Zhou et al. *Boundary Value Problems* (2024) 2024:26
https://doi.org/10.1186/s13661-024-01832-8

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Existence and multiplicity of solutions of fractional differential equations on infinite intervals

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Abstract

In this research, we investigate the existence and multiplicity of solutions for fractional differential equations on infinite intervals. By using monotone iteration, we identify two solutions, and the multiplicity of solutions is demonstrated by the Leggett–Williams fixed point theorem.

Keywords: Fractional differential equation; Infinite intervals; Monotone iteration; Leggett–Williams fixed point theorem

1 Introduction

Fractional differential equations' theory has been widely employed in astronomy, biology, economics, and other domains, such as described in [1, 2, 4, 8, 11–13, 15, 17, 19, 20, 26]. In recent years, many authors have combined fractional derivative operators with problems of p-Laplacian type, Kirchhoff-type, etc. [21–24]. Their work made important contributions to enriching the study of fractional derivative problems. Research has investigated many difficulties of solutions of fractional differential equations on infinite intervals in addition to those on finite intervals.

In [29], the authors studied the BVP

$$\begin{cases} D^{\nu}w(\zeta)+h(\zeta,w(\zeta))=0, & \zeta\in(0,\infty),\nu\in(1,2),\\ w(0)=0, & \lim_{\zeta\to\infty}D_{0^+}^{\nu-1}w(\zeta)=\beta w(\xi). \end{cases} \tag{1}$$

The authors discovered the presence of solutions by employing the Leray–Schauder nonlinear theorem. In [25], Guotao Wang studied the BVP

$$\begin{cases} D_{0^+}^{\delta}w(\varsigma)+k(\varsigma,w(\varsigma))=0, & 2<\delta\le 3,\\ w(0)=w'(0)=0, & \\ D^{\delta-1}w(\infty)=\rho I^{\gamma}w(\varrho), & \gamma>0, \end{cases} \tag{2}$$

where $\varsigma\in K=[0,\infty)$, $k\in C(K\times\mathbb{R},\mathbb{R})$, $\rho\in\mathbb{R}$, $\varrho\in K$. The author obtained the existence and uniqueness of solutions by the monotone iterative technique. In [18], the Leggett–

Williams fixed point theorem and the Guo–Krasnoselskii fixed point theorem were used by Phollakrit Thiramanus et al. to investigate

$$\begin{cases} {}^{H}D^{\delta}w(\tau) + k(\tau)r(w(\tau)) = 0, \quad \delta \in (1,2), \tau \in (1,\infty), \\ w(1) = 0, \\ {}^{H}D^{\delta-1}w(\infty) = \sum_{p=1}^{n} \zeta_p I^{\gamma_p} w(\upsilon), \end{cases} \tag{3}$$

where $\upsilon \in (1,\infty)$, γ_p, $p = 1,\dots,n$, and $\zeta_p \geq 0$, $p = 1,\dots,n$ are given constants. In [9], the authors studied the existence of solutions to the problem

$$\begin{cases} D_{0^+}^{\alpha} w(\zeta) + k(\zeta)r(w(\zeta)) = 0, \quad \zeta \in (0,\infty), \\ w(0) = w'(0) = 0, \\ D^{\alpha-1}w(\infty) = \sum_{i=1}^{n-2} \varsigma_i w(\delta_i), \end{cases} \tag{4}$$

where $2 < \alpha < 3$, $0 < \delta_1 < \delta_2 < \cdots < \delta_{n-2} < \infty$, $\varsigma_i \geq 0$, $i = 1,\dots,n-2$ satisfy $0 < \sum_{i=1}^{n-2} \varsigma_i \delta_i^{\alpha-1} < \Gamma(\alpha)$. From the Leggett–Williams fixed point theorem, the existence of at least three positive solutions was demonstrated. In [27], the authors investigated the family of BVPs

$$\begin{cases} {}^{H}D_{1^+}^{\alpha} w(\theta) + k(\theta)h(\theta, w(\theta)) = 0, \quad 2 < \alpha < 3, \theta \in (1,\infty), \\ w(1) = w'(1) = 0, \\ {}^{H}D_{1^+}^{\alpha-1}w(\infty) = \sum_{m=1}^{p} \alpha_m {}^{H}I_{1^+}^{\gamma_m} w(\upsilon) + b\sum_{n=1}^{q} \delta_n w(\zeta_n), \end{cases} \tag{5}$$

where $1 < \upsilon < \zeta_1 < \zeta_2 < \cdots < \zeta_p < \infty$, b, α_m, $\zeta_n \geq 0$, $m = 1,\dots,p$, $n = 1,\dots,q$ are given constants. Various fixed point theorems were used to prove the results. The generalized Avery–Henderson fixed point theorem was used to demonstrate the presence of many positive solutions to problem (5) in [28].

In this paper, motivated by the previous results, we investigate the BVP

$$\begin{cases} D^{\delta}x(t) + b(t)r(t, x(t)) = 0, \quad t \in [0,\infty), m-1 < \delta \leq m, \\ x^{(n)}(0) = 0, \quad n = 0,1,\dots,m-2, \\ D^{\delta-1}x(\infty) = \sum_{i=1}^{q} \nu_i I^{\beta_i} x(\varrho) + \lambda \sum_{j=1}^{p} \kappa_j x(\varsigma_j), \end{cases} \tag{6}$$

where $m \geq 2$, D^{δ} is the Riemann–Liouville fractional derivative of order δ, $D^{\delta-1}x(\infty) = \lim_{t\to\infty} D^{\delta-1}x(t)$, I^{β_j} is the Riemann–Liouville fractional integral of order $\beta_i > 0$, $i = 1,2,\dots,q$, $0 < \varrho < \infty$, $\nu_i, \lambda, \kappa_j, \varsigma_j \geq 0$, $i = 1,\dots,q$, $j = 1,\dots,p$ are given constants; $r : I \times \mathbb{R}^+ \to \mathbb{R}^+$, with $I = [0,\infty)$, is continuous and $b : I \to \mathbb{R}^+$ is integrable, and

$$\Delta = \Gamma(\delta) - \sum_{i=1}^{q} \nu_i \frac{\Gamma(\delta)}{\Gamma(\delta+\beta_i)} \varrho^{\delta+\beta_j-1} - \lambda \sum_{j=1}^{p} \kappa_j \varsigma_j^{\delta-1} \neq 0.$$

Using the monotone iteration method, we prove that two solutions of problem (6) exist. Using the Leggett–Williams fixed point theorem, we determine that problem (6) has at least three solutions.

Compared with [27, 28], we use different methods to study multiple solutions. Compared with [9, 18, 25, 27–29], we study fractional differential equations of arbitrary order

$m \geq 2$. It is obvious that our problem is more general. Our boundary conditions have more general forms and the boundary conditions of [9, 18, 25, 29] are our special cases. When $m = 2$, $\nu_i = 0$, $\lambda = 1$, $j = 1$, we know that problem (1) is a special case of problem (6). When $m = 3$, $i = 1$, $\lambda = 0$, we have that problem (2) is a special case of problem (6). When $\lambda = 0$, we obtain that the boundary conditions of problem (3) are a special case of the boundary conditions of problem (6). When $m = 3$, $\nu_i = 0$, $\lambda = 1$, $j = 1, 2, \ldots, p-2$, we get that the boundary conditions of problem (4) are a special case of the boundary conditions of problem (6).

In this paper, the following four conditions will be used:

(H_1) $r \in C(I \times \mathbb{R}^+, \mathbb{R}^+)$ and $r(t, \cdot) \not\equiv 0$ on any subinterval of $\mathbb{R}^+$, and when x is bounded, $r(t, (1 + t^{\delta-1})x)$ is bounded on $\mathbb{R}^+$.

(H_2) $b : I \to \mathbb{R}^+$ does not identically vanish on any subinterval of $\mathbb{R}^+$ and

$$0 < \int_0^\infty b(s)\, ds < \infty.$$

(H_3) r is nondecreasing with respect to the second variable.

(H_4) There exists a positive constant Λ such that

$$r\bigl(t, \bigl(1 + t^{\delta-1}\bigr)x\bigr) \leq \frac{\Lambda}{\frac{1}{\Lambda}\int_0^\infty b(s)\, ds} \quad \text{for any } (t, x) \in [0, \infty) \times [0, \Lambda].$$

Assumptions (H_1) and (H_2) will be applied in Lemma 3.2 and Theorem 3.5, while (H_1)–(H_4) will be used in Theorem 3.4.

The remainder of this article is organized as follows. Section 2 contains the definitions and lemmas required to prove our results. Section 3 presents the existence and multiplicity results for the boundary value problem (6). Section 4 provides examples relevant to the key findings of this paper.

2 Preliminaries

We present several definitions and lemmas here for the reader's convenience, as they will be utilized to prove our primary results.

Definition 2.1 (see [14, 16]) The Riemann–Liouville fractional derivative of order $\delta > 0$ of a function $h : (0, \infty) \to \mathbb{R}$ is given by

$$D^\delta h(\varsigma) = \frac{1}{\Gamma(m - \delta)} \left(\frac{d}{d\varsigma}\right)^m \int_0^\varsigma \frac{h(\eta)}{(\varsigma - \eta)^{\delta - m + 1}}\, d\eta,$$

where $m - 1 < \nu \leq m$.

Definition 2.2 (see [14, 16]) The Riemann–Liouville fractional integral of order $\eta > 0$ of a function $h : (0, \infty) \to \mathbb{R}$ is given by

$$I^\eta h(\varrho) = \frac{1}{\Gamma(\eta)} \int_0^\varrho (\varrho - \varsigma)^{\eta - 1} h(\varsigma)\, d\varsigma.$$

Lemma 2.3 (see [5]) *If $\varrho, \theta > 0$, then*

$$I^{\varrho}\left(t^{\theta}\right) = \frac{\Gamma(\theta+1)}{\Gamma(\varrho+\theta+1)} t^{\varrho+\theta},$$
$$D^{\varrho}\left(t^{\theta}\right) = \frac{\Gamma(\theta+1)}{\Gamma(\theta-\varrho+1)} t^{\theta-\varrho}.$$

Lemma 2.4 *If $e \in C([0,\infty),\mathbb{R})$, then the problem*

$$\begin{cases} D^{\delta}x(t) + e(t) = 0, \quad m-1 < \delta \le m, \\ x^{(n)}(0) = 0, \quad n = 0,1,\dots,m-2, \\ D^{\delta-1}x(\infty) = \sum_{i=1}^{q} \nu_i I^{\beta_i} x(\varrho) + \lambda \sum_{j=1}^{p} \kappa_j x(\varsigma_j) \end{cases} \tag{7}$$

has a unique solution

$$x(t) = \int_0^{\infty} \Pi(t,s)e(s)\,ds,$$

where

$$\begin{aligned} \Pi(t,s) &= \pi(t,s,\delta) + \frac{t^{\delta-1}}{\Delta} \sum_{i=1}^{q} \nu_i \pi(\varrho,s,\delta+\beta_i) + \frac{t^{\delta-1}}{\Delta} \lambda \sum_{j=1}^{p} \kappa_j \pi(\varsigma_j,s,\delta), \\ \pi(t,s,\delta) &= \frac{1}{\Gamma(\delta)} \begin{cases} t^{\delta-1} - (t-s)^{\delta-1}, & 0 \le s \le t < \infty, \\ t^{\delta-1}, & 0 \le t \le s < \infty. \end{cases} \end{aligned} \tag{8}$$

Proof Since $D^{\delta}x(t) + e(t) = 0$, we obtain

$$x(t) = l_1 t^{\delta-1} + l_2 t^{\delta-2} + \cdots + l_m t^{\delta-m} - I^{\delta}e(t).$$

Due to $x^{(n)}(0) = 0$, $n = 0,\dots,m-2$, we have $l_2 = \cdots = l_m = 0$, that is,

$$x(t) = l_1 t^{\delta-1} - I^{\delta}e(t).$$

Since $D^{\delta-1}x(\infty) = \sum_{i=1}^{q} \nu_i I^{\beta_i} x(\varrho) + \lambda \sum_{j=1}^{p} \kappa_j x(\varsigma_j)$, we have

$$\begin{aligned} l_1 &= \frac{1}{\Delta} \int_0^{\infty} e(s)\,ds - \frac{\sum_{i=1}^{q} \nu_i}{\Delta\Gamma(\delta+\beta_i)} \int_0^{\varrho} (\varrho-s)^{\delta+\beta_i-1} e(s)\,ds \\ &\quad - \frac{\lambda}{\Delta} \sum_{j=1}^{p} \kappa_j \frac{1}{\Gamma(\delta)} \int_0^{\varsigma_j} (\varsigma_j - s)^{\delta-1} e(s)\,ds \\ &= \frac{1}{\Gamma(\delta)} \int_0^{\infty} e(s)\,ds + \frac{\sum_{i=1}^{q} \nu_i}{\Delta} \int_0^{\infty} \pi(\varrho,s,\delta+\beta_i) e(s)\,ds \\ &\quad + \frac{\lambda}{\Delta} \sum_{j=1}^{p} \kappa_j \int_0^{\infty} \pi(\varsigma_j,s,\delta) e(s)\,ds. \end{aligned}$$

Then

$$\begin{aligned}
x(t) &= \frac{1}{\Gamma(\delta)}\int_0^\infty t^{\delta-1}e(s)\,ds + \frac{\sum_{i=1}^q \nu_i t^{\delta-1}}{\Delta}\int_0^\infty \pi(\varrho,s,\delta+\beta_i)e(s)\,ds \\
&\quad + \frac{\lambda}{\Delta}t^{\delta-1}\sum_{j=1}^p \kappa_j \int_0^\infty \pi(\varsigma_j,s,\delta)e(s)\,ds - \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}e(s)\,ds \\
&= \int_0^\infty \pi(t,s,\delta)e(s)\,ds + \frac{t^{\delta-1}}{\Delta}\sum_{i=1}^q \nu_i \int_0^\infty \pi(\varrho,s,\delta+\beta_i)e(s)\,ds \\
&\quad + \frac{t^{\delta-1}}{\Delta}\lambda\sum_{j=1}^p \kappa_j \int_0^\infty \pi(\varsigma_j,s,\delta)e(s)\,ds \\
&= \int_0^\infty \Pi(t,s)e(s)\,ds.
\end{aligned}$$

The proof is complete. □

Lemma 2.5 *If $\Delta > 0$, the following properties apply to the Green function $\Pi(t,s)$ defined by (8) for all $(t,s) \in I \times I$:*

(i) *$\Pi(t,s)$ is nonnegative and continuous.*
(ii) *$\Pi(t,s)$ is increasing with respect to t.*
(iii) $\frac{\Pi(t,s)}{1+t^{\delta-1}} \le \frac{1}{\Delta}$.
(iv) $\Pi(t,s) \le \frac{1}{\Delta}t^{\delta-1}$.

Proof (i) According to the definition of $\Pi(t,s)$, this is clearly true.

(ii) We just have to prove that $\pi(t,s,\delta)$ increases as t increases. Let

$$\begin{aligned}
&\eta_1(t) = t^{\delta-1} - (t-s)^{\delta-1}, \quad 0 \le s \le t < \infty, \\
&\eta_2(t) = t^{\delta-1}, \quad 0 \le t \le s < \infty.
\end{aligned}$$

We have

$$\begin{aligned}
&\eta_1'(t) = (\delta-1)t^{\delta-2} - (\delta-1)(t-s)^{\delta-2} \ge 0, \\
&\eta_2'(t) = (\delta-1)t^{\delta-2} \ge 0.
\end{aligned}$$

Then

$$\eta_1(t_1) < \eta_1(t_2), \qquad \eta_2(t_1) < \eta_2(t_2), \qquad \eta_2(t_1) \le \eta_2(s) = \eta_1(s) \le \eta_1(t_2).$$

(iii) We have

$$\begin{aligned}
\frac{\Pi(t,s)}{1+t^{\delta-1}} &= \frac{\pi(t,s,\delta)}{1+t^{\delta-1}} + \frac{t^{\delta-1}}{1+t^{\delta-1}}\frac{1}{\Delta}\sum_{i=1}^q \nu_i \pi(\varrho,s,\delta+\beta_i) \\
&\quad + \frac{t^{\delta-1}}{1+t^{\delta-1}}\frac{1}{\Delta}\lambda\sum_{j=1}^p \kappa_j \pi(\varsigma_j,s,\delta)
\end{aligned} \tag{9}$$

$$\leq \frac{1}{\Gamma(\delta)} + \frac{1}{\Delta}\sum_{i=1}^{q} \nu_i \frac{\varrho^{\delta+\beta_i-1}}{\Gamma(\delta+\beta_i)} + \frac{1}{\Delta}\lambda \sum_{j=1}^{p} \kappa_j \frac{\varsigma_j^{\delta-1}}{\Gamma(\delta)}$$
$$= \frac{1}{\Delta}.$$

(iv) We have

$$\Pi(t,s) \leq t^{\delta-1}\left(\frac{1}{\Gamma(\delta)} + \frac{1}{\Delta}\sum_{i=1}^{q} \nu_i \frac{\varrho^{\delta+\beta_i-1}}{\Gamma(\delta+\beta_i)} + \frac{1}{\Delta}\lambda \sum_{j=1}^{p} \kappa_j \frac{\varsigma_j^{\delta-1}}{\Gamma(\delta)}\right) = t^{\delta-1}\frac{1}{\Delta}.$$

The proof is complete. □

Let X be a Banach space endowed with norm $\|\cdot\|_X$. Let $0 < v < w$ be given, and let ϑ be a nonnegative continuous concave functional on K. Define the convex sets K_μ and $K(\vartheta, v, w)$ by $K_\mu = \{x \in K : \|x\|_X < \mu\}$ and $K(\vartheta, v, w) = \{x \in K : \vartheta(x) \geq v, \|x\|_X \leq w\}$.

Lemma 2.6 (see [6]) *Let $H : \overline{K_\sigma} \to \overline{K_\sigma}$ be a completely continuous operator, and let ϑ be a nonnegative continuous concave functional on K such that $\vartheta(x) \leq \|x\|$ for all $x \in \overline{K_\sigma}$. Assume there exist $0 < \tau_1 < \tau_2 < \tau_3 \leq \sigma$ such that*

(B_1) $\{x \in K(\vartheta, \tau_2, \tau_3) | \vartheta(x) > \tau_2\} \neq \emptyset$ *and* $\vartheta(Hx) > \tau_2$ *for* $x \in K(\vartheta, \tau_2, \tau_3)$;

(B_2) $\|Hx\| \leq \tau_1$ *for* $\|x\| \leq \tau_1$;

(B_3) $\vartheta(Hx) > \tau_2$ *for* $x \in K(\vartheta, \tau_2, \sigma)$ *with* $\|Hx\| > \tau_3$.

Then H has at least three fixed points x_1, x_2, and x_3 such that

$$\|x_1\| < \tau_1, \qquad \varsigma(x_2) > \tau_2, \qquad \|x_3\| > \tau_1, \quad \textit{and} \quad \varsigma(x_3) < \tau_2.$$

3 Main results

Let $X = \{x \in C(I, \mathbb{R}^+) : \sup_{t\in I} \frac{|x(t)|}{1+t^{\delta-1}} < \infty\}$ be the Banach space with norm $\|x\|_X = \sup_{t\in I} \frac{|x(t)|}{1+t^{\delta-1}}$. We define a cone $\Upsilon \subset X$ by

$$\Upsilon = \{x \in X : x(t) \geq 0, \forall t \in I\},$$

and an operator $\Phi : \Upsilon \to X$ by

$$\Phi x(t) = \int_0^\infty \Pi(t,s) b(s) r\big(s, x(s)\big)\, ds. \tag{10}$$

It is simple to demonstrate that $\Phi : \Upsilon \to \Upsilon$.

Lemma 3.1 (see [3, 10]) *Let $\Omega \subset X$ be a bounded set. Then Ω is relatively compact in X if the following conditions hold:*

(i) *For any $x \in \Omega$, $\frac{x(t)}{1+t^{\alpha-1}}$ is equicontinuous on any compact interval of $[0, \infty)$;*

(ii) *For any $\varepsilon > 0$, there exists a constant $N > 0$ such that*

$$\left|\frac{x(t_1)}{1+t_1^{\alpha-1}} - \frac{x(t_2)}{1+t_2^{\alpha-1}}\right| < \varepsilon$$

for any $t_1, t_2 > N$ and $x \in \Omega$.

Lemma 3.2 (see [7]) *If* (H_1) *and* (H_2) *hold, then* $\Phi : \Upsilon \to \Upsilon$ *is completely continuous.*

Remark 3.3 To prove that Φx is equiconvergent at infinity, we will give another method. For any $\varepsilon > 0$, there exists a constant $N_1 > 0$ such that

$$0 < \int_{N_1}^{\infty} b(s)M_\lambda \, ds < \varepsilon,$$

where

$$M_\lambda = \sup\bigl\{r\bigl(s, \bigl(1 + s^{\delta-1}\bigr)x\bigr) : (s,x) \in I \times [0,\lambda]\bigr\}.$$

Note

$$\lim_{t\to\infty} \frac{t^{\delta-1}}{1+t^{\delta-1}} = 1, \qquad \lim_{t\to\infty} \frac{\pi(t,N_1,\delta)}{1+t^{\delta-1}} = 0.$$

Then for the above $\varepsilon > 0$, there exist constants $N_2 > 0$, $N_3 > N_1$ such that for any $t_1, t_2 > N_2$, we have

$$\left|\frac{t_1^{\delta-1}}{1+t_1^{\delta-1}} - \frac{t_2^{\delta-1}}{1+t_2^{\delta-1}}\right| \le \left|1 - \frac{t_1^{\delta-1}}{1+t_1^{\delta-1}}\right| + \left|1 - \frac{t_2^{\delta-1}}{1+t_2^{\delta-1}}\right| < \varepsilon$$

and for any $t_1, t_2 > N_3$, $0 \le s \le N_1$, we have

$$\left|\frac{\pi(t_1,s,\delta)}{1+t_1^{\delta-1}} - \frac{\pi(t_2,s,\delta)}{1+t_2^{\delta-1}}\right| \le \left|\frac{\pi(t_1,N_1,\delta)}{1+t_1^{\delta-1}}\right| + \left|\frac{\pi(t_2,N_1,\delta)}{1+t_2^{\delta-1}}\right| < \frac{\varepsilon}{\Gamma(\delta)}.$$

Choose $N > \max\{N_2, N_3\}$. Then for any $t_1, t_2 > N$, we have

$$\begin{aligned}
\left|\frac{\Phi x(t_1)}{1+t_1^{\delta-1}} - \frac{\Phi x(t_2)}{1+t_2^{\delta-1}}\right| &\le \int_0^\infty \left|\frac{\Pi(t_1,s)}{1+t_1^{\delta-1}} - \frac{\Pi(t_2,s)}{1+t_2^{\delta-1}}\right| b(s)r\bigl(s,x(s)\bigr)\, ds \\
&\le \int_0^{N_1} \left|\frac{\Pi(t_1,s)}{1+t_1^{\delta-1}} - \frac{\Pi(t_2,s)}{1+t_2^{\delta-1}}\right| b(s)M_\lambda \, ds \\
&\quad + \int_{N_1}^\infty \left|\frac{\Pi(t_1,s)}{1+t_1^{\delta-1}} - \frac{\Pi(t_2,s)}{1+t_2^{\delta-1}}\right| b(s)M_\lambda \, ds \\
&< \frac{\varepsilon}{\Delta}\int_0^{N_1} b(s)M_\lambda \, ds + \frac{2}{\Delta}\int_{N_1}^\infty b(s)M_\lambda \, ds \\
&< \left(\frac{1}{\Delta}\int_0^\infty b(s)M_\lambda \, ds + \frac{2}{\Delta}\right)\varepsilon.
\end{aligned}$$

Thus, Φx is equiconvergent at infinity.

Theorem 3.4 *If* (H_1)–(H_4) *hold, then two explicit monotone iterative sequences can yield two positive solutions* x^*, y^* *of problem* (6), *namely*

$$\begin{cases} x_{n+1} = \int_0^\infty \Pi(t,s)b(s)r(s,x_n(s))\, ds, & x_0(t) = 0, \\ y_{n+1} = \int_0^\infty \Pi(t,s)b(s)r(s,y_n(s))\, ds, & y_0(t) = \Lambda t^{\delta-1}, t \in I, \Lambda > 0, \end{cases} \tag{11}$$

in the interval $(0, \Lambda t^{\delta-1}]$.

Proof We define $W = \{x \in X : \|x\|_X \leq \Lambda\}$, while Φ is defined by (10). Then we show that $\Phi(W) \subset W$. For any $x \in W$, by (H_4) and (9), we have

$$\begin{aligned}\|\Phi x\|_X &= \sup_{t\in I} \int_0^\infty \frac{\Pi(t,s)}{1+t^{\delta-1}} b(s) r\bigl(s,x(s)\bigr)\,ds \\ &\leq \frac{\Lambda}{\frac{1}{\Delta}\int_0^\infty b(s)\,ds} \frac{1}{\Delta} \int_0^\infty b(s)\,ds \leq \Lambda.\end{aligned}$$

Due to the definition of the operator Φ and assumption (H_3), we have that Φ is a nondecreasing operator. Define $x_0(t) = 0$, $x_1 = \Phi x_0$, $x_2 = \Phi x_1 = \Phi^2 x_0$ for any $t \in I$. In view of $x_0(t) = 0 \in W$ and $\Phi(W) \subset W$, we have $x_1 \in W$, $x_2 \in W$ and

$$x_1(t) = \Phi x_0(t) = \Phi 0(t) \geq 0 = x_0(t) \quad \text{for any } t \in I.$$

Considering the nondecreasing nature of the operator Φ, we get

$$x_2(t) = \Phi x_1(t) \geq \Phi x_0(t) = x_1(t) \quad \text{for any } t \in I.$$

Define a sequence $\Phi x_n = x_{n+1}$, $n \in \mathbb{N}$. Clearly, the sequence $\{x_n\}_{n=1}^\infty \subset W$ and it satisfies

$$x_{n+1}(t) \geq x_n(t) \quad \text{for any } t \in I, n \in \mathbb{N}. \tag{12}$$

Because of the complete continuity of the operator Φ, there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty \subset W$, $x^* \subset W$ such that $x_{n_k} \to x^*$, $k \to \infty$. This, together with the monotone nature of $\{x_n\}_{n=1}^\infty$, implies that $\lim_{n\to\infty} x_n = x^*$. Since Φ is continuous and $\Phi x_n = x_{n+1}$, we have $\Phi x^* = x^*$, i.e., x^* is a fixed point of the operator Φ.

Define $y_0(t) = \Lambda t^{\alpha-1}$, $y_1 = \Phi y_0$, $y_2 = \Phi y_1 = \Phi^2 y_0$ for any $t \in I$. In view of $y_0(t) = \Lambda t^{\alpha-1} \in W$ and $\Phi(W) \subset W$, we have $y_1 \in W$, $y_2 \in W$. By Lemma 2.5 and (H_4), we obtain

$$\begin{aligned}y_1(t) = \Phi y_0(t) &= \int_0^\infty \Pi(t,s) b(s) r\bigl(s, y_0(s)\bigr)\,ds \\ &\leq \int_0^\infty t^{\delta-1} b(s) \frac{1}{\Delta} \frac{\Lambda}{\frac{1}{\Delta}\int_0^\infty b(s)\,ds}\,ds \\ &\leq \Lambda t^{\delta-1} = y_0(t).\end{aligned}$$

Due to the nondecreasing nature of the operator Φ, we have

$$y_2(t) = \Phi y_1(t) \leq \Phi y_0(t) = y_1(t) \quad \text{for any } t \in I.$$

Define a sequence $\Phi y_n = y_{n+1}$, $n \in \mathbb{N}$. Clearly, the sequence $\{y_n\}_{n=1}^\infty \subset W$ and it satisfies

$$y_{n+1}(t) \leq y_n(t) \quad \text{for any } t \in I, n \in \mathbb{N}. \tag{13}$$

As before, we can conclude that there exists $y^* \in W$ such that $\lim_{n\to\infty} y_n = y^*$. Since Φ is continuous and $\Phi y_n = y_{n+1}$, we have $\Phi y^* = y^*$, i.e., y^* is a fixed point of the operator Φ.

Since for any $t \in I$, $r(t,\cdot) \not\equiv 0$, 0 is not a solution of problem (6). According to the above process, we know that x^* and y^* are two positive solutions of problem (6) in $(0, \Lambda t^{\delta-1}]$,

which can be established using two explicit monotonic iterative sequences (11), respectively. □

Theorem 3.5 *If* (H_1), (H_2) *hold, then there exist numbers* $\delta_1, \delta_2, \delta_3 > 0$, *and* $0 < \theta < 1$ *such that* $0 < \delta_1 < \delta_2 < \frac{\delta_2}{\theta} \le \delta_3$. *In addition, assume*

(A_1) $r(t,(1+t^{\delta-1})x) \le \delta_3 M_1$, $(t,x) \in I \times [0,\delta_3]$, *where* $M_1 = (\frac{1}{\Delta}\int_0^\infty b(s)\,ds)^{-1}$;

(A_2) $r(t,(1+t^{\delta-1})x) \le \delta_1 M_1$, $(t,x) \in I \times [0,\delta_1]$;

(A_3) $r(t,(1+t^{\delta-1})x) \ge \delta_2 M_2$, $(t,x) \in [\frac{1}{k},k] \times [\delta_2,\delta_3]$, *where* $M_2 = (\frac{k^{1-\delta}}{1+k^{\delta-1}}\int_{\frac{1}{k}}^k b(s)\,ds)^{-1}$.

Then problem (6) *has at least three positive solutions* x_1^*, x_2^*, *and* x_3^* *such that*

$$\|x_1^*\|_X \le \delta_1, \qquad \omega(x_2^*) \ge \delta_2, \qquad \|x_3^*\|_X \ge \delta_1, \quad \textit{and} \quad \omega(x_3^*) < \delta_2.$$

Proof Let $k > 1$, $\omega(x) = \min_{t\in[\frac{1}{k},k]} \frac{x(t)}{1+t^{\delta-1}}$, while Φ is defined by (10). The proof will be broken down into four steps.

Step 1.

For any $x \in \overline{\Upsilon_{\delta_3}}$, by the condition (A_1) and (9), we have

$$\begin{aligned}
\|\Phi x\|_X &= \sup_{t\in I}\int_0^\infty \frac{\Pi(t,s)}{1+t^{\delta-1}} b(s) r\bigl(s,x(s)\bigr)\,ds \\
&= \sup_{t\in I}\int_0^\infty \frac{\Pi(t,s)}{1+t^{\delta-1}} b(s) r\biggl(s,\bigl(1+s^{\delta-1}\bigr)\frac{x(s)}{1+s^{\delta-1}}\biggr)\,ds \\
&\le \delta_3 M_1 \sup_{t\in I}\int_0^\infty \frac{\Pi(t,s)}{1+t^{\delta-1}} b(s)\,ds \\
&\le \delta_3 M_1 \frac{1}{\Delta}\int_0^\infty b(s)\,ds = \delta_3.
\end{aligned}$$

Then $\Phi : \overline{\Upsilon_{\delta_3}} \to \overline{\Upsilon_{\delta_3}}$. According to Lemma 3.2, we get that $\Phi : \overline{\Upsilon_{\delta_3}} \to \overline{\Upsilon_{\delta_3}}$ is completely continuous.

Step 2.

Let $x_0(t) = 0.5(\delta_2 + \frac{\delta_2}{\theta})(1+t^{\delta-1})$, then we obtain that

$$\omega(x_0) = 0.5\biggl(\delta_2 + \frac{\delta_2}{\theta}\biggr) > \delta_2$$

and

$$\|x_0\|_X = 0.5\biggl(\delta_2 + \frac{\delta_2}{\theta}\biggr) < \frac{\delta_2}{\theta},$$

which shows that

$$x_0 \in \biggl\{x \in \Upsilon\biggl(\omega,\delta_2,\frac{\delta_2}{\theta}\biggr)\biggm|\omega(x) > \delta_2\biggr\},$$

and thus

$$\biggl\{x \in \Upsilon\biggl(\omega,\delta_2,\frac{\delta_2}{\theta}\biggr)\biggm|\omega(x) > \delta_2\biggr\} \ne \emptyset.$$

By the condition (A_3), Lemma 2.5, and (8), for any $x \in \Upsilon(\omega, \delta_2, \frac{\delta_2}{\theta})$, we have

$$\begin{aligned}
\omega(\Phi x) &= \min_{t\in[\frac{1}{k},k]} \int_0^\infty \frac{\Pi(t,s)}{1+t^{\delta-1}} b(s) r\bigl(s, x(s)\bigr)\, ds \\
&= \min_{t\in[\frac{1}{k},k]} \int_0^\infty \frac{\Pi(t,s)}{1+t^{\delta-1}} b(s) r\left(s, \bigl(1+s^{\delta-1}\bigr)\frac{x(s)}{1+s^{\delta-1}}\right) ds \\
&\geq \int_0^\infty \min_{t\in[\frac{1}{k},k]} \frac{\Pi(t,s)}{1+t^{\delta-1}} b(s) r\left(s, \bigl(1+s^{\delta-1}\bigr)\frac{x(s)}{1+s^{\delta-1}}\right) ds \\
&> \delta_2 M_2 \int_{\frac{1}{k}}^k \frac{\pi(\frac{1}{k}, s, \delta)}{1+k^{\delta-1}} b(s)\, ds = \delta_2.
\end{aligned} \tag{14}$$

Step 3.

By the condition (A_2) and (9), we have

$$\begin{aligned}
\|\Phi x\|_X &= \sup_{t\in I} \int_0^\infty \frac{\Pi(t,s)}{1+t^{\delta-1}} b(s) r\bigl(s, x(s)\bigr)\, ds \\
&= \sup_{t\in I} \int_0^\infty \frac{\Pi(t,s)}{1+t^{\delta-1}} b(s) r\left(s, \bigl(1+s^{\delta-1}\bigr)\frac{x(s)}{1+s^{\delta-1}}\right) ds \\
&\leq \delta_1 M_1 \frac{1}{\Delta} \int_0^\infty b(s)\, ds = \delta_1.
\end{aligned}$$

Step 4.

Similar to (14), for any $x \in \Upsilon(\omega, \delta_2, \delta_3)$ and $\|\Phi x\|_X > \frac{\delta_2}{\theta}$, by condition (A_3), Lemma 2.5, and (8), we have $\omega(\Phi x) > \delta_2$.

Conclusion.

Thus, by Lemma 2.6, problem (6) has at least three positive solutions x_1^*, x_2^*, and x_3^* such that $\|x_1^*\|_X \leq \delta_1$, $\omega(x_2^*) \geq \delta_2$, $\|x_3^*\|_X \geq \delta_1$, and $\omega(x_3^*) < \delta_2$. □

4 Examples

Example 4.1 We consider the problem

$$\begin{cases}
D^{\frac{3}{2}} x(t) + \frac{1}{(1+t)^2} r(t, x(t)) = 0, \quad t \in [0, \infty), \\
x(0) = 0, \\
D^{\frac{1}{2}} x(\infty) = \sum_{i=1}^2 \nu_i I^{\beta_i} x(\varrho) + \lambda\kappa x(\varsigma^{\frac{1}{2}}),
\end{cases} \tag{15}$$

where

$$\delta = \frac{3}{2}, \qquad m = 2, \qquad q = 2, \qquad p = 1, \qquad \nu_1 = 1, \qquad \nu_2 = \frac{1}{2}, \qquad \Lambda = 1,$$
$$\beta_1 = \frac{1}{2}, \qquad \beta_2 = \frac{1}{4}, \qquad \varrho = \frac{1}{2}, \qquad \lambda = \frac{1}{100}, \qquad \kappa = 1, \qquad \varsigma = \frac{1}{3},$$

and

$$r\bigl(t, x(t)\bigr) = \frac{x(t)}{10(1+t^{\frac{1}{2}})},$$
$$b(t) = \frac{1}{(1+t)^2}.$$

We can show that

$$\Delta = \Gamma\left(\frac{3}{2}\right) - \sum_{i=1}^{2} \nu_i \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2}+\beta_i)} \varrho^{\frac{3}{2}+\beta_i-1} - \frac{1}{100}\frac{1}{3}^{\frac{1}{2}} \approx 0.2219 > 0.$$

When $x \in [0,1]$, we have

$$r\bigl(t, \bigl(1+t^{\frac{1}{2}}\bigr)x\bigr) \le \frac{1}{10} < \frac{1}{\frac{1}{\Delta}\int_0^\infty c(s)\,ds} \approx 0.2219 < 1.$$

Thus, we have that (H_1)–(H_4) hold. We can obtain that problem (15) has two positive solutions x^*, y^* in $(0, t^{\alpha-1}]$ by Theorem 3.4, which can be approximated by the iterative sequences

$$\begin{cases} x_{n+1} = \int_0^\infty \Pi(t,s)b(s)r(s,x_n(s))\,ds, & x_0(t) = 0, \\ y_{n+1} = \int_0^\infty \Pi(t,s)b(s)r(s,y_n(s))\,ds, & y_0(t) = t^{\delta-1}, t \in I. \end{cases}$$

Example 4.2 We consider the problem

$$\begin{cases} D^{\frac{5}{2}}x(t) + e^{-t}r(t,x(t)) = 0, & t \in [0,\infty), \\ x(0) = x'(0) = 0, \\ D^{\frac{3}{2}}x(\infty) = \sum_{i=1}^{2} \nu_i I^{\beta_i}x(\varrho) + \lambda\kappa x(\varsigma^{\frac{3}{2}}), \end{cases} \tag{16}$$

where

$$\delta = \frac{5}{2}, \quad m = 3, \quad q = 2, \quad p = 1, \quad \nu_1 = \frac{1}{3}, \quad \nu_2 = \frac{1}{4},$$
$$\beta_1 = \frac{1}{2}, \quad \beta_2 = \frac{1}{4}, \quad \varrho = \frac{1}{4}, \quad k = 2, \quad \lambda = \frac{1}{20}, \quad \kappa = \frac{1}{50}, \quad \varsigma = 1,$$

and

$$r(t,x) = \begin{cases} \frac{x}{(1+t^{\frac{3}{2}})(1+t)}, & x < 1, \\ \frac{t}{e^t} + \frac{x}{1+t^{\frac{3}{2}}} + 15, & x \ge 1, \end{cases}$$
$$b(t) = e^{-t}.$$

We have

$$M_1 = \left(\frac{1}{\Delta}\int_0^\infty b(s)\,ds\right)^{-1} \approx 1.281 > 1,$$
$$M_2 = \left(\frac{k^{1-\alpha}}{1+k^{\alpha-1}}\int_{\frac{1}{k}}^{k} b(s)\,ds\right)^{-1} \approx 11.7055.$$

Choosing $\delta_1 = \frac{1}{2}$, $\delta_2 = 1.01$, $\delta_3 = 101$, $\theta = \frac{1}{10}$, we have

$$r\bigl(t, \bigl(1+t^{\frac{3}{2}}\bigr)x\bigr) \le \frac{1}{2} \le \delta_1 M_1 \approx 0.6405, \quad (t,x) \in [0,\infty) \times \left[0, \frac{1}{2}\right],$$

$$r\big(t,\big(1+t^{\frac{3}{2}}\big)x\big) \geq 16 \geq \delta_2 M_2 \approx 11.822555, \quad (t,x) \in \left[\frac{1}{2},2\right] \times [1.01,101],$$

$$r\big(t,\big(1+t^{\frac{3}{2}}\big)x\big) \leq 113.369 \leq \delta_3 M_1 \approx 129.381, \quad (t,x) \in [0,\infty) \times [0,101].$$

From Theorem 3.5, the BVP (16) has at least three positive solutions x_1^*, x_2^*, and x_3^* such that $\|x_1^*\|_X \leq \frac{1}{2}$, $\omega(x_2^*) \geq 1.01$, $\|x_3^*\|_X \geq \frac{1}{2}$, and $\omega(x_3^*) < 1.01$.

Acknowledgements
The authors would like to thank the referee for pertinent comments and valuable suggestions.

Funding
This research is supported by NSFC (11871302), the Fund of the Natural Science of Shandong Province (ZR2014AM034), and colleges and universities of Shandong province science and technology plan projects (J13LI01).

Availability of data and materials
Date sharing not applicable to this paper as no datasets were generated or analyzed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

Author contributions
All authors have equal contributions. All authors read and approved the final manuscript.

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Received: 29 October 2023 Accepted: 23 January 2024 Published online: 08 February 2024

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