}(\mathrm{p}))\right| d \mathrm{p}\right\} \\
& \leq e^{\lambda} \widehat{\mho} \frac{\eta}{e^{\lambda} \widehat{\mho}}<\eta .
\end{aligned}
$$

So, $\$$ is continuous on $\mathcal{S}$ for every $\mathrm{r} \in \mathrm{Q}$.
Now, we have

$$
\begin{aligned}
\mu_{c^{\ell}}(\$ \mathcal{S})= & \lim _{j \rightarrow \infty}\left\{\operatorname { s u p } _ { z ( \mathrm { r } ) \in \mathcal { S } } \operatorname { s u p } _ { m \geq j } \left[\mid \ell_{m} \int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) e^{-\lambda(\mathrm{r}-\mathrm{p})} \psi_{m}(\mathrm{p}, \mathrm{z}(\mathrm{p})) d \mathrm{p}\right.\right. \\
& \left.\left.-\lim _{n \rightarrow \infty}\left(\ell_{n} \int_{0}^{1} \mho(\mathrm{r}, \mathrm{p}) e^{-\lambda(\mathrm{r}-\mathrm{p})} \psi_{n}(\mathrm{p}, \mathrm{z}(\mathrm{p})) d \mathrm{p}\right) \mid\right]\right\} \\
\leq & e^{\lambda} \widehat{\widetilde{\delta}} \lim _{j \rightarrow \infty}\left\{\sup _{z(\mathrm{r}) \in \mathcal{S}} \sup _{m \geq j}\left[\int_{0}^{\mathrm{T}}\left|\ell_{m} \psi_{m}(\mathrm{p}, \mathrm{z}(\mathrm{p}))-\lim _{n \rightarrow \infty} \ell_{n} \psi_{n}(\mathrm{p}, \mathrm{z}(\mathrm{p}))\right| d \mathrm{p}\right]\right\} \\
\leq & e^{\lambda} \widehat{\mho} \lim _{j \rightarrow \infty}\left\{\sup _{z(\mathrm{r}) \in \mathcal{S} \mathcal{S}} \sup _{m \geq j}\left[\int_{0}^{\mathrm{T}}\left|\ell_{m} \mathfrak{z}_{m}(\mathrm{p}) \mathrm{z}_{m}(\mathrm{p})-\lim _{n \rightarrow \infty} \ell_{n} \mathfrak{z}_{n}(\mathrm{p}) \mathrm{z}_{n}(\mathrm{p})\right| d \mathrm{p}\right]\right\} \\
\leq & e^{\lambda} \widehat{\mho} \lim _{j \rightarrow \infty}\left\{\operatorname { s u p } _ { z ( \mathrm { r } ) \in \mathcal { S } } \operatorname { s u p } _ { m \geq j } \left[\int _ { 0 } ^ { \mathrm { T } } \left(\left|\mathfrak{z}_{m}(\tau)\right|\left|\ell_{m} \mathrm{z}_{m}(\tau)-\lim _{n \rightarrow \infty} \ell_{n} z_{n}(\tau)\right|\right.\right.\right. \\
& \left.\left.\left.-\left|\lim _{n \rightarrow \infty} \ell_{n} \mathrm{z}_{n}(\tau)\left(\mathfrak{z}_{m}(\tau)-\mathfrak{z}_{n}(\tau)\right)\right|\right) d \tau\right]\right\} \\
\leq & e^{\lambda} \widehat{\mathcal{J}}_{\mathfrak{z}^{\star}} \mu_{c^{\ell}}(\mathcal{S}) .
\end{aligned}
$$

Thus,

It follows that,

$$
\mu_{\mathrm{C}\left(0, \mathrm{c}^{\ell}\right)}(\$ \mathcal{S}) \leq e^{\lambda} \widehat{\mathcal{J}}_{\mathfrak{z}^{\star}} \mu_{\mathrm{C}\left(\mathrm{Q}, \mathrm{c}^{\ell}\right)}(\mathcal{S})<\eta \quad \Longrightarrow \quad \mu_{\mathrm{C}\left(\mathrm{Q}, \mathrm{c}^{\ell}\right)}(\mathcal{S})<\frac{\eta}{e^{\lambda} \widehat{\mho}_{\mathfrak{z}^{\star}}} .
$$

Setting $\xi=\frac{\eta}{e^{\lambda} \hat{\mathcal{J}}_{\mathfrak{z}^{\star}}}\left[1-e^{\lambda} \widehat{\mathcal{J}}_{\mathfrak{z}^{\star}}\right]$, we get $\eta \leq \mu_{\mathrm{C}\left(\mathrm{( }, \mathrm{c}^{\ell}\right)}(\mathcal{S})<\eta+\xi$.
Therefore, $\$$ is a Meir-Keeler condensing operator on $\mathcal{S}$. Since $r$ is arbitrary, so for every $r \in Q, \$$ satisfies all the conditions of Theorem 2.14, i.e., $\$$ has a fixed point in $\mathcal{S}$. Hence, BVP (1) has a solution in $\complement\left(Q, c^{\ell}\right)$.

## 5 Applications

In this section, we provide two examples to check the validity of our main results.
Example 5.1 Consider the following BVP

$$
\begin{equation*}
{ }_{0}^{R} D_{r}^{\frac{5}{2}, \frac{1}{6}} z_{j}(r)+\left[\frac{e^{-j r} \cos (j r)}{j}+\sum_{i=j}^{\infty} \frac{z_{j}(r)}{i^{2}}\right]=0, \quad 0<r<1, \tag{11}
\end{equation*}
$$

$$
\left\{\begin{array}{l}
z(0)=0, \quad{ }_{0}^{\mathrm{R}} D_{r}^{\frac{3}{2}, \frac{1}{6}} z(0)=0  \tag{12}\\
0.1 z(1)+\frac{2}{3 \sqrt{\pi}}{ }^{R} D_{r}^{\frac{3}{2}, \frac{1}{6}} z(1)=0
\end{array}\right.
$$

So, $\varrho=\frac{5}{2}, m=\frac{3}{2}, \lambda=\frac{1}{6}, \mathrm{~b}_{1}=0.1, \mathrm{~b}_{2}=\frac{2}{3 \sqrt{\pi}}, Q=[0,1]$, and $\psi_{j}(\mathrm{r}, \mathrm{z}(\mathrm{r}))=\frac{e^{-\mathrm{jr} \cos (\mathrm{jr})}}{j}+$ $\sum_{i=j}^{\infty} \frac{z_{j}(r)}{i^{2}}$.

Let $\ell_{j}=\frac{1}{j}$ for all $j \in \mathbb{N}$. Now, for $z(r) \in C\left(Q, c_{0}^{\ell}\right)$, we have

$$
\lim _{j \rightarrow \infty} \ell_{j} \psi_{j}(r, z(r))=\lim _{j \rightarrow \infty}\left[\frac{e^{-j r} \cos (j r)}{j}+\frac{1}{j} \sum_{i=j}^{\infty} \frac{z_{j}(r)}{i^{2}}\right]=0 .
$$

Next, let $v(r)=\left(\nu_{j}(r)\right)_{j=1}^{\infty} \in C\left(Q, C_{0}^{\ell}\right)$. Let $\eta>0$ be given and $\delta=\frac{24 \eta}{\pi^{2}}$ such that $\| z(r)-$ $v(r) \|_{C\left(Q, c_{0}^{\ell}\right)}<\delta$. Then,

$$
\begin{aligned}
\|(\psi z)(r)-(\psi v)(r)\|_{c_{0}^{\ell}} & =\sup _{j \in \mathbb{N}}\left\{\ell_{j}\left|\psi_{j}(\mathrm{r}, \mathrm{z}(\mathrm{r}))-\psi_{j}(\mathrm{r}, v(\mathrm{r}))\right|\right\} \\
& =\sup _{j \in \mathbb{N}}\left\{\frac{1}{j} \sum_{\mathrm{i}=\mathrm{j}}^{\infty} \frac{1}{\mathrm{i}^{2}}\left|z_{j}(\mathrm{r})-v_{j}(\mathrm{r})\right|\right\} \\
& \leq \frac{\pi^{2}}{6}\left\|z_{j}(\mathrm{r})-v_{j}(\mathrm{r})\right\|_{c_{0}^{\ell}<\eta} .
\end{aligned}
$$

Thus, $((\psi z)(r))_{r \in Q}$ is equicontinuous on $c_{0}^{\ell}$. For $r \in Q$ and $j \in \mathbb{N}$, we also have

$$
\begin{aligned}
\left|\psi_{j}(\mathrm{r}, \mathrm{z}(\mathrm{r}))\right| & \leq \frac{e^{-\mathrm{jr}}|\cos (\mathrm{jr})|}{\mathrm{j}}+\sum_{\mathrm{i}=\mathrm{j}}^{\infty} \frac{1}{\mathrm{i}^{2}}\left|\mathrm{z}_{\mathrm{j}}(\mathrm{r})\right| \\
& \leq \frac{1}{j}+\frac{\pi^{2}}{6}\left|z_{j}(\mathrm{r})\right|
\end{aligned}
$$

where $\xi_{j}(r)=\frac{1}{j}$ and $\zeta_{j}(r)=\frac{\pi^{2}}{6}$. So, $\zeta^{\star}=\frac{\pi^{2}}{6}$. We note that $\left(\ell \xi_{j}(r)\right)=\left(1 / j^{2}\right) \xrightarrow{\text { uniformly }} 0$ on $Q$, and the sequence $\zeta_{j}(r)$ is equibounded on $Q$. Also, $\mathbb{k}=\left[b+c \frac{\Gamma(\rho)}{\Gamma(\rho-m)}\right]^{-1}=1.67$,

$$
\begin{aligned}
U^{\star}(\mathrm{p}) & =\mathbb{k} \mathrm{b} \frac{(1-\mathrm{p})^{\varrho-1}}{\Gamma(\varrho)}+\mathbb{k} \mathrm{c} \frac{(1-\mathrm{p})^{\varrho-\mathrm{m}-1}}{\Gamma(\varrho-\mathrm{m})} \\
& =0.126(1-\mathrm{p})^{1.5}+0.628
\end{aligned}
$$

Then, $\widehat{\mho}=0.754$. So,

$$
e^{\lambda} \widehat{\mho}=0.754 \sqrt[6]{e}<1 .
$$

Hence, by Theorem 3.1, BVP (11)-(12) has a solution in C( $\left.\mathrm{Q}, \mathrm{c}_{0}^{\ell}\right)$.
Example 5.2 Consider the BVP

$$
\begin{equation*}
{ }_{0}^{R} D_{r}^{\frac{7}{3}, \frac{1}{8}} z_{j}(r)+\left[\frac{1}{j}+\sum_{i=j}^{\infty} \frac{z_{j}(r)}{2 i^{2}}\right]=0, \quad 0<r<1, \tag{13}
\end{equation*}
$$

$$
\left\{\begin{array}{l}
z(0)=0, \quad{ }_{0}^{R} D_{r}^{\frac{4}{3}, \frac{1}{8}} z(0)=0  \tag{14}\\
0.2 z(1)+\frac{27}{4 \sqrt{3} \pi} 0^{R} D_{r}^{\frac{4}{3}, \frac{1}{6}} z(1)=0
\end{array}\right.
$$

So, $\varrho=\frac{7}{3}, m=\frac{4}{3}, \lambda=\frac{1}{8}, Q=[0,1], b_{1}=0.2, b_{2}=\frac{27}{4 \sqrt{3} \pi}$, and $\psi_{j}(r, z(r))=\frac{1}{j}+\sum_{i=j}^{\infty} \frac{z_{j}(r)}{2 i^{2}}$.
Let $\ell_{j}=\frac{1}{j}$ for all $j \in \mathbb{N}$. Now, for $z(r) \in C\left(Q, c^{\ell}\right)$, we have

$$
\lim _{j \rightarrow \infty} \ell_{j} \psi_{j}(r, z(r))=\lim _{j \rightarrow \infty}\left[\frac{1}{j^{2}}+\frac{1}{j} \sum_{i=j}^{\infty} \frac{z_{j}(r)}{2 \mathrm{i}^{2}}\right]=0
$$

This shows that $\left(\psi_{j}(r, z(r))\right) \in c^{\ell}$. Next, let $w(r)=\left(w_{j}(r)\right)_{j=1}^{\infty} \in \complement\left(Q, c^{\ell}\right)$. Let $\eta>0$ be given and $\delta=\frac{12 \eta}{\pi^{2}}$ such that $\|z(r)-w(r)\|_{C\left(\Omega, c^{\ell}\right)}<\xi$. Then

$$
\begin{aligned}
\|(\psi z)(r)-(w z)(r)\|_{c^{\ell}} & =\sup _{j \in \mathbb{N}}\left\{\ell_{j}\left|\psi_{j}(r, z(r))-\psi_{j}(r, w(r))\right|\right\} \\
& =\sup _{j \in \mathbb{N}}\left\{\frac{1}{j} \sum_{i=j}^{\infty} \frac{1}{2 \mathrm{i}^{2}}\left|z_{j}(r)-w_{j}(r)\right|\right\} \\
& \leq \frac{\pi^{2}}{12}\left\|z_{j}(r)-w_{j}(r)\right\|_{c_{0}^{\ell}}<\eta .
\end{aligned}
$$

Thus, $((\psi z)(r))_{r \in Q}$ is equicontinuous on $c^{\ell}$. For $r \in Q$ and $j \in \mathbb{N}$, we also have $\eta_{j}(r)=\frac{1}{j}$ and $\mathfrak{z}_{j}(\mathrm{r})=\frac{\pi^{2}}{12}$. So, $\mathfrak{z}^{\star}=\frac{\pi^{2}}{12}$. We note that $\left(\ell \eta_{j}(\mathrm{r})\right)=\left(\frac{1}{\mathrm{j}^{2}}\right) \xrightarrow{\text { uniformly }} 0$ on $Q$, and the sequence $\mathfrak{z}_{j}(\mathrm{r})$ is convergent on $Q$. Also, $\mathbb{k}=\left[\mathrm{b}+\mathrm{c} \frac{\Gamma(\varrho)}{\Gamma(\varrho-m)}\right]^{-1}=1.06$,

$$
\begin{aligned}
\mho^{\star}(\mathrm{p}) & =\mathbb{k} \mathrm{b} \frac{(1-\mathrm{p})^{\varrho-1}}{\Gamma(\varrho)}+\mathbb{k} \mathrm{C} \frac{(1-\mathrm{p})^{\varrho-m-1}}{\Gamma(\varrho-\mathrm{m})} \\
& =0.178(1-\mathrm{p})^{4 / 3}+0.662 .
\end{aligned}
$$

Then, $\widehat{\mho}=0.84$. So,

$$
e^{\lambda}{\widehat{\delta_{\mathfrak{z}}}}^{\star}=0.84 \times \sqrt[8]{e} \times \frac{\pi^{2}}{12}<1 .
$$

Hence, by Theorem 4.1, BVP (13)-(14) has a solution in C( $\left.\mathrm{Q}, \mathrm{c}^{\ell}\right)$.

## 6 Conclusion

The present paper considers a boundary value problem with an infinite system of tempered fractional order. A variation of the well-known RL-fractional derivative, the socalled tempered fractional RL-derivative, is the fractional derivative used in our case. Using the HMNC technique and the Meir-Keeler fixed point theorem, we looked into whether there is a solution to an infinite system. This study was conducted in two brandnew sequence spaces: tempered sequence spaces $\complement\left(Q, c_{0}^{\ell}\right)$ and $C\left(Q, c^{\ell}\right)$. Finally, numerical examples are also given to demonstrate the results we achieved. Future research could focus on the following areas:
(1) To investigate infinite system of singular TFBVP, further research is required.
(2) Is it possible to expand the concept used in this paper to investigate infinite systems of fractional difference equations and dynamic equations on time scales.

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## Declarations

## Competing interests

The authors declare no competing interests.

## Author contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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