Zhao et al. *Boundary Value Problems* (2024) 2024:22
https://doi.org/10.1186/s13661-024-01831-9

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Nonexistence of positive solutions for the weighted higher-order elliptic system with Navier boundary condition

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Abstract

We establish a Liouville-type theorem for a weighted higher-order elliptic system in a wider exponent region described via a critical curve. We first establish a Liouville-type theorem to the involved integral system and then prove the equivalence between the two systems by using superharmonic properties of the differential systems. This improves the results in (Complex Var. Elliptic Equ. 5:1436–1450, 2013) and (Abstr. Appl. Anal. 2014:593210, 2014).

Mathematics Subject Classification: 35J58; 35J61

Keywords: Weighted higher-order elliptic system; Liouville-type theorem; Navier boundary value; Superharmonic property

1 Introduction

In this paper, we establish a Liouville-type theorem for the weighted $2m$th-order elliptic equations coupled via the Navier boundary conditions in the half-space $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$:

$$\begin{cases} (-\Delta)^m(|x|^\alpha u(x)) = |x|^{-\beta} v^q & \text{in } \mathbb{R}^n_+, \\ (-\Delta)^m(|x|^\beta v(x)) = |x|^{-\alpha} u^p & \text{in } \mathbb{R}^n_+, \\ u = \Delta u = \cdots = \Delta^{m-1} u = 0 & \text{on } \partial\mathbb{R}^n_+, \\ v = \Delta v = \cdots = \Delta^{m-1} v = 0 & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{1.1}$$

where m is a positive integer satisfying $0 < 2m < n$, $p, q \geq 1$, and $\alpha, \beta \geq 0$, which is closely related to the following integral system:

$$\begin{cases} u(x) = C_n \int_{\mathbb{R}^n_+} \frac{1}{|x|^\alpha |y|^\beta} \left(\frac{1}{|x-y|^{n-2m}} - \frac{1}{|\bar{x}-y|^{n-2m}} \right) v^q(y)\, dy, \\ v(x) = C_n \int_{\mathbb{R}^n_+} \frac{1}{|x|^\beta |y|^\alpha} \left(\frac{1}{|x-y|^{n-2m}} - \frac{1}{|\bar{x}-y|^{n-2m}} \right) u^p(y)\, dy, \end{cases} \tag{1.2}$$

where $C_n > 0$, and $\bar{x} = (x_1, \ldots, x_{n-1}, -x_n)$ is the reflection of the point x about the $\partial\mathbb{R}^n_+$. Similar to some integral systems or partial differential systems, the integral system (1.2) is

usually divided into three cases according to the value of the exponents (p,q). We introduce the critical curve

$$\frac{1}{p+1}+\frac{1}{q+1}=\frac{n-2m+\alpha+\beta}{n} \tag{1.3}$$

for (1.2) to determine a Liouville-type theorem.

The well-known classical Hardy–Littlewood–Sobolev inequality states that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x-y|^\nu}\,dx\,dy\le C_{l,\nu,n}\|f\|_h\|g\|_l$$

for all $f\in L^h(R^n)$ and $g\in L^l(R^n)$, where $1<h,l<\infty$, $0<\nu<n$, and $\frac{1}{h}+\frac{1}{l}+\frac{\nu}{n}=2$. Hardy and Littlewood also introduced the double weighted inequality, which was generalized by Stein and Weiss [13]. This inequality is called the double weighted Hardy–Littlewood–Sobolev (WHLS) inequality

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x|^\tau|x-y|^\nu|y|^\kappa}\,dx\,dy\right|\le C_{\tau,\kappa,l,\nu,n}\|f\|_h\|g\|_l, \tag{1.4}$$

where $1<l$, $h<\infty$, $0<\nu<n$, $\tau+\kappa\ge 0$, and τ and κ satisfy $1-\frac{1}{h}-\frac{\nu}{n}<\frac{\tau}{n}<1-\frac{1}{h}$ with $\frac{1}{l}+\frac{1}{h}+\frac{\nu+\kappa+\tau}{n}=2$. To obtain the best constant in the weighted inequality (1.4), we can maximize the functional

$$J(f,g)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x|^\tau|x-y|^\nu|y|^\kappa}\,dx\,dy$$

under the constrains $\|f\|_h=\|g\|_l=1$. The corresponding Euler–Lagrange equations are the following system of integral equations:

$$\begin{cases}\lambda_1 hf(x)^{h-1}=\int_{\mathbb{R}^n}\frac{g(y)}{|x|^\tau|y|^\kappa|x-y|^\mu}\,dy,\\ \lambda_2 lg(x)^{l-1}=C_n\int_{\mathbb{R}^n}\frac{f(y)}{|x|^\kappa|y|^\tau|x-y|^\mu}\,dy,\end{cases} \tag{1.5}$$

where $f,g\ge 0$, $x\in\mathbb{R}^n$, and $\lambda_1 h=\lambda_2 l=J(f,g)$. Let $u=c_1f^{h-1}$, $v=c_2g^{l-1}$, $p=\frac{1}{h-1}$, $q=\frac{1}{l-1}$ with $pq\ne 1$. Then by a proper choice of constants c_1 and c_2 system (1.5) becomes

$$\begin{cases}u(x)=C_n\int_{\mathbb{R}^n}\frac{1}{|x|^\tau|y|^\kappa}\frac{1}{|x-y|^\mu}v^q(y)\,dy,\\ v(x)=C_n\int_{\mathbb{R}^n}\frac{1}{|x|^\kappa|y|^\tau}\frac{1}{|x-y|^\mu}u^p(y)\,dy,\end{cases} \tag{1.6}$$

where $u,v\ge 0$, $0<p,q<\infty$, $0<\mu<n$, $\frac{\tau}{n}<\frac{1}{p+1}<\frac{\mu+\tau}{n}$, and $\frac{1}{p+1}+\frac{1}{q+1}=\frac{\mu+\tau+\kappa}{n}$.

Jin and Li [10] derived that the positive solution of systems (1.6) is symmetric and monotonic. In [6] and [9], they also discussed the regularity of solutions to (1.6). Lei and Lü [11] proved that system (1.6) and the related differential systems are equivalent to each other under the condition $\max\{\tau(p+1),\kappa(q+1)\}\le n-\mu$ with $pq>1$ and $\tau,\kappa\ge 0$, and the positive locally bounded solutions are symmetric and decreasing about some axis. The Liouville-type theorem to the whole space problem was established by Ma and Chen [8]. In recent years, the nonlocal fractional Laplacian $(0<m<1)$ on the whole space has received much attention from researchers. Zhuo and Li [17] had proved the nonexistence

of an antisymmetric solution in the case $0 < p \leq \frac{n+2m}{n-2m}$, whereas Li and Zhuo [18] have proved the consequence of systems in the case $0 < pq < 1$ or $p + 2m > 1$ and $q + 2m > 1$ with $0 < p, q \leq \frac{n+2m}{n-2m}$. For more related results, see [19–23] and the references therein.

For $\alpha = \beta = 0$ in system (1.2), Zhuo and Li [14] established the symmetry of solutions to an integral system, and Cao and Dai [3] obtained the nonexistence of nontrivial solutions. Zhao, Yang, and Zheng proved the nonexistence of nontrivial solutions for partial differential equations (1.1) in [15] and considered the general nonlinear source in [16].

For $\alpha, \beta \neq 0$ in system (1.2), Cao and Dai [4] obtained a Liouville-type theorem in the super- and subcritical cases under some integrability conditions by the Pohozaev-type identity of integral form, and in the critical case, they showed that a pair of positive solutions to the system is rotationally symmetric about the x_n-axis. Also, we mention the recent important works on the existence and asymptotic analysis of nontrivial solutions for some elliptic systems; see [24–28].

In the present paper, instead of (1.1), we will first establish a Liouville-type theorem for the integral system (1.2) in the supercritical case and then prove that systems (1.2) and (1.1) are equivalent by using the superharmonic properties, that is, the following two propositions.

Proposition 1 *Let $(u, v) \in L^{q_1}(\mathbb{R}^n_+) \times L^{q_2}(\mathbb{R}^n_+)$ be a nonnegative solution of system (1.2), and let $q_1 := \frac{n(pq-1)}{(2m-\alpha-\beta)(1+q)}$ and $q_2 := \frac{n(pq-1)}{(2m-\alpha-\beta)(1+p)}$ with $p, q \geq 1$, $pq \neq 1$, and $\alpha + \beta < 2m$. If*

$$\frac{1}{p+1} + \frac{1}{q+1} < \frac{n - 2m + \alpha + \beta}{n}, \tag{1.7}$$

then $(u, v) \equiv (0, 0)$.

Proposition 2 *Let $p, q \geq 1$ with $pq \neq 1$, and let $\alpha + \beta < 2m$. Then the differential system (1.1) is equivalent to the integral system (1.2).*

Remark 1 Without the growth conditions

$$\left|(-\Delta)^{m-1}u\right|, \left|(-\Delta)^{m-1}v\right| = O\left(|x|^a\right), \quad a \in (0,1), |x| \to \infty,$$

in [12, Theorem 1], we can arrive at the same result by using the proof of Proposition 2.

Remark 2 By Proposition 2 we can show that the conclusions of [4, Theorems 1.2 and 1.3] hold for the partial differential system (1.1). Moreover, the conditions $\frac{1}{p+1} < \frac{n-2m}{2n} + \frac{\alpha}{n}$ and $\frac{1}{q+1} < \frac{n-2m}{2n} + \frac{\beta}{n}$ in [4, Theorem 1.2] are covered by condition (1.7).

Based on Propositions 1 and 2, the main result of the paper is the following theorem.

Theorem 1 *Under the conditions of Proposition 1, the classical nonnegative solutions of system (1.1) must be trivial.*

To prove Proposition 1, we will explore the moving plane method in integral forms by Chen, Li, and Ou [5]. For the proof of Proposition 2, we first prove the superharmonic properties of systems (1.1) and then establish the equivalence between the two systems by using a technique introduced in [7] for the scalar case of higher-order equations.

Next, we will prove Propositions 1 and 2 in Sects. 2 and 3, respectively.

2 Proof of Proposition 1

We introduce three lemmas for the integral system (1.2) as preliminaries, and let $C_n = 1$ there for simplicity.

Denote

$$G(x,y) := \frac{1}{|x-y|^{n-2m}} - \frac{1}{|\bar{x}-y|^{n-2m}}, \quad x,y \in \mathbb{R}^n_+,$$

with $\bar{x}$ reflecting x about the $\partial\mathbb{R}^n_+$. Let $x^\lambda = (x_1, x_2, \dots, 2\lambda - x_n)$ be the reflection of the point x about the plane $T_\lambda = \{x \in \mathbb{R}^n_+ | x_n = \lambda\}$, and denote $u_\lambda(x) = u(x^\lambda)$, $v_\lambda(x) = v(x^\lambda)$. Define $\Sigma_\lambda := \{x \in \mathbb{R}^n_+ | 0 < x_n < \lambda\}$ and $\tilde{\Sigma}_\lambda := \{x^\lambda | x \in \Sigma_\lambda\}$, $\Sigma^c_\lambda = \mathbb{R}^n_+ \setminus \Sigma_\lambda$. The following lemma on the Green function $G(x,y)$ in Σ_λ is known.

Lemma 2.1 ([2, *Lemma* 2.1]) (*i*) *For all* $x, y \in \Sigma_\lambda$, $x \neq y$, *we have*

$$G(x^\lambda, y^\lambda) > \max\{G(x^\lambda, y), G(x, y^\lambda)\},$$
$$G(x^\lambda, y^\lambda) - G(x,y) > |G(x^\lambda, y) - G(x, y^\lambda)|.$$

(*ii*) *For all* $x \in \Sigma_\lambda$, $y \in \Sigma^c_\lambda$, *we have*

$$G(x^\lambda, y) > G(x,y).$$

Lemma 2.2 *Let* (u,v) *be a nonnegative solution of* (1.2). *For all* $x \in \Sigma_\lambda$, *we have*

$$u(x) - u_\lambda(x) \le \int_{\Sigma_\lambda} G(x^\lambda, y^\lambda) \frac{[v^q - v^q_\lambda](y)}{|x|^\alpha |y|^\beta}\, dy,$$
$$v(x) - v_\lambda(x) \le \int_{\Sigma_\lambda} G(x^\lambda, y^\lambda) \frac{[u^p - u^p_\lambda](y)}{|x|^\beta |y|^\alpha}\, dy.$$

Proof Since

$$u(x) = \int_{\Sigma_\lambda} G(x,y) \frac{v^q(y)}{|x|^\alpha |y|^\beta}\, dy + \int_{\Sigma_\lambda} G(x, y^\lambda) \frac{v^q_\lambda(y)}{|x|^\alpha |y^\lambda|^\beta}\, dy + \int_{\Sigma^c_\lambda \setminus \tilde{\Sigma}_\lambda} G(x,y) \frac{v^q(y)}{|x|^\alpha |y|^\beta}\, dy,$$
$$u_\lambda(x) = \int_{\Sigma_\lambda} G(x^\lambda, y) \frac{v^q(y)}{|x^\lambda|^\alpha |y|^\beta}\, dy + \int_{\Sigma_\lambda} G(x^\lambda, y^\lambda) \frac{v^q_\lambda(y)}{|x^\lambda|^\alpha |y^\lambda|^\beta}\, dy + \int_{\Sigma^c_\lambda \setminus \tilde{\Sigma}_\lambda} G(x^\lambda, y) \frac{v^q(y)}{|x^\lambda|^\alpha |y|^\beta}\, dy,$$

we have by Lemma 2.1 that

$$\begin{aligned} u(x) - u_\lambda(x) \le{}& \int_{\Sigma_\lambda} \left[G(x^\lambda, y^\lambda) - G(x, y^\lambda)\right] \frac{v^q(y)}{|x|^\alpha |y|^\beta}\, dy \\ &- \int_{\Sigma_\lambda} \left[G(x^\lambda, y^\lambda) - G(x, y^\lambda)\right] \frac{v^q_\lambda(y)}{|x^\lambda|^\alpha |y^\lambda|^\beta}\, dy \\ &+ \int_{\Sigma^c_\lambda \setminus \tilde{\Sigma}_\lambda} \left[G(x,y) - G(x^\lambda, y)\right] \frac{v^q(y)}{|x|^\alpha |y|^\beta}\, dy \end{aligned}$$

$$\leq \int_{\Sigma_\lambda} \left[G(x^\lambda, y^\lambda) - G(x, y^\lambda)\right] \frac{[v^q - v_\lambda^q](y)}{|x|^\alpha |y|^\beta}\, dy$$
$$\leq \int_{\Sigma_\lambda} G(x^\lambda, y^\lambda) \frac{[v^q - v_\lambda^q](y)}{|x|^\alpha |y|^\beta}\, dy.$$

The second inequality can be obtained in the same way. □

In addition, we also need the weighted Hardy–Littlewood–Sobolev inequality.

Lemma 2.3 ([10]) *Let* $1 < l, m < \infty$, $0 < \nu < n$, $\tau + \kappa \geq 0$, $\frac{1}{l} + \frac{1}{m} + \frac{\nu+\kappa+\tau}{n} = 2$, *and* $1 - \frac{1}{m} - \frac{\nu}{n} < \frac{\tau}{n} < 1 - \frac{1}{m}$. *Then*

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x|^\tau |x-y|^\nu |y|^\kappa}\, dx\, dy \right| \leq C \|f\|_m \|g\|_l$$

with $C = C(\tau, \kappa, l, \nu, n) > 0$, *or, equivalently,*

$$\left\| Tg(x) \right\|_\gamma := \sup_{\|f\|_m = 1} \left\langle Tg(x), f(x) \right\rangle \leq C \|g\|_l$$

with $Tg(x) = \int_{\mathbb{R}^n} \frac{g(y)}{|x|^\tau |x-y|^\nu |y|^\kappa}\, dy$, $\frac{1}{l} + \frac{\nu+\kappa+\tau}{n} = 1 + \frac{1}{\gamma}$ *and* $\frac{1}{m} + \frac{1}{\gamma} = 1$.

Now we can prove Proposition 1.

Proof of Proposition 1 We apply the moving-plane method in two steps.

1. *Determine the starting position*

Start from the very low end of $\mathbb{R}^n_+$, i.e., near $x_n = 0$. We will show that for λ sufficiently small,

$$w_\lambda(x) := u(x) - u_\lambda(x) \leq 0, \qquad g_\lambda(x) := v(x) - v_\lambda(x) \leq 0 \quad \text{a.e. in } \Sigma_\lambda. \tag{2.1}$$

Denote

$$B_\lambda^u := \left\{ x \in \Sigma_\lambda \,|\, w_\lambda(x) > 0 \right\}, \qquad B_\lambda^v := \left\{ x \in \Sigma_\lambda \,|\, g_\lambda(x) > 0 \right\}.$$

We will prove that B_λ^u and B_λ^v must be of zero measure, provided that λ sufficiently small. In fact, by Lemma 2.2 with the mean value theorem we have that for sufficiently small λ and $x \in B_\lambda^u$,

$$0 \leq w_\lambda(x) = \int_{B_\lambda^v} + \int_{\Sigma_\lambda \setminus B_\lambda^v} G(x^\lambda, y^\lambda) \frac{[v^q - v_\lambda^q](y)}{|x|^\alpha |y|^\beta}\, dy$$
$$\leq \int_{B_\lambda^v} G(x^\lambda, y^\lambda) \frac{[v^q - v_\lambda^q](y)}{|x|^\alpha |y|^\beta}\, dy$$
$$\leq q \int_{B_\lambda^v} \frac{[v^{q-1}(v - v_\lambda)](y)}{|x-y|^{n-2m} |x|^\alpha |y|^\beta}\, dy.$$

Furthermore, by Lemma 2.3 with Hölder's inequality and $q_1^* = \frac{q_1}{q_1-1}$

$$\begin{aligned}\|w_\lambda\|_{q_1,B_\lambda^u} &\le \sup_{\|f\|_{q_1^*}=1} \int_{B_\lambda^v} \frac{[v^{q-1}(v-v_\lambda)](y)}{|x-y|^{n-2m}|x|^\alpha|y|^\beta}\,dy \\ &= C\left\|v^{q-1}g_\lambda\right\|_{l,B_\lambda^v} \\ &\le C\|v\|_{q_2,B_\lambda^v}^{q-1}\|g_\lambda\|_{q_2,B_\lambda^v}\end{aligned} \tag{2.2}$$

with the universal constant $C > 0$, where the supercritical inequality (1.7) with $p, q \ge 1$ and $\alpha + \beta < 2m$ implies

$$\begin{aligned}q_1 &= \frac{n(pq-1)}{(2m-\alpha-\beta)(1+q)} > p+1 > 1, \\ q_2 &= \frac{n(pq-1)}{(2m-\alpha-\beta)(1+p)} > q+1 > 1,\end{aligned}$$

and

$$\frac{1}{l} = 1 + \frac{1}{q_1} - \frac{n-2m+\alpha+\beta}{n} = \frac{(2m-\alpha-\beta)(1+p)q}{n(pq-1)} < \frac{q}{q+1} < 1.$$

Similarly, we have

$$\|g_\lambda\|_{q_2,B_\lambda^v} \le C\|u\|_{q_1,B_\lambda^u}^{p-1}\|w_\lambda\|_{q_1,B_\lambda^u}. \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$\|w_\lambda\|_{q_1,B_\lambda^u} \le C\|u\|_{q_1,B_\lambda^u}^{p-1}\|v\|_{q_2,B_\lambda^v}^{q-1}\|w_\lambda\|_{q_1,B_\lambda^u}. \tag{2.4}$$

Since $(u, v) \in L^{q_1}(\mathbb{R}_+^n) \times L^{q_2}(\mathbb{R}_+^n)$, we can choose λ small enough such that

$$C\|u\|_{q_1,B_\lambda^u}^{p-1}\|v\|_{q_2,B_\lambda^v}^{q-1} < \frac{1}{2},$$

and thus $\|w_\lambda\|_{q_1,B_\lambda^u} = 0$ by (2.4). In the same way, $\|g_\lambda\|_{q_2,B_\lambda^v} = 0$. This proves (2.1).

2. *Move the plane to the infinity*

Inequalities (2.1) provide a starting point to move the plane T_λ. We start from a neighborhood of λ and move the plane up as long as (2.1) holds. Define

$$\lambda_0 := \sup\{\lambda | w_\rho, g_\rho \le 0, \rho \le \lambda \text{ for a.e. } x \in \Sigma_\rho\}. \tag{2.5}$$

We first prove that $\lambda_0 = \infty$. Assume for contradiction that $\lambda_0 < \infty$. We claim that

$$w_{\lambda_0}(x) = g_{\lambda_0}(x) = 0 \quad \text{a.e.in } \Sigma_{\lambda_0}. \tag{2.6}$$

Otherwise, for such λ_0, e.g.,

$$E_0 := \left\{x | g_{\lambda_0}(x) < 0, x \in \Sigma_{\lambda_0}\right\} \quad \text{has a positive measure.} \tag{2.7}$$

By Lemma 2.2

$$\begin{aligned} u(x) - u_{\lambda_0}(x) &\le \int_{\Sigma_{\lambda_0}} G\big(x^{\lambda_0}, y^{\lambda_0}\big) \frac{[v^q - v^q_{\lambda_0}](y)}{|x|^\alpha |y|^\beta}\, dy \\ &= \int_{E_0} G\big(x^{\lambda_0}, y^{\lambda_0}\big) \frac{[v^q - v^q_{\lambda_0}](y)}{|x|^\alpha |y|^\beta}\, dy, \end{aligned}$$

Consequently,

$$w_{\lambda_0}(x) < 0 \quad \text{a.e. in } \Sigma_{\lambda_0}. \tag{2.8}$$

Denote $\lambda_\epsilon := \lambda + \epsilon$ with $\epsilon > 0$ to be determined. For any small $\eta > 0$, choose R sufficiently large such that

$$\int_{\mathbb{R}^n_+ \setminus B_R(0)} |u|^{q_1}(y)\, dy \le \eta.$$

It follows from Lusin's theorem and (2.8) that for any $\delta > 0$, there exists a closed set $F_\delta \subset E := \Sigma_{\lambda_0} \bigcap B_R(0)$ with $m(E \setminus F_\delta) < \delta$ such that $w_{\lambda_0}(x) < 0$ and is continuous in F_δ. Choosing $\epsilon > 0$ sufficiently small, we have

$$w_{\lambda_\epsilon}(x) < 0 \quad \text{for all } x \in F_\delta$$

by continuity. Denote $D_{\lambda_\epsilon} := (\Sigma_{\lambda_\epsilon} \setminus \Sigma_{\lambda_0}) \cap B_R(0)$. Then

$$B^u_{\lambda_\epsilon} \subset M := \big(\mathbb{R}^n_+ \setminus B_R(0)\big) \cup (E \setminus F_\delta) \cup D_{\lambda_\epsilon}.$$

Let R be large and δ and ϵ small such that $\int_{B^u_{\lambda_\epsilon}} |u|^{q_1}(y)\, dy \le \int_M |u|^{q_1}(y)\, dy \le \frac{1}{2}$. Similarly, $\int_{B^v_{\lambda_\epsilon}} |v|^{q_2}(y)\, dy \le \frac{1}{2}$.

By (2.4) with $\lambda = \lambda_\epsilon$ we can get

$$\|w_{\lambda_\epsilon}\|_{q_1, B^u_{\lambda_\epsilon}} \le C\|u\|^{p-1}_{q_1, B^u_{\lambda_\epsilon}} \|v\|^{q-1}_{q_2, B^v_{\lambda_\epsilon}} \|w_{\lambda_\epsilon}\|_{q_1, B^u_{\lambda_\epsilon}} \le \frac{1}{4}\|w_{\lambda_\epsilon}\|_{q_1, B^u_{\lambda_\epsilon}},$$

which implies $\|w_{\lambda_\epsilon}\|_{q_1, B^u_{\lambda_\epsilon}} \equiv 0$. Thus

$$w_{\lambda_\epsilon}(x) \le 0 \quad \text{a.e. in } \Sigma_{\lambda_\epsilon},$$

and, similarly,

$$g_{\lambda_\epsilon}(x) \le 0 \quad \text{a.e. in } \Sigma_{\lambda_\epsilon}.$$

This contradicts (2.7) with (2.5). Thus (2.6) holds. This yields the contradiction that $u(x) = v(x) \equiv 0$ on the plane $\{x_n = 2\lambda_0\}$. We conclude that $\lambda_0 = +\infty$, which implies that both u and v are strictly monotonically increasing with respect to x_n. Moreover, we know that $u \in L^{q_1}(\mathbb{R}^n_+)$ and $v \in L^{q_2}(\mathbb{R}^n_+)$and for any $a > 0$,

$$\int_{\mathbb{R}^n_+} \big|u(x', x_n)\big|^{q_1}\, dx'\, dx_n \ge \int_{\mathbb{R}^{n-1}} \int_a^\infty \big|u(x', a)\big|^{q_1}\, dx_n\, dx',$$

$$\int_{\mathbb{R}^n_+} \left|v(x',x_n)\right|^{q_2} dx'\,dx_n \geq \int_{\mathbb{R}^{n-1}} \int_a^\infty \left|v(x',a)\right|^{q_2} dx_n\,dx',$$

and hence $u(x',a) = v(x',a) = 0$ for all $x' \in \mathbb{R}^{n-1}$, a contradiction. □

3 Proof of Proposition 2

Denote by $B_R(0) := \{x \in \mathbb{R}^n, |x| < R\}$ the ball of radius R centered at the origin in $\mathbb{R}^n$ with $B_R^+(0) := B_R(0) \cap \mathbb{R}^n_+$ and $\partial B_R^+(0) := \Gamma_R = \bar{\Gamma}_R \cup \widehat{\Gamma}_R$, the union of the flat and hemisphere parts of Γ_R. Let $x* := \frac{x}{|x^2|}R^2$ be the reflection of x about $\partial B_R(0)$, and let

$$\tilde{G}_R(x,y) := \left(\frac{1}{|x-y|^{n-2}} - \frac{1}{(\frac{|x|}{R}|x^*-y|)^{n-2}}\right) - \left(\frac{1}{|\bar{x}-y|^{n-2}} - \frac{1}{(\frac{|\bar{x}|}{R}|\bar{x}^*-y|)^{n-2}}\right).$$

We begin with the well-known lemma.

Lemma 3.1 ([1, *Lemma* 2.1])
(*i*) *For* $x \in B_R^+(0)$, $\tilde{G}_R(x,y)$ *satisfies the equation*

$$\begin{cases} -\Delta \tilde{G}_R(x,y) = \delta(x-y) & \text{in } B_R^+(0), \\ \tilde{G}_R(x,y) = 0 & \text{on } \partial B_R^+(0). \end{cases} \tag{3.1}$$

(*ii*) *For* $x, y \in B_R^+(0)$,

$$\tilde{G}_R(x,y) \to G(x,y,2) = \frac{1}{|x-y|^{n-2}} - \frac{1}{|\bar{x}-y|^{n-2}} \quad \text{as } R \to \infty. \tag{3.2}$$

(*iii*) *For* $x \in B_R^+(0)$ *and* $y \in \widehat{\Gamma}_R$,

$$\frac{\partial \tilde{G}_R}{\partial \nu}(x,y) = (2-n)R\left(1 - \frac{|x|^2}{R^2}\right)\left(\frac{1}{|x-y|^n} - \frac{1}{|\bar{x}-y|^n}\right),$$

where ν *is the outward unit normal vector of* $\widehat{\Gamma}_R$.

We follow the main idea of Chen, Fang, and Li [7] to give superharmonic properties of system (1.1). This result plays a key role in the proof of Proposition 2.

Lemma 3.2 *If* (u,v) *is a positive solution of* (1.1), *then*

$$(-\Delta)^i\left(|x|^\alpha u\right) > 0, \qquad (-\Delta)^i\left(|x|^\beta v\right) > 0, \quad i = 1,\dots,m-1, x \in \mathbb{R}^n_+.$$

Proof We make an odd extension of u and v to the whole space. Define

$$u(x',x_n) = -u(x',-x_n), \qquad v(x',x_n) = -v(x',-x_n), \quad x_n < 0,$$

with $x' = (x_1,\dots,x_{n-1})$. Then (u,v) satisfy

$$\begin{cases} (-\Delta)^m(|x|^\alpha u(x)) = |x|^{-\beta}|v|^{q-1}v & \text{in } \mathbb{R}^n, \\ (-\Delta)^m(|x|^\beta v(x)) = |x|^{-\alpha}|u|^{p-1}u & \text{in } \mathbb{R}^n. \end{cases} \tag{3.3}$$

Write $u_i(x) := (-\Delta)^i(|x|^\alpha u)$ and $v_i(x) := (-\Delta)^i(|x|^\beta v)$. We will prove that $u_i(x), v_i(x) > 0$, $x \in \mathbb{R}^n_+$, $i = 1, 2, \ldots, m-1$.

Step 1. We claim that $u_{m-1}(x) \geq 0$, $x \in \mathbb{R}^n_+$. Otherwise, there exists $x_1 \in \mathbb{R}^n_+$ such that

$$u_{m-1}(x_1) < 0. \tag{3.4}$$

We will deduce a contradiction by two substeps.

(i) We first claim

$$(-1)^i \hat{u}_{m-i}(r) > 0, \quad \forall r \geq 0, i = 1, 2, \ldots, m-1, \tag{3.5}$$

where $\hat{u}_{m-i}$ is the $(m-i)$th average of u_{m-i}.

Denote by $B_r(x_1)$ the ball of radius r centered at x_1, and define the first averages of u and v on $\partial B_r(x_1)$ as

$$\bar{u}(r) := \frac{1}{|\partial B_r(x_1)|} \int_{\partial B_r(x_1)} |x|^\alpha u(x)\, ds; \qquad \bar{v}(r) := \frac{1}{|\partial B_r(x_1)|} \int_{\partial B_r(x_1)} |x|^\beta v(x)\, ds$$

and

$$\bar{u}_i(r) := \frac{1}{|\partial B_r(x_1)|} \int_{\partial B_r(x_1)} u_i(x)\, ds; \qquad \bar{v}_i(r) := \frac{1}{|\partial B_r(x_1)|} \int_{\partial B_r(x_1)} v_i(x)\, ds$$

with $i = 2, 3, \ldots, m-1$. Then for $r > 0$, we have by (3.3) that for $x \in \mathbb{R}^n$,

$$\begin{cases} -\Delta \bar{u} = \bar{u}_1, & -\Delta \bar{v} = \bar{v}_1, \\ -\Delta \bar{u}_1 = \bar{u}_2, & -\Delta \bar{v}_1 = \bar{v}_2, \\ \ldots \\ -\Delta \bar{u}_{m-2} = \bar{u}_{m-1}, & -\Delta \bar{v}_{m-2} = \bar{v}_{m-1}, \\ -\Delta \bar{u}_{m-1} = f(r), & -\Delta \bar{v}_{m-1} = g(r), \end{cases} \tag{3.6}$$

where $f(r) := \overline{|x|^{-\beta}|v|^{q-1}v}$ and $g(r) := \overline{|x|^{-\alpha}|u|^{p-1}u}$. Integrate the last equation for u in (3.6) from 0 to r. Notice that $x_1 \in \mathbb{R}^n_+$ implies that more than half of $B_r(x_1)$ is contained in $\mathbb{R}^n_+$. By the odd symmetry of v with respect to $\partial\mathbb{R}^n_+$ we have

$$\begin{aligned} -r^{n-1}\bar{u}'_{m-1}(r) &= \int_0^r s^{n-1} f(s)\, ds = \frac{1}{n\alpha(n)} \int_0^r \int_{\partial B_s(x_1)} |x|^{-\beta}|v|^{q-1} v d\sigma\, ds \\ &= \frac{1}{n\alpha(n)} \int_{B_r(x_1)} |x|^{-\beta}|v|^{q-1} v dx > 0, \end{aligned} \tag{3.7}$$

where $\alpha(n)$ denotes the surface area of the unit sphere $\partial B_1(0)$ in $\mathbb{R}^n$.

By (3.4) and (3.7) we deduce that

$$\bar{u}'_{m-1}(r) < 0 \quad \text{and} \quad \bar{u}_{m-1}(r) \leq \bar{u}_{m-1}(0) = u_{m-1}(x_1) < 0 \quad \forall r \geq 0. \tag{3.8}$$

Then by the second to the last equation in (3.6) we have

$$-\frac{1}{r^{n-1}}\left(r^{n-1}\bar{u}'_{m-2}(r)\right)' = \bar{u}_{m-1}(r) \leq \bar{u}_{m-1}(0) \equiv -c_0 < 0 \quad \forall r > 0$$

with universal positive constant c_0, that is,

$$\left(r^{n-1}\bar{u}'_{m-2}(r)\right)' > r^{n-1}c_0 \quad \forall r > 0,$$

and hence

$$\bar{u}_{m-2}(r) \geq \bar{u}_{m-2}(0) + \frac{c_0}{2n}r^2 \quad \forall r > 0 \tag{3.9}$$

after integrating. So we find a suitably large $r_1 > 0$ such that $\bar{u}_{m-2}(r_1) > 0$. In view of the definition of the average, there exists $x_2 \in (\partial B_{r_1}(x_1) \cap \mathbb{R}^n_+)$ such that

$$u_{m-2}(x_2) > 0. \tag{3.10}$$

Moreover, we deduce by (3.8) that

$$u_{m-1}(x_2) < 0. \tag{3.11}$$

Define the second averages of u and v on $\partial B_r(x_2)$:

$$\tilde{u}(r) := \frac{1}{|\partial B_r(x_2)|}\int_{\partial B_r(x_2)} |x|^\alpha u(x)\,ds; \qquad \tilde{v}(r) := \frac{1}{|\partial B_r(x_2)|}\int_{\partial B_r(x_2)} |x|^\beta v(x)\,ds,$$
$$\tilde{u}_i(r) := \frac{1}{|\partial B_r(x_2)|}\int_{\partial B_r(x_2)} u_i(x)\,ds; \qquad \tilde{v}_i(r) := \frac{1}{|\partial B_r(x_2)|}\int_{\partial B_r(x_2)} v_i(x)\,ds,$$

where $i = 2, 3, \ldots, m-1$. By (3.7) and (3.11) we have

$$\tilde{u}_{m-1}(r) \leq \tilde{u}_{m-1}(0) = u_{m-1}(x_2) < 0, \quad r \geq 0.$$

Similarly to (3.9) and (3.11), we have

$$\tilde{u}_{m-2}(r) \geq \tilde{u}_{m-2}(0) + cr^2 = u_{m-2}(x_2) + cr^2 > 0, \quad r \geq 0.$$

Repeating the same argument to u_{m-3}, we also obtain the third average on $\partial B_r(x_3)$:

$$\check{u}_{m-1}(r) < 0, \qquad \check{u}_{m-2}(r) > 0, \qquad \check{u}_{m-3}(r) < 0, \quad r \geq 0.$$

By induction we can get the claim (3.5) for the component u.

(ii) Taking the scaling transformations

$$u_\mu(x) = \mu^{\frac{(2m-\alpha-\beta)(q+1)}{pq-1}} u(\mu x), \qquad v_\mu(x) = \mu^{\frac{(2m-\alpha-\beta)(p+1)}{pq-1}} v(\mu x),$$

we find that u_μ and v_μ are also nonnegative solutions of (3.3). This implies that by repeating step 2 in Part I of [7, Sect. 2] a suitably large $\mu > 0$ ensures

$$\hat{u}(r) \geq a_0(r-1)^{b_0}, \quad r \in [1, 2], \tag{3.12}$$

with $b_0 := p + q + 2m + n$ and $a_0 > 0$ sufficiently large.

Next, we treat the component v. Set $U^+ = B_\tau(x_{m-1}) \cap \mathbb{R}^N_+$ and $U^- = B_\tau(x_{m-1}) \cap (\mathbb{R}^N \setminus \mathbb{R}^N_+)$. Let $\tilde{U}^-$ be the reflection of U^- with respect to the boundary $\partial\mathbb{R}^N_+$, and let $U_\tau = U^+ \setminus \tilde{U}^-$. By Jensen's inequality and the equations for v in (3.6), we derive that for all $0 \le r \le 2$,

$$\begin{aligned}
-\hat{v}_{m-1}(r) &= \int_0^r \frac{1}{\tau^{n-1}} \int_0^\tau s^{n-1} g(s)\,ds\,d\tau - \hat{v}_{m-1}(0) \\
&= \int_0^r \frac{1}{\tau^{n-1}} \left(\int_0^\tau s^{n-1} \left[\frac{1}{|\partial B_r(x_{m-1})|} \int_{\partial B_r(x_{m-1})} |x|^{-\alpha} |u|^{p-1} u(x)\,d\sigma \right] ds \right) d\tau \\
&\quad - \hat{v}_{m-1}(0) \\
&= \int_0^r \frac{1}{\tau^{n-1}} \left(\int_0^\tau s^{n-1} \frac{1}{n\alpha(n)} \int_{\partial B_r(x_{m-1})} |x|^{-\alpha} |u|^{p-1} u(x)\,d\sigma\,ds \right) d\tau - \hat{v}_{m-1}(0) \\
&\ge \frac{1}{n\alpha(n)(2+|x_{m-1}|)^\alpha} \int_0^r \frac{1}{\tau^{n-1}} \left(\int_{B_\tau(x_{m-1})} |u|^{p-1} u(x)\,dx \right) d\tau - \hat{v}_{m-1}(0) \\
&= \frac{1}{n\alpha(n)(2+|x_{m-1}|)^\alpha} \int_0^r \frac{1}{\tau^{n-1}} \left(\int_{U_\tau} u^p(x)\,dx \right) d\tau - \hat{v}_{m-1}(0) \\
&\ge \frac{1}{n\alpha(n)(2+|x_{m-1}|)^\alpha} \int_0^r \frac{|U_\tau|}{\tau^{n-1}} \left(\frac{1}{|U_\tau|} \int_{U_\tau} u^p(x)\,dx \right) d\tau - \hat{v}_{m-1}(0) \\
&\ge \frac{1}{n\alpha(n)(2+|x_{m-1}|)^\alpha} \int_0^r \left(\frac{|B_\tau(x_{m-1})|}{|U_\tau|} \right)^{p-1} \\
&\quad \times \frac{1}{\tau^{n-1}|B_\tau(x_{m-1})|^{p-1}} \left(\int_{U_\tau} u(x)\,dx \right)^p d\tau - \hat{v}_{m-1}(0) \\
&\ge \frac{1}{(n\alpha(n))^p(2+|x_{m-1}|)^\alpha} \int_0^r \frac{1}{\tau^{np-1}} \left(\int_{U_\tau} |x|^{-\alpha} \cdot |x|^\alpha u(x)\,dx \right)^p d\tau - \hat{v}_{m-1}(0) \\
&\ge \frac{1}{(n\alpha(n))^p(2+|x_{m-1}|)^{\alpha(p+1)}} \int_0^r \frac{1}{\tau^{np-1}} \left(\int_{B_\tau(x_{m-1})} |x|^\alpha u(x)\,dx \right)^p d\tau - \hat{v}_{m-1}(0) \\
&\ge \frac{1}{(n\alpha(n))^p(2+|x_{m-1}|)^{\alpha(p+1)}} \int_0^r \frac{1}{\tau^{np-1}} \left(\int_0^\tau \int_{\partial B_s(x_{m-1})} |x|^\alpha u(x)\,dx\,ds \right)^p d\tau \\
&\quad - \hat{v}_{m-1}(0) \\
&= \frac{1}{(2+|x_{m-1}|)^{\alpha(p+1)}} \int_0^r \frac{1}{\tau^{np-1}} \left(\int_0^\tau s^{n-1} \hat{u}(s)\,ds \right)^p d\tau - \hat{v}_{m-1}(0).
\end{aligned} \tag{3.13}$$

Choosing $r = 2$ in (3.12) and substituting the latter into (3.13), we get

$$\hat{v}_{m-1}(r) \le -\frac{a_0}{(2+|x_{m-1}|)^{\alpha(p+1)}} r^2 + \hat{v}_{m-1}(0).$$

Thus $\hat{v}_{m-1}(r)$ must be negative whenever a_0 is large. Now we can repeat the above procedure for u with $m-1$ times recenters to deduce for the component v that

$$(-1)^i \hat{v}_{m-i}(r) > 0, \quad r \in [0,2], i = 1,2,\ldots,m-1. \tag{3.14}$$

For any $1 \le r \le 2$, we get by (3.12)–(3.14) that

$$-\hat{v}_{m-1}(r) \ge \frac{1}{(2+|x_{m-1}|)^{\alpha(p+1)}} \int_0^r \frac{1}{\tau^{np-1}} \left(\int_0^\tau s^{n-1} \hat{u}(s)\,ds \right)^p d\tau - \hat{v}_{m-1}(0)$$

$$
\begin{aligned}
&\geq \frac{1}{(2+|x_{m-1}|)^{\alpha(p+1)}} \int_1^r \frac{1}{\tau^{np-1}} \left(\int_1^\tau s^{n-1} a_0 (s-1)^{b_0}\, ds \right)^p d\tau \\
&\geq \frac{1}{(2+|x_{m-1}|)^{\alpha(p+1)}} \int_1^r \frac{1}{\tau^{np-1}} \left(\frac{a_0}{b_0+n} (\tau-1)^{b_0+1} \tau^{n-1} \right)^p d\tau) \\
&= \frac{1}{(2+|x_{m-1}|)^{\alpha(p+1)}} \cdot \frac{a_0^p}{(n+b_0)^p} \int_1^r (\tau-1)^{(b_0+1)p} \tau^{1-p}\, d\tau \\
&\geq \frac{1}{(2+|x_{m-1}|)^{\alpha(p+1)}} \cdot \frac{a_0^p}{(n+b_0)^p[(b_0+1)p+1]} (r-1)^{(b_0+1)p+1} r^{1-p} \\
&\geq \frac{1}{(2+|x_{m-1}|)^{\alpha(p+1)}} \cdot \frac{a_0^p}{2^{p-1}(n+b_0)^p[(b_0+1)p+1]} (r-1)^{(b_0+1)p+1} \\
&\geq \frac{1}{2^p(2+|x_{m-1}|)^{\alpha(p+1)}} \cdot \frac{a_0^p}{(n+b_0)^p[(b_0+1)p+1]} (r-1)^{(b_0+1)p+1},
\end{aligned} \tag{3.15}
$$

where $\int_1^r (s-1)^\theta s^\iota\, ds \geq \frac{1}{\theta+\iota+1}(r-1)^{\theta+1} r^\iota$, and $\theta, \iota > 0$ by an elementary calculation. This implies

$$
\hat{v}_{m-1}(r) \leq -\frac{(Ma_0)^p}{(2b_0p)^{p+1}} (r-1)^{(b_0+1)p+1}
$$

with $M^p := \frac{1}{2^p(2+|x_{m-1}|)^{\alpha(p+1)}}$. So we get

$$
-\frac{1}{r^{n-1}} \left(r^{n-1} \hat{v}'_{m-2}(r) \right)' = \hat{v}'_{m-1}(r) \leq -\frac{(Ma_0)^p}{(2b_0p)^{p+1}} (r-1)^{(b_0+1)p+1} < 0.
$$

Integrating twice from 0 to r and from 0 to τ, by (3.14) we obtain that

$$
\begin{aligned}
\hat{v}_{m-2}(r) &\geq \int_0^r \frac{1}{\tau^{n-1}} \int_0^\tau s^{n-1} \frac{(Ma_0)^p}{(2b_0p)^{p+1}} (s-1)^{(b_0+1)p+1}\, ds\, d\tau \\
&\geq \frac{(Ma_0)^p}{(2b_0p)^{p+1}} \int_1^r \frac{1}{\tau^{n-1}} \frac{1}{(b_0+1)p+n+1} \tau^{n-1} (\tau-1)^{(b_0+1)p+2}\, d\tau \\
&\geq \frac{(Ma_0)^p}{(2b_0p)^{p+1}} \frac{1}{(b_0+1)p+n+1} \frac{1}{(b_0+1)p+3} (r-1)^{(b_0+1)p+3} \\
&\geq \frac{(Ma_0)^p}{(2b_0p)^{p+3}} (r-1)^{(b_0+1)p+3}.
\end{aligned}
$$

We know by repeating step 1 in Part I of [7, Sect. 2] that m is even. Continuing this way, for m even, we obtain

$$
\hat{v}(r) \geq \frac{(Ma_0)^p}{(2b_0p)^{p+2m-1}} (r-1)^{(b_0+1)p+2m-1} \geq A_0 (r-1)^{B_0} \tag{3.16}
$$

with $A_0 := c_0 (Ma_0)^p (2pb_0)^{-p-2m}$ and $B_0 := 2pb_0 \geq (b_0+1)p + 2m$.

Similarly to (3.13), by (3.5) we have

$$
\begin{aligned}
-\hat{u}_{m-1}(r) &\geq \int_0^r \frac{1}{\tau^{n-1}} \int_0^\tau s^{n-1} g(s)\, ds - \hat{u}_{m-1}(0) \\
&\geq \frac{1}{(2+|x_{m-1}|)^{\beta(q+1)}} \int_0^r \frac{1}{\tau^{nq-1}} \left(\int_0^\tau s^{n-1} \hat{v}(s)\, ds \right)^q d\tau.
\end{aligned}
$$

Together with (3.16), we obtain

$$\hat{u}_{m-1}(r) \le -\frac{(M_0A_0)^q}{(2B_0q)^{q+1}}(r-1)^{(B_0+1)q+1}$$

with $M_0^q := \frac{1}{2^q(2+|x_{m-1}|)^{\beta(q+1)}}$. By this and (3.6) we obtain

$$\hat{u}(r) \ge \frac{(M_0A_0)^q}{(2B_0q)^{q+2m}}(r-1)^{(B_0+1)q+2m} \ge a_1(r-1)^{b_1}$$

with $a_1 := c_0(M_0A_0)^q(2B_0q)^{-q-2m}$ and $b_1 := 2B_0q$. Then

$$b_1 \ge (B_0+1)q + 2m. \tag{3.17}$$

By induction we can obtain by k steps that $\hat{u}(r) \ge a_k(r-1)^{b_k}$ with $b_{k+1} = 4pqb_k$, $a_{k+1} = \frac{M^{pq}M_k^q a_k^{pq}}{(2pb_k)^{q(p+2m)+(q+2m)}(2q)^{q+2m}}$, $k = 0, 1, \ldots$.

Set $h := \max\{q+2m, p+2m\}$ and $M_k^q := \min\{\frac{1}{2^p(2+|x_{m-1}|)^{\alpha(p+1)}}, \frac{1}{2^q(2+|x_{m-1}|)^{\beta(q+1)}}\}$. Choose z such that $z \ge \frac{h(q+1)+1}{pq-1}$ and thus $b_0 \ge c(4pq)^{pq(q+1)+z}$ with

$$c := 2^{pq+q+2(q+2m)+q(p+2m)}\bigl(2+|x_{m-1}|\bigr)^{\beta(q+1)+\alpha q(p+1)}p^{q(p+2m)+(q+2m)}q^{q+2m} > 0.$$

We claim that

$$a_k^{pq} \ge cb_k^{h(q+1)+z}, \quad k = 0, 1, \ldots. \tag{3.18}$$

Obviously, (3.18) holds for $k = 0$ by choosing a_0 sufficiently large. Assume that (3.18) is true for k. We have

$$\begin{aligned}\frac{a_{k+1}^{pq}}{cb_{k+1}^{h(q+1)+z}} &= \left[\frac{a_k^{pq}}{b_k^{h(q+1)}}\right]^{pq}\frac{1}{c(4pqb_k)^{h(q+1)+z}} \\ &\ge \frac{b_k^{z(pq-1)-h(q+1)}}{c(4pq)^{pq(q+1)+z}} \\ &\ge \frac{b_0}{c(4pq)^{pq(q+1)+z}}.\end{aligned}$$

Therefore (3.18) holds for all integer k.

Now choose $r = 2$. Then (3.17) and (3.18) yield a contradiction that

$$\hat{u}(2) \ge a_k \ge cb_k^{(pq(q+1)+z)/(pq)} \to \infty \quad \text{as } k \to \infty.$$

This excludes (3.4).

Step 2. We furthermore claim that $u_{m-1}(x) > 0$.

Otherwise, there exists $x_1 \in \mathbb{R}_+^n$ such that $u_{m-1}(x_1) = 0$. Thus $-\Delta u_{m-1}(x_1) \le 0$, since x_1 is a local minimum of $u_{m-1}(x)$. This contradicts $-\Delta u_{m-1}(x) = u^p > 0$, $x \in \mathbb{R}_+^n$.

Similarly, we can get $v_{m-1}(x) > 0$ for $\mathbb{R}_+^n$ by similar arguments in Steps 1 and 2 for $u_{m-1}(x)$.

Step 3. We show that $u_{m-i}(x), v_{m-i}(x) > 0$, $x \in \mathbb{R}_+^n$ for $i = 2, 3, \ldots, m-1$.

Based on the positivity of $v_{m-1}(x)$, we first show that $u_{m-i}(x) > 0$, $x \in \mathbb{R}^n_+$, for $i = 2, 3, \ldots, m-1$. Otherwise, there exists $x_0 \in \mathbb{R}^n_+$ with $i \in \{2, 3, \ldots, m-1\}$ such that

$$\begin{aligned} &u_{m-1}(x) > 0, u_{m-2}(x) > 0, \ldots, u_{m-i+1}(x) > 0, \quad x \in \mathbb{R}^n_+, \\ &u_{m-i}(x_0) < 0. \end{aligned} \tag{3.19}$$

(i) Assume that $m-i$ is even. Then $\bar{u}_{m-i}(r)$ for $x \in \mathbb{R}^n$ satisfies that

$$\begin{cases} -\Delta \bar{u} = \bar{u}_1, \\ \cdots \\ -\Delta \bar{u}_{m-i-1} = \bar{u}_{m-i}, \\ -\Delta \bar{u}_{m-i} = \bar{u}_{m-i+1}. \end{cases} \tag{3.20}$$

Integrating the last equation in (3.20), we arrive at

$$\begin{aligned} -r^{n-1}\bar{u}'_{m-i}(r) &= \int_0^r s^{n-1}\bar{u}_{m-i+1}(s)\,ds = \frac{1}{n\alpha(n)}\int_0^r \int_{\partial B_s(x_0)} u_{m-i+1}(s)\,d\sigma\,ds \\ &= \frac{1}{n\alpha(n)}\int_{B_r(x_0)} u_{m-i+1}(s)\,ds > 0. \end{aligned}$$

Here we have used the odd symmetry of $u_{m-i+1}(x)$ with respect to $\partial\mathbb{R}^n_+$ and the fact that more than half of $B_r(x_0)$ is contained in $\mathbb{R}^n_+$. Together with (3.19), we deduce

$$\bar{u}'_{m-i}(r) < 0 \quad \text{and} \quad \bar{u}_{m-i}(r) \le \bar{u}_{m-i}(0) = u_{m-i}(x_0) < 0, \quad r \ge 0.$$

Then by the second to the last equation in (3.20) we have

$$-\frac{1}{r^{n-1}}\left(r^{n-1}\bar{u}'_{m-i-1}(r)\right)' = \bar{u}_{m-i}(r) \le \bar{u}_{m-i}(0) \equiv -c_0 < 0, \quad r > 0.$$

This yields

$$\bar{u}'_{m-i-1}(r) > \frac{c_0}{n}r,$$

and hence

$$\bar{u}_{m-i-1}(r) \ge \bar{u}_{m-i-1}(0) + \frac{c_0}{2n}r^2 \ge \frac{c_0}{2n}r^2 + c_1, \quad r > 0.$$

Continuing this way with $m-i$ even, we derive that

$$\bar{u}(r) \le -c_0 r^{2(m-i)} + \sum_{j=1}^{m-i} c_j r^{2(m-i-j)}, \quad r > 0. \tag{3.21}$$

This yields a contraction that

$$0 < \frac{1}{n\alpha(n)}\int_{B_r(x_0)} |x|^\alpha u(x)\,dx = \int_0^r s^{n-1}\bar{u}(s)\,ds$$

$$\leq -c_0 r^{n+2(m-i)} + \sum_{j=1}^{m-i} c_j r^{n+2(m-i-j)}$$

$$< 0 \quad \text{as } r \text{ is sufficiently large.}$$

(ii) Assume that $m-i$ is odd.

Similarly to (3.21) with $m-i$ odd, by (3.5) we deduce

$$\hat{u}(r) \geq c_0 r^{2(m-i)}, \quad r > 0. \tag{3.22}$$

By parallel arguments for (3.13), by (3.22) we have

$$\begin{aligned} -r^{n-1}\hat{v}'_{m-1}(r) &= \frac{1}{n\alpha(n)} \int_{B_r(x_{m-1})} |x|^{-\alpha} |u|^{p-1} u(x)\, dx \\ &\geq \frac{c}{r^{np-n}} \left(\int_0^r s^{n-1} \hat{u}(s)\, ds \right)^p \\ &\geq c r^{n+2p(m-i)} \end{aligned}$$

with $c := \frac{c_0}{(1+|x_{m-1}|)^{\alpha(p+1)}}$ or, equivalently, $\hat{v}'_{m-1}(r) \leq -c r^{2p(m-i)+1}$, and, consequently,

$$\hat{v}_{m-1}(r) \leq -c_0 r^{2p(m-i)+2} + \hat{v}_{m-1}(0).$$

Combining this with $v_{m-1}(x) > 0$ for $x \in \mathbb{R}^n_+$, we obtain by (3.14) a contradiction that

$$\begin{aligned} 0 &< \frac{1}{n\alpha(n)} \int_{B_r(x_{m-1})} v_{m-1}(x)\, dx = \int_0^r s^{n-1} \hat{v}_{m-1}(s)\, ds \\ &\leq -c_0 r^{n+2p(m-i)+2} \leq 0. \end{aligned}$$

Combining (i) and (ii), we exclude (3.19).

Similarly, we can prove that $v_{m-i}(x) > 0$, $i = 2, 3, \ldots, m-1$, by $u_{m-1}(x) > 0$ for $\mathbb{R}^n_+$. □

Proof of Proposition 2 First, we show that the classical solutions of (1.2) must solve (1.1).

When $x \in \partial R^n_+$, we have

$$\Delta^j \left(\frac{1}{|x-y|^{n-2m}} \right) = \Delta^j \left(\frac{1}{|\bar{x}-y|^{n-2m}} \right)$$

due to $x = \bar{x}$, and thus by system (1.2) that $\Delta^j u(x) = 0$, $j = 0, 1, \ldots, m-1$. For $x \in R^n_+$, we have

$$\begin{aligned} (-\Delta)^m \big(|x|^\alpha u(x) \big) &= \int_{\mathbb{R}^n_+} (-\Delta)^m \left(\frac{1}{|x-y|^{n-2m}} - \frac{1}{|\bar{x}-y|^{n-2m}} \right) |y|^{-\beta} v^q(y)\, dy \\ &= C \int_{\mathbb{R}^n_+} \delta(x-y) |y|^{-\beta} v^q(y)\, dy \\ &= C |x|^{-\beta} v^q(x). \end{aligned}$$

Similarly, $\Delta^j v(x) = 0$, $j = 0, 1, \ldots, m-1$, on $\partial \mathbb{R}^n_+$ and $(-\Delta)^m (|x|^\beta v(x)) = C|x|^{-\alpha} u^p(x)$ in $\mathbb{R}^n_+$.

Next, we should prove that if (u,v) is a smooth positive solution of (1.1) with $p,q \geq 1$ and $\alpha,\beta > 0$, then a constant multiple of (u,v) satisfies (1.2).

Rewrite the higher-order PDEs problem (1.1) as the following second-order system:

$$\begin{cases} -\Delta u_i = u_{i+1}, u_i|_{\partial\mathbb{R}^n_+} = 0, \\ \quad i = 0,1,\dots,m-1, \text{ with } u_0 = |x|^\alpha u, u_m = |x|^{-\beta} v^q, \\ -\Delta v_i = v_{i+1}, v_i|_{\partial\mathbb{R}^n_+} = 0, \\ \quad i = 0,1,\dots,m-1, \text{ with } v_0 = |x|^\beta v, v_m = |x|^{-\alpha} u^p. \end{cases} \tag{3.23}$$

On the other hand, rewrite the integral system (1.3) as

$$\begin{cases} u_i = \int_{\mathbb{R}^n_+} G(x,y,2) u_{i+1}(y)\,dy, \\ \quad i = 0,1,\dots,m-1, \text{ with } u_0 = |x|^\alpha u, u_m = |x|^{-\beta} v^q, \\ v_i = \int_{\mathbb{R}^n_+} G(x,y,2) v_{i+1}(y)\,dy, \\ \quad i = 0,1,\dots,m-1, \text{ with } v_0 = |x|^\beta v, v_m = |x|^{-\alpha} u^p. \end{cases} \tag{3.24}$$

By [1, Theorem 2.4] we know that

$$G(x,y,2j+2) = \int_{\mathbb{R}^n_+} G(x,y,2j) G(y,z,2)\,dz, \quad j = 1,\dots,m-1.$$

This yields the equivalence between the integral systems (1.2) and (3.24).

Now let $(u_0,\dots,u_{m-1},v_0,\dots,v_{m-1})$ be a positive classical solution of (3.23). It suffices to show that $(u_0,\dots,u_{m-1},v_0,\dots,v_{m-1})$ does satisfy (3.24).

Let $(u_0,\dots,u_{m-1},v_0,\dots,v_{m-1})$ be a positive solution of (3.23). Multiply (3.23) by $\tilde{G}_R(x,y)$ and then integrate over $B^+_R(0)$ to get by (3.1) that

$$\begin{cases} \int_{B^+_R} \tilde{G}_R(x,y) u_{i+1}(y)\,dy = u_i(x) + \int_{\widehat{\Gamma}_R \cup \bar{\Gamma}_R} u_i(y) \frac{\partial \tilde{G}_R}{\partial \nu}(x,y)\,ds - \int_{\widehat{\Gamma}_R \cup \bar{\Gamma}_R} \tilde{G}_R \frac{\partial u_i(y)}{\partial \nu}(x,y)\,ds, \\ \int_{B^+_R} \tilde{G}_R(x,y) v_{i+1}(y)\,dy = v_i(x) + \int_{\widehat{\Gamma}_R \cup \bar{\Gamma}_R} v_i(y) \frac{\partial \tilde{G}_R}{\partial \nu}(x,y)\,ds - \int_{\widehat{\Gamma}_R \cup \bar{\Gamma}_R} \tilde{G}_R(x,y) \frac{\partial v_i(y)}{\partial \nu}\,ds, \end{cases}$$

which $i = 0,1,\dots,m-1$. By $\frac{\partial \tilde{G}_R}{\partial \nu}|_{\bar{\Gamma}_R} = 0$ and $\tilde{G}_R|_{\widehat{\Gamma}_R \cup \bar{\Gamma}_R} = 0$ we have

$$\begin{cases} \int_{B^+_R} \tilde{G}_R(x,y) u_{i+1}(y)\,dy = u_i(x) + \int_{\widehat{\Gamma}_R} u_i(y) \frac{\partial \tilde{G}_R}{\partial \nu}(x,y)\,ds, \quad i = 0,1,\dots,m-1, \\ \int_{B^+_R} \tilde{G}_R(x,y) v_{i+1}(y)\,dy = v_i(x) + \int_{\widehat{\Gamma}_R} v_i(y) \frac{\partial \tilde{G}_R}{\partial \nu}(x,y)\,ds, \quad i = 0,1,\dots,m-1, \end{cases} \tag{3.25}$$

which implies by Lemma 3.2 that

$$\int_{B^+_R} \tilde{G}_R(x,y) u_{i+1}(y)\,dy \leq u_i(x), \int_{B^+_R} \tilde{G}_R(x,y) v_{i+1}(y)\,dy \leq v_i(x), \quad i = 0,1,\dots,m-1.$$

Letting $R \to \infty$, we deduce with (3.2) that

$$\int_{\mathbb{R}^n_+} G(x,y,2) u_{i+1}(y)\,dy < +\infty, \int_{\mathbb{R}^n_+} G(x,y,2) v_{i+1}(y)\,dy < +\infty, \quad i = 0,1,\dots,m-1,$$

and hence there exists a sequence $R_k \to \infty$ such that

$$\int_{\widehat{\Gamma}_{R_k}} G(x,y,2)u_{i+1}(y)\,ds \to 0, \int_{\widehat{\Gamma}_{R_k}} G(x,y,2)v_{i+1}(y)\,ds \to 0,$$
$$\text{as } k \to \infty, i = 0,1,\ldots,m-2.$$

For fixed $x \in B_R^+(0)$, we have

$$G(x,y,2)|_{y\in\widehat{\Gamma}_R} = \left(\frac{1}{|x-y|^{n-2}} - \frac{1}{|\bar{x}-y|^{n-2}}\right)|_{y\in\widehat{\Gamma}_R} = O\left(\frac{y_n}{R^n}\right), \quad R \to \infty, \tag{3.26}$$

and thus

$$\frac{\partial \widetilde{G}_R}{\partial \nu}(x,y) = O\left(\frac{y_n}{R^{n+1}}\right), \quad R \to \infty. \tag{3.27}$$

By (3.26) we derive

$$\begin{aligned} &\frac{1}{R_k^n}\int_{\widehat{\Gamma}_R} y_n u_{i+1}(y)\,ds \to 0, \\ &\frac{1}{R_k^n}\int_{\widehat{\Gamma}_R} y_n v_{i+1}(y)\,ds \to 0 \quad \text{as } k \to \infty, i = 0,1,\ldots,m-2. \end{aligned} \tag{3.28}$$

Similarly, there exists a sequence of $\{R_k\}$ such that

$$\frac{1}{R_k^{n+\beta}}\int_{\widehat{\Gamma}_R} y_n v^q(y)\,ds \to 0, \qquad \frac{1}{R_k^{n+\alpha}}\int_{\widehat{\Gamma}_R} y_n u^p(y)\,ds \to 0 \quad \text{as } k \to \infty. \tag{3.29}$$

To show that the boundary terms in (3.25) approach 0 as $R \to \infty$, by (3.27) we only need to derive that there exists a sequence $R_k \to \infty$ such that

$$\begin{aligned} &\frac{1}{R_k^{n+1}}\int_{\widehat{\Gamma}_R} y_n u_{i+1}(y)\,ds \to 0, \\ &\frac{1}{R_k^{n+1}}\int_{\widehat{\Gamma}_R} y_n v_{i+1}(y)\,ds \to 0, \quad k \to \infty, i = 0,1,\ldots,m-2, \end{aligned} \tag{3.30}$$

and

$$\frac{1}{R_k^{n+\alpha+1}}\int_{\widehat{\Gamma}_R} y_n u(y)\,ds \to 0, \qquad \frac{1}{R_k^{n+\beta+1}}\int_{\widehat{\Gamma}_R} y_n v(y)\,ds \to 0 \quad \text{as } k \to \infty. \tag{3.31}$$

Obviously, (3.30) is a direct consequence of (3.28).

By Jensen's inequality with $p \geq 1$ and (3.29) we have

$$\begin{aligned} \left(\frac{1}{R_k^{n+\alpha}}\int_{\widehat{\Gamma}_{R_k}} y_n^{\frac{1}{p}} u(y)\,ds\right)^p &\leq \frac{1}{R_k^{\alpha p+n}}\int_{\widehat{\Gamma}_{R_k}} y_n u^p(y)\,ds \\ &\leq \frac{1}{R_k^{\alpha+n}}\int_{\widehat{\Gamma}_{R_k}} y_n u^p(y)\,ds \to 0 \quad \text{as } R_k \to \infty, \end{aligned}$$

and hence

$$\frac{1}{R_k^{n+\alpha}}\int_{\widehat{\Gamma}_{R_k}} y_n^{\frac{1}{p}} u(y)\,ds \to 0, \quad R_k \to \infty. \tag{3.32}$$

Denote $\widehat{\Gamma}^1_{R_k} = \{y \in \widehat{\Gamma}_{R_k}, y_n \le 1\}$ and $\widehat{\Gamma}^2_{R_k} = \{y \in \widehat{\Gamma}_{R_k}, y_n > 1\}$. Then

$$\begin{aligned}
\int_{\widehat{\Gamma}_{R_k}} \frac{y_n}{R_k^{n+\alpha+1}} u(y)\,ds &= \int_{\widehat{\Gamma}^1_{R_k}} \frac{y_n}{R_k^{n+\alpha+1}} u(y)\,ds + \int_{\widehat{\Gamma}^2_{R_k}} \frac{y_n}{R_k^{n+\alpha+1}} u(y)\,ds \\
&\le \int_{\widehat{\Gamma}^1_{R_k}} \frac{y_n^{\frac{1}{p}}}{R_k^{n+\alpha+1}} u(y)\,ds + \int_{\widehat{\Gamma}^2_{R_k}} \frac{1}{R_k^{n+\alpha}} u(y)\,ds \\
&\le \frac{1}{R_k}\int_{\widehat{\Gamma}^1_{R_k}} \frac{y_n^{\frac{1}{p}}}{R_k^{n+\alpha}} u(y)\,ds + \int_{\widehat{\Gamma}^2_{R_k}} \frac{y_n^{\frac{1}{p}}}{R_k^{n+\alpha}} u(y)\,ds \\
&\le \frac{2}{R_k^{n+\alpha}}\int_{\widehat{\Gamma}_{R_k}} y_n^{\frac{1}{p}} u(y)\,ds,
\end{aligned}$$

which vanishes as $R_k \to \infty$ by (3.32). This proves (3.31) for u. The argument for v is similar.

Substituting (3.30) and (3.31) into (3.25), we arrive at (3.24). □

Acknowledgements
The authors thank a lot to the editor and reviewer for valuable suggestions, which helped us improve the quality of the paper.

Funding
This work is supported by the National Natural Science Foundation of China (No. 62262012), the Hainan Provincial Natural Science Foundation of China and the Foundation of Hainan University (No. KYQD22094, No. KYQD23050).

Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

Author contributions
Weiwei Zhao proposed the idea of this paper and performed all the steps of the proofs. Changhui Hu and Xiaoling Shao wrote the main manuscript text. All authors read and approved the final manuscript.

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Received: 15 December 2023 Accepted: 19 January 2024 Published online: 26 January 2024

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