# Least energy nodal solutions for a weighted ( $N, p$ )-Schrödinger problem involving a continuous potential under exponential growth nonlinearity 

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#### Abstract

This article aims to investigate the existence of nontrivial solutions with minimal energy for a logarithmic weighted ( $N, p$ )-Laplacian problem in the unit ball $B$ of $\mathbb{R}^{N}$, $N>2$. The nonlinearities of the equation are critical or subcritical growth, which is motivated by weighted Trudinger-Moser type inequalities. Our approach is based on constrained minimization within the Nehari set, the quantitative deformation lemma, and degree theory results.


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## 1 Introduction

This paper is devoted to the existence of sign-changing solutions for the following problem involving the logarithmic weighted ( $N, p$ )-Laplacian:

$$
\begin{cases}-\nabla \cdot\left(w_{\beta}(x)|\nabla u|^{N-2} \nabla u\right)-\Delta_{p} u+V(x)|u|^{N-2} u=f(x, u) & \text { in } B,  \tag{1.1}\\ u=0 & \text { on } \partial B,\end{cases}
$$

where $B$ is the unit ball of $\mathbb{R}^{N}, N>p \geq 2$, the nonlinearity $f(x, t)$ is continuous in $B \times \mathbb{R}$ and behaves like $e^{\alpha t^{\frac{N}{N-1)(1-\beta)}}}, \beta \in[0,1)$, as $t \rightarrow \infty$, for some $\alpha>0, \Delta_{p}$ denotes the $p$-Laplacian the second-order operator defined by $\Delta_{p} u=\nabla \cdot\left(|\nabla u|^{p-2} \nabla u\right)$, and the weight $w_{\beta}(x)$ is given by

$$
\begin{equation*}
w_{\beta}(x)=(1-\log |x|)^{\beta(N-1)}, \quad \beta \in[0,1) . \tag{1.2}
\end{equation*}
$$

The potential $V: \bar{B} \rightarrow \mathbb{R}$ is a positive continuous function and verifies

[^0]$\left(V_{1}\right) \quad V(x) \geq V_{0}>0$ in $B$ for some $V_{0}>0$.
Condition $\left(V_{1}\right)$ implies that the function $\frac{1}{V}$ belongs to $L^{\frac{1}{N-1}}(B)$.
The weighted $(N, p)$-Laplacian operator is a generalization of the nonlinear $(N, p)$ Laplacian operator. Indeed, the weight function serves as a versatile tool for modeling various effects, including the presence of a medium or the influence of an external field. This versatility makes it particularly suitable for representing a wide range of physical phenomena, such as diffusion, fluid flow, reaction-diffusion equations [2, 13], biophysics [23], plasma physics [39], and specific elementary particle models [6, 7, 18]. It is essential to note that the origins of the ( $N, p$ )-Laplacian equation can be traced back to the study of stationary solutions in the context of the following reaction-diffusion system:
$$
u_{t}=\nabla \cdot\left(|\nabla u|^{N-2} \nabla u+\Delta_{p} u\right)-V(x)|u|^{N-2} u+f(x, u) .
$$

In this equation, the function $u$ represents a concentration. The first term characterizes diffusion with a diffusion coefficient of $\left(|\nabla u|^{N-2}+|\nabla u|^{p-2}\right)$. The continuous potential $V(x)$ is a function influencing the system's energy, and $f(x, u)$ corresponds to the reaction term, linking to source and loss processes [5,13]. As a result, quasilinear elliptic boundary value problems involving the ( $N, p$ )-Laplacian operator with various nonlinearities have been extensively investigated by numerous researchers, as evident in works such as $[3,22,33$, 36] and the associated references.
When $N=q \geq 2$, much attention from researchers has been directed towards this particular limit in the Sobolev embedding, widely recognized as the Trudinger-Moser case. More precisely, consider a smooth bounded domain $\Omega$ in $\mathbb{R}^{N}$, where $N \geq 2$. Let $W_{0}^{1, N}(\Omega)$ represent the standard first-order Sobolev space defined as follows:

$$
W_{0}^{1, N}(\Omega)=\operatorname{closure}\left\{\left.u \in C_{0}^{\infty}(\Omega)\left|\int_{\Omega}\right| \nabla u\right|^{N} d x<\infty\right\}
$$

equipped with the norm

$$
\|u\|_{W_{0}^{1, N}(\Omega)}=\left(\int_{\Omega}|\nabla u|^{N} d x\right)^{\frac{1}{N}}
$$

This space represents a critical limit in the Sobolev embedding theorem. While the theorem establishes that the space $W_{0}^{1, N}(\Omega) \hookrightarrow L^{p}(\Omega)$ for all $1 \leq p<\infty$, it is well known, through simple examples, that $W_{0}^{1, N}(\Omega) \nsubseteq L^{\infty}(\Omega)$. Consequently, the natural question that arises is to identify the maximum growth function $\Phi: \mathbb{R} \rightarrow \mathbb{R}^{+}$such that

$$
\int_{\Omega} \Phi(u) d x<\infty \quad \text { for } u \in W_{0}^{1, N}(\Omega) \text { while satisfying }\|u\|_{W_{0}^{1, N}(\Omega)} \leq 1 .
$$

This question was conclusively answered by Yudovich [42], Pohozaev [37], and Trudinger [38]. Their collective work has established that the maximal growth function is defined as $\Phi(t)=e^{|t|^{\frac{N}{N-1}}}$. Moser further refined this result in his work [35]. More specifically, he demonstrated that for all $u \in W_{0}^{1, N}(\Omega)$, the function $\exp \left(\alpha|u|^{N-1}\right)$ belongs to $L^{1}(\Omega)$ for some $\alpha>0$. In fact, he established that

$$
\begin{equation*}
\sup _{\|u\|_{W_{0}^{1, N}(\Omega)} \leq 1} \int_{\Omega} e^{\alpha|u|^{N-1}} d x<C(N) \Longleftrightarrow \alpha \leq \alpha_{N}:=N \omega_{N-1}^{\frac{1}{N-1}}, \tag{1.3}
\end{equation*}
$$

where $\omega_{N-1}$ represents the area of the unit sphere in $\mathbb{R}^{N}$. It is worth noting that the constant $\alpha_{N}$ is a critical threshold; for values of $\alpha$ greater than $\alpha_{N}$, the supremum in (1.3) becomes infinite. Such estimates, akin to (1.3), are now commonly referred to as TrudingerMoser type inequalities. For further exploration and applications in this field, the reader is directed to works such as $[15-17,31,32]$ and the associated references.
It is essential to highlight that considerable research efforts have been dedicated to the investigation of the impact of weight functions on the limiting inequalities of TrudingerMoser type. Interested readers can delve into studies like [1, 12] to explore the effect of power weights within the integral term on maximal growth and [8-11] for insights into the influence of weights within the Sobolev norm. Kufner, in his work [29], introduced the concept of weighted Sobolev spaces and developed an embedding theory for such spaces with general weight functions. Let $\Omega \subset \mathbb{R}$ and $w \in L^{1}(\Omega)$ be a nonnegative function, then the weighted Sobolev space can be introduced as follows:

$$
\begin{equation*}
W_{0}^{1, N}(\Omega, w)=\operatorname{closure}\left\{\left.u \in C_{0}^{\infty}(\Omega)\left|\int_{\Omega} w(x)\right| \nabla u\right|^{N} d x \text { finite }\right\} . \tag{1.4}
\end{equation*}
$$

When the weight function $w$ takes the form of the logarithmic function, the weighted Sobolev spaces defined as in (1.4) hold particular significance. These spaces deal with limiting scenarios of such embeddings. Nevertheless, to obtain meaningful results, we find it necessary to confine our attention to radial functions. Thus, we turn our focus to the weighted Sobolev space of radial functions.

$$
W_{0, \text { rad }}^{1, N}(\Omega, w)=\text { closure }\left\{\left.u \in C_{0, \text { rad }}^{\infty}(\Omega)\left|\int_{B} w(x)\right| \nabla u\right|^{N} d x \text { finite }\right\}
$$

endowed with the norm

$$
\begin{equation*}
\|u\|_{W_{0, \text { rad }}^{1, N}(\Omega, w)}:=|\nabla u|_{N, w}=\left(\int_{B} w(x)|\nabla u|^{N} d x\right)^{\frac{1}{N}}, \tag{1.5}
\end{equation*}
$$

when $w$ is given by (1.2).
The initial exploration of Trudinger-Moser inequalities for Sobolev spaces with logarithmic weights was conducted by Calanchi and Ruf, as documented in [8]. Their work focused on the case when $N=2$, considering a Sobolev norm of logarithmic type. Subsequently, they expanded their research to encompass the general case, as outlined in [9]. To be more specific, they established the following result.

Theorem 1.1 [9] Let $\beta \in[0,1)$ and let $w_{\beta}$ be given by (1.2), then

$$
\begin{align*}
& \int_{B} e^{|u|^{\gamma}} d x \text { is finite, for every } u \in W_{0, \mathrm{rad}}^{1}\left(B, w_{\beta}\right), \quad \text { if and only if } \\
& \gamma \leq \gamma_{N, \beta}=\frac{N}{(N-1)(1-\beta)}=\frac{N^{\prime}}{1-\beta} \tag{1.6}
\end{align*}
$$

and

$$
\begin{equation*}
\sup _{\substack{u \in W_{0, \text { rad }}^{1}\left(B, w_{\beta}\right) \\|\nabla u|_{N, w_{\beta}} \leq 1}} \int_{B} e^{\alpha|u|^{\gamma N, \beta}} d x<+\infty \quad \Leftrightarrow \quad \alpha \leq \alpha_{N, \beta}=N\left[\omega_{N-1}^{\frac{1}{N-1}}(1-\beta)\right]^{\frac{1}{1-\beta}}, \tag{1.7}
\end{equation*}
$$

where $\omega_{N-1}$ is the area of the unit sphere $S^{N-1}$ in $\mathbb{R}^{N}$ and $N^{\prime}$ is the Hölder conjugate of $N$.

Let $\gamma:=\gamma_{N, \beta}=\frac{N^{\prime}}{1-\beta}$, in view of inequalities (1.6) and (1.7), we say that $f$ has subcritical growth at $+\infty$ if

$$
\begin{equation*}
\lim _{|s| \rightarrow \infty} \frac{|f(x, s)|}{e^{\alpha s^{\gamma}}}=0 \quad \text { for all } \alpha \text { such that } \alpha_{N, \beta} \geq \alpha>0 \tag{1.8}
\end{equation*}
$$

and $f$ has critical growth at $\infty$ if there exists some $0<\alpha_{0} \leq \alpha_{N, \beta}$,

$$
\begin{align*}
& \lim _{|s| \rightarrow \infty} \frac{|f(x, s)|}{e^{\alpha s^{\gamma}}}=0, \quad \forall \alpha \text { such that } \alpha_{0} \leq \alpha \leq \alpha_{N, \beta} \quad \text { and } \\
& \lim _{|s| \rightarrow \infty} \frac{|f(x, s)|}{e^{\alpha s^{\gamma}}}=\infty, \quad \forall \alpha<\alpha_{0} . \tag{1.9}
\end{align*}
$$

In the case $p<q=N$, problem (1.1) has attracted substantial attention from various researchers, each addressing it with distinct nonlinearities. This work is inspired by the work of authors such as $[24,25,28,30,41,44]$ and the references therein. Notably, in the work by Figueredo and Nunes [24], the focus was on investigating the existence of positive solutions within a specific class of quasilinear problems.

$$
\begin{cases}-\operatorname{div}\left(a\left(|\nabla u|^{p}\right)|\nabla u|^{p-2} \nabla u\right)=f(u) & \text { in } \Omega \subset \mathbb{R}^{N}  \tag{1.10}\\ u=0 & \text { on } \partial \Omega\end{cases}
$$

in which the hypotheses on function $a$ included the case $-\Delta_{N} u-\Delta_{p} u$. The nonlinearity $f: \mathbb{R} \rightarrow \mathbb{R}$ is a superlinear continuous function with exponential subcritical or critical growth, and the function $a$ is $C^{1}$. By using the minimization argument and deformation lemma, the authors proved the existence of a least energy nodal solutions for equation (1.10) with two nodal domains. Moreover, Zhang and Yang [44] considered the problem

$$
\begin{cases}-\Delta_{N} u-\Delta_{p} u=\lambda u^{N-1} e^{\beta u^{N^{\prime}}}-\mu & \text { in } \Omega \\ u>0 & \text { in } \Omega \\ u=0 & \text { on } \partial \Omega\end{cases}
$$

where $1<N / 2<p<N, N^{\prime}=\frac{N}{N-1}, \Omega$ is an open bounded domain containing the origin in $\mathbb{R}^{N}$ with $C^{2}$ boundary and $\lambda, \mu>0$ are positive real parameters. To be more specific, the authors demonstrated the existence of positive solutions for the above problem by combining variational techniques with regularity arguments. Alternatively, [4, 10, 11, 14, 19, 20, 43] investigated elliptic equations with weighted $N$-Laplacian operator and critical Trudinger-Moser nonlinearities, while this paper will focus on a different class of problems.
Inspired by the above results, this paper embarks on an inquiry into the presence of sign-changing solutions possessing minimal energy for weighted problems akin to those of Shrödinger type. We study both subcritical and critical exponential growth patterns at infinity by using the constraint minimization argument and topological degree theory.

With this objective in mind, we consider the space

$$
X_{\beta}:=W_{0, \mathrm{rad}}^{1, N}\left(B, w_{\beta}\right),
$$

which is a reflexive and Banach space provided condition $\left(V_{1}\right)$ holds. $X_{\beta}$ is endowed with the norm

$$
\begin{equation*}
\|u\|:=\left(\int_{B} w_{\beta}(x)|\nabla u|^{N}+V(x)|u|^{N} d x\right)^{\frac{1}{N}} \tag{1.11}
\end{equation*}
$$

which is equivalent to the following norm:

$$
\|u\|_{W_{0, \text { rad }}^{1, N}\left(B, w_{\beta}\right)}=\left(\int_{B} w_{\beta}(x)|\nabla u|^{N} d x\right)^{\frac{1}{N}}
$$

Note that the embedding $X_{\beta} \hookrightarrow L^{q}(B)$ is continuous for all $q \geq 1$. Moreover, the embed$\operatorname{ding} X_{\beta} \hookrightarrow L^{q}(B)$ is compact for all $q \geq N$, see [26]. For this work, we impose the following conditions on the nonlinearity $f(x, t)$ :
$\left(A_{1}\right) f: B \times \mathbb{R} \rightarrow \mathbb{R}$ is $C^{1}$ and radial in $x$;
$\left(A_{2}\right)$ There exists $\theta>N$ such that

$$
0<\theta F(x, t) \leq t f(x, t), \quad \forall(x, t) \in B \times \mathbb{R} \backslash\{0\},
$$

where $F(x, t)=\int_{0}^{t} f(x, s) d s$;
$\left(A_{3}\right)$ For each $x \in B, t \mapsto \frac{f(x, t)}{|t|^{N-1}}$ is increasing for all $t \in \mathbb{R} \backslash\{0\}$;
$\left(A_{4}\right) \lim _{t \rightarrow 0} \frac{f(x, t)}{|t|^{N-1}}=0$, uniformly in $x \in B$;
$\left(A_{5}\right)$ There exist $r>N$ and $C_{r}>1$ such that

$$
\operatorname{sgn}(t) f(x, t) \geq C_{r}|t|^{r-1} \quad \text { for all }(x, t) \in B \times \mathbb{R}
$$

where $\operatorname{sgn}(t)=1$ if $t>0, \operatorname{sgn}(t)=0$ if $t=0$, and $\operatorname{sgn}(t)=-1$ if $t<0$.
A typical example of a function $f$ satisfying conditions $\left(A_{1}\right),\left(A_{2}\right),\left(A_{3}\right),\left(A_{4}\right)$, and $\left(A_{5}\right)$ is given by

$$
f(t)=C_{r}|t|^{r-2} t+|t|^{r-2} t e^{t^{\gamma}} \quad \text { with } r>N .
$$

The energy functional, sometimes referred to as the Euler-Lagrange functional associated with problem (1.1), is defined as follows:

$$
\begin{equation*}
\mathcal{J}(u):=\frac{1}{N} \int_{B} w_{\beta}(x)|\nabla u|^{N}+V(x)|u|^{N} d x+\frac{1}{p} \int_{B}|\nabla u|^{p} d x-\int_{B} F(x, u) d x, \tag{1.12}
\end{equation*}
$$

where $F(x, u)=\int_{0}^{u} f(x, t) d t$. It is evident that the search for nontrivial weak solutions to problem (1.1) is equivalent to identifying nonzero critical points within the functional $\mathcal{J}$. Since the reaction term $f$ is of critical or subcritical growth, there exist positive constants $c_{1}$ and $c_{2}$ such that

$$
\begin{equation*}
|f(x, t)| \leq c_{1} \exp \left\{c_{2}|t|^{\gamma}\right\}, \quad \forall x \in B, \quad \forall t \in \mathbb{R} \tag{1.13}
\end{equation*}
$$

and so, by using $\left(A_{1}\right)$, we have that $\mathcal{J} \in C^{1}\left(X_{\beta}, \mathbb{R}\right)$ and

$$
\begin{aligned}
\left\langle\mathcal{J}^{\prime}(u), \varphi\right\rangle:= & \int_{B}\left(w_{\beta}(x)|\nabla u|^{N-2}+|\nabla u|^{p-2}\right) \nabla u \nabla \varphi d x+\int_{B} V(x)|u|^{N-2} u \varphi d x \\
& -\int_{B} f(x, u) \varphi d x
\end{aligned}
$$

for all $u$ and $\varphi \in X_{\beta}$. Clearly, the critical points of functional $\mathcal{J}$ are weak solutions of problem (1.1).

Definition 1.1 A function $u$ is called a solution to (1.1) if $u \in X_{\beta}$ and

$$
\begin{align*}
& \int_{B}\left(w_{\beta}(x)|\nabla u|^{N-2}+|\nabla u|^{p-2}\right) \nabla u \nabla \varphi d x+\int_{B} V(x)|u|^{N-2} u \varphi d x \\
& \quad=\int_{B} f(x, u) \varphi d x \quad \text { for all } \varphi \in X_{\beta} . \tag{1.14}
\end{align*}
$$

Definition $1.2 v \in X_{\beta}$ is called nodal or sign-changing solution of problem (1.1) if $v$ is a solution of problem (1.1) and $v^{ \pm} \neq 0$ a.e. in $B$.
$v \in X_{\beta}$ is called least energy sign-changing solution of problem (1.1) if $v$ is a signchanging solution of (1.1) and

$$
\mathcal{J}(v)=\inf \left\{\mathcal{J}(u): \mathcal{J}^{\prime}(u)=0, u^{ \pm} \neq 0 \text { a.e. in } B\right\} .
$$

Our approach revolves around the pursuit of sign-changing solutions that minimize the associated energy functional $\mathcal{J}$ among the ensemble of all sign-changing solutions to problem (1.1). To this end, we introduce the sign-changing Nehari set defined as follows:

$$
\mathcal{N}:=\left\{u \in X_{\beta}, u^{ \pm} \neq 0 \text { and }\left\langle\mathcal{J}^{\prime}(u), u^{+}\right\rangle=\left\langle\mathcal{J}^{\prime}(u), u^{-}\right\rangle=0\right\},
$$

where $u^{+}(x):=\max \{u(x), 0\}$ and $u^{-}(x):=\min \{u(x), 0\}$. It is evident that any sign-changing solution of problem (1.1) resides in the set $\mathcal{N}$. According to (1.14), we have

$$
\left\langle\mathcal{J}^{\prime}(u), u^{ \pm}\right\rangle=\int_{B} w_{\beta}(x)\left|\nabla u^{ \pm}\right|^{N}+V(x)\left|u^{ \pm}\right|^{N} d x+\int_{B}\left|\nabla u^{ \pm}\right|^{p} d x-\int_{B} f\left(x, u^{ \pm}\right) u^{ \pm} d x .
$$

It is important to note that, for every $u=u^{+}+u^{-} \in \mathcal{N}$, it is readily observed that

$$
\begin{aligned}
& \mathcal{J}(u)=\mathcal{J}\left(u^{+}\right)+\mathcal{J}\left(u^{-}\right), \quad\left\langle\mathcal{J}^{\prime}(u), u^{+}\right\rangle=\left\langle\mathcal{J}^{\prime}\left(u^{+}\right), u^{+}\right\rangle \quad \text { and } \\
& \left\langle\mathcal{J}^{\prime}(u), u^{-}\right\rangle=\left\langle\mathcal{J}^{\prime}\left(u^{-}\right), u^{-}\right\rangle .
\end{aligned}
$$

In our initial theorem, we prove the existence of sign-changing solutions for (1.1) in the exponential subcritical case.

Theorem 1.2 Assume that $f(x, t)$ has a subcritical growth at $\infty$ and satisfies conditions $\left(A_{1}\right),\left(A_{2}\right),\left(A_{3}\right)$, and $\left(A_{4}\right)$. If in addition condition $\left(V_{1}\right)$ holds, then problem (1.1) admits a least energy sign-changing solution $v \in \mathcal{N}$ with precisely two nodal domains.

In our second theorem, we establish the existence of a sign-changing solution for (1.1) in the exponential critical case.

Theorem 1.3 Assume that $f(x, t)$ has a critical growth at $\infty$ and satisfies conditions $\left(A_{1}\right)$, $\left(A_{2}\right),\left(A_{3}\right),\left(A_{4}\right)$, and $\left(A_{5}\right)$. If in addition condition $\left(V_{1}\right)$ holds, then there exists $\delta>0$ such that problem (1.1) admits a least energy sign-changing solution $v \in \mathcal{N}$ with precisely two nodal domains provided

$$
\begin{equation*}
C_{p}>\max \left(1,\left(\frac{N^{2} \theta(r-p) c_{\mathcal{N}_{r}}}{p(r-N)(\theta-N)}\left(\frac{2\left(\alpha_{0}+\delta\right)}{\alpha_{N, \beta}}\right)^{(N-1)(1-\beta)}\right)^{\frac{r-p}{p}}\right)>0 \tag{1.15}
\end{equation*}
$$

where $c_{\mathcal{N}_{r}}=\inf _{\mathcal{N}_{r}} \mathcal{J}_{r}(u)>0$,

$$
\mathcal{J}_{r}(u):=\frac{1}{N}\|u\|^{N}+\frac{1}{p}\|u\|_{p}^{p}-\frac{1}{r} \int_{B}|u|^{r} d x
$$

and

$$
\mathcal{N}_{r}:=\left\{u \in X_{\beta}, u^{ \pm} \neq 0 \text { and }\left\langle\mathcal{J}_{r}^{\prime}(u), u^{+}\right\rangle=\left\langle\mathcal{J}_{r}^{\prime}(u), u^{-}\right\rangle=0\right\} .
$$

The rest of this paper proceeds as follows. In Sects. 2, we introduce preliminaries for the compactness analysis and some useful lemmas. Section 3 is devoted to proving Theorem 1.2. Finally, in Sect. 4, we establish some estimates and prove Theorem 1.3.

Notation Throughout this paper, we use the following notations:

- $C$ denotes a positive constant that may change from one line to another, and we sometimes index the constant to show how they change.
- $|u|_{p}$ denotes the norm in the Lebesgue space $L^{p}(B)$ for $p \geq 1$.
- $|u|_{p, \omega}$ denotes the norm in the weighted Lebesgue space $L^{p}(B, \omega)$, which is defined by

$$
|u|_{p, \omega}=\left(\int_{B} w_{\beta}(x)|u|^{p} d x\right)^{\frac{1}{p}}
$$

- $\|u\|_{p}$ denotes the norm in the usual Sobolev space $W_{0}^{1, p}(B)$, which is defined by

$$
\|u\|_{p}=\left(\int_{B}|\nabla u|^{p} d x\right)^{\frac{1}{p}}
$$

## 2 Some useful lemmas

In this section, we prove some lemmas that are important to obtain the desired results. To this end, let $u \in X_{\beta}$ with $u^{ \pm} \neq 0$, we define the function $\mathcal{G}: \mathbb{R}_{+} \times \mathbb{R}_{+} \rightarrow \mathbb{R}$ and the mapping $\mathcal{K}: \mathbb{R}_{+} \times \mathbb{R}_{+} \rightarrow \mathbb{R}^{2}$, where

$$
\begin{equation*}
\mathcal{G}(s, t)=\mathcal{J}\left(s u^{+}+t u^{-}\right) \tag{2.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{K}(s, t)=\left(\left\langle\mathcal{J}^{\prime}\left(s u^{+}+s u^{-}\right), s u^{+}\right\rangle,\left\langle\mathcal{J}^{\prime}\left(s u^{+}+t u^{-}\right), t u^{-}\right\rangle\right) . \tag{2.2}
\end{equation*}
$$

Lemma 1 For any $u \in X_{\beta}$ with $u^{ \pm} \neq 0$, there is the unique maximum point pair $\left(s_{u}, t_{u}\right) \in$ $\mathbb{R}_{+} \times \mathbb{R}_{+}$of the function $\mathcal{G}$ such that $s_{u} u^{+}+t_{u} u^{-} \in \mathcal{N}$.

Proof The proof of this lemma is obtained in three steps. The first step consists in showing that there exists a pair of positive numbers $\left(s_{u}, t_{u}\right)$ such that $s_{u} u^{+}+t_{u} u^{-} \in \mathcal{N}$ for any $u \in X_{\beta}$ with $u^{ \pm} \neq 0$. Note that

$$
\begin{equation*}
\left\langle\mathcal{J}^{\prime}\left(s u^{+}+t u^{-}\right), s u^{+}\right\rangle=\left\|s u^{+}\right\|^{N}+\left\|s u^{+}\right\|_{p}^{p}-\int_{B} f\left(x, s u^{+}\right) s u^{+} d x \tag{2.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle\mathcal{J}^{\prime}\left(s u^{+}+t u^{-}\right), t u^{-}\right\rangle=\left\|t u^{-}\right\|^{N}+\left\|t u^{-}\right\|_{p}^{p}-\int_{B} f\left(x, t u^{-}\right) t u^{-} d x \tag{2.4}
\end{equation*}
$$

Note that, from $\left(A_{4}\right)$, for any $\varepsilon>0$, there exists $\delta>0$ such that

$$
\begin{equation*}
f(x, t) \leq \varepsilon|t|^{N-1} \quad \text { for }|t| \leq \delta \text { and } x \in B \tag{2.5}
\end{equation*}
$$

Since $f$ is subcritical or critical, for every $\varepsilon>0$, there exist constants $c_{1}=c_{1}(\varepsilon)>0$ and $K>0$ such that, for all $q>N$,

$$
\begin{equation*}
f(x, t) t \leq c_{1}|t|^{q} \exp \left(\alpha|t|^{\gamma}\right), \quad \forall|t| \geq K, \alpha>\alpha_{0} \forall x \in B . \tag{2.6}
\end{equation*}
$$

Then, from (2.5) and (2.6), we have

$$
f(x, t) t \leq \varepsilon|t|^{N}+C|t|^{q} \exp \left(\alpha|t|^{\gamma}\right) \quad \text { for all } \alpha>\alpha_{0}, q>N .
$$

So, we get

$$
\begin{equation*}
\int_{B} f\left(x, s u^{+}\right) s u^{+} d x \leq \varepsilon \int_{B}\left|s u^{+}\right|^{N} d x+C \int_{B}\left|s u^{+}\right|^{q} \exp \left(\alpha\left(s u^{+}\right)^{\gamma}\right) d x \tag{2.7}
\end{equation*}
$$

$$
\text { for all } \alpha>\alpha_{0}, q>N
$$

Now, from (2.7), (1.6), the Sobolev embedding theorem, and Hôlder's inequality, we have for $s>0$ small enough satisfying $s \leq \frac{\alpha_{N, \beta}^{\frac{1}{V}}}{(2 \alpha)^{\frac{1}{\gamma}}\left\|u^{+}\right\|}$:

$$
\begin{align*}
& \left\langle\mathcal{J}^{\prime}\left(s u^{+}+t u^{-}\right), s u^{+}\right\rangle \\
& \quad \geq\left\|s u^{+}\right\|^{N}+\left\|s u^{+}\right\|_{p}^{p}-\varepsilon s^{N} \int_{B}\left|u^{+}\right|^{N} d x-C_{\varepsilon} s^{q} \int_{B}\left|u^{+}\right|^{q} \exp \left(\alpha\left(s u^{+}\right)^{\gamma}\right) d x \\
& \quad \geq\left\|s u^{+}\right\|^{N}+\left\|s u^{+}\right\|_{p}^{p}-\varepsilon s^{N} \int_{B}\left|u^{+}\right|^{N} d x  \tag{2.8}\\
& \quad-C s^{q}\left(\int_{B}\left|u^{+}\right|^{2 q} d x\right)^{\frac{1}{2}}\left(\int_{B} \exp \left(2 \alpha s^{\gamma}\left\|u^{+}\right\|^{\gamma}\left(\frac{u^{+}}{\left\|u^{+}\right\|}\right)^{\gamma}\right) d x\right)^{\frac{1}{2}} \\
& \quad \geq s^{N}\left\|u^{+}\right\|^{N}-\varepsilon s^{N} C_{1}\left\|u^{+}\right\|^{N}-C_{2} s^{q}\left\|u^{+}\right\|^{q} .
\end{align*}
$$

Choose $\varepsilon>0$ small enough such that $\left(1-\varepsilon C_{1}\right)>0$. Since $q>N$, we have that

$$
\left\langle\mathcal{J}^{\prime}\left(s u^{+}+t u^{-}\right), s u^{+}\right\rangle>0 \quad \text { for } s \text { small enough and all } t \geq 0 .
$$

Similarly, according to (2.4) and (2.7), we get

$$
\left\langle\mathcal{J}^{\prime}\left(s u^{+}+t u^{-}\right), t u^{-}\right\rangle>0 \quad \text { for } t \text { small enough and all } s \geq 0
$$

Hence, there exists $r>0$ such that

$$
\begin{equation*}
\left\langle\mathcal{J}^{\prime}\left(r u^{+}+t u^{-}\right), r u^{+}\right\rangle>0 \quad \text { and } \quad\left\langle\mathcal{J}^{\prime}\left(s u^{+}+r u^{-}\right), r u^{-}\right\rangle>0 \quad \text { for all } s, t \geq 0 . \tag{2.9}
\end{equation*}
$$

On the other hand, by $\left(A_{2}\right)$, we can find positive constants $C_{3}$ and $C_{4}$ such that

$$
\begin{equation*}
f(x, t) t \geq C_{3}|t|^{\theta}-C_{4}, \quad \forall(x, t) \in(B, \mathbb{R} \backslash\{0\}) \tag{2.10}
\end{equation*}
$$

Thus, we get

$$
\begin{align*}
& \left\langle\mathcal{J}^{\prime}\left(s u^{+}+t u^{-}\right), s u^{+}\right\rangle \\
& \quad=\left\|s u^{+}\right\|^{N}+\left\|s u^{+}\right\|_{p}^{p}-\int_{B} f\left(x, s u^{+}\right) s u^{+} d x  \tag{2.11}\\
& \quad \leq s^{N}\left\|u^{+}\right\|^{N}+s^{p}\left\|u^{+}\right\|_{p}^{p}-C_{3} s^{\theta} \int_{B}\left|u^{+}\right|^{\theta}+\frac{\omega_{N-1}}{N} C_{4}
\end{align*}
$$

Since $\theta>N$, there exists $R>r$ large enough such that

$$
\begin{equation*}
\left\langle\mathcal{J}^{\prime}\left(R u^{+}+t u^{-}\right), R u^{+}\right\rangle<0 \quad \text { and } \quad\left\langle\mathcal{J}^{\prime}\left(s u^{+}+R u^{-}\right), R u^{-}\right\rangle<0 \quad \text { for all } s, t \in[r, R] \tag{2.12}
\end{equation*}
$$

In view of Miranda's theorem [34], together with (2.9) and (2.12), we can conclude that there exists $\left(s_{u}, t_{u}\right) \in \mathbb{R}_{+} \times \mathbb{R}_{+}$such that $\mathcal{K}\left(s_{u}, t_{u}\right)=(0,0)$, i.e., $s_{u} u^{+}+t_{u} u^{-} \in \mathcal{N}$.
In the second step, we prove the uniqueness of the pair ( $s_{u}, t_{u}$ ). First, we assume that $u=u^{+}+u^{-} \in \mathcal{N}$. Then, we have

$$
\left\langle\mathcal{J}^{\prime}(u), u^{+}\right\rangle=0 \quad \text { and } \quad\left\langle\mathcal{J}^{\prime}(u), u^{-}\right\rangle=0,
$$

that is,

$$
\begin{equation*}
\left\|u^{+}\right\|^{N}+\left\|u^{+}\right\|_{p}^{p}=\int_{B} f\left(x, u^{+}\right) u^{+} d x \quad \text { and } \quad\left\|u^{-}\right\|^{N}+\left\|u^{-}\right\|_{p}^{p}=\int_{B} f\left(x, u^{-}\right) u^{-} d x . \tag{2.13}
\end{equation*}
$$

By Claim 1, we know that there exists at least one positive pair $\left(s_{0}, t_{0}\right)$ satisfying $s_{0} u^{+}+$ $t_{0} u^{-} \in \mathcal{N}$. Now, we show that $\left(s_{0}, t_{0}\right)=(1,1)$ is the unique pair of numbers. Without loss of generality, let us assume that $s_{0} \leq t_{0}$. It follows from (2.3) that

$$
\begin{equation*}
s_{0}^{N}\left\|u^{+}\right\|^{N}+s_{0}^{p}\left\|u^{+}\right\|_{p}^{p}=\int_{B} f\left(x, s_{0} u^{+}\right) s_{0} u^{+} d x \tag{2.14}
\end{equation*}
$$

If $s_{0}<1$, then from (2.13), (2.14), and $\left(A_{3}\right)$, we have

$$
\begin{align*}
0 & <\left(s_{0}^{-N}-1\right)\left\|u^{+}\right\|^{N}+\left(s_{0}^{p-N}-1\right)\left\|u^{+}\right\|_{p}^{p} \\
& =\int_{B}\left(\frac{f\left(x, s_{0} u^{+}\right)}{\left(s_{0} u^{+}\right)^{N-1}}-\frac{f\left(x, u^{+}\right)}{\left(u^{+}\right)^{N-1}}\right)\left(u^{+}\right)^{N} d x \leq 0, \tag{2.15}
\end{align*}
$$

which is a contradiction. Hence, $1 \leq s_{0} \leq t_{0}$.
Arguing similarly by using the equations $\left\langle\mathcal{J}^{\prime}\left(s u^{+}+t u^{-}\right), t u^{-}\right\rangle=0$ and $\left\langle\mathcal{J}^{\prime}(u), u^{-}\right\rangle=0$, we obtain that $s_{0} \leq t_{0} \leq 1$, which implies that $s_{0}=t_{0}=1$, and the proof is complete.
For the general case, we suppose that $u \notin \mathcal{N}$. Assume that there exist two other pairs of positive numbers $\left(s_{1}, t_{1}\right)$ and $\left(s_{2}, t_{2}\right)$ such that

$$
\sigma_{1}=s_{1} u^{+}+t_{1} u^{-} \in \mathcal{N} \quad \text { and } \quad \sigma_{2}=s_{2} u^{+}+t_{2} u^{-} \in \mathcal{N} .
$$

Then

$$
\sigma_{2}=\left(\frac{s_{2}}{s_{1}}\right) s_{1} u^{+}+\left(\frac{t_{2}}{t_{1}}\right) t_{1} u^{-}=\left(\frac{s_{2}}{s_{1}}\right) \sigma_{1}^{+}+\left(\frac{t_{2}}{t_{1}}\right) \sigma_{1}^{-} \in \mathcal{N} .
$$

Since $\sigma_{1} \in \mathcal{N}$, it is clear that

$$
\frac{s_{2}}{s_{1}}=\frac{t_{2}}{t_{1}}=1
$$

which means that $s_{1}=s_{2}$ and $t_{1}=t_{2}$.
Finally, we prove that the pair $\left(s_{u}, t_{u}\right)$ is the unique maximum point of the function $\mathcal{G}$ on $\mathbb{R}_{+} \times \mathbb{R}_{+}$. We know from the above that $\left(s_{u}, t_{u}\right)$ is the unique critical point of $\mathcal{G}$ on $\mathbb{R}_{+} \times \mathbb{R}_{+}$. By definition and (2.10), we obtain

$$
\begin{aligned}
\mathcal{G}(s, t)= & \mathcal{J}\left(s u^{+}+t u^{-}\right) \\
= & \frac{1}{N}\left\|s u^{+}+t u^{-}\right\|^{N}+\frac{1}{p}\left\|s u^{+}+t u^{-}\right\|^{p}-\int_{B} F\left(x, s u^{+}+t u^{-}\right) d x \\
\leq & \frac{1}{N}\left\|s u^{+}+t u^{-}\right\|^{N}+\frac{1}{p}\left\|s u^{+}+t u^{-}\right\|^{p}-C_{1} s^{\theta} \int_{B}\left|u^{+}\right|^{\theta} d x-C_{1} t^{\theta} \int_{B}\left|u^{-}\right|^{\theta} d x+C|B| \\
\leq & \frac{1}{N}\left(s^{N}\left\|u^{+}\right\|^{N}+t^{N}\left\|u^{-}\right\|^{N}\right)+\frac{1}{p}\left(s^{p}\left\|u^{+}\right\|^{p}+t^{p}\left\|u^{-}\right\|^{p}\right)-C_{1} s^{\theta} \int_{B}\left|u^{+}\right|^{\theta} d x \\
& -C_{1} t^{\theta} \int_{B}\left|u^{-}\right|^{\theta} d x+C|B|,
\end{aligned}
$$

which implies that $\lim _{|(s, t)| \rightarrow \infty} \mathcal{G}(s, t)=-\infty$ because $\theta>N$. Hence, it suffices to show that the maximum point cannot be achieved on the boundary of $\mathbb{R}_{+} \times \mathbb{R}_{+}$. We carry out the proof by contradiction. Assuming $(0, \bar{t})$ is the global maximum point of $\mathcal{G}$ with $\bar{t} \geq 0$, we have

$$
\mathcal{G}(s, \bar{t})=\frac{1}{N}\left\|s u^{+}+\bar{t} u^{-}\right\|^{N}+\frac{1}{p}\left\|s u^{+}+\bar{t} u^{-}\right\|^{p}-\int_{B} F\left(x, s u^{+}+\bar{t} u^{-}\right) d x .
$$

Hence, by (2.8) it is clear that

$$
\mathcal{G}_{s}^{\prime}(s, \bar{t})=s^{N-1}\left\|u^{+}\right\|^{N}+s^{p-1}\left\|u^{+}\right\|^{p}-\int_{B} f\left(x, s u^{+}\right) u^{+} d x>0
$$

for small enough $s$. This means that $\mathcal{G}$ is an increasing function with respect to $s$ if $s$ is small enough, which is a contradiction. In a similar way, we can deduce that $\mathcal{G}$ cannot achieve its global maximum at $(s, 0)$ with $s \geq 0$. Thus, we have completed the proof.

Lemma 2 Assume that $\left(A_{1}\right)-\left(A_{4}\right)$ and $\left(V_{1}\right)$ hold. Then, for any $u \in X_{\beta}$ with $u^{ \pm} \neq 0$ such that $\left\langle\mathcal{J}^{\prime}(u), u^{ \pm}\right\rangle \leq 0$, the unique maximum point pair of $\mathcal{G}$ on $\mathbb{R}_{+} \times \mathbb{R}_{+}$verifies $0<s_{u}, t_{u} \leq 1$.

Proof Without loss of generality, we may suppose that $0<t_{u} \leq s_{u}$. Since $s_{u} u^{+}+t_{u} u^{-} \in \mathcal{N}$, we have

$$
\begin{equation*}
\left\|s_{u} u^{+}\right\|^{N}+\left\|s_{u} u^{+}\right\|_{p}^{p}=\int_{B} f\left(x, s u^{+}\right) s u^{+} d x . \tag{2.16}
\end{equation*}
$$

Furthermore, since $\left\langle\mathcal{J}^{\prime}(u), u^{+}\right\rangle \leq 0$, we have

$$
\begin{equation*}
\left\|u^{+}\right\|^{N}+\left\|u^{+}\right\|_{p}^{p} \leq \int_{B} f\left(x, u^{+}\right) u^{+} d x \tag{2.17}
\end{equation*}
$$

Then, from (2.16) and (2.17), we get

$$
\begin{equation*}
\left(s_{u}^{-N}-1\right)\left\|u^{+}\right\|^{N}+\left(s_{u}^{p-N}-1\right)\left\|u^{+}\right\|_{p}^{p} \geq \int_{B}\left(\frac{f\left(x, s_{u} u^{+}\right)}{\left(s_{u} u^{+}\right)^{N-1}}-\frac{f\left(x, u^{+}\right)}{\left(u^{+}\right)^{N-1}}\right)\left(u^{+}\right)^{N} d x . \tag{2.18}
\end{equation*}
$$

From $\left(A_{3}\right)$ and $\left(M_{2}\right)$, the left-hand side of (2.18) is negative for $s_{u}>1$, whereas the righthand side is positive, which is a contradiction. Therefore $0<s_{u}, t_{u} \leq 1$.

Lemma 3 Suppose that hypotheses $\left(A_{1}\right),\left(A_{2}\right)$, and $\left(A_{3}\right)$ are satisfied. Then, for each $x \in B$, we have

$$
t f(x, t)-N F(x, t) \quad \text { is increasing for } t>0 \text { and decreasing for } t<0 .
$$

In particular, $t f(x, t)-N F(x, t)>0 \quad$ for all $(x, t) \in B \times \mathbb{R} \backslash\{0\}$.

Proof To prove this lemma, it is enough to analyze the derivative of $t f(x, t)-N F(x, t)$ together with assumptions $\left(A_{1}\right)$ and $\left(A_{3}\right)$.

Lemma 4 Assume that $\left(A_{2}\right),\left(A_{4}\right)$, and $\left(V_{1}\right)$ hold. Then, for all $u \in \mathcal{N}$, we have:
i) There exists $\kappa>0$ such that $\left\|u^{+}\right\|,\left\|u^{-}\right\| \geq \kappa$;
ii) $\mathcal{J}(u) \geq\left(\frac{1}{N}-\frac{1}{\theta}\right)\|u\|^{N}$.

Proof $i$ ) We only prove that there exists $\kappa>0$ such that $\left\|u^{+}\right\| \geq \kappa$ for all $u \in \mathcal{N}$ and the proof for $\left\|u^{-}\right\|$is similar. By contradiction, we suppose that there exists a sequence $\left\{u_{n}^{+}\right\} \subset$ $\mathcal{M}$ such that $\left\|u_{n}^{+}\right\| \rightarrow 0$ as $n \rightarrow \infty$. Since $u_{n} \in \mathcal{N}$, we have $\left\langle\mathcal{J}^{\prime}\left(u_{n}\right), u_{n}^{+}\right\rangle=0$. Thus, from (2.7), we get

$$
\begin{align*}
\left\|u_{n}^{+}\right\|^{N} & <\left\|u_{n}^{+}\right\|^{N}+\left\|u_{n}^{+}\right\|_{p}^{p}=\int_{B} f\left(x, u_{n}^{+}\right) u_{n}^{+} d x  \tag{2.19}\\
& \leq \varepsilon \int_{B}\left|u_{n}^{+}\right|^{N} d x+C \int_{B}\left|u_{n}^{+}\right|^{q} \exp \left(\alpha\left(u_{n}^{+}\right)^{\gamma}\right) d x
\end{align*}
$$

for all $n \in \mathbb{N}, q>N$, and $\alpha>\alpha_{0}$. Since $\left\|u_{n}^{+}\right\| \rightarrow 0$ as $n \rightarrow \infty$, there exists $n_{0} \in \mathbb{N}$ such that $\left\|u_{n}^{+}\right\|^{\gamma} \leq\left(\frac{\alpha_{N, \beta}}{2 \alpha}\right)$ for all $n \geq n_{0}$. From Hôlder's inequality and (1.6), we get

$$
\begin{align*}
\int_{B}\left|u_{n}^{+}\right|^{q} \exp \left(\alpha\left(u_{n}^{+}\right)^{\gamma}\right) d x & \leq\left(\int_{B}\left|u_{n}^{+}\right|^{2 q} d x\right)^{\frac{1}{2}}\left(\int_{B} \exp \left(2 \alpha\left\|u_{n}^{+}\right\|^{\gamma}\left(\frac{u_{n}^{+}}{\left\|u_{n}^{+}\right\|}\right)^{\gamma}\right) d x\right)^{\frac{1}{2}}  \tag{2.20}\\
& \leq C\left\|u_{n}^{+}\right\|_{2 q^{-}}^{q}
\end{align*}
$$

Combining (2.19) with the last inequality, we can deduce from the Sobolev embedding theorem that when $n \geq n_{0}$,

$$
\begin{equation*}
\left\|u_{n}^{+}\right\|^{N} \leq C_{3} \varepsilon\left\|u_{n}^{+}\right\|^{N}+C_{4}\left\|u_{n}^{+}\right\|^{q} \tag{2.21}
\end{equation*}
$$

We can choose $\varepsilon>0$ such that $\left(1-C_{3} \varepsilon\right)>0$, and since $q>N$, we can deduce that (2.21) contradicts $\left\|u_{n}^{+}\right\| \rightarrow 0$ as $n \rightarrow \infty$. The proof is complete.
ii) Given $u \in \mathcal{N}$, by the definition of $\mathcal{N}$ and $\left(A_{2}\right)$, we obtain

$$
\begin{align*}
\mathcal{J}(u) & =\mathcal{J}(u)-\frac{1}{\theta}\left\langle\mathcal{J}^{\prime}(u), u\right\rangle \\
& =\left(\frac{1}{N}-\frac{1}{\theta}\right)\|u\|^{N}+\left(\frac{1}{p}-\frac{1}{\theta}\right)\|u\|_{p}^{p}-\frac{1}{\theta} \int_{B}(f(x, u) u-\theta F(x, u)) d x  \tag{2.22}\\
& \geq\left(\frac{1}{N}-\frac{1}{\theta}\right)\|u\|^{N} .
\end{align*}
$$

So, we have $\mathcal{J}(u)>0$ for all $u \in \mathcal{N}$. Therefore, $\mathcal{J}(u)$ is bounded below on $\mathcal{N}$, that is, $c_{\mathcal{M}}=\inf _{u \in \mathcal{N}} \mathcal{J}(u)$ is well defined.

Lemma 5 If $u_{0} \in \mathcal{N}$ satisfies $\mathcal{J}\left(u_{0}\right)=c_{\mathcal{N}}$, then $\mathcal{J}^{\prime}\left(u_{0}\right)=0$.

Proof Suppose by contradiction that $\mathcal{J}^{\prime}\left(u_{0}\right) \neq 0$. By the continuity of $\mathcal{J}^{\prime}$, it follows that there exist $\delta>0$ and $\iota>0$ such that

$$
\left\|\mathcal{J}^{\prime}(v)\right\| \geq \iota \quad \text { for all }\left\|v-u_{0}\right\| \leq 3 \delta
$$

Choose $\tau \in\left(0, \min \left\{1 / 2, \frac{\delta}{\sqrt{2}\left\|u_{0}\right\|}\right\}\right)$. Let $D:=(1-\tau, 1+\tau) \times(1-\tau, 1+\tau)$ and

$$
k(s, t):=s u_{0}^{+}+t u_{0}^{-} \quad \text { for all }(s, t) \in D .
$$

In view of Lemma 1, we have

$$
\begin{equation*}
c_{\mathcal{N}}^{-}:=\max _{\partial D}(\mathcal{J} \circ k)<c_{\mathcal{N}} . \tag{2.23}
\end{equation*}
$$

Let $\varepsilon:=\min \left\{\left(c_{\mathcal{N}}-c_{\mathcal{N}}^{-}\right) / 3, \iota \delta / 8\right\}$ and $S_{\delta}:=B\left(u_{0}, \delta\right)$. According to Lemma 2.3 in [40], there exists a deformation $\eta \in C\left([0,1] \times X_{\beta}, X_{\beta}\right)$ such that
(a) $\eta(1, v)=v$ if $v \notin\left(\mathcal{J}^{-1}\left(\left[c_{\mathcal{N}}-2 \varepsilon, c_{\mathcal{N}}+2 \varepsilon\right]\right) \cap S_{2 \delta}\right)$,
(b) $\eta\left(1, \mathcal{J}^{c} \mathcal{N}^{+\varepsilon} \cap S_{\delta}\right) \subset \mathcal{J}^{c} \mathcal{N}^{-\varepsilon}$,
(c) $\mathcal{J}(\eta(1, v)) \leq \mathcal{J}(v)$ for all $v \in \mathcal{W}$.

## Clearly,

$$
\begin{equation*}
\max _{(s, t) \in \bar{D}} \mathcal{J}(\eta(1, k(s, t)))<c_{\mathcal{N}} . \tag{2.24}
\end{equation*}
$$

Therefore we claim that $\eta(1, k(D)) \cap \mathcal{M} \neq \emptyset$, which contradicts the definition of $c_{\mathcal{N}}$.
We define $\bar{k}(s, t):=\eta(1, k(s, t))$,

$$
\begin{aligned}
\Phi_{0}(s, t) & :=\left(\left\langle\mathcal{J}^{\prime}(k(s, t)), s u_{0}^{+}\right\rangle,\left\langle\mathcal{J}^{\prime}(k(s, t)), t u_{0}^{-}\right\rangle\right) \\
& =\left(\left\langle\mathcal{J}^{\prime}\left(s u_{0}^{+}+t u_{0}^{-}\right), s u_{0}^{+}\right\rangle,\left\langle\mathcal{J}^{\prime}\left(s u_{0}^{+}+t u_{0}^{-}\right), t u_{0}^{-}\right\rangle\right) \\
& =\left(\phi_{u}^{1}(s, t), \phi_{u}^{2}(s, t)\right)
\end{aligned}
$$

and

$$
\Phi_{1}(s, t):=\left(\frac{1}{s}\left\langle\mathcal{J}^{\prime}(\bar{k}(s, t)),(\bar{k}(s, t))^{+}\right\rangle, \frac{1}{t}\left\langle\mathcal{J}^{\prime}(\bar{k}(s, t)),(\bar{k}(s, t))^{-}\right\rangle\right) .
$$

By a straightforward computation, we get

$$
\begin{align*}
& \left.\frac{\partial \phi_{u}^{1}(s, t)}{\partial s}\right|_{(1,1)}:=N\left\|u_{0}^{+}\right\|^{N}+p\left\|u_{0}^{+}\right\|^{p}-\int_{B}\left(f^{\prime}\left(x, u_{0}^{+}\right)\left(u_{0}^{+}\right)^{2}+f\left(x, u_{0}^{+}\right) u_{0}^{+}\right) d x,  \tag{2.25}\\
& \left.\frac{\partial \phi_{u}^{2}(s, t)}{\partial t}\right|_{(1,1)}:=N\left\|u_{0}^{-}\right\|^{N}+p\left\|u_{0}^{-}\right\|^{p}-\int_{B}\left(f^{\prime}\left(x, u_{0}^{-}\right)\left(u_{0}^{-}\right)^{2}+f\left(x, u_{0}^{-}\right) u_{0}^{-}\right) d x \tag{2.26}
\end{align*}
$$

and

$$
\left.\frac{\partial \phi_{u}^{1}(s, t)}{\partial t}\right|_{(1,1)}=\left.\frac{\partial \phi_{u}^{2}(s, t)}{\partial s}\right|_{(1,1)}:=0
$$

Let

$$
H=\left[\begin{array}{ll}
\left.\frac{\partial \phi_{u}^{1}(s, t)}{\partial s}\right|_{(1,1)} & \left.\frac{\partial \phi_{u}^{2}(s, t)}{\partial s}\right|_{(1,1)} \\
\left.\frac{\partial \phi_{u}^{1}(s, t)}{\partial t}\right|_{(1,1)} & \left.\frac{\partial \phi_{u}^{2}(s, t)}{\partial t}\right|_{(1,1)}
\end{array}\right] .
$$

Then we have that $\operatorname{det} H \neq 0$. Hence, $\Phi_{0}(s, t)$ is a $C^{1}$ function and $(1,1)$ is the unique isolated zero point of $\Phi_{0}$; by using the degree theory, we deduce that $\operatorname{deg}\left(\Phi_{0}, D, 0\right)=1$.

Hence, combining (2.23) with (a), we obtain

$$
k(s, t)=\bar{k}(s, t) \quad \text { on } \partial D .
$$

Therefore, by the degree theory (see [21, Theorem 4.5]), we get $\operatorname{deg}\left(\Phi_{1}, D, 0\right)=\operatorname{deg}\left(\Phi_{0}\right.$, $D, 0)=1$. Hence, again by the degree theory, $\Phi_{1}\left(s_{0}, t_{0}\right)=0$ for some $\left(s_{0}, t_{0}\right) \in D$ so that

$$
\eta\left(1, k\left(s_{0}, t_{0}\right)\right)=\bar{k}\left(s_{0}, t_{0}\right) \in \mathcal{N},
$$

which contradicts (2.24). Hence, $\mathcal{J}^{\prime}\left(u_{0}\right)=0$, which implies $u_{0}$ is a critical point of $\mathcal{J}$.

In the following lemma, we prove that $w$ has exactly two nodal domains.

Lemma 6 If $w$ is a least energy sign-changing solution of problem (1.1), then $w$ has exactly two nodal domains.

Proof Assume by contradiction that $w=w_{1}+w_{2}+w_{3}$ satisfies

$$
\begin{array}{ll}
w_{i} \neq 0, \quad i=1,2,3, \quad w_{1} \geq 0, \quad w_{2} \leq 0, & \text { a.e. in } B, \\
B_{1} \cap B_{2}=\emptyset, \quad B_{1}=\left\{x \in B: w_{1}(x)>0\right\}, & B_{2}=\left\{x \in B: w_{2}(x)<0\right\}, \\
\left.w_{1}\right|_{B \backslash B_{1} \cup B_{2}}=\left.w_{2}\right|_{B \backslash B_{2} \cup B_{1}}=\left.w_{3}\right|_{B_{1} \cup B_{2}}=0 &
\end{array}
$$

and

$$
\begin{equation*}
\left\langle\mathcal{J}^{\prime}(w), w_{i}\right\rangle=0 \quad \text { for } i=1,2,3 . \tag{2.27}
\end{equation*}
$$

Setting $v=w_{1}+w_{2}$, we have that $v^{+}=w_{1}$ and $v^{-}=w_{2}$, i.e., $v^{ \pm} \neq 0$. From Lemma 1 , it follows that there exists a unique point pair $\left(s_{v}, t_{v}\right) \in \mathbb{R}_{+} \times \mathbb{R}_{+}$such that $s_{v} w_{1}+t_{v} w_{2} \in \mathcal{N}$. Hence, $\mathcal{J}\left(s_{\nu} w_{1}+t_{\nu} w_{2}\right) \geq c_{\mathcal{N}}$. Moreover, from (2.27), we obtain $\left\langle\mathcal{J}^{\prime}(v), v^{ \pm}\right\rangle \leq 0$. Then, by Lemma 2 , we have that

$$
\left(s_{v}, t_{v}\right) \in(0,1] \times(0,1] .
$$

On the other hand, by $\left(A_{2}\right)$ we have that

$$
0=\frac{1}{N}\left\langle\mathcal{J}^{\prime}(w), w_{3}\right\rangle=\frac{1}{N}\left\langle\mathcal{J}^{\prime}\left(w_{3}\right), w_{3}\right\rangle\left\langle\mathcal{J}\left(w_{3}\right) .\right.
$$

Hence, by Lemma 3, we can obtain that

$$
\begin{aligned}
c_{\mathcal{M}} \leq & \mathcal{J}\left(s_{v} w_{1}+t_{v} w_{2}\right)=\mathcal{J}\left(s_{v} w_{1}+t_{v} w_{2}\right)-\frac{1}{N}\left\langle\mathcal{J}^{\prime}\left(s_{v} w_{1}+t_{\nu} w_{2}\right), s_{v} w_{1}+t_{\nu} w_{2}\right\rangle \\
= & \frac{s_{v}^{N}}{N}\left\|w_{1}\right\|^{N}+\frac{t_{v}^{N}}{N}\left\|w_{2}\right\|^{N}+\left(\frac{1}{p}-\frac{1}{N}\right) s_{v}^{p}\left\|w_{1}\right\|_{p}^{N}+\left(\frac{1}{p}-\frac{1}{N}\right) t_{v}^{p}\left\|w_{2}\right\|_{p}^{N} \\
& +\frac{1}{N} \int_{B}\left(f\left(x, s_{v} w_{1}\right) s_{v} w_{1}-N F\left(x, s_{v} w_{1}\right)\right) d x \\
& +\frac{1}{N} \int_{B}\left(f\left(x, t_{v} w_{2}\right) t_{v} w_{2}-N F\left(x, t_{v} w_{2}\right)\right) d x \\
\leq & \frac{1}{N}\left\|w_{1}\right\|^{N}+\frac{1}{N}\left\|w_{2}\right\|^{N}+\left(\frac{1}{p}-\frac{1}{N}\right)\left\|w_{1}\right\|_{p}^{N}+\left(\frac{1}{p}-\frac{1}{N}\right)\left\|w_{2}\right\|_{p}^{N} \\
& +\frac{1}{N} \int_{B}\left(f\left(x, w_{1}\right) w_{1}-N F\left(x, w_{1}\right)\right) d x \\
& +\frac{1}{N} \int_{B}\left(f\left(x, w_{2}\right) w_{2}-N F\left(x, w_{2}\right)\right) d x \\
= & \mathcal{J}\left(w_{1}+w_{2}\right)-\frac{1}{N}\left\langle\mathcal{J}^{\prime}\left(w_{1}+w_{2}\right), w_{1}+w_{2}\right\rangle \\
= & \mathcal{J}\left(w_{1}+w_{2}\right)+\frac{1}{N}\left\langle\mathcal{J}^{\prime}(w), w_{3}\right\rangle \\
< & \mathcal{J}\left(w_{1}+w_{2}\right)+\mathcal{J}\left(w_{3}\right)=\mathcal{J}(w)=c_{\mathcal{N}},
\end{aligned}
$$

which is a contradiction, that is, $w_{3}=0$ and $w$ has exactly two nodal domains.

## 3 Proof of Theorem 1.2

Lemma 7 There exists $w \in \mathcal{N}$ such that $\mathcal{J}(w)=c_{\mathcal{N}}$.

Proof Let the sequence $\left(w_{n}\right) \subset \mathcal{N}$ satisfy $\lim _{n \rightarrow \infty} \mathcal{J}\left(w_{n}\right)=c_{\mathcal{N}}$. It is clear that $\left(w_{n}\right)$ is bounded by Lemma 4 . Then, up to a subsequence, there exists $w \in E$ such that

$$
\begin{array}{ll}
w_{n}^{ \pm} \rightharpoonup w^{ \pm} & \text {in } X_{\beta} \\
w_{n}^{ \pm} \rightarrow w^{ \pm} & \text {in } L^{q}(B), \forall q \geq N  \tag{3.1}\\
w_{n}^{ \pm} \rightarrow w^{ \pm} & \text {a.e. in } B
\end{array}
$$

We claim that

$$
\begin{equation*}
\int_{B} f\left(x, w_{n}^{ \pm}\right) w_{n}^{ \pm} d x \rightarrow \int_{B} f\left(x, w^{ \pm}\right) w^{ \pm} d x . \tag{3.2}
\end{equation*}
$$

Indeed, by (2.7), we have

$$
\begin{align*}
& \int_{B} f\left(x, w_{n}^{ \pm}\right) w_{n}^{ \pm} d x \leq \varepsilon \int_{B}\left|w_{n}^{ \pm}\right|^{N} d x+C \int_{B}\left|w_{n}^{ \pm}\right|^{q} \exp \left(\alpha\left|w_{n}^{ \pm}\right|^{\gamma}\right) d x  \tag{3.3}\\
& \quad \text { for all } \alpha>0 \text { and } q>N
\end{align*}
$$

We define $g\left(w_{n}^{ \pm}(x)\right)$ as follows:

$$
\begin{equation*}
g\left(w_{n}^{ \pm}(x)\right):=\varepsilon\left|w_{n}^{ \pm}\right|^{N}+C\left|w_{n}^{ \pm}\right|^{q} \exp \left(\alpha\left|w_{n}^{ \pm}\right|^{\gamma}\right) \tag{3.4}
\end{equation*}
$$

We will prove that $g\left(w_{n}^{ \pm}(x)\right)$ is convergent in $L^{1}(B)$. First note that

$$
\begin{equation*}
\left|w_{n}\right|^{N} \rightarrow|w|^{N} \quad \text { in } L^{1}(B) . \tag{3.5}
\end{equation*}
$$

Considering $s, s^{\prime}>1$ such that $\frac{1}{s}+\frac{1}{s^{\prime}}=1$, we get

$$
\begin{equation*}
\left|w_{n}\right|^{q} \rightarrow|w|^{q} \quad \text { in } L^{s^{\prime}}(B) . \tag{3.6}
\end{equation*}
$$

Moreover, choosing $\alpha>0$ enough small such that $s \alpha\left(\max _{n}\left\|w_{n}^{ \pm}\right\|^{\gamma}\right) \leq \alpha_{N, \beta}$, we conclude from Theorem 1.1 that

$$
\begin{equation*}
\int_{B} \exp \left(s \alpha\left|w_{n}^{ \pm}\right|^{\gamma}\right) d x \leq M \tag{3.7}
\end{equation*}
$$

Since

$$
\begin{equation*}
\exp \left(\alpha\left|w_{n}^{ \pm}\right|^{\gamma}\right) \rightarrow \exp \left(\alpha\left|w^{ \pm}\right|^{\gamma}\right) \quad \text { a.e. in } B \tag{3.8}
\end{equation*}
$$

Then, from (3.7) and [27, Lemma 4.8, Chap. 1], we get that

$$
\begin{equation*}
\exp \left(\alpha \mid w_{n}^{ \pm} \gamma^{\gamma}\right) \rightharpoonup \exp \left(\alpha\left|w^{ \pm}\right|^{\gamma}\right) \quad \text { in } L^{s}(B) \tag{3.9}
\end{equation*}
$$

Then it follows from Hölder's inequality, (3.5), (3.6), (3.9), and the Trudinger-Moser inequality that

$$
\begin{aligned}
& \int_{B}\left(g\left(w_{n}^{ \pm}(x)\right)-g\left(w^{ \pm}(x)\right)\right) d x \\
&= \varepsilon \int_{B}\left(\left|w_{n}^{ \pm}\right|^{N}-\left|w^{ \pm}\right|^{N}\right) d x \\
&+C \int_{B}\left(\left|w_{n}^{ \pm}\right|^{q}-\left|w^{ \pm}\right|^{q}\right) \exp \left(\alpha\left|w_{n}^{ \pm}\right|^{\gamma}\right) d x \\
&+C \int_{B}\left|w^{ \pm}\right|^{q}\left(\exp \left(\alpha\left|w_{n}^{ \pm}\right|^{\gamma}\right)-\exp \left(\alpha\left|w^{ \pm}\right|^{\gamma}\right)\right) d x \\
& \leq \varepsilon \int_{B}\left(\left|w_{n}^{ \pm}\right|^{N}-\left|w^{ \pm}\right|^{N}\right) d x+C\left(\int_{B}\left(\left|w_{n}^{ \pm}\right|^{q}-\left|w^{ \pm}\right|^{q}\right)^{s^{\prime}} d x\right)^{\frac{1}{s^{\prime}}}\left(\int_{B} \exp \left(s \alpha\left|w^{ \pm}\right|^{\gamma}\right) d x\right)^{\frac{1}{s}} \\
&+C \int_{B}\left|w^{ \pm}\right|^{q}\left(\exp \left(\alpha\left|w_{n}^{ \pm}\right|^{\gamma}\right)-\exp \left(\alpha\left|w^{ \pm}\right|^{\gamma}\right)\right) d x \\
& \leq \varepsilon \int_{B}\left(\left|w_{n}^{ \pm}\right|^{N}-\left|w^{ \pm}\right|^{N}\right) d x+C M\left(\int_{B}\left(\left|w_{n}^{ \pm}\right|^{q}-\left|w^{ \pm}\right|^{q}\right)^{s^{\prime}} d x\right)^{\frac{1}{s^{\prime}}} \\
&+C \int_{B}\left|w^{ \pm}\right|^{q}\left(\exp \left(\alpha\left|w_{n}^{ \pm}\right|^{\gamma}\right)-\exp \left(\alpha\left|w^{ \pm}\right|^{\gamma}\right)\right) d x \\
& \rightarrow 0 \quad \text { as } n \rightarrow \infty,
\end{aligned}
$$

which closes the proof of (3.2). Thus, as a direct consequence,

$$
\begin{equation*}
\int_{B} F\left(x, w_{n}^{ \pm}\right) d x \rightarrow \int_{B} F\left(x, w^{ \pm}\right) d x \tag{3.10}
\end{equation*}
$$

holds.
Now, we claim that $w^{ \pm} \neq 0$. Suppose, by contradiction, $w^{+}=0$. From the definition of $\mathcal{N},(3.1)$, and (3.2), we have that $\lim _{n \rightarrow+\infty}\left\|w_{n}^{+}\right\|=0$, which contradicts Lemma 4. Hence, $w^{+} \neq 0$ and $w^{-} \neq 0$.

From the lower semicontinuity of norm and (3.1) it follows that

$$
\begin{equation*}
\left\|w^{+}\right\|^{N}+\left\|w^{+}\right\|_{p}^{p} \leq \liminf _{n \rightarrow+\infty}\left(\left\|w_{n}^{+}\right\|^{N}+\left\|w_{n}^{+}\right\|_{p}^{p}\right) . \tag{3.11}
\end{equation*}
$$

On the other hand, by using $\left\langle\mathcal{J}^{\prime}\left(w_{n}\right), w_{n}^{+}\right\rangle=0$ and (3.2), we have

$$
\begin{equation*}
\liminf _{n \rightarrow+\infty}\left(\left\|w_{n}^{+}\right\|^{N}+\left\|w_{n}^{+}\right\|_{p}^{p}\right)=\liminf _{n \rightarrow+\infty} \int_{B} f\left(x, w_{n}^{+}\right) w_{n}^{+} d x=\int_{B} f\left(x, w^{+}\right) w^{+} d x \tag{3.12}
\end{equation*}
$$

From (3.11) and (3.12) we deduce that $\left\langle\mathcal{J}^{\prime}(w), w^{+}\right\rangle \leq 0$, and similarly we can prove $\left\langle\mathcal{J}^{\prime}(w), w^{-}\right\rangle \leq 0$. Then, Lemma 2 implies that there exists $\left(s_{u}, t_{u}\right) \in(0,1] \times(0,1]$ such that $s_{u} w^{+}+t_{u} w^{-} \in \mathcal{N}$. Thus, by the lower semicontinuity of norm, (3.2), (3.10), and Lemma 3,
we get that

$$
\begin{aligned}
c_{\mathcal{N}} \leq & \mathcal{J}\left(s_{u} w^{+}+t_{u} w^{-}\right)=\mathcal{J}\left(s_{u} w^{+}+t_{u} w^{-}\right)-\frac{1}{N}\left\langle\mathcal{J}^{\prime}\left(s_{u} w^{+}+t_{u} w^{-}\right), s_{u} w^{+}+t_{u} w^{-}\right\rangle \\
\leq & \mathcal{J}(w)-\frac{1}{N}\left\langle\mathcal{J}^{\prime}(w), w\right\rangle \\
= & \frac{1}{N}\|w\|^{N}+\left(\frac{1}{p}-\frac{1}{N}\right)\|w\|_{p}^{p}+\frac{1}{N} \int_{B}(f(x, w) w-N F(x, w)) d x \\
\leq & \liminf _{n \rightarrow+\infty}\left[\frac{1}{N}\left\|w_{n}\right\|^{N}+\left(\frac{1}{p}-\frac{1}{N}\right)\left\|w_{n}\right\|_{p}^{p}\right. \\
& \left.+\frac{1}{N} \int_{B}\left(f\left(x, w_{n}\right) w_{n}-N F\left(x, w_{n}\right)\right) d x\right] \\
\leq & \liminf _{n \rightarrow+\infty}\left[\mathcal{J}\left(w_{n}\right)-\frac{1}{N}\left\langle\mathcal{J}^{\prime}\left(w_{n}\right), w_{n}\right\rangle\right]=c_{\mathcal{N}} .
\end{aligned}
$$

Therefore, we get that $\mathcal{J}(w)=c_{\mathcal{N}}$, which is the desired conclusion.

Thus, from Lemmas 5 and 6, $w$ is a least energy sign-changing solution of problem (1.1) with exactly two nodal domains.

## 4 Proof of Theorem 1.3

To prove Theorem 1.3, we need to consider the auxiliary problem

$$
\begin{cases}-\nabla \cdot\left(w_{\beta}(x)|\nabla u|^{N-2} \nabla u\right)-\Delta_{p} u+V(x)|u|^{N-2} u=|u|^{r-2} u & \text { in } B  \tag{4.1}\\ u=0 & \text { on } \partial B\end{cases}
$$

where $r$ is the constant that appears in assumption $\left(A_{5}\right)$. The energy functional $\mathcal{J}_{r}$ associated with (4.1) is given by

$$
\mathcal{J}_{r}(u):=\frac{1}{N}\|u\|^{N}+\frac{1}{p}\|u\|_{p}^{p}-\frac{1}{r} \int_{B}|u|^{r} d x
$$

and the sign-changing Nehari set is defined by

$$
\mathcal{N}_{r}:=\left\{u \in X_{\beta}, u^{ \pm} \neq 0 \text { and }\left\langle\mathcal{J}_{r}^{\prime}(u), u^{+}\right\rangle=\left\langle\mathcal{J}_{r}^{\prime}(u), u^{-}\right\rangle=0\right\} .
$$

Let $c_{\mathcal{N}_{r}}=\inf _{\mathcal{N}_{r}} \mathcal{J}_{r}(u)$, we have the following result.

Lemma 8 There exists $w \in \mathcal{N}_{r}$ such that $\mathcal{J}_{r}(w)=c_{\mathcal{N}_{r}}$.

Proof The proof of this lemma is obtained in four steps:
Step 1. For any $u \in X_{\beta}$ with $u^{ \pm} \neq 0$, similar to Lemma 1 , there is the unique maximum point pair $\left(s_{u}, t_{u}\right) \in \mathbb{R}_{+} \times \mathbb{R}_{+}$of the function $\mathcal{J}_{r}$ such that $s_{u} u^{+}+t_{u} u^{-} \in \mathcal{N}_{r}$.
Step 2. If $u \in X_{\beta}$ with $u^{ \pm} \neq 0$, such that $\left\langle\mathcal{J}_{r}^{\prime}(u), u^{ \pm}\right\rangle \leq 0$, then, similar to Lemma 2, the unique maximum point pair $\left(s_{u}, t_{u}\right)$ in Step (1) satisfies $0<s_{u}, t_{u} \leq 1$.
Step 3. Similar to Lemma 4 , for all $u \in \mathcal{N}_{r}$, there exists $\kappa>0$ such that $\left\|u^{+}\right\|,\left\|u^{-}\right\| \geq \kappa$.

Step 4. Now, let the sequence $\left(w_{n}\right) \subset \mathcal{N}_{p}$ satisfy $\lim _{n \rightarrow+\infty} \mathcal{J}_{r}\left(w_{n}\right)=c_{\mathcal{M}_{r}}$. Similar to Lemma 7, we can show that, up to a subsequence, $w_{n}^{ \pm} \rightharpoonup w^{ \pm}$in $X_{\beta}$. From Step (3), we show that $w^{ \pm} \neq 0$. Using Steps (1), (2) and again similar to Lemma 7, we get $w \in \mathcal{N}_{r}$ such that $\mathcal{J}_{r}(w)=c_{\mathcal{N}_{r}}$.
Now, we will obtain an important estimate for the nodal level $c_{\mathcal{N}}$. That will be a powerful tool to obtain an appropriate bound of the norm of a minimizing sequence for $c_{\mathcal{N}}$ in $\mathcal{N}$.

Lemma 9 Assume that $\left(A_{1}\right),\left(A_{5}\right),\left(V_{1}\right)$, and (1.15) are satisfied. It holds that

$$
\begin{equation*}
c_{\mathcal{N}} \leq \frac{\theta-N}{N \theta}\left(\frac{\alpha_{N, \beta}}{2\left(\alpha_{0}+\delta\right)}\right)^{(N-1)(1-\beta)} \tag{4.2}
\end{equation*}
$$

Proof From Lemma 8, there exists $w \in \mathcal{N}_{r}$ such that $\mathcal{J}_{r}(w)=c_{\mathcal{N}_{r}}$ and $\mathcal{J}_{r}^{\prime}(w)=0$. Consequently, we get

$$
\begin{equation*}
\frac{1}{N}\|w\|^{N}+\frac{1}{p}\|w\|_{p}^{p}-\frac{1}{r} \int_{B}|w|^{r} d x=c_{\mathcal{N}_{r}} \quad \text { and } \quad\left\|w^{ \pm}\right\|^{N}+\left\|w^{ \pm}\right\|_{p}^{p}=\int_{B}\left|w^{ \pm}\right|^{r} d x \tag{4.3}
\end{equation*}
$$

From $\left(A_{5}\right)$ and (4.3), we get $\left\langle\mathcal{J}^{\prime}(w), w^{ \pm}\right\rangle \leq 0$, which together with Lemma 2 yields that there is a unique pair $(s, t) \in(0,1] \times(0,1]$ such that $s w^{+}+t w^{-} \in \mathcal{N}$. Using $\left(A_{5}\right)$ and (4.3), we obtain

$$
\begin{aligned}
c_{\mathcal{N}} \leq & \mathcal{J}\left(s w^{+}+t w^{-}\right) \\
\leq & \frac{s^{N}}{N}\left\|w^{+}\right\|^{N}+\frac{t^{N}}{N}\left\|w^{-}\right\|^{N}+\frac{s^{p}}{p}\left\|w^{+}\right\|_{p}^{p}+\frac{t^{p}}{p}\left\|w^{-}\right\|_{p}^{p}-\frac{C_{r}}{r} s^{r}\left|w^{+}\right|_{r}^{r}-\frac{C_{r}}{r} t^{r}\left|w^{-}\right|_{r}^{r} \\
\leq & \frac{s^{p}}{p}\left(\left|w^{+}\right|_{r}^{r}-\left\|w^{+}\right\|_{p}^{p}\right)+\frac{t^{p}}{p}\left(\left|w^{-}\right|_{r}^{r}-\left\|w^{-}\right\|_{p}^{p}\right)+\frac{s^{p}}{p}\left\|w^{+}\right\|_{p}^{p}+\frac{t^{p}}{p}\left\|w^{-}\right\|_{p}^{p} \\
& -\frac{C_{r}}{r} s^{r}\left|w^{+}\right|_{r}^{r}-\frac{C_{r}}{r} t^{r}\left|w^{-}\right|_{r}^{r} \\
\leq & \max _{\xi>0}\left(\frac{\xi^{p}}{p}-C_{r} \frac{\xi^{r}}{r}\right)|w|_{r}^{r} .
\end{aligned}
$$

By some straightforward algebraic manipulations, we get

$$
\begin{equation*}
c_{\mathcal{N}} \leq C_{r}^{\frac{-p}{r-p}} \frac{r-p}{r p}|w|_{r}^{r} . \tag{4.4}
\end{equation*}
$$

Note that from (4.3) we have

$$
\begin{align*}
\left(\frac{1}{N}-\frac{1}{r}\right)|w|_{r}^{r} & =\frac{1}{N}\|w\|^{N}+\frac{1}{N}\|w\|_{p}^{p}-\frac{1}{r} \int_{B}|w|^{r} d x \\
& <\frac{1}{N}\|w\|^{N}+\frac{1}{p}\|w\|_{p}^{p}-\frac{1}{r} \int_{B}|w|^{r} d x=c_{\mathcal{N}_{r}} . \tag{4.5}
\end{align*}
$$

Thus, by combining (4.4) and (4.5), we obtain

$$
\begin{equation*}
c_{\mathcal{N}}<C_{r}^{\frac{-p}{r-p}} \frac{N(r-p)}{p(r-N)} c_{\mathcal{N}_{r}} . \tag{4.6}
\end{equation*}
$$

Therefore, by (1.15) and (4.6), we obtain that (4.2) holds.

The next result gives us some compactness properties of minimizing sequences.

## Lemma 10

(i) If $\left(w_{n}\right) \subset \mathcal{N}$ is a minimizing sequence for $c_{\mathcal{N}}$, then up to a subsequence there exists $w \in X_{\beta}$ such that

$$
\begin{aligned}
& w_{n}^{ \pm} \rightharpoonup w^{ \pm} \quad \text { in } X_{\beta}, \quad w_{n}^{ \pm} \rightarrow w^{ \pm} \quad \text { in } L^{q}(B), \forall q \geq N, \quad \text { and } \\
& w_{n}^{ \pm} \rightarrow w^{ \pm} \quad \text { a.e. in } B
\end{aligned}
$$

and

$$
\begin{equation*}
\int_{B} f\left(x, w_{n}^{ \pm}\right) w_{n}^{ \pm} d x \rightarrow \int_{B} f\left(x, w^{ \pm}\right) w^{ \pm} d x \tag{4.7}
\end{equation*}
$$

(ii) There exists $w \in \mathcal{N}$ such that $\mathcal{J}(w)=c_{\mathcal{N}}$.

Proof (i) Let the sequence $\left(w_{n}\right) \subset \mathcal{N}$ satisfy $\lim _{n \rightarrow \infty} \mathcal{J}\left(w_{n}\right)=c_{\mathcal{N}}$. It is clear that $\left(w_{n}\right)$ is bounded by Lemma 4. Then, up to a subsequence, there exists $w \in X_{\beta}$ such that

$$
w_{n}^{ \pm} \rightharpoonup w^{ \pm} \quad \text { in } X_{\beta}, \quad w_{n}^{ \pm} \rightarrow w^{ \pm} \quad \text { in } L^{q}(B), \forall q \geq N, \quad \text { and } \quad w_{n}^{ \pm} \rightarrow w^{ \pm} \quad \text { a.e. in } B .
$$

Note that, according to (2.7), we have

$$
f\left(x, w_{n}^{ \pm}\right) w_{n}^{ \pm} \leq \varepsilon\left|w_{n}^{ \pm}\right|^{N}+C\left|w_{n}^{ \pm}\right|^{q} \exp \left(\alpha\left|w_{n}^{ \pm}\right|^{\gamma}\right)=: g\left(w_{n}^{ \pm}(x)\right) \quad \text { for all } \alpha>\alpha_{0} \text { and } q>N .
$$

We will prove that $g\left(w_{n}^{ \pm}(x)\right)$ is convergent in $L^{1}(B)$. First note that

$$
\begin{equation*}
\left|w_{n}\right|^{N} \rightarrow|w|^{N} \quad \text { in } L^{1}(B) . \tag{4.8}
\end{equation*}
$$

Considering $s, s^{\prime}>1$ such that $\frac{1}{s}+\frac{1}{s^{\prime}}=1$ and $s$ close to 1 , we get

$$
\begin{equation*}
\left|w_{n}\right|^{q} \rightarrow|w|^{q} \quad \text { in } L^{s^{\prime}}(B) . \tag{4.9}
\end{equation*}
$$

On the other hand, using Lemma 3, we obtain that

$$
\begin{align*}
c_{\mathcal{N}}= & \limsup _{n \rightarrow+\infty} \mathcal{J}\left(u_{n}\right) \\
= & \limsup _{n \rightarrow+\infty}\left(\mathcal{J}\left(u_{n}\right)-\frac{1}{\theta}\left\langle\mathcal{J}^{\prime}\left(u_{n}\right), u_{n}\right\rangle\right) \\
= & \limsup _{n \rightarrow+\infty}\left(\left(\frac{1}{N}-\frac{1}{\theta}\right)\left\|u_{n}\right\|^{N}+\left(\frac{1}{p}-\frac{1}{\theta}\right)\left\|u_{n}\right\|_{p}^{p}\right.  \tag{4.10}\\
& +\frac{1}{\theta}\left(\int_{B}\left(f\left(x, u_{n}\right) u_{n}-\theta F\left(x, u_{n}\right) d x\right)\right. \\
> & \limsup _{n \rightarrow+\infty}\left(\frac{1}{N}-\frac{1}{\theta}\right)\left\|u_{n}\right\|^{N},
\end{align*}
$$

which together with Lemma 9 gives that $\lim \sup _{n \rightarrow+\infty}\left\|w_{n}\right\|^{\gamma}<\frac{\alpha_{N, \beta}}{2\left(\alpha_{0}+\delta\right)}$.

Now, choosing $\alpha=\alpha_{0}+\delta$, we get that

$$
\begin{align*}
\int_{B} \exp \left(s \alpha\left|w_{n}^{ \pm}\right|^{\gamma}\right) d x & \leq \int_{B} \exp \left(s\left(\alpha_{0}+\delta\right)\left\|w_{n}\right\|^{\gamma}\left(\frac{w_{n}}{\left\|w_{n}\right\|}\right)^{\gamma}\right) d x \\
& \leq \int_{B} \exp \left(s\left(\alpha_{0}+\delta\right) \frac{\alpha_{N, \beta}}{2\left(\alpha_{0}+\delta\right)}\left(\frac{w_{n}}{\left\|w_{n}\right\|}\right)^{\gamma}\right) d x \tag{4.11}
\end{align*}
$$

Since $s>1$ and is sufficiently close to 1 , we get $\frac{s}{2} \alpha_{N, \beta} \leq \alpha_{N, \beta}$. Then it follows by Theorem 1.1 that there is $M>0$ such that

$$
\begin{equation*}
\int_{B} \exp \left(s \alpha\left|w_{n}^{ \pm}\right|^{\gamma}\right) d x \leq M \tag{4.12}
\end{equation*}
$$

Since

$$
\begin{equation*}
\exp \left(\alpha\left|w_{n}^{ \pm}\right|^{\gamma}\right) \rightarrow \exp \left(\alpha\left|w^{ \pm}\right|^{\gamma}\right) \quad \text { a.e. in } B . \tag{4.13}
\end{equation*}
$$

Then, from (4.12) and [27, Lemma 4.8, Chap. 1], we get that

$$
\begin{equation*}
\exp \left(\alpha\left|w_{n}^{ \pm}\right|^{\gamma}\right) \rightharpoonup \exp \left(\alpha\left|w^{ \pm}\right|^{\gamma}\right) \quad \text { in } L^{s}(B) \tag{4.14}
\end{equation*}
$$

Now, using (4.8), (4.9), (4.14) and proceeding as in Lemma 7, we will complete the proof of (4.7).
(ii) Now, proceeding in the similar way to the proof of Lemma 7, there exists $w \in \mathcal{N}$ such that $\mathcal{J}(w)=c_{\mathcal{N}}$, which is the conclusion we want.

Therefore, from Lemmas 5 and 6, we deduce that $w$ is a least energy sign-changing solution for problem (1.1) with exactly two nodal domains.

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## Data Availability

No datasets were generated or analysed during the current study.

## Declarations

## Competing interests

The authors declare no competing interests.

## Author contributions

All the authors wrote the main manuscript and reviewed it.

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