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Existence and stability of a q -Caputo fractional jerk differential equation having anti-periodic boundary conditions

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Abstract

In this work, we analyze a q -fractional jerk problem having anti-periodic boundary conditions. The focus is on investigating whether a unique solution exists and remains stable under specific conditions. To prove the uniqueness of the solution, we employ a Banach fixed point theorem and a mathematical tool for establishing the presence of distinct fixed points. To demonstrate the availability of a solution, we utilize Leray–Schauder's alternative, a method commonly employed in mathematical analysis. Furthermore, we examine and introduce different kinds of stability concepts for the given problem. In conclusion, we present several examples to illustrate and validate the outcomes of our study.

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1 Introduction

Recently, a lot of researchers have shown a great interest in the field of q -calculus (\mathcal{QC}) and problems involving fractional q -differential equations (q -DDEs). The roots of \mathcal{QC} can be traced back to 1908 with the work of Jackson in [1]. Additionally, q -DDEs were developed to characterize the variety of physical processes that emerged, such as discrete stochastic processes, discrete dynamical systems, quantum dynamics, and so on [2]. As the theory of \mathcal{QC} progressed, some associated ideas have been presented and examined, including q -integral transform theory, q -Mittag-Leffler functions, q -gamma, q -beta functions, q -Laplace transform, and so forth (for more details, see [3–9]). These concepts find applications in understanding and solving problems related to \mathcal{QC} . The reader may refer to [10–17] for more details on \mathcal{QC} .

In 1978, Schot [18] introduced the concept of “jerk” \mathcal{J} , which is essentially the rate at which acceleration changes. It involves the third derivative of quantity represented by u . The idea of \mathcal{J} has proven in several scientific fields, including acoustics, electrical circuits, mechanics, and dynamical processes. It also helps us to understand how acceleration is changing over time, providing valuable insights into the behavior of systems in various

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applications [19–25]. In three dimensions, a dynamic system can be represented as

$$v'(\chi) = a, \quad a'(\chi) = e, \quad e'(\chi) = f(v, a, e),$$

and can be well written in the form of $v''' = f(v, v', v'')$. The JIE is third order autonomous IDE that has found applications in various scientific fields, such as signal processing, secure communication, electrical engineering, control systems, bio-mechanics, and economic systems [17, 22]. Marcelo and Silva [26] employed the algebraic techniques in 2020 to ascertain the exact structure for a polynomial \mathcal{J} function, hence guaranteeing the non-chaotic behavior of the subsequent JIE:

$$v''' = \mathcal{J}(v, v', v'').$$

They also provided the proof for nonchaotic behavior. It can also be useful to investigate the different kinds of ordinary IDEs and their nonchaotic behavior. The authors in [27] addressed an initial value problem of nonlinear 3rd order JIE:

$$\begin{cases} v''' + f(v, v', v'') = 0, \\ v(0) = 0, \quad v'(0) = \mathcal{B}, \quad v''(0) = 0. \end{cases}$$

By employing analytical methodologies, the authors were able to enhance the method known as the global error minimization method GEMM to generate estimations using analytical techniques. Their developed approaches were known to be more successful and efficient than previously known current methods when compared to known solutions and accurate numerical ones. The authors in [28] utilized the modified harmonic balance technique for the subsequent nonlinear JIE:

$$\mathbb{D}^3 v(\chi) + \xi(v(\chi), \mathbb{D}^1 v(\chi), {}^C\mathbb{D}^2 v(\chi)) = 0,$$

under conditions $v(0) = 0$, $\mathbb{D}^1 v(0) = \mathcal{B}$, and $\mathbb{D}^2 v(0) = 0$. Sousa *et al.*, by employing fixed point approach, studied stability of the modified impulsive fractional IDEs

$$\begin{cases} {}^H\mathbb{D}_{0+}^{\alpha, \beta, \psi} v(\chi) = \xi(\chi, v(\chi)), \quad \chi \in (s_i, t_{i+1}), i = 0, 1, \dots, m, \\ v(\chi) = \tau_i(\chi, v(t_i^+)), \quad \chi \in (t_i, s_i], i = 1, 2, \dots, m, \end{cases}$$

where ${}^H\mathbb{D}_{0+}^{\alpha, \beta, \psi}(\cdot)$ is the ψ -Hilfer fractional derivative with $\alpha \in (0, 1]$, $\beta \in [0, 1]$, and

$$0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \dots < t_m \leq s_m < t_{m+1} = T$$

are prefixed numbers, $\xi \in C(\Lambda \times \mathbb{R})$ and $\tau_i \in C([t_i, s_i] \times \mathbb{R})$ for all $i = 1, 2, \dots, m$, which are noninstantaneous impulses, here $\Lambda := [0, T]$ with $T > 0$ [29]. Wang *et al.* in [30] studied the various forms of Ulam stability (\mathcal{US}) and existence, uniqueness (\mathcal{EU}) for the following nonlinear implicit fractional integro-differential equations involving Caputo derivative (${}^C\mathcal{D}$) of fractional order:

$$\begin{cases} {}^C\mathbb{D}^\alpha v(\chi) = \xi(\chi, v(\chi), {}^C\mathbb{D}^\alpha v(\chi)) + \int_0^\chi \frac{(\chi-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, v(s), {}^C\mathbb{D}^\alpha v(s)) ds, \quad \chi \in \Lambda, \\ v(\chi)|_{\chi=0} = -v(\chi)|_{\chi=T}, \quad {}^C\mathbb{D}^\beta v(\chi)|_{\chi=0} = -{}^C\mathbb{D}^\beta v(\chi)|_{\chi=T}, \end{cases}$$

where $\nu, \zeta > 0$, $1 < \alpha \leq 2$, $0 \leq \beta \leq 2$ and continuous functions are represented as $\xi, g : \Lambda \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. The authors introduced the ψ -Hilfer pseudo-fractional operator, motivated by the ψ -Hilfer fractional derivative and the theory of pseudo-analysis, and investigated a new class of important and essential results for pseudo-fractional calculus in a semi-ring $([a, b], \oplus, \odot)$, and some particular cases were discussed (for more instances, see related research works [31–37]). Houas *et al.*, by using Riemann–Liouville (RL) and q -fractional \mathcal{CD} , examined the $\mathfrak{E}\mathfrak{U}$, Ulam–Hyers ($\mathcal{U}\mathcal{H}$), and Ulam–Hyers–Rassias ($\mathcal{U}\mathcal{H}\mathcal{R}$) stability of the solution to q -fractional problem (FJP) as follows:

$$\begin{cases} {}^{\mathcal{RL}}\mathbb{D}_q^\alpha({}^{\mathcal{C}}\mathbb{D}_q^\omega({}^{\mathcal{C}}\mathbb{D}_q^\theta v(\chi))) = \xi(\chi, v(\chi), {}^{\mathcal{C}}\mathbb{D}_q^\theta v(\chi), ({}^{\mathcal{C}}\mathbb{D}_q^\omega({}^{\mathcal{C}}\mathbb{D}_q^\theta v(\chi)))), & \chi \in \Lambda, \\ v(0) - {}^{\mathcal{I}_q^\beta}v(T) = 0, \quad {}^{\mathcal{C}}\mathbb{D}_q^\theta v(\delta) = 0, \quad {}^{\mathcal{C}}\mathbb{D}_q^\omega({}^{\mathcal{C}}\mathbb{D}_q^\theta v(T)) = 0, \end{cases}$$

where $\chi \in \Lambda$, $\{\alpha, \omega, \theta\} \in (0, 1]$, $\beta \geq 1$, $0 < \delta < T$, ${}^{\mathcal{RL}}\mathbb{D}_q^\alpha$, ${}^{\mathcal{C}}\mathbb{D}_q^\mu$, $\mu \in \{\omega, \theta\}$ are the q -fractional RL and \mathcal{CD} s respectively [38]. The q -FI is ${}^{\mathcal{I}_q^\beta}$ having RL type and $\xi : \Lambda \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is given an appropriate function [38].

Influenced by the aforementioned works, we present the following q -Caputo fractional JDEs with anti-periodic boundary conditions (ABCs):

$$\begin{cases} {}^{\mathcal{C}}\mathbb{D}_q^\alpha({}^{\mathcal{C}}\mathbb{D}_q^\omega({}^{\mathcal{C}}\mathbb{D}_q^\theta v(\chi))) \\ = \xi(\chi, v(\chi), {}^{\mathcal{C}}\mathbb{D}_q^\theta v(\chi), ({}^{\mathcal{C}}\mathbb{D}_q^\omega({}^{\mathcal{C}}\mathbb{D}_q^\theta v(\chi)))) \\ + \int_0^\chi \frac{(x-qs)^{\nu-1}}{\Gamma_q(\zeta)} g(s, v(s), {}^{\mathcal{C}}\mathbb{D}_q^\theta v(s), ({}^{\mathcal{C}}\mathbb{D}_q^\omega({}^{\mathcal{C}}\mathbb{D}_q^\theta v(s)))) d_qs, & \chi \in \Lambda, \\ v(\chi)|_{\chi=0} = -v(\chi)|_{\chi=T}, \quad ({}^{\mathcal{C}}\mathbb{D}_q^\omega({}^{\mathcal{C}}\mathbb{D}_q^\theta v(\chi)))|_{\chi=\delta} = 0, \\ {}^{\mathcal{C}}\mathbb{D}_q^\beta v(\chi)|_{\chi=0} = -{}^{\mathcal{C}}\mathbb{D}_q^\beta v(\chi)|_{\chi=T}, \end{cases} \quad (1)$$

where $0 < \{\alpha, \omega, \theta\} \leq 1$, $\beta \in (0, 1]$, $0 < \delta < T$, q -fractional \mathcal{CD} is ${}^{\mathcal{C}}\mathbb{D}_q^\mu$, $\mu \in \{\alpha, \omega, \theta, \beta\}$ of order μ on Λ , $\xi, g : \Lambda \times \mathbb{R}^3 \rightarrow \mathbb{R}$ are appropriate functions and $\nu, \zeta > 0$.

We list the important points of this manuscript:

- 1: We implement Caputo q -fractional JDE having ABCs for the first time in the literature.
- 2: In this manuscript, we established the $\mathfrak{E}\mathfrak{U}$ and $\mathcal{U}\mathcal{S}$ results for the suggested Problem (1).
- 3: Different from previous papers that used nonlinear implicit fractional integrodifferential equations in [30] and RL and q -fractional \mathcal{CD} [38], we get better results by employing q -fractional JDE having ABCs.
- 4: We also show the graphical representation of JDE having ABCs.

This research article is organized in the following manner: Sect. 2 clarifies some basic ideas in \mathcal{QC} and provides related lemmas. In Sect. 3, we establish the $\mathfrak{E}\mathfrak{U}$ of solution for the proposed system (1) by employing the Leray–Schauder alternative and the Banach fixed point theorem. Various types of $\mathcal{U}\mathcal{S}$ have been discussed in Sect. 4. In Sect. 5 an example is also presented at the end to verify our results. Finally, conclusion is also provided in Sect. 6.

2 Basic concepts

The following Banach space $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ is needed to analyze the q -FJP:

$$\mathcal{F} = \{v : v, {}^{\mathcal{C}}\mathbb{D}_q^\theta v, ({}^{\mathcal{C}}\mathbb{D}_q^\omega({}^{\mathcal{C}}\mathbb{D}_q^\theta v)) \in \mathcal{C}(\Lambda, \mathbb{R})\},$$

supplied with the norm

$$\begin{aligned}\|v\|_{\mathcal{F}} &= \|v\| + \|\mathcal{D}_q^\theta v\| + \|\mathcal{D}_q^\omega(\mathcal{D}_q^\theta v)\| \\ &= \sup_{\chi \in \Lambda} |v(\chi)| + \sup_{\chi \in \Lambda} |\mathcal{D}_q^\theta v(\chi)| + \sup_{\chi \in \Lambda} |(\mathcal{D}_q^\omega(\mathcal{D}_q^\theta v))(\chi)|.\end{aligned}$$

The fractional \mathcal{QC} is examined on $\mathfrak{T}_{\chi_0} = \{0\} \cup \{\chi : \chi = \chi_0 q^N\}$ for $N \in N_0$, $\chi_0 \in \mathbb{R}$ and $0 < q < 1$ in [39]. We shall denote \mathcal{T}_{χ_0} by \mathcal{T} . Let $\mu \in \mathbb{R}$. Define $\lceil \mu \rceil_q = \frac{1-q^\mu}{1-q}$ in [40].

Definition 2.1 ([39]) The $(\chi - s)_q^N$ is a q -factorial function. The expression $N \in N_0$ is given by

$$(\chi - s)_q^N = \prod_{l=0}^{N-1} (\chi - sq^l), \quad (\chi, s \in \mathbb{R}), \quad (2)$$

and $(\chi - s)_q^{(0)} = 1$, where $N_0 := \{0, 1, 2, \dots\}$. Also, for $\mu \in \mathbb{R}$, we obtain

$$(\chi - s)_q^{(\mu)} = \chi^\mu \prod_{l=0}^{\infty} \frac{\chi - sq^l}{\chi - sq^{\mu+l}}. \quad (3)$$

Algorithm 1 is useful in this regard [41]. The q -gamma function is defined by $\Gamma_q(\mu) = (1 - q)^{(\mu-1)} / (1 - q)^{\mu-1}$, where $\mu \in \mathbb{R} \setminus (-\infty, 0]$ and satisfies $\Gamma_q(\mu+1) = \lceil \mu \rceil_q \Gamma_q(\mu)$ s.t. $\lceil \mu \rceil_q = (1 - q^\mu)(1 - q)^{-1}$ [39]. Algorithm 2, written using MATLAB commands, calculates q -gamma well [41].

Definition 2.2 ([42]) The q -derivative of a function $v : \mathcal{T} \rightarrow \mathbb{R}$ is expressed by

$$\mathbb{D}_q v(\chi) = \left(\frac{d}{d\chi} \right)_q v(\chi) = \frac{v(\chi) - v(q\chi)}{(1 - q\chi)}, \quad \forall \chi \in \mathcal{T} \setminus \{0\}, \quad (4)$$

and $\mathbb{D}_q v(0) = \lim_{\chi \rightarrow 0} \mathbb{D}_q v(\chi)$. Also the higher q -derivative of function v is defined by $\mathbb{D}_q^n v(\chi) = \mathbb{D}_q[\mathbb{D}_q^{n-1} v(\chi)]$, $\forall n \geq 1$, here $\mathbb{D}_{\chi q}^0 v(\chi) = v(\chi)$.

Definition 2.3 ([42]) The q -integral of the function v is expressed by

$$\mathcal{I}_q v(\chi) = \int_0^\chi v(s) d_qs = \chi(1 - q) \sum_{l=0}^{\infty} q^l v(\chi q^l), \quad 0 \leq \chi \leq b, \quad (5)$$

provided the series absolutely converges. If $\chi_1 \in [0, r]$, then

$$\int_{\chi_1}^r v(s) d_qs = \mathcal{I}_q v(r) - \mathcal{I}_q v(\chi_1) = (1 - q) \sum_{l=0}^{\infty} q^l [r - v(rq^l) - \chi_1 v(\chi_1 q^l)], \quad (6)$$

whenever the series exists (see Algorithm 3 and [41]). The operator \mathcal{I}_q^n is given as $\mathcal{I}_q^0 v(\chi) = v(\chi)$ and $\mathcal{I}_q^n v(\chi) = \mathcal{I}_q[\mathcal{I}_q^{n-1} v(\chi)]$ for $n \geq 1$ and $v \in \mathcal{C}([0, r])$.

It has been verified that $\mathbb{D}_q[\mathcal{I}_q v(\chi)] = v(\chi)$ and $\mathcal{I}_q[\mathbb{D}_q v(\chi)] = v(\chi) - v(0)$ whenever the function v is continuous at $\chi = 0$ in [42]. The fractional RLL type q -integral of the function v is given by

$$\mathcal{I}_q^\mu v(\chi) = \int_0^\chi \frac{(\chi - qs)^{\mu-1}}{\Gamma_q(\mu)} v(s) d_qs, \quad \chi > 0, \mu > 0,$$

$$\mathcal{I}_q^0 v(\chi) = v(\chi) \text{ [43].}$$

Definition 2.4 ([43]) The operator ${}^C\mathbb{D}_q^\mu$ is the fractional q -CD of order μ given by

$${}^C\mathbb{D}_q^\mu v(\chi) = \mathcal{I}_q^{[\mu]-\mu} \mathbb{D}_q^{[\mu]} v(\chi), \quad \mu > 0,$$

and ${}^C\mathbb{D}_q^0 v(\chi) = v(\chi)$ where $[\mu]$ is the smallest integer greater than μ .

Lemma 2.5 ([28]) Let $\mu, \sigma \geq 0$ and v be a function defined in Λ . Then (i) $\mathcal{I}_q^\mu [\mathcal{I}_q^\sigma v(\chi)] = \mathcal{I}_q^{\mu+\sigma} v(\chi)$; (ii) ${}^C\mathbb{D}_q^\mu [\mathcal{I}_q^\mu v(\chi)] = v(\chi)$; (iii) ${}^C\mathbb{D}_q^\mu [\mathcal{I}_q^\sigma v(\chi)] = \mathcal{I}_q^{\sigma-\mu} v(\chi)$.

Lemma 2.6 ([43]) Let $\mu \in \mathbb{R}^+ \setminus \mathbb{N}$. Then the following equality

$$\mathcal{I}_q^\mu {}^C\mathbb{D}_q^\mu v(\chi) = v(\chi) - \sum_{k=0}^{n-1} \frac{\chi^k}{\Gamma_q(k+1)} {}^C\mathcal{D}_q^k v(0)$$

is satisfied, and n is the smallest integer greater than or equal to μ . Equivalently, we can also write it as $n = [\mu] + 1$, $n-1 < \mu \leq n$.

Lemma 2.7 ([43]) (a) For $\mu \in \mathbb{R}_+$ and $\sigma > -1$, we obtain

$$\mathcal{I}_q^\mu [\chi^{(\sigma)}] = \frac{\Gamma_q(\sigma+1)}{\Gamma_q(\mu+\sigma+1)} \chi^{(\mu+\sigma)}.$$

If $\sigma = 0$, we obtain $\mathcal{I}_q^\mu [1] = \frac{1}{\Gamma_q(\mu+1)} \chi^{(\mu)}$. (b) Similarly, for derivative, $\sigma > -1$, we get

$${}^C\mathbb{D}_q^\mu [\chi^{(\sigma)}] = \frac{\Gamma_q(\sigma+1)}{\Gamma_q(\sigma-\mu+1)} \chi^{(\sigma-\mu)}.$$

If $\sigma = 0$, we obtain ${}^C\mathbb{D}_q^\mu [1] = 0$.

We also point out formulas in [14], which will be used in our results.

$$\begin{aligned} [a(\chi-s)]^{(\alpha)} &= a^\alpha (\chi-s)^\alpha, \\ {}_x\mathbb{D}_q(\chi-s)^\alpha &= [\alpha]_q (-s)^{(\alpha-1)}, \\ {}_s\mathbb{D}_q(-s)^\alpha &= -[\alpha]_q (\chi-qs)^{(\alpha-1)}. \end{aligned}$$

Lemma 2.8 (Leray-Schauder alternative [44]) Let $\rho : \mathcal{F} \rightarrow \mathcal{F}$ be a completely continuous operator (i.e., a map restricted to any bounded set in \mathcal{F} is compact). Let

$$\Phi(\rho) = \{v \in \mathcal{F} : v = \pi\rho(v) \text{ for some } 0 < \pi < 1\}. \quad (7)$$

Then the set $\Phi(\rho)$ is unbounded, or ρ has at least one fixed point.

Lemma 2.9 (Banach fixed point theorem [45]) *Let \mathcal{F} be a Banach space and mapping $\rho : \mathcal{F} \rightarrow \mathcal{F}$ be a contraction on \mathcal{F} . Hence ρ has a unique fixed point.*

We now examine the $\mathcal{U}\mathcal{S}$ for the q -FJIP (1), as discussed in [46]. For $\bar{x} > 0$ and $h : \Lambda \rightarrow \mathbb{R}_+$, we get

$$|{}^C\mathbb{D}_q^\alpha({}^C\mathbb{D}_q^\omega({}^C\mathbb{D}_q^\theta v(\chi))) - \Theta_{v,\omega,\theta}^*(\chi)| \leq \bar{x} \quad (8)$$

and

$$|{}^C\mathbb{D}_q^\alpha({}^C\mathbb{D}_q^\omega({}^C\mathbb{D}_q^\theta v(\chi))) - \Theta_{v,\omega,\theta}^*(\chi)| \leq \bar{x}h(\chi) \quad (9)$$

for $\chi \in \Lambda$, where

$$\begin{aligned} \Theta_{v,\omega,\theta}^*(\chi) = & \xi(\chi, v(\chi), {}^C\mathbb{D}_q^\theta v(\chi), ({}^C\mathbb{D}_q^\omega({}^C\mathbb{D}_q^\theta v(\chi)))) \\ & + \int_0^\chi \frac{(\chi - qs)^{\nu-1}}{\Gamma_q(\zeta)} g(s, v(s), {}^C\mathbb{D}_q^\theta v(s), ({}^C\mathbb{D}_q^\omega({}^C\mathbb{D}_q^\theta v(s)))) d_qs. \end{aligned}$$

Definition 2.10 ([46]) The q -FJIP (1) demonstrates the stability as:

- 1: In $\mathcal{U}\mathcal{H}$ sense, if there is a positive real number $\mathbb{E}_{\Theta_{\omega,\theta}^*} > 0$ such that there is a solution b of the q -FJIP (1) for each $\bar{x} > 0$ and for each solution v of inequality (8) having

$$\|\chi - p\|_{\mathcal{F}} \leq \mathbb{E}_{\Theta_{\omega,\theta}^*} \bar{x};$$

- 2: In $\mathcal{U}\mathcal{H}\mathcal{R}$ sense, concerning $h \in \mathcal{C}(\Lambda, \mathbb{R}_+)$, if there is a real number $\mathbb{E}_{\Theta_{\omega,\theta}^*, h} > 0$ such that for each $\bar{x} > 0$ and for each solution v of inequality (9) there \exists a solution \hat{v} of q -FJIP (1) with

$$\|v - \hat{v}\|_{\mathcal{F}} \leq \mathbb{E}_{\Theta_{\omega,\theta}^*, h} \bar{x}h(\chi).$$

Remark 2.1 A function $v \in \mathcal{F}$ is considered a solution of inequality (8) iff \exists another function $\varrho : \Lambda \rightarrow \mathbb{R}$ (which relies on v) s.t. $|\varrho(\chi)| \leq \bar{x}$ for every $\chi \in \Lambda$ and

$$\|{}^C\mathbb{D}_q^\alpha({}^C\mathbb{D}_q^\omega({}^C\mathbb{D}_q^\theta v(\chi))) - \Theta_{v,\omega,\theta}^*(\chi)\| \leq \|\varrho(\chi)\|, \quad \chi \in \Lambda.$$

3 Existence and uniqueness results

In this section, we investigate the 1EU of solution of problem (1).

Lemma 3.1 Consider $\phi \in \mathcal{C}(\Lambda)$. Thus, the solution of problem

$$\begin{cases} {}^C\mathbb{D}_q^\alpha({}^C\mathbb{D}_q^\omega({}^C\mathbb{D}_q^\theta v(\chi))) = \phi(\chi), & \chi \in \Lambda = [0, \mathbb{T}], \\ v(\chi)|_{\chi=0} = -v(\chi)|_{\chi=\mathbb{T}}, \quad ({}^C\mathbb{D}_q^\omega({}^C\mathbb{D}_q^\theta v(\chi)))|_{\chi=\delta} = 0, \\ {}^C\mathbb{D}_q^\beta v(\chi)|_{\chi=0} = -{}^C\mathbb{D}_q^\beta v(\chi)|_{\chi=\mathbb{T}} \end{cases} \quad (10)$$

for $0 < \max\{\alpha, \omega, \theta\} \leq 1$, $0 < \delta < \mathbb{T}$ is given as

$$v(\chi) = \int_0^\chi \frac{(\chi - qs)^{\alpha+\omega+\theta-1}}{\Gamma_q(\alpha + \omega + \theta)} \phi(s) d_qs - \frac{1}{2} \int_0^{\mathbb{T}} \frac{(\mathbb{T} - qs)^{\alpha+\omega+\theta-1}}{\Gamma_q(\alpha + \omega + \theta)} \phi(s) d_qs$$

$$\begin{aligned}
& + \int_0^T \frac{(\mathbb{T} - qs)^{\alpha+\omega+\theta-\beta-1}}{\Gamma_q(\alpha+\omega+\theta-\beta)} \left(\frac{\mathbb{T}^\theta}{2\Delta\Gamma_q(\theta+1)} - \frac{\chi^\theta}{\Delta\Gamma_q(\theta+1)} \right) \phi(s) d_qs \\
& + \int_0^\delta \frac{(\delta - qs)^{\alpha-1}}{\Gamma_q(\alpha)} \left[\frac{-\chi^{\omega+\theta}}{\Gamma_{\chi q}(\omega+\theta+1)} + \frac{\chi^\theta T^{\omega+\theta-\beta}}{\Delta\Gamma_q(\theta+1)\Gamma_q(\omega+\theta-\beta+1)} \right. \\
& \left. + \frac{\mathbb{T}^{\omega+\theta}}{2\Gamma_q(\omega+\theta+1)} - \frac{\mathbb{T}^{\omega+2\theta-\beta}}{2\Delta\Gamma_q(\theta+1)\Gamma_q(\omega+\theta-\beta+1)} \right] \phi(s) d_qs, \tag{11}
\end{aligned}$$

where $\phi \in \mathcal{F}$ is given as

$$\begin{aligned}
\phi(\chi) = & \xi(\chi, v(\chi), {}^C\mathbb{D}_q^\theta v(\chi), ({}^C\mathbb{D}_q^\omega({}^C\mathbb{D}_q^\theta v(\chi)))) \\
& + \int_0^\chi \frac{(\chi - qs)^{\nu-1}}{\Gamma_q(\nu)} g(s, v(s), {}^C\mathbb{D}_q^\theta v(s), ({}^C\mathbb{D}_q^\omega({}^C\mathbb{D}_q^\theta v(s)))) d_qs,
\end{aligned}$$

$$\text{and } \Delta = 1 + \frac{\mathbb{T}^{\theta-\beta}}{\Gamma_q(\theta-\beta+1)}.$$

Proof Now, let us consider

$${}^C\mathbb{D}_q^\alpha({}^C\mathbb{D}_q^\omega({}^C\mathbb{D}_q^\theta v(\chi))) = \phi(\chi), \quad \chi \in \Lambda. \tag{12}$$

Applying the operator \mathcal{I}_q^α on both sides of (12) and employing Lemma 2.6 with $n = 1$, we obtain

$${}^C\mathbb{D}_q^\omega({}^C\mathbb{D}_q^\theta v(\chi)) = \mathcal{I}_q^\alpha \phi(\chi) + c_0, \quad c_0 \in \mathbb{R}. \tag{13}$$

Now, using the operator \mathcal{I}_q^ω , (1) of Lemma 2.5, (a) of Lemma 2.7, and applying the same procedure on both sides of (13), we get

$$({}^C\mathbb{D}_q^\theta v(\chi)) = \mathcal{I}_q^{\alpha+\omega} \phi(\chi) + c_0 \frac{\chi^\omega}{\Gamma_q(\omega+1)} + c_1, \quad c_j \in \mathbb{R}, j = 0, 1. \tag{14}$$

It follows that

$$v(\chi) = \mathcal{I}_q^{\alpha+\omega+\theta} \phi(\chi) + c_0 \frac{\chi^{\omega+\theta}}{\Gamma_{\chi q}(\omega+\theta+1)} + c_1 \frac{\chi^\theta}{\Gamma_q(\theta+1)} + c_2, \tag{15}$$

where $c_j \in \mathbb{R}$, ($j = 0, 1, 2$). Using boundary constraints

$$v(\chi)|_{\chi=0} = -v(\chi)|_{\chi=T}. \tag{16}$$

Now, using the L.H.S of (16) in (15), we obtain

$$v(\chi)|_{\chi=0} = \mathcal{I}_q^{\alpha+\omega+\theta} \phi(\chi) + c_0 \frac{\chi^{\omega+\theta}}{\Gamma_q(\omega+\theta+1)} + c_1 \frac{\chi^\theta}{\Gamma_q(\theta+1)} + c_2,$$

$$v(\chi)|_{\chi=0} = c_2.$$

Similarly, using the R.H.S of (16) in (15), we obtain

$$-v(\chi)|_{\chi=T} = -\mathcal{I}_q^{\alpha+\omega+\theta} \phi(T) - c_0 \frac{\mathbb{T}^{\omega+\theta}}{\Gamma_q(\omega+\theta+1)} - c_1 \frac{\mathbb{T}^\theta}{\Gamma_q(\theta+1)} - c_2.$$

Thus (16) becomes

$$c_2 = -\frac{1}{2} \left[\mathcal{I}_q^{\alpha+\omega+\theta} \phi(\mathbb{T}) + c_0 \frac{\mathbb{T}^{\omega+\theta}}{\Gamma_q(\omega+\theta+1)} + c_1 \frac{\mathbb{T}^\theta}{\Gamma_q(\theta+1)} \right].$$

By the 2nd boundary condition,

$$(\mathcal{C}\mathbb{D}_q^\omega (\mathcal{C}\mathbb{D}_q^\theta v(\chi)))|_{\chi=\delta} = 0. \quad (17)$$

Applying $\mathcal{C}\mathbb{D}_q^\theta$, (3) of Lemma 2.5 and (b) of Lemma 2.7 on both sides of (15), we get

$$\mathcal{C}\mathbb{D}_q^\theta v(\chi) = \mathcal{I}_q^{\alpha+\omega} \phi(\chi) + c_0 \frac{\chi^\omega}{\Gamma_q(\omega+1)} + c_1. \quad (18)$$

Now, applying $\mathcal{C}\mathbb{D}_q^\omega$ and the same procedure on both sides of (18), we get

$$\mathcal{C}\mathbb{D}_q^\omega (\mathcal{C}\mathbb{D}_q^\theta v(\chi)) = \mathcal{I}_q^\alpha \phi(\chi) + c_0. \quad (19)$$

So, Eq. (19) becomes $\mathcal{C}\mathbb{D}_q^\omega (\mathcal{C}\mathbb{D}_q^\theta v(\chi))|_{\chi=\delta} = \mathcal{I}_q^\alpha \phi(\delta) + c_0$. By Eq. (17), we get $c_0 = -\mathcal{I}_q^\alpha \phi(\delta)$. Using the 3rd boundary condition,

$$\mathcal{C}\mathbb{D}_q^\beta (\chi)|_{\chi=0} = -\mathcal{C}\mathbb{D}_q^\beta v(\chi)|_{\chi=\mathbb{T}}. \quad (20)$$

Now, using the L.H.S of (20) in (15), we get

$$\mathcal{C}\mathbb{D}_q^\beta v(\chi) = \mathcal{I}_q^{\alpha+\omega+\theta-\beta} \phi(\chi) + c_0 \frac{\chi^{\omega+\theta-\beta}}{\Gamma_q(\omega+\theta-\beta+1)} + c_1 \frac{\chi^{\theta-\beta}}{\Gamma_q(\theta-\beta+1)}.$$

So, at $\mathcal{C}\mathbb{D}_q^\beta v(\chi)|_{\chi=0} = c_1$, since $\theta-\beta \leq 0$ by Eq. (2). Now, using the R.H.S of (20) in (15), we have

$$-\mathcal{C}\mathbb{D}_q^\beta v(\chi)|_{\chi=\mathbb{T}} = -\mathcal{I}_q^{\alpha+\omega+\theta-\beta} \phi(\mathbb{T}) - c_0 \frac{\mathbb{T}^{\omega+\theta-\beta}}{\Gamma_q(\omega+\theta-\beta+1)} - c_1 \frac{\mathbb{T}^{\theta-\beta}}{\Gamma_q(\theta-\beta+1)}.$$

So, (20) becomes

$$c_1 = \frac{1}{\Delta} \left[-\mathcal{I}_q^{\alpha+\omega+\theta-\beta} \phi(\mathbb{T}) - c_0 \frac{\mathbb{T}^{\omega+\theta-\beta}}{\Gamma_q(\omega+\theta-\beta+1)} \right].$$

Putting all values in (15), we obtain

$$\begin{aligned} v(\chi) &= \mathcal{I}_q^{\alpha+\omega+\theta} \phi(\chi) - \mathcal{I}_q^\alpha \phi(\delta) \frac{\chi^{\omega+\theta}}{\Gamma_q(\omega+\theta+1)} \\ &\quad + \frac{\chi^\theta}{\Delta \Gamma_q(\theta+1)} \left[-\mathcal{I}_q^{\alpha+\omega+\theta-\beta} \phi(\mathbb{T}) + \mathcal{I}_q^\alpha(\delta) \frac{\mathbb{T}^{\omega+\theta-\beta}}{\Gamma_q(\omega+\theta-\beta+1)} \right] \\ &\quad + \frac{\mathbb{T}^\theta}{\Delta \Gamma_q(\theta+1)} \left(-\mathcal{I}_q^{\alpha+\omega+\theta-\beta} \phi(\mathbb{T}) + \mathcal{I}_q^\alpha(\delta) \frac{\mathbb{T}^{\omega+\theta-\beta}}{\Gamma_q(\omega+\theta-\beta+1)} \right) \\ &= \mathcal{I}_q^{\alpha+\omega+\theta} \phi(\chi) - \frac{1}{2} \mathcal{I}_q^{\alpha+\omega+\theta} \phi(\mathbb{T}) \end{aligned}$$

$$\begin{aligned}
& + \mathcal{I}_q^{\alpha+\omega+\theta-\beta} \phi(\mathbb{T}) \left(\frac{\mathbb{T}^\theta}{2\Delta\Gamma_q(\theta+1)} - \frac{\chi^\theta}{\Delta\Gamma_q(\theta+1)} \right) \\
& + \mathcal{I}_q^\alpha \phi(\delta) \left[\frac{-\chi^{\omega+\theta}}{\Gamma_q(\omega+\theta+1)} + \frac{\chi^\theta \mathbb{T}^{\omega+\theta-\beta}}{\Delta\Gamma_q(\theta+1)\Gamma_q(\omega+\theta-\beta+1)} \right. \\
& \left. + \frac{\mathbb{T}^{\omega+\theta}}{2\Gamma_q(\omega+\theta+1)} - \frac{\mathbb{T}^{\omega+2\theta-\beta}}{2\Delta\Gamma_q(\theta+1)\Gamma_q(\omega+\theta-\beta+1)} \right]
\end{aligned}$$

and

$$\begin{aligned}
v(\chi) = & \int_0^\chi \frac{(\chi - qs)^{\alpha+\omega+\theta-1}}{\Gamma_q(\alpha+\omega+\theta)} \phi(s) d_qs - \frac{1}{2} \int_0^\mathbb{T} \frac{(\mathbb{T} - qs)^{\alpha+\omega+\theta-1}}{\Gamma_q(\alpha+\omega+\theta)} \phi(s) d_qs \\
& + \int_0^\mathbb{T} \frac{(\mathbb{T} - qs)^{\alpha+\omega+\theta-\beta-1}}{\Gamma_q(\alpha+\omega+\theta-\beta)} \left[\frac{\mathbb{T}^\theta}{2\Delta\Gamma_q(\theta+1)} - \frac{\chi^\theta}{\Delta\Gamma_q(\theta+1)} \right] \phi(s) d_qs \\
& + \int_0^\delta \frac{(\delta - qs)^{\alpha-1}}{\Gamma_q(\alpha)} \left[\frac{-\chi^{\omega+\theta}}{\Gamma_q(\omega+\theta+1)} + \frac{\chi^\theta \mathbb{T}^{\omega+\theta-\beta}}{\Delta\Gamma_q(\theta+1)\Gamma_q(\omega+\theta-\beta+1)} \right. \\
& \left. + \frac{\mathbb{T}^{\omega+\theta}}{2\Gamma_q(\omega+\theta+1)} - \frac{\mathbb{T}^{\omega+2\theta-\beta}}{2\Delta\Gamma_q(\theta+1)\Gamma_q(\omega+\theta-\beta+1)} \right] \phi(s) d_qs. \quad \square
\end{aligned}$$

We define an operator $\rho : \mathcal{F} \rightarrow \mathcal{F}$ by applying Lemma 3.1 as follows:

$$\begin{aligned}
\rho v(\chi) = & \int_0^\chi \frac{(\chi - qs)^{\alpha+\omega+\theta-1}}{\Gamma_q(\alpha+\omega+\theta)} \Theta_{v,\omega,\theta}^*(s) d_qs - \frac{1}{2} \int_0^\mathbb{T} \frac{(\mathbb{T} - qs)^{\alpha+\omega+\theta-1}}{\Gamma_q(\alpha+\omega+\theta)} \Theta_{v,\omega,\theta}^*(s) d_qs \\
& + \int_0^\mathbb{T} \frac{(\mathbb{T} - qs)^{\alpha+\omega+\theta-\beta-1}}{\Gamma_q(\alpha+\omega+\theta-\beta)} \left(\frac{\mathbb{T}^\theta}{2\Delta\Gamma_q(\theta+1)} - \frac{\chi^\theta}{\Delta\Gamma_q(\theta+1)} \right) \Theta_{v,\omega,\theta}^*(s) d_qs \\
& + \int_0^\delta \frac{(\delta - qs)^{\alpha-1}}{\Gamma_q(\alpha)} \left[\frac{-\chi^{\omega+\theta}}{\Gamma_q(\omega+\theta+1)} + \frac{\chi^\theta \mathbb{T}^{\omega+\theta-\beta}}{\Delta\Gamma_q(\theta+1)\Gamma_q(\omega+\theta-\beta+1)} \right. \\
& \left. + \frac{\mathbb{T}^{\omega+\theta}}{2\Gamma_q(\omega+\theta+1)} - \frac{\mathbb{T}^{\omega+2\theta-\beta}}{2\Delta\Gamma_q(\theta+1)\Gamma_q(\omega+\theta-\beta+1)} \right] \Theta_{v,\omega,\theta}^*(s) d_qs.
\end{aligned}$$

The following assumptions will be used in our upcoming results:

$$(H_1) \quad \Delta = 1 + \frac{\mathbb{T}^{\theta-\beta}}{\Gamma_q(\theta-\beta+1)} \neq 0;$$

(H₂) $\xi, g : \Lambda \times \mathbb{R}^3 \rightarrow \mathbb{R}$ are continuous;

(H₃) \exists constant $\bar{y} > 0$ in such a way that $\forall \chi \in \Lambda$ and $v, \hat{v} \in \mathbb{R}$, $m = \{1, 2, 3\}$, we get

$$|\xi(\chi, v_1, v_2, v_3) - \xi(\chi, \hat{v}_1, \hat{v}_2, \hat{v}_3)| \leq \sum_{m=1}^3 \bar{y}_m |v_m - \hat{v}_m|;$$

(H₄) \exists constant $\bar{z} > 0$ in such a way that $\forall \chi \in \Lambda$ and $v, \hat{v} \in \mathbb{R}$, $v = \{1, 2, 3\}$, we have

$$|g(v, v_1, v_2, v_3) - g(\chi, \hat{v}_1, \hat{v}_2, \hat{v}_3)| \leq \sum_{v=1}^3 \bar{z}_v |v_v - \hat{v}_v|;$$

(H₅) \exists real constants $\varphi_m \geq 0$ ($m = 1, 2, 3$) and $\varphi_0 > 0$ in such a way that for any $v_m \in \mathbb{R}$ ($m = 1, 2, 3$) we have

$$|\xi(\chi, v_1, v_2, v_3)| \leq \varphi_0 + \varphi_1 |v_1| + \varphi_2 |v_2| + \varphi_3 |v_3|;$$

(H₆) \exists real constants $\varphi_\nu \geq 0$ ($\nu = 1, 2, 3$) and $\varphi_0 > 0$ in such a way that for any $v_\nu \in \mathbb{R}$ ($\nu = 1, 2, 3$) we have

$$|g(\chi, v_1, v_2, v_3)| \leq \varphi_0 + \varphi_1|v_1| + \varphi_2|v_2| + \varphi_3|v_3|;$$

(H₇) \exists an increasing $\vartheta \in \mathcal{C}(\Omega, \mathbb{R}_+)$ and $\vartheta_h > 0$, then the following inequality

$$\mathcal{I}_q^{\alpha+\omega+\theta} h(\chi) \leq \vartheta_h h(\chi), \quad \chi \in \Lambda,$$

is satisfied.

In the following sections, we will employ the fixed point theory to confirm the existence of solution of q -fractional \mathcal{J} problem outlined in (1). For simplicity, the following notations will be used in our upcoming results:

$$\begin{aligned} \varpi_1 &= \frac{3}{2} \frac{T^{\alpha+\omega+\theta}}{\Gamma_q(\alpha+\omega+\theta+1)} + \frac{T^{\alpha+\omega+\theta-\beta}}{\Gamma_q(\alpha+\omega+\theta-\beta+1)} \left(\frac{3T^\theta}{2|\Delta|\Gamma_q(\theta+1)} \right) \\ &\quad + \frac{\delta^\alpha}{\Gamma_q(\alpha+1)} \left(\frac{3T^{\omega+\theta}}{2\Gamma_q(\omega+\theta+1)} + \frac{3T^{\omega+2\theta-\beta}}{|\Delta|\Gamma_q(\theta+1)\Gamma_q(\omega+\theta-\beta+1)} \right), \\ \varpi_2 &= \frac{T^{\alpha+\omega}}{\Gamma_q(\alpha+\omega+1)} + \frac{T^{\alpha+\omega+\theta-\beta}}{|\Delta|\Gamma_q(\alpha+\omega+\theta-\beta+1)} \\ &\quad + \frac{\delta^\alpha}{\Gamma_q(\alpha+1)} \left(\frac{T^\omega}{\Gamma_q(\omega+1)} + \frac{T^{\omega+\theta-\beta}}{|\Delta|\Gamma_q(\omega+\theta-\beta+1)} \right), \\ \varpi_3 &= \frac{T^\alpha}{\Gamma_q(\alpha+1)} + \frac{\delta^\alpha}{\Gamma_q(\alpha+1)}. \end{aligned} \tag{21}$$

Theorem 3.2 Suppose that assumptions (H₂), (H₃), and (H₄) hold. Thus, q-FJP (1) has a unique solution if

$$\left[\sum_{m=1}^3 \bar{y}_m + \sum_{\nu=1}^3 \bar{z}_\nu \right] \left(\sum_{i=1}^3 \varpi_i \right) < 1, \tag{22}$$

where ϖ_i , $i = 1, 2, 3$, are given by (21).

Proof First, we demonstrate that $\rho \mathcal{W}_\epsilon \subset \mathcal{W}_\epsilon$, where $\mathcal{W}_\epsilon = \{v \in \mathcal{F} : \|v\|_{\mathcal{F}} \leq \epsilon\}$ with

$$\epsilon \geq \frac{(\Pi + \psi) \sum_{i=1}^3 \varpi_i}{1 - (\sum_{m=1}^3 \bar{y}_m + \sum_{\nu=1}^3 \bar{z}_\nu) \sum_{i=1}^3 \varpi_i},$$

s.t. $\Pi = \sup_{\chi \in \Lambda} |\xi(\chi, 0, 0, 0)|$, $\psi = \sup_{\chi \in \Lambda} |g(\chi, 0, 0, 0)|$, and ϖ_i , $i = 1, 2, 3$, are given by (21). Using (H₃) and (H₄), we get

$$\begin{aligned} \Theta_{v,\omega,\theta}^*(\chi) &= \left| \xi(\chi, v(\chi), {}^C\mathbb{D}_q^\theta v(\chi), ({}^C\mathbb{D}_q^\omega({}^C\mathbb{D}_q^\theta v(\chi)))) \right. \\ &\quad \left. + \int_0^\chi \frac{(\chi - qs)^{\nu-1}}{\Gamma_q(\nu)} g(s, v(s), {}^C\mathbb{D}_q^\theta v(s), ({}^C\mathbb{D}_q^\omega({}^C\mathbb{D}_q^\theta v(s)))) \, d_qs \right| \\ &\leq \left| \xi(\chi, v(\chi), {}^C\mathbb{D}_q^\theta v(\chi), ({}^C\mathbb{D}_q^\omega({}^C\mathbb{D}_q^\theta v(\chi)))) \right. \\ &\quad \left. + \int_0^\chi \frac{(\chi - qs)^{\nu-1}}{\Gamma_q(\nu)} g(s, v(s), {}^C\mathbb{D}_q^\theta v(s), ({}^C\mathbb{D}_q^\omega({}^C\mathbb{D}_q^\theta v(s)))) \, d_qs \right| \end{aligned}$$

$$\begin{aligned}
& + \int_0^\chi \frac{(\chi - qs)^{\nu-1}}{\Gamma_q(\zeta)} g(s, v(s), {}^C\mathbb{D}_q^\theta v(s), ({}^C\mathbb{D}_q^\omega ({}^C\mathbb{D}_q^\theta v(s)))) d_qs \\
& - \xi(\chi, 0, 0, 0) - \int_0^\chi \frac{(\chi - qs)^{\nu-1}}{\Gamma_q(\zeta)} g(s, 0, 0, 0) d_qs \Big| \\
& + |\xi(\chi, 0, 0, 0)| + \left| \int_0^\chi \frac{(\chi - qs)^{\nu-1}}{\Gamma_q(\zeta)} g(s, 0, 0, 0) d_qs \right| \\
& \leq \sum_{m=1}^3 \bar{y}_m (\|v\| + \|{}^C\mathbb{D}_q^\theta v\| + \|{}^C\mathbb{D}_q^\omega ({}^C\mathbb{D}_q^\theta v)\|) + \Pi \\
& + \sum_{\nu=1}^3 \bar{z}_\nu (\|v\| + \|{}^C\mathbb{D}_q^\theta v\| + \|{}^C\mathbb{D}_q^\omega ({}^C\mathbb{D}_q^\theta v)\|) + \psi \\
& \leq \sum_{m=1}^3 \bar{y}_m \|v\|_{\mathcal{F}} + \Pi + \sum_{\nu=1}^3 \bar{z}_\nu \|v\|_{\mathcal{F}} + \psi \\
& \leq \sum_{m=1}^3 \bar{y}_m \epsilon + \Pi + \sum_{\nu=1}^3 \bar{z}_\nu \epsilon + \psi \\
& = \left(\sum_{m=1}^3 \bar{y}_m + \sum_{\nu=1}^3 \bar{z}_\nu \right) \epsilon + \Pi + \psi. \tag{23}
\end{aligned}$$

Then we get

$$\begin{aligned}
|\rho v(\chi)| & \leq \int_0^\chi \frac{(\chi - qs)^{\alpha+\omega+\theta-1}}{\Gamma_q(\alpha + \omega + \theta)} |\Theta_{v,\omega,\theta}^*(s)| d_qs + \frac{1}{2} \int_0^T \frac{(T - qs)^{\alpha+\omega+\theta-1}}{\Gamma_q(\alpha + \omega + \theta)} |\Theta_{v,\omega,\theta}^*(s)| d_qs \\
& + \int_0^T \frac{(T - qs)^{\alpha+\omega+\theta-\beta-1}}{\Gamma_q(\alpha + \omega + \theta - \beta)} \left[\frac{T^\theta}{2|\Delta|\Gamma_q(\theta+1)} + \frac{\chi^\theta}{|\Delta|\Gamma_q(\theta+1)} \right] |\Theta_{v,\omega,\theta}^*(s)| d_qs \\
& + \int_0^\delta \frac{(\delta - qs)^{\alpha-1}}{\Gamma_q(\alpha)} \left[\frac{\chi^{\omega+\theta}}{\Gamma_q(\omega+\theta+1)} + \frac{\chi^\theta T^{\omega+\theta-\beta}}{|\Delta|\Gamma_q(\theta+1)\Gamma_q(\omega+\theta-\beta+1)} \right. \\
& \left. + \frac{T^{\omega+\theta}}{2\Gamma_q(\omega+\theta+1)} + \frac{T^{\omega+2\theta-\beta}}{2|\Delta|\Gamma_q(\theta+1)\Gamma_q(\omega+\theta-\beta+1)} \right] |\Theta_{v,\omega,\theta}^*(s)| d_qs.
\end{aligned}$$

Now, using (23), we obtain

$$\begin{aligned}
\|\rho(v)\| & \leq \left[\frac{T^{\alpha+\omega+\theta}}{\Gamma_q(\alpha + \omega + \theta + 1)} + \frac{1}{2} \frac{T^{\alpha+\omega+\theta}}{\Gamma_q(\alpha + \omega + \theta + 1)} \right. \\
& + \frac{T^{\alpha+\omega+\theta-\beta}}{\Gamma_q(\alpha + \omega + \theta - \beta + 1)} \left(\frac{T^\theta}{2|\Delta|\Gamma_q(\theta+1)} + \frac{T^\theta}{|\Delta|\Gamma_q(\theta+1)} \right) \\
& + \frac{\delta^\alpha}{\Gamma_q(\alpha + 1)} \left(\frac{T^{\omega+\theta}}{\Gamma_q(\omega+\theta+1)} + \frac{T^{\omega+2\theta-\beta}}{|\Delta|\Gamma_q(\theta+1)\Gamma_q(\omega+\theta-\beta+1)} \right. \\
& \left. + \frac{T^{\omega+\theta}}{2\Gamma_q(\omega+\theta+1)} + \frac{T^{\omega+2\theta-\beta}}{2|\Delta|\Gamma_q(\theta+1)\Gamma_q(\omega+\theta-\beta+1)} \right) \Big] (\bar{y}_m + \bar{z}_\nu) \epsilon + \Pi + \psi \\
& = \frac{3}{2} \frac{T^{\alpha+\omega+\theta}}{\Gamma_q(\alpha + \omega + \theta + 1)} + \frac{T^{\alpha+\omega+\theta-\beta}}{\Gamma_q(\alpha + \omega + \theta - \beta + 1)} \left(\frac{3T^\theta}{2|\Delta|\Gamma_q(\theta+1)} \right) \\
& + \frac{\delta^\alpha}{\Gamma_q(\alpha + 1)} \left(\frac{3T^{\omega+\theta}}{2\Gamma_q(\omega+\theta+1)} + \frac{3T^{\omega+2\theta-\beta}}{|\Delta|\Gamma_q(\theta+1)\Gamma_q(\omega+\theta-\beta+1)} \right) (\bar{y}_m + \bar{z}_\nu) \epsilon
\end{aligned}$$

$$\begin{aligned}
& + \Pi + \psi \\
& = \left[\sum_{m=1}^3 \bar{y}_m + \sum_{v=1}^3 \bar{z}_v \right] \varpi_1 \epsilon + (\Pi + \psi) \varpi_1.
\end{aligned}$$

Also, we have

$$\begin{aligned}
\| {}^C \mathbb{D}_q^\theta \rho v(\chi) \| & \leq \left[\frac{\mathbb{T}^{\alpha+\omega}}{\Gamma_q(\alpha+\omega+1)} + \frac{\mathbb{T}^{\alpha+\omega+\theta-\beta}}{|\Delta| \Gamma_q(\alpha+\omega+\theta-\beta+1)} \right. \\
& \quad \left. + \frac{\delta^\alpha}{\Gamma_q(\alpha+1)} \left(\frac{\mathbb{T}^\omega}{\Gamma_q(\omega+1)} + \frac{\mathbb{T}^{\omega+\theta-\beta}}{|\Delta| \Gamma_q(\omega+\theta-\beta+1)} \right) \right] (\bar{y}_m + \bar{z}_v) \epsilon \\
& \quad + \Pi + \psi \\
& = \left[\sum_{m=1}^3 \bar{y}_m + \sum_{v=1}^3 \bar{z}_v \right] \varpi_2 \epsilon + (\Pi + \psi) \varpi_2
\end{aligned}$$

and

$$\begin{aligned}
\| {}^C \mathbb{D}_q^\omega ({}^C \mathbb{D}_q^\theta) \rho v(\chi) \| & \leq \left[\frac{\mathbb{T}^\alpha}{\Gamma_q(\alpha+1)} + \frac{\delta^\alpha}{\Gamma_q(\alpha+1)} \right] (\bar{y}_m + \bar{z}_v) \epsilon + \Pi + \psi \\
& = \left(\sum_{m=1}^3 \bar{y}_m + \sum_{v=1}^3 \bar{z}_v \right) \varpi_3 \epsilon + (\Pi + \psi) \varpi_3.
\end{aligned}$$

From the definition of $\|\cdot\|_{\mathcal{F}}$, we have

$$\begin{aligned}
\| \rho(v) \|_{\mathcal{F}} & = \| \rho(v) \| + \| {}^C \mathbb{D}_q^\theta \rho(v) \| + \| {}^C \mathbb{D}_q^\omega ({}^C \mathbb{D}_q^\theta \rho(v)) \| \\
& \leq (\bar{y} + \bar{z}) \epsilon \varpi_1 + (\Pi + \psi) \varpi_1 + (\bar{y} + \bar{z}) \epsilon \varpi_2 \\
& \quad + (\Pi + \psi) \varpi_2 + (\bar{y} + \bar{z}) \epsilon \varpi_3 + (\Pi + \psi) \varpi_3 \\
& = \left[\sum_{m=1}^3 \bar{y}_m + \sum_{v=1}^3 \bar{z}_v \right] \sum_{i=1}^3 \varpi_i \epsilon + (\Pi + \psi) \sum_{i=1}^3 \varpi_i \leq \epsilon,
\end{aligned}$$

which means that $\rho \mathcal{W}_\epsilon \subset \mathcal{W}_\epsilon$. We now demonstrate that the ρ is an operator for a contraction mapping. Now $v, \hat{v} \in \mathcal{W}_\epsilon$ and $\chi \in \Lambda$, we obtain

$$\begin{aligned}
& | \rho v(\chi) - \rho \hat{v}(\chi) | \\
& \leq \int_0^\chi \frac{(\chi - qs)^{\alpha+\omega+\theta-1}}{\Gamma_q(\alpha+\omega+\theta)} | \Theta_{v,\omega,\theta}^*(s) - \Theta_{\hat{v},\omega,\theta}^*(s) | d_qs \\
& \quad + \frac{1}{2} \int_0^\mathbb{T} \frac{(\mathbb{T} - qs)^{\alpha+\omega+\theta-1}}{\Gamma_q(\alpha+\omega+\theta)} | \Theta_{v,\omega,\theta}^*(s) - \Theta_{\hat{v},\omega,\theta}^*(s) | d_qs \\
& \quad + \int_0^\mathbb{T} \frac{(\mathbb{T} - qs)^{\alpha+\omega+\theta-\beta-1}}{\Gamma_q(\alpha+\omega+\theta-\beta)} \left[\frac{\mathbb{T}^\theta}{2|\Delta| \Gamma_q(\theta+1)} \right. \\
& \quad \left. + \frac{\chi^\theta}{|\Delta| \Gamma_q(\theta+1)} \right] | \Theta_{v,\omega,\theta}^*(s) - \Theta_{\hat{v},\omega,\theta}^*(s) | d_qs \\
& \quad + \int_0^\delta \frac{(\delta - qs)^{\alpha-1}}{\Gamma_q(\alpha)} \left[\frac{\chi^{\omega+\theta}}{\Gamma_q(\omega+\theta+1)} + \frac{\chi^\theta \mathbb{T}^{\omega+\theta-\beta}}{|\Delta| \Gamma_q(\theta+1) \Gamma_q(\omega+\theta-\beta+1)} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{\mathbb{T}^{\omega+\theta}}{2\Gamma_q(\omega+\theta+1)} \\
& + \frac{\mathbb{T}^{\omega+2\theta-\beta}}{2|\Delta|\Gamma_q(\theta+1)\Gamma_q(\omega+\theta-\beta+1)} \Big] \left| \Theta_{v,\omega,\theta}^*(s) - \Theta_{v,\omega,\theta}^*(s) \right| d_qs.
\end{aligned}$$

By (H₃) and (H₄), we obtain

$$\begin{aligned}
& \|\rho(v) - \rho(\hat{v})\| \\
& \leq \frac{3}{2} \frac{\mathbb{T}^{\alpha+\omega+\theta}}{\Gamma_q(\alpha+\omega+\theta+1)} + \frac{\mathbb{T}^{\alpha+\omega+\theta-\beta}}{\Gamma_q(\alpha+\omega+\theta-\beta+1)} \left(\frac{3\mathbb{T}^\theta}{2|\Delta|\Gamma_q(\theta+1)} \right) \\
& \quad + \frac{\delta^\alpha}{\Gamma_q(\alpha+1)} \left(\frac{3\mathbb{T}^{\omega+\theta}}{2\Gamma_q(\omega+\theta+1)} + \frac{3\mathbb{T}^{\omega+2\theta-\beta}}{|\Delta|\Gamma_q(\theta+1)\Gamma_q(\omega+\theta-\beta+1)} \right) (\bar{y} + \bar{z}) \|v - \hat{v}\|_{\mathcal{F}} \\
& = \left[\sum_{m=1}^3 \bar{y}_m + \sum_{v=1}^3 \bar{z}_v \right] \varpi_1 \|v - \hat{v}\|_{\mathcal{F}}.
\end{aligned}$$

Also, by using (H₃) and (H₄), we obtain

$$\begin{aligned}
& \|{}^C\mathbb{D}_q^\theta \rho(v) - {}^C\mathbb{D}_q^\theta \rho(\hat{v})\| \\
& \leq \left[\frac{\mathbb{T}^{\alpha+\omega}}{\Gamma_q(\alpha+\omega+1)} + \frac{\mathbb{T}^{\alpha+\omega+\theta-\beta}}{|\Delta|\Gamma_q(\alpha+\omega+\theta-\beta+1)} \right. \\
& \quad \left. + \frac{\delta^\alpha}{\Gamma_q(\alpha+1)} \left(\frac{\mathbb{T}^\omega}{\Gamma_q(\omega+1)} + \frac{\mathbb{T}^{\omega+\theta-\beta}}{|\Delta|\Gamma_q(\omega+\theta-\beta+1)} \right) \right] (\bar{y}_m + \bar{z}_v) \|v - \hat{v}\|_{\mathcal{F}} \\
& = \left[\sum_{m=1}^3 \bar{y}_m + \sum_{v=1}^3 \bar{z}_v \right] \varpi_2 \|v - \hat{v}\|_{\mathcal{F}}
\end{aligned}$$

and

$$\begin{aligned}
& \|{}^C\mathbb{D}_q^\omega({}^C\mathbb{D}_q^\theta)\rho(v) - {}^C\mathbb{D}_q^\omega({}^C\mathbb{D}_q^\theta)\rho(\hat{v})\| \leq \left[\frac{\mathbb{T}^\alpha}{\Gamma_q(\alpha+1)} + \frac{\delta^\alpha}{\Gamma_q(\alpha+1)} \right] (\bar{y} + \bar{z}) \|v - \hat{v}\|_{\mathcal{F}} \\
& = \left[\sum_{m=1}^3 \bar{y}_m + \sum_{v=1}^3 \bar{z}_v \right] \varpi_3 \|v - \hat{v}\|_{\mathcal{F}}.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
& \|\rho(v) - \rho(\hat{v})\|_{\mathcal{F}} = \|\rho(v) - \rho(\hat{v})\| + \|{}^C\mathbb{D}_q^\theta \rho(v) - {}^C\mathbb{D}_q^\theta \rho(\hat{v})\| \\
& \quad + \|({}^C\mathbb{D}_q^\omega({}^C\mathbb{D}_q^\theta \rho(v))) - ({}^C\mathbb{D}_q^\omega({}^C\mathbb{D}_q^\theta \rho(\hat{v})))\| \\
& \leq \left[\sum_{m=1}^3 \bar{y}_m + \sum_{v=1}^3 \bar{z}_v \right] \sum_{i=1}^3 \varpi_i \|v - \hat{v}\|_{\mathcal{F}}.
\end{aligned}$$

We observe that ρ is a contraction operator by using (22). We infer that ρ has a unique fixed point that is a solution of (1) as a result of Lemma 2.9. \square

By applying Lemma 2.8, we explore certain conditions where q -FJP (1) has at least one solution in Theorem 3.2.

Theorem 3.3 Assume that hypotheses (H₅) and (H₆) hold. If

$$\left[\sum_{m=1}^3 \varphi_m + \sum_{\nu=1}^3 \wp_\nu \right] \left(\sum_{i=1}^3 \varpi_i \right) < 1 \quad (24)$$

is satisfied, then the proposed problem described by (1) has at least one solution within the domain Λ .

Proof Our initial goal is to investigate the complete continuity of an operator $\rho : \mathcal{F} \rightarrow \mathcal{F}$. Considering function's continuity Θ , we can also conclude that the operator ρ is also continuous. Assume that $\kappa \subset \mathcal{F}$ is bounded. Then there exists a positive constant \mathfrak{P} s.t. $|\Theta_{v,\omega,\theta}^*(s)| \leq \mathfrak{P}$ for each $v \in \kappa$. Then, for any $v \in \kappa$ and using (21), we can find that

$$\|\rho(v)\|_{\mathcal{F}} = \|\rho(v)\| + \|{}^c\mathbb{D}_q^\theta \rho(v)\| + \|{}^c\mathbb{D}_q^\omega({}^c\mathbb{D}_q^\theta \rho(v))\| \leq \mathfrak{P} \sum_{i=1}^3 \varpi_i.$$

The inequalities indicate that an operator ρ remains uniformly bounded. Furthermore, we will verify that ρ is equicontinuous. For $v \in \Lambda$ and $0 < \chi_1 < \chi_2 \leq \mathbb{T}$, we get

$$\begin{aligned} & |\rho v(\chi_1) - \rho v(\chi_2)| \\ & \leq \mathfrak{P} \left[\frac{|\chi_1^{\alpha+\omega+\theta} - \chi_2^{\alpha+\omega+\theta}|}{\Gamma_q(\alpha+\omega+\theta+1)} + \frac{|\chi_2^\theta - \chi_1^\theta| \mathbb{T}^{\alpha+\omega+\theta-\beta}}{|\Delta| \Gamma_q(\theta+1) \Gamma_q(\alpha+\omega+\theta-\beta+1)} \right] \\ & \quad + \mathfrak{P} \left(\frac{\delta^\alpha}{\Gamma_q(\alpha+1)} \left[\frac{|\chi_2^{\omega+\theta} - \chi_1^{\omega+\theta}|}{\Gamma_q(\omega+\theta+1)} + \frac{|\chi_1^\theta - \chi_2^\theta| \mathbb{T}^{\alpha+\omega+\theta-\beta}}{|\Delta| \Gamma_q(\theta+1) \Gamma_q(\alpha+\omega+\theta-\beta+1)} \right] \right). \end{aligned} \quad (25)$$

Also, we obtain

$$|{}^c\mathbb{D}_q^\theta \rho v(\chi_1) - {}^c\mathbb{D}_q^\theta \rho v(\chi_2)| \leq \mathfrak{P} \left[\frac{|\chi_1^{\alpha+\omega} - \chi_2^{\alpha+\omega}|}{\Gamma_q(\alpha+\omega+1)} + \frac{\delta^\alpha}{\Gamma_q(\alpha+1)} \left(\frac{|\chi_2^\omega - \chi_1^\omega|}{\Gamma_q(\omega+1)} \right) \right] \quad (26)$$

and

$$|{}^c\mathbb{D}_q^\omega({}^c\mathbb{D}_q^\theta \rho v(\chi_1)) - {}^c\mathbb{D}_q^\omega({}^c\mathbb{D}_q^\theta \rho v(\chi_2))| \leq \mathfrak{P} \left[\frac{\chi_1^\alpha - \chi_2^\alpha}{\Gamma_q(\alpha+1)} \right]. \quad (27)$$

The right-hand sides of (25), (26), (27) tend to zero independently of v as $\chi_1 \rightarrow \chi_2$. Therefore, an operator $\rho : \mathcal{F} \rightarrow \mathcal{F}$ is completely continuous by Arzelà–Ascoli theorem. Finally, we show that a set $\Upsilon = \{v \in \mathcal{F} : v = \varepsilon \rho(v), 0 < \varepsilon < 1\}$ is bounded. Let $v \in \Upsilon$, thus $v = \varepsilon \rho(v)$. For every $\chi \in \Lambda$, we have $v(\chi) = \varepsilon \rho v(\chi)$. Then

$$\begin{aligned} |v(\chi)| & \leq \varpi_1 [(\varphi_1 + \wp_1) \|v\| + (\varphi_2 + \wp_2) \|{}^c\mathbb{D}_q^\theta(v)\| \\ & \quad + (\varphi_3 + \wp_3) \|{}^c\mathbb{D}_q^\omega({}^c\mathbb{D}_q^\theta(v))\|] + \varpi_1 (\varphi_0 + \wp_0). \end{aligned}$$

We also have

$$\begin{aligned} |\mathcal{C}\mathbb{D}_q^\theta v(\chi)| &\leq \varpi_2 [(\varphi_1 + \wp_1)\|v\| + (\varphi_2 + \wp_2)\|\mathcal{C}\mathbb{D}_q^\theta(v)\| \\ &\quad + (\varphi_3 + \wp_3)\|\mathcal{C}\mathbb{D}_q^\omega(\mathcal{C}\mathbb{D}_q^\theta(v))\|] + \varpi_2(\varphi_0 + \wp_0), \\ |\mathcal{C}\mathbb{D}_q^\omega(\mathcal{C}\mathbb{D}_q^\theta v(\chi))| &\leq \varpi_3 [(\varphi_1 + \wp_1)\|v\| + (\varphi_2 + \wp_2)\|\mathcal{C}\mathbb{D}_q^\theta(v)\| \\ &\quad + (\varphi_3 + \wp_3)\|\mathcal{C}\mathbb{D}_q^\omega(\mathcal{C}\mathbb{D}_q^\theta(v))\|] + \varpi_3(\varphi_0 + \wp_0), \end{aligned}$$

which implies that

$$\|v\|_{\mathcal{F}} \leq \left[\sum_{m=1}^3 \varphi_m + \sum_{\nu=1}^3 \wp_\nu \right] \sum_{i=1}^3 \varpi_i \|v\|_{\mathcal{F}} + \sum_{i=1}^3 \varpi_i (\varphi_0 + \wp_0).$$

Consequently,

$$\|v\|_{\mathcal{F}} \leq \frac{\sum_{i=1}^3 \varpi_i (\varphi_0 + \wp_0)}{1 - [\sum_{m=1}^3 \varphi_m + \sum_{\nu=1}^3 \wp_\nu] \sum_{i=1}^3 \varpi_i}, \quad (28)$$

where $\varpi_i, i = 1, 2, 3$, are given by (21). From (28), we see that $\|v\|_{\mathcal{F}} \leq \infty$. As a result, Υ is bounded. We deduce that an operator ρ has a fixed point, which is the solution of q -FJJP (1) as a result of Lemma 2.8. \square

4 Stability results

We study the \mathcal{UH} and \mathcal{UHR} stability [46] of q -FJJP in this section.

Theorem 4.1 Assume that (H₂)–(H₄) and (22) hold. Then the q -FJJP (1) is \mathcal{UH} stable.

Proof Consider $\hat{v} \in \mathcal{F}$ to be the only solution to the problem

$$\begin{cases} \mathcal{C}\mathbb{D}_q^\alpha(\mathcal{C}\mathbb{D}_q^\omega(\mathcal{C}\mathbb{D}_q^\theta \hat{v}(\chi))) = \Theta_{v,\omega,\theta}^*, \\ \hat{v}(\chi)|_{\chi=0} = v(\chi)|_{\chi=0}, \quad -\hat{v}'(\chi)|_{\chi=T} = -v'(\chi)|_{\chi=T}, \\ (\mathcal{C}\mathbb{D}_q^\omega(\mathcal{C}\mathbb{D}_q^\theta \hat{v}(\chi)))|_{\chi=\delta} = (\mathcal{C}\mathbb{D}_q^\omega(\mathcal{C}\mathbb{D}_q^\theta v(\chi)))|_{\chi=\delta}, \\ \mathcal{C}\mathbb{D}_q^\beta \hat{v}(\chi)|_{\chi=0} = \mathcal{C}\mathbb{D}_q^\beta v(\chi)|_{\chi=0}, \quad -\mathcal{C}\mathbb{D}_q^\beta \hat{v}(\chi)|_{\chi=T} = -\mathcal{C}\mathbb{D}_q^\beta v(\chi)|_{\chi=T} \end{cases} \quad (29)$$

for $\chi \in \Lambda$, WHERE $0 < \alpha, \omega, \theta \leq \Lambda$. So that inequality (8) can be solved with v in \mathcal{F} . Utilizing Remark 2.1, we obtain

$$v(\chi) = \mathcal{I}_q^{\alpha+\omega+\theta} \phi_v(\chi) + c_0 \frac{\chi^{\omega+\theta}}{\Gamma_q(\omega+\theta+1)} + c_1 \frac{\chi^\theta}{\Gamma_q(\theta+1)} + c_2 + \mathcal{I}_q^{\alpha+\omega+\theta} \varrho(\chi),$$

where $c_j \in \mathbb{R}, j = \{0, 1, 2\}$, $\phi_v(\chi) = \Theta_{v,\omega,\theta}^*(\chi)$, and $|\varrho(\chi)| \leq \bar{x}, \chi \in \Lambda$. Thanks to Lemma 3.1,

$$|v(\chi) - \hat{v}(\chi)| = |\mathcal{I}_q^{\alpha+\omega+\theta} \varrho(\chi)| \leq \frac{\bar{x} T^{\alpha+\omega+\theta}}{\Gamma_q(\alpha+\omega+\theta+1)}.$$

Also, we have

$$\begin{aligned} |v(\chi) - \hat{v}(\chi)| &= \left| v(\chi) - \mathcal{I}_q^{\alpha+\omega+\theta} \phi_{\hat{v}}(\chi) + c_0 \frac{\chi^{\omega+\theta}}{\Gamma_q(\omega+\theta+1)} \right. \\ &\quad \left. + c_1 \frac{\chi^\theta}{\Gamma_q(\theta+1)} + c_2 + \mathcal{I}_q^{\alpha+\omega+\theta} \varrho(\chi) \right| \\ &= |v(\chi) - \rho v(\chi) + \rho v(\chi) - \rho \hat{v}(\chi)| \\ &\leq |v(\chi) - \rho v(\chi)| + |\rho v(\chi) - \rho \hat{v}(\chi)|. \end{aligned}$$

(H₃) and (H₄) imply that

$$\begin{aligned} \|v - \hat{v}\|_{\mathcal{F}} &\leq \|v - \rho v\|_{\mathcal{F}} + \|\rho v - \rho \hat{v}\|_{\mathcal{F}} \\ &\leq \frac{\bar{x} T^{\alpha+\omega+\theta}}{\Gamma_q(\alpha+\omega+\theta+1)} + \left[\sum_{m=1}^3 \bar{y}_m + \sum_{\nu=1}^3 \bar{z}_{\nu} \right] \sum_{i=1}^3 \varpi_i \|u - \hat{u}\|_{\mathcal{F}}, \end{aligned}$$

where Eq. (21) provides ϖ_i , $i = \{1, 2, 3\}$. Next

$$\|v - \hat{v}\|_{\mathcal{F}} \leq \frac{T^{\alpha+\omega+\theta}}{\Gamma_q(\alpha+\omega+\theta+1)[(1 - (\sum_{m=1}^3 \bar{y}_m + \sum_{\nu=1}^3 \bar{z}_{\nu}) \sum_{i=1}^3 \varpi_i)]} \bar{x}.$$

If we put

$$\mathbb{E}_{\Theta_{\omega,\theta}^*} := \frac{T^{\alpha+\omega+\theta}}{\Gamma_q(\alpha+\omega+\theta+1)[1 - (\sum_{m=1}^3 \bar{y}_m + \sum_{\nu=1}^3 \bar{z}_{\nu}) \sum_{i=1}^3 \varpi_i]},$$

we obtain $\|v - \hat{v}\|_{\mathcal{F}} \leq \mathbb{E}_{\Theta_{\omega,\theta}^*} \bar{x}$. As a result, the q -FJP (1) is \mathcal{UH} stable. \square

Theorem 4.2 Suppose that (H₂)–(H₄), (H₇), and (22) hold. Then q-FJP (1) is \mathcal{UHR} stable in relation to h .

Proof We have

$$v(\chi) = \mathcal{I}_q^{\alpha+\omega+\theta} \phi_v(\chi) + c_0 \frac{\chi^{\omega+\theta}}{\Gamma_q(\omega+\theta+1)} + c_1 \frac{\chi^\theta}{\Gamma_q(\theta+1)} + c_2 + \mathcal{I}_q^{\alpha+\omega+\theta} \varrho(\chi),$$

where $\chi \in \Lambda$, $c_j \in \mathbb{R}$, $j = 0, 1, 2$, and $|\varrho(\chi)| \leq \bar{x} h(\chi)$, and inequality (9) can be solved by using $v \in \mathcal{F}$. Taking $\hat{v} \in \mathcal{F}$ as the singular solution of (29), by Lemma 3.1, we have

$$|v(\chi) - \rho v(\chi)| = |\mathcal{I}_q^{\alpha+\omega+\theta} \varrho(\chi)| \leq \bar{x} \mathcal{I}_q^{\alpha+\omega+\theta} [h(\chi)] \leq \bar{x} \vartheta_h h(\chi).$$

Also, we have

$$\begin{aligned} |v(\chi) - \hat{v}(\chi)| &= \left| v(\chi) - \mathcal{I}_q^{\alpha+\omega+\theta} \phi_{\hat{v}}(\chi) + c_0 \frac{\chi^{\omega+\theta}}{\Gamma_q(\omega+\theta+1)} \right. \\ &\quad \left. + c_1 \frac{\chi^\theta}{\Gamma_q(\theta+1)} + c_2 + \mathcal{I}_q^{\alpha+\omega+\theta} \varrho(\chi) \right| \end{aligned}$$

$$\begin{aligned}
&= |v(\chi) - \rho v(\chi) + \rho v(\chi) - \rho \hat{v}(\chi)| \\
&\leq |v(\chi) - \rho v(\chi)| + |\rho v(\chi) - \rho \hat{v}(\chi)|.
\end{aligned}$$

So, by (H₃), (H₄), and (H₇), we obtain

$$\|v - \hat{v}\|_{\mathcal{F}} \leq \bar{x} \vartheta_h h(\chi) + \left[\sum_{m=1}^3 \bar{y}_m + \sum_{v=1}^3 \bar{z}_v \right] \sum_{i=1}^3 \varpi_i \|v - \hat{v}\|_{\mathcal{F}}.$$

Then we get

$$\|v - \hat{v}\|_{\mathcal{F}} \leq \frac{\vartheta_h}{1 - [\sum_{m=1}^3 \bar{y}_m + \sum_{v=1}^3 \bar{z}_v] \sum_{i=1}^3 \varpi_i} \bar{x} h(\chi), \quad \chi \in \Lambda.$$

If we take

$$\mathbb{E}_{\Theta_{\omega,\theta}^*, h} := \frac{\vartheta_h}{1 - [\sum_{m=1}^3 \bar{y}_m + \sum_{v=1}^3 \bar{z}_v] \sum_{i=1}^3 \varpi_i},$$

we can obtain $\|v - \hat{v}\|_{\mathcal{F}} \leq \mathbb{E}_{\Theta_{\omega,\theta}^*, h} \bar{x} h(\chi)$ considering $\chi \in \Lambda$. Consequently, the \mathcal{UHR} stability is achieved by q -FJJP (1). \square

5 Examples and illustrative results

In this section, we check the correctness of the results by showing several examples. In the first example, we test q -Caputo fractional JDEs with ABCs (1) for the changes of q in the range of zero and one according to the proposed theorems.

Example 5.1 Let

$$\left\{
\begin{aligned}
&\mathcal{C}D_q^{\frac{1}{3}} (\mathcal{C}D_q^{\frac{4}{5}} (\mathcal{C}D_q^{\frac{3}{4}} v(\chi))) \\
&= \frac{\sinh(e^\chi + 2)}{4} + \frac{\sqrt{15}e^{-\chi} |v(\chi)|}{41(\chi + 3)(|v(\chi)| + 1)} \\
&+ \frac{\cos(\mathcal{C}D_q^{\frac{3}{4}} v(\chi))}{333\sqrt{\ln(\chi + 12)}} + \frac{\arctan(\mathcal{C}D_q^{\frac{4}{5}} (\mathcal{C}D_q^{\frac{3}{4}} v(\chi)))}{22(\chi + 3)} \\
&+ \int_0^\chi \frac{(\chi - qs)^{\frac{3}{2}-1}}{\Gamma_q(\frac{3}{2})} \left[\frac{\sqrt{e^{2s}} |v(s)|}{29(s+3)(|v(s)|+5)} + \frac{\cos(\mathcal{C}D_q^{\frac{3}{4}} v(s))}{345\sqrt{e^{s+19}}} \right. \\
&\left. + \frac{\sin(\mathcal{C}D_q^{\frac{4}{5}} (\mathcal{C}D_q^{\frac{3}{4}} v(s)))}{137\ln(\sqrt{s+34})} \right] d_qs, \quad \chi \in [0, 1], \\
&v(0) = -v(1), \quad (\mathcal{C}D_q^{\frac{4}{5}} (\mathcal{C}D_q^{\frac{3}{4}} v(\frac{7}{11}))) = 0, \\
&\mathcal{C}D_q^{\frac{5}{9}} v(0) = -\mathcal{C}D_q^{\frac{5}{9}} v(1),
\end{aligned} \tag{30}
\right.$$

where $q \in \{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}\} \subseteq (0, 1)$, $\alpha = \frac{1}{3} \in (0, 1]$, $\omega = \frac{4}{5} \in (0, 1]$, $v = \zeta = \frac{3}{2}$, $\theta = \frac{3}{4} \in (0, 1]$, $\delta = \frac{7}{11} \in (0, \mathbb{T})$, $\beta = \frac{5}{9} \in (0, 1]$, $\mathbb{T} = 1$, and

$$|\mathcal{C}D_q^{\frac{1}{3}} (\mathcal{C}D_q^{\frac{4}{5}} (\mathcal{C}D_q^{\frac{3}{4}} v(\chi))) - \Theta_{v, \frac{4}{5}, \frac{3}{4}}^*(\chi)| \leq \bar{x},$$

$$|\mathcal{C}D_q^{\frac{1}{3}} (\mathcal{C}D_q^{\frac{4}{5}} (\mathcal{C}D_q^{\frac{3}{4}} v(\chi))) - \Theta_{v, \frac{4}{5}, \frac{3}{4}}^*(\chi)| \leq \bar{x} h(\chi),$$

where $\bar{x} > 0$, $h : \Lambda \rightarrow \mathbb{R}^+$, and

$$\begin{aligned}\Theta_{v,\frac{4}{5},\frac{3}{4}}^*(\chi) = & \frac{\sinh(e^\chi + 2)}{4} + \frac{\sqrt{15}e^{-\chi}|v(\chi)|}{41(\chi + 3)(|v(\chi)| + 1)} + \frac{\cos(\mathcal{C}\mathbb{D}_q^{\frac{3}{4}}v(\chi))}{333\sqrt{\ln(\chi + 12)}} \\ & + \frac{\arctan(\mathcal{C}\mathbb{D}_q^{\frac{4}{5}}(\mathcal{C}\mathbb{D}_q^{\frac{3}{4}}v(\chi)))}{22(\chi + 3)} + \int_0^\chi \frac{(\chi - qs)^{\frac{3}{2}-1}}{\Gamma_q(\frac{3}{2})} \left[\frac{\sqrt{e^{2s}}|v(s)|}{29(s+3)(|v(s)|+5)} \right. \\ & \left. + \frac{\cos(\mathcal{C}\mathbb{D}_q^{\frac{3}{4}}v(s))}{345\sqrt{e^{s+19}}} + \frac{\sin(\mathcal{C}\mathbb{D}_q^{\frac{4}{5}}(\mathcal{C}\mathbb{D}_q^{\frac{3}{4}}v(s)))}{137\ln(\sqrt{s+34})} \right] d_qs.\end{aligned}\quad (31)$$

For $\chi \in \Lambda$ and $(v_m, \hat{v}_m) \in \mathbb{R}^2$, $m = 1, 2, 3$, we obtain

$$\begin{aligned}|\xi(\chi, v_1, v_2, v_3) - \xi(\chi, \hat{v}_1, \hat{v}_2, \hat{v}_3)| & \leq \frac{\sqrt{15}}{123}|v_1 - \hat{v}_1| + \frac{1}{333\sqrt{\ln(12)}}|v_2 - \hat{v}_2| + \frac{1}{66}|v_3 - \hat{v}_3|,\end{aligned}$$

and similarly for $(v_\nu, \hat{v}_\nu) \in \mathbb{R}^2$, $\nu = 1, 2, 3$, we get

$$\begin{aligned}|g(\chi, v_1, v_2, v_3) - g(\chi, \hat{v}_1, \hat{v}_2, \hat{v}_3)| & \leq \frac{\sqrt{e^2}}{435}|v_1 - \hat{v}_1| + \frac{1}{345\sqrt{e^{19}}}|v_2 - \hat{v}_2| + \frac{1}{137\ln\sqrt{35}}|v_3 - \hat{v}_3|.\end{aligned}$$

Therefore, conditions (H_3) and (H_4) are satisfied with

$$\begin{aligned}\bar{y}_1 &= \frac{\sqrt{15}}{123}, \quad \bar{y}_2 = \frac{1}{333\sqrt{\ln(12)}}, \quad \bar{y}_3 = \frac{1}{66}, \\ \bar{z}_1 &= \frac{\sqrt{e^2}}{435}, \quad \bar{z}_2 = \frac{1}{345\sqrt{e^{19}}}, \quad \bar{z}_3 = \frac{1}{137\ln\sqrt{35}}.\end{aligned}$$

Furthermore, thanks to Eq. (21), we get

$$\Delta = 1 + \frac{\mathbb{T}^{\theta-\beta}}{\Gamma_q(\theta-\beta+1)} \approx \begin{cases} 2.036, & q = \frac{1}{5}, \\ 2.054, & q = \frac{2}{5}, \\ 2.067, & q = \frac{3}{5}, \end{cases} \quad (32)$$

and

$$\begin{aligned}\varpi_1 &= \frac{3}{2} \frac{\mathbb{T}^{\alpha+\omega+\theta}}{\Gamma_q(\alpha+\omega+\theta+1)} + \frac{\mathbb{T}^{\alpha+\omega+\theta-\beta}}{\Gamma_q(\alpha+\omega+\theta-\beta+1)} \left(\frac{3\mathbb{T}^\theta}{2|\Delta|\Gamma_q(\theta+1)} \right) \\ &+ \frac{\delta^\alpha}{\Gamma_q(\alpha+1)} \left(\frac{3\mathbb{T}^{\omega+\theta}}{2\Gamma_q(\omega+\theta+1)} + \frac{3\mathbb{T}^{\omega+2\theta-\beta}}{|\Delta|\Gamma_q(\theta+1)\Gamma_q(\omega+\theta-\beta+1)} \right) \\ &\approx \begin{cases} 4.600, & q = \frac{1}{5}, \\ 4.412, & q = \frac{2}{5}, \\ 4.265, & q = \frac{3}{5}, \end{cases}\end{aligned}$$

$$\varpi_2 = \frac{\mathbb{T}^{\alpha+\omega}}{\Gamma_q(\alpha+\omega+1)} + \frac{\mathbb{T}^{\alpha+\omega+\theta-\beta}}{|\Delta|\Gamma_q(\alpha+\omega+\theta-\beta+1)} \\ + \frac{\delta^\alpha}{\Gamma_q(\alpha+1)} \left(\frac{\mathbb{T}^\omega}{\Gamma_q(\omega+1)} + \frac{\mathbb{T}^{\omega+\theta-\beta}}{|\Delta|\Gamma_q(\omega+\theta-\beta+1)} \right) \\ \approx \begin{cases} 2.812, & q = \frac{1}{5}, \\ 2.821, & q = \frac{2}{5}, \\ 2.828, & q = \frac{3}{5}, \end{cases} \\ \varpi_3 = \frac{\mathbb{T}^\alpha}{\Gamma_q(\alpha+1)} + \frac{\delta^\alpha}{\Gamma_q(\alpha+1)} \\ \approx \begin{cases} 1.947, & q = \frac{1}{5}, \\ 1.994, & q = \frac{2}{5}, \\ 2.029, & q = \frac{3}{5}. \end{cases}$$

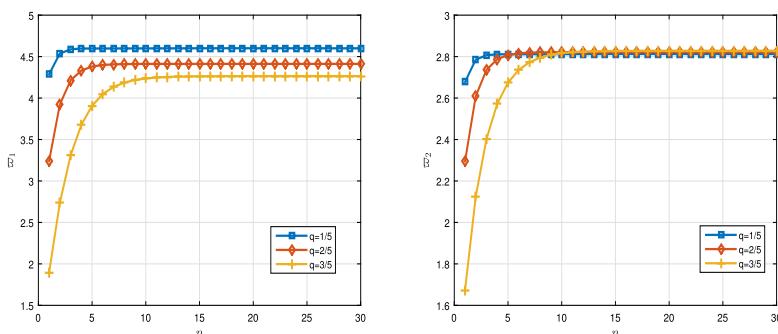
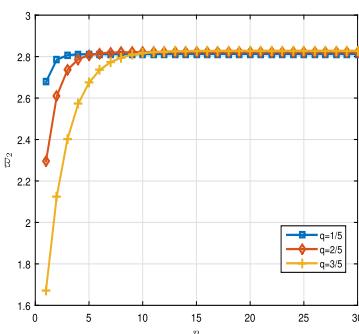
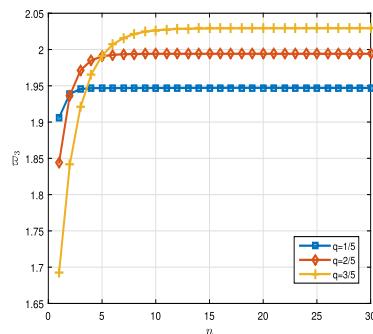
The data in Table 1 show the values of w_i , $i = 1, 2, 3$, for three different values q . Because the relations of q -calculators depend on the number of repetitions n , after several steps, their value is fixed. This mathematical performance can be clearly seen in Tables 1 and 2. The approach is similar to each group of curves in Figs. 1a, 1b, and 1c, aligning with each other and reaching a stable value that precisely determines the correctness of the argument. By (22), we get

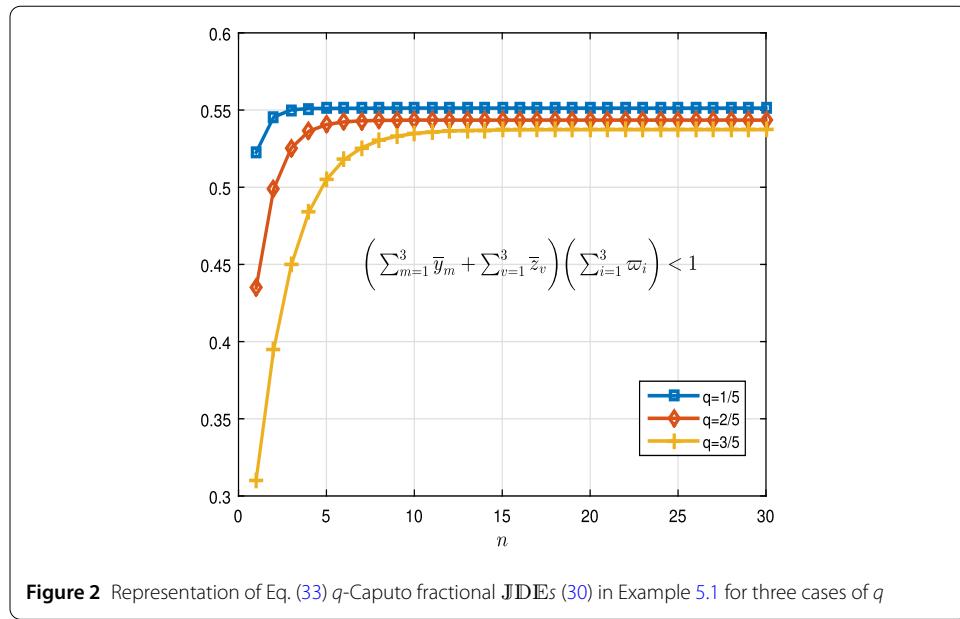
$$\left[\sum_{m=1}^3 \bar{y}_m + \sum_{\nu=1}^3 \bar{z}_{\nu} \right] \sum_{i=1}^3 \varpi_i \approx \begin{cases} 0.5513, & q = \frac{1}{5}, \\ 0.5435, & q = \frac{2}{5}, \\ 0.5373, & q = \frac{3}{5}, \end{cases} < 1. \quad (33)$$

Table 1 Numerical results for Δ and $\varpi_i, i = 1, 2, 3$ in Example 5.1 for three cases of q

Table 2 Numerical results for Eq. (33) in Example 5.1 for three cases of q

n	Eq. (33)		
	$q = \frac{1}{5}$	$q = \frac{2}{5}$	$q = \frac{3}{5}$
1	0.5226	0.4347	0.3100
2	0.5455	0.4988	0.3948
3	0.5501	0.5254	0.4496
4	0.5510	0.5362	0.4839
5	0.5512	0.5406	0.5050
6	0.5512	0.5423	0.5178
7	0.5512	0.5430	0.5256
8	0.5512	0.5433	0.5303
9	0.5512	0.5434	0.5331
10	<u>0.5513</u>	<u>0.5435</u>	0.5348
11	0.5513	0.5435	0.5358
12	0.5513	0.5435	0.5364
13	0.5513	0.5435	0.5368
14	0.5513	0.5435	0.5370
15	0.5513	0.5435	0.5371
16	0.5513	0.5435	0.5372
17	0.5513	0.5435	<u>0.5373</u>
18	0.5513	0.5435	0.5373
19	0.5513	0.5435	0.5373
20	0.5513	0.5435	0.5373
:	:	:	:
:	:	:	:

(a) w_1 (b) w_2 (c) w_3 **Figure 1** 2D plot of w_i , $i = 1, 2, 3$ for q -Caputo fractional JDEs (30) in Example 5.1 for three cases of q



The numerical values of relation (33) are shown in Table 2. It can be seen that after stabilizing the data of each column, these results are less than one (see Fig. 2). Therefore, the given q -FJIP (30) is addressed in Theorem 3.2, asserting that it possesses a unique solution within the interval Λ . Additionally, Theorem 4.1 states that the same q -FJIP (30) is \mathcal{UH} stable having

$$\begin{aligned} \|v - \hat{v}\|_{\mathcal{F}} &\leq \frac{T^{\alpha+\omega+\theta}}{\Gamma_q(\alpha+\omega+\theta+1)[1 - (\sum_{m=1}^3 \bar{y}_m + \sum_{v=1}^3 \bar{z}_v) \sum_{i=1}^3 \varpi_i]} \bar{x} \\ &\leq \begin{cases} 1.9020\bar{x}, & q = \frac{1}{5}, \\ 1.6398\bar{x}, & q = \frac{2}{5}, \quad \bar{x} > 0, \\ 1.4460\bar{x}, & q = \frac{3}{5}, \end{cases} \end{aligned}$$

In general, as q approaches 1, we will achieve stability of the results with a higher number of iterations. For $h(\chi) = \chi^{-\frac{\ln(3)}{5}}$, we obtain

$$\begin{aligned} \mathcal{I}_q^{\frac{1}{3} + \frac{4}{5} + \frac{3}{4}} [h(\chi)] &= \mathcal{I}_q^{\frac{1}{3} + \frac{4}{5} + \frac{3}{4}} [\chi^{-\frac{\ln(3)}{5}}] \\ &\approx \begin{cases} 0.0834\chi^{-\frac{\ln(3)}{5}}, & q = \frac{1}{5}, \\ 0.1173\chi^{-\frac{\ln(3)}{5}}, & q = \frac{2}{5}, \\ 0.1066\chi^{-\frac{\ln(3)}{5}}, & q = \frac{3}{5}, \end{cases} \\ &= \vartheta_h h(\chi). \end{aligned}$$

Table 3 shows these results. In addition, the curves drawn in Figs. 3a and 3b confirm the existence of ϑ_h and Ineq. (34) variables. Therefore, condition (H₇) is fulfilled with $h(\chi) = \chi^{-\frac{\ln(3)}{5}}$ and $\vartheta_h = 0.0834, 0.1173, 0.1066$ whenever $q = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}$, respectively. Theo-

Table 3 Numerical results of ϑ_h in $\vartheta_h^{\alpha+\omega+\theta} h(\chi) \leq \vartheta_h h(\chi)$, in Example 5.1 for three cases of q and $\chi \in \Lambda$

n	$q = \frac{1}{5}$		$q = \frac{2}{5}$		$q = \frac{3}{5}$	
	ϑ_h	Ineq. (34)	ϑ_h	Ineq. (34)	ϑ_h	Ineq. (34)
1	0.0707	0.1480	0.0747	0.1321	0.0450	0.0652
2	0.0818	0.1800	0.1029	0.2053	0.0718	0.1187
3	<u>0.0834</u>	0.1854	0.1127	0.2374	0.0875	0.1589
4	0.0836	0.1862	0.1159	0.2498	0.0963	0.1866
5	0.0836	<u>0.1864</u>	0.1169	0.2544	0.1011	0.2043
6	0.0836	0.1864	0.1172	0.2561	0.1037	0.2151
7	0.0836	0.1864	<u>0.1173</u>	0.2567	0.1051	0.2216
8	0.0836	0.1864	0.1173	0.2569	0.1058	0.2253
9	0.0836	0.1864	0.1173	<u>0.2570</u>	0.1062	0.2275
10	0.0836	0.1864	0.1173	0.2570	0.1064	0.2287
11	0.0836	0.1864	0.1173	0.2570	0.1065	0.2294
12	0.0836	0.1864	0.1173	0.2570	0.1065	0.2298
13	0.0836	0.1864	0.1173	0.2570	<u>0.1066</u>	0.2300
14	0.0836	0.1864	0.1173	0.2570	0.1066	0.2302
15	0.0836	0.1864	0.1173	0.2570	0.1066	0.2302
16	0.0836	0.1864	0.1173	0.2570	0.1066	<u>0.2303</u>
17	0.0836	0.1864	0.1173	0.2570	0.1066	0.2303
18	0.0836	0.1864	0.1173	0.2570	0.1066	0.2303
19	0.0836	0.1864	0.1173	0.2570	0.1066	0.2303
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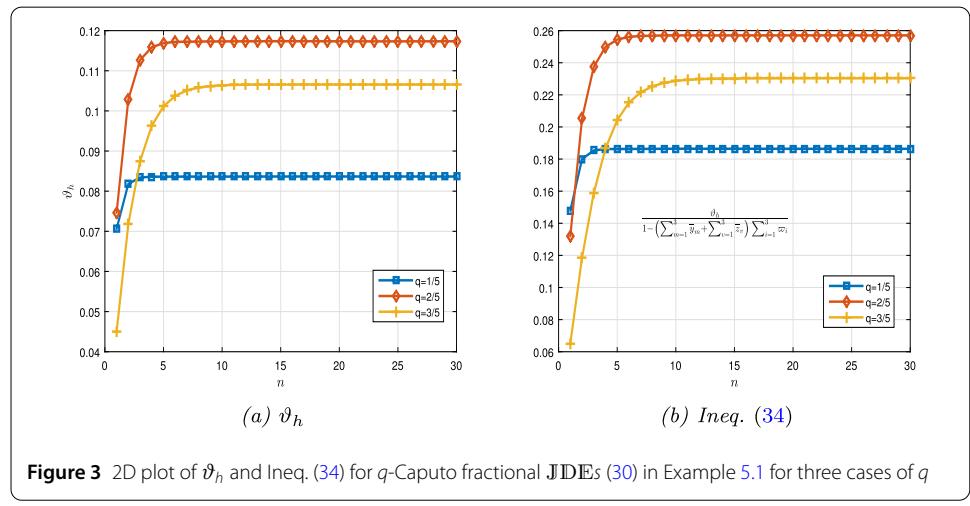


Figure 3 2D plot of ϑ_h and Ineq. (34) for q -Caputo fractional JDEs (30) in Example 5.1 for three cases of q

rem 4.2 indicates that the q -FJP is \mathcal{UHR} (30) stable s.t.

$$\begin{aligned} \|v - \hat{v}\|_{\mathcal{F}} &\leq \frac{\vartheta_h}{1 - (\sum_{m=1}^3 \bar{y}_m + \sum_{v=1}^3 \bar{z}_v) \sum_{i=1}^3 \bar{w}_i} \bar{x} h(\chi) \\ &\approx \begin{cases} 0.1864, & q = \frac{1}{5}, \\ 0.2570, & q = \frac{2}{5}, \\ 0.2303, & q = \frac{3}{5}, \end{cases} \times \bar{x} h(\chi), \quad \bar{x} > 0, \chi \in \Lambda. \end{aligned} \quad (34)$$

The next example shows the proven facts for changes in the order of the derivative α .

Example 5.2 We consider the q -Caputo fractional JDEs with ABCs (30) in Example 5.1

$$\left\{ \begin{array}{l} {}^C\mathbb{D}_{\frac{3}{5}}^\alpha ({}^C\mathbb{D}_{\frac{3}{5}}^{\frac{4}{5}} ({}^C\mathbb{D}_{\frac{3}{5}}^{\frac{3}{5}} v(\chi))) \\ = \frac{\sinh(e^\chi+2)}{4} + \frac{\sqrt{15}e^{-\chi}|v(\chi)|}{41(\chi+3)(|v(\chi)|+1)} \\ + \frac{\cos({}^C\mathbb{D}_{\frac{3}{5}}^{\frac{3}{5}} v(\chi))}{333\sqrt{\ln(\chi+12)}} + \frac{\arctan({}^C\mathbb{D}_{\frac{3}{5}}^{\frac{4}{5}} ({}^C\mathbb{D}_{\frac{3}{5}}^{\frac{3}{5}} v(\chi)))}{22(\chi+3)} \\ + \int_0^\chi \frac{(x-\frac{3}{5}s)^{\frac{8}{5}-1}}{\Gamma_{\frac{3}{5}}^{\frac{8}{5}}} \left[\frac{\sqrt{e^{2s}}|v(s)|}{29(s+3)(|v(s)|+5)} + \frac{\cos({}^C\mathbb{D}_{\frac{3}{5}}^{\frac{4}{5}} v(s))}{345\sqrt{e^{s+19}}} \right. \\ \left. + \frac{\sin({}^C\mathbb{D}_{\frac{3}{5}}^{\frac{4}{5}} ({}^C\mathbb{D}_{\frac{3}{5}}^{\frac{3}{5}} v(s)))}{137\ln(\sqrt{s+34})} \right] d_{\frac{3}{5}} s, \\ v(0) = -v(1), \quad ({}^C\mathbb{D}_{\frac{3}{5}}^{\frac{4}{5}} ({}^C\mathbb{D}_q^{\frac{3}{5}} v(\frac{7}{11}))) = 0, \\ {}^C\mathbb{D}_{\frac{3}{5}}^{\frac{5}{9}} v(0) = -{}^C\mathbb{D}_{\frac{3}{5}}^{\frac{5}{9}} v(1), \end{array} \right. \quad \chi \in [0, 1], \quad (35)$$

with the difference that $q = \frac{3}{5}$ is fixed and α chooses $\{\frac{1}{8}, \frac{1}{6}, \frac{1}{3}\} \subseteq (0, 1)$, $\omega = \frac{4}{5}$, $\nu = \zeta = \frac{8}{5}$, $\theta = \frac{3}{4}$, $\delta = \frac{7}{11}$, $\beta = \frac{5}{9}$, $T = 1$, and

$$\begin{aligned} & |{}^C\mathbb{D}_{\frac{3}{5}}^\alpha ({}^C\mathbb{D}_{\frac{3}{5}}^{\frac{4}{5}} ({}^C\mathbb{D}_{\frac{3}{5}}^{\frac{3}{5}} v(\chi))) - \Theta_{v, \frac{4}{5}, \frac{3}{4}}^*(\chi)| \leq \bar{x}, \\ & |{}^C\mathbb{D}_{\frac{3}{5}}^\alpha ({}^C\mathbb{D}_{\frac{3}{5}}^{\frac{4}{5}} ({}^C\mathbb{D}_{\frac{3}{5}}^{\frac{3}{5}} v(\chi))) - \Theta_{v, \frac{4}{5}, \frac{3}{4}}^*(\chi)| \leq \bar{x}h(\chi), \end{aligned}$$

where $\bar{x} > 0$, $h : \Lambda \rightarrow \mathbb{R}^+$, and $\Theta_{v, \frac{4}{5}, \frac{3}{4}}^*(\chi)$ is defined by (31). It was found that conditions (H₃) and (H₄) are satisfied with $\bar{y}_1 = \frac{\sqrt{15}}{123}$, $\bar{y}_2 = \frac{1}{333\sqrt{\ln 12}}$, $\bar{y}_3 = \frac{1}{66}$, and $\bar{z}_1 = \frac{\sqrt{e^2}}{435}$, $\bar{z}_2 = \frac{1}{345\sqrt{e^{19}}}$, $\bar{z}_3 = \frac{1}{137\ln(\sqrt{35})}$. Thanks to Eq. (21), by using these data, we obtain $\Delta = 1 + \frac{T^{\theta-\beta}}{\Gamma_q(\theta-\beta+1)} \approx 2.0674$ and

$$\begin{aligned} w_1 &= \frac{3}{2} \frac{T^{\alpha+\omega+\theta}}{\Gamma_q(\alpha+\omega+\theta+1)} + \frac{T^{\alpha+\omega+\theta-\beta}}{\Gamma_q(\alpha+\omega+\theta-\beta+1)} \left(\frac{3T^\theta}{2|\Delta|\Gamma_q(\theta+1)} \right) \\ &\quad + \frac{\delta^\alpha}{\Gamma_q(\alpha+1)} \left(\frac{3T^{\omega+\theta}}{2\Gamma_q(\omega+\theta+1)} + \frac{3T^{\omega+2\theta-\beta}}{|\Delta|\Gamma_q(\theta+1)\Gamma_q(\omega+\theta-\beta+1)} \right) \\ &\approx \begin{cases} 4.588, & \alpha = \frac{1}{8}, \\ 4.533, & \alpha = \frac{1}{6}, \\ 4.265, & \alpha = \frac{1}{3}, \end{cases} \\ w_2 &= \frac{T^{\alpha+\omega}}{\Gamma_q(\alpha+\omega+1)} + \frac{T^{\alpha+\omega+\theta-\beta}}{|\Delta|\Gamma_q(\alpha+\omega+\theta-\beta+1)} \\ &\quad + \frac{\delta^\alpha}{\Gamma_q(\alpha+1)} \left(\frac{T^\omega}{\Gamma_q(\omega+1)} + \frac{T^{\omega+\theta-\beta}}{|\Delta|\Gamma_q(\omega+\theta-\beta+1)} \right) \\ &\approx \begin{cases} 3.011, & \alpha = \frac{1}{8}, \\ 2.981, & \alpha = \frac{1}{6}, \\ 2.828, & \alpha = \frac{1}{3}, \end{cases} \end{aligned}$$

Table 4 Numerical results of $\varpi_i, i = 1, 2, 3$, in Example 5.2 for three cases of derivative order α

n	$\alpha = \frac{1}{8}$			$\alpha = \frac{1}{6}$			$\alpha = \frac{1}{3}$		
	ϖ_1	ϖ_2	ϖ_3	ϖ_1	ϖ_2	ϖ_3	ϖ_1	ϖ_2	ϖ_3
1	2.222	1.941	1.896	2.156	1.887	1.857	1.898	1.673	1.692
2	3.084	2.372	1.961	3.017	2.325	1.942	2.736	2.125	1.841
3	3.656	2.628	1.995	3.592	2.588	1.986	3.309	2.403	1.921
4	4.018	2.782	2.013	3.957	2.745	2.010	3.677	2.572	1.966
5	4.243	2.874	2.024	4.183	2.839	2.024	3.907	2.674	1.992
6	4.380	2.929	2.030	4.322	2.896	2.032	4.049	2.736	2.007
7	4.463	2.962	2.034	4.406	2.930	2.037	4.134	2.773	2.016
8	4.513	2.981	2.036	4.456	2.950	2.040	4.186	2.795	2.021
9	4.543	2.993	2.037	4.487	2.963	2.041	4.218	2.808	2.025
10	4.561	3.000	2.038	4.505	2.970	2.042	4.237	2.816	2.027
11	4.572	3.005	2.038	4.516	2.974	2.043	4.248	2.821	2.028
12	4.578	3.007	2.038	4.523	2.977	2.043	4.255	2.824	2.028
13	4.582	3.009	<u>2.039</u>	4.527	2.979	2.043	4.259	2.826	<u>2.029</u>
14	4.585	3.010	2.039	4.529	2.980	<u>2.044</u>	4.261	2.827	2.029
15	4.586	3.010	2.039	4.531	2.980	2.044	4.263	2.827	2.029
16	4.587	<u>3.011</u>	2.039	4.532	2.980	2.044	4.264	<u>2.828</u>	2.029
17	<u>4.588</u>	3.011	2.039	4.532	<u>2.981</u>	2.044	4.264	2.828	2.029
18	4.588	3.011	2.039	4.532	2.981	2.044	4.264	2.828	2.029
19	4.588	3.011	2.039	<u>4.533</u>	2.981	2.044	<u>4.265</u>	2.828	2.029
20	4.588	3.011	2.039	<u>4.533</u>	2.981	2.044	4.265	2.828	2.029
21	4.588	3.011	2.039	4.533	2.981	2.044	4.265	2.828	2.029
:	:	:	:	:	:	:	:	:	:

$$\varpi_3 = \frac{\mathbb{T}^\alpha}{\Gamma_q(\alpha + 1)} + \frac{\delta^\alpha}{\Gamma_q(\alpha + 1)} \approx \begin{cases} 2.039, & \alpha = \frac{1}{8}, \\ 2.044, & \alpha = \frac{1}{6}, \\ 2.029, & \alpha = \frac{1}{3}. \end{cases}$$

The data in Table 4 show the values of $\varpi_i, i = 1, 2, 3$, for three different values of derivative order α . The approach is similar to each group of curves in Figs. 4a, 4b, and 4c, aligning with each other and reaching a stable value that precisely determines the correctness of the argument. By (22), we get

$$\left[\sum_{m=1}^3 \bar{y}_m + \sum_{v=1}^3 \bar{z}_v \right] \sum_{i=1}^3 \varpi_i \approx \begin{cases} 0.568, & \alpha = \frac{1}{8}, \\ 0.563, & \alpha = \frac{1}{6}, \\ 0.537, & \alpha = \frac{1}{3}, \end{cases} < 1. \quad (36)$$

The numerical values of relation (36) are shown in Table 5. It can be seen that after stabilizing the data of each column, these results are less than one (see Fig. 5). Therefore, the given q -FJIP (35) is addressed in Theorem 3.2, asserting that it possesses a unique solution within the interval Λ . Additionally, Theorem 4.1 states that the same q -FJIP (35) is \mathcal{UH} stable having

$$\begin{aligned} \|v - \hat{v}\|_{\mathcal{F}} &\leq \frac{\mathbb{T}^{\alpha+\omega+\theta}}{\Gamma_q(\alpha + \omega + \theta + 1)[1 - (\sum_{m=1}^3 \bar{y}_m + \sum_{v=1}^3 \bar{z}_v) \sum_{i=1}^3 \varpi_i]} \bar{x} \\ &\leq \begin{cases} 1.7349\bar{x}, & \alpha = \frac{1}{8}, \\ 1.6786\bar{x}, & \alpha = \frac{1}{6}, \quad \bar{x} > 0, \\ 1.4460\bar{x}, & \alpha = \frac{1}{3}, \end{cases} \end{aligned}$$

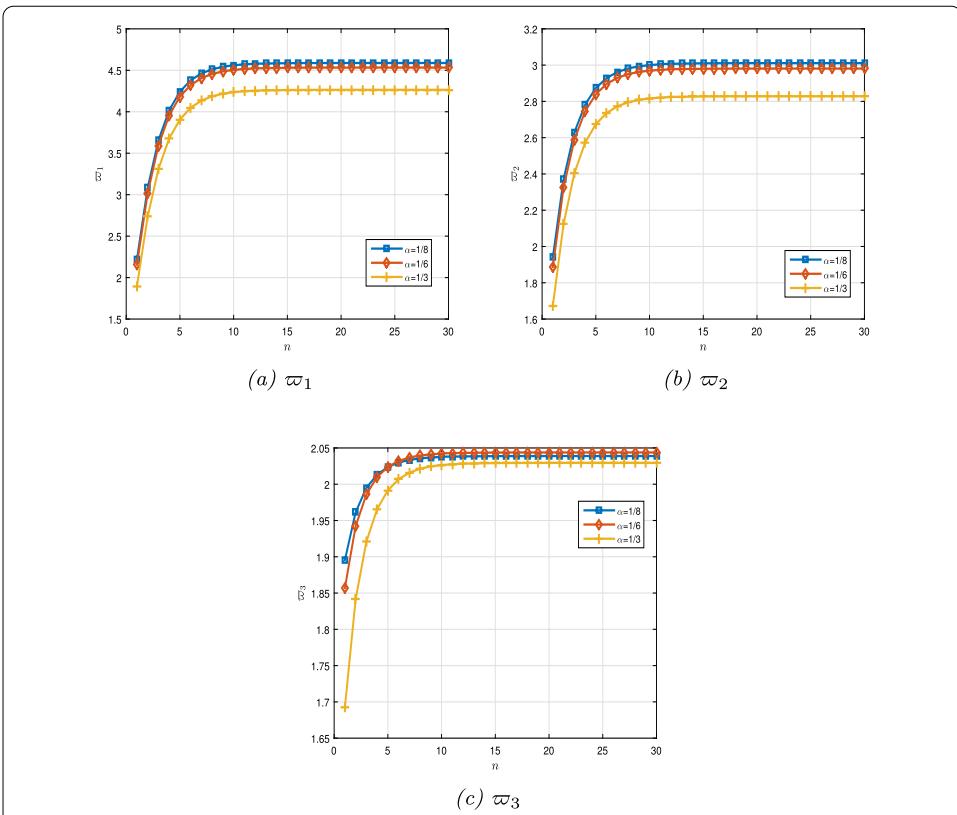


Figure 4 2D plot of ϖ_i , $i = 1, 2, 3$, for q -Caputo fractional JDEs (35) in Example 5.2 for three cases of derivative order α

Table 5 Numerical results of Eq. (36), ϑ_h and Ineq. (37) in Example 5.2 for three cases of derivative order α

For $h(\chi) = \chi^{\frac{\ln(3)}{5}}$, we have

$$\mathcal{I}_a^{\frac{1}{3} + \frac{4}{5} + \frac{3}{4}} [h(\chi)] = \mathcal{I}_a^{\frac{1}{3} + \frac{4}{5} + \frac{3}{4}} \left[\chi^{\frac{\ln(3)}{5}} \right]$$

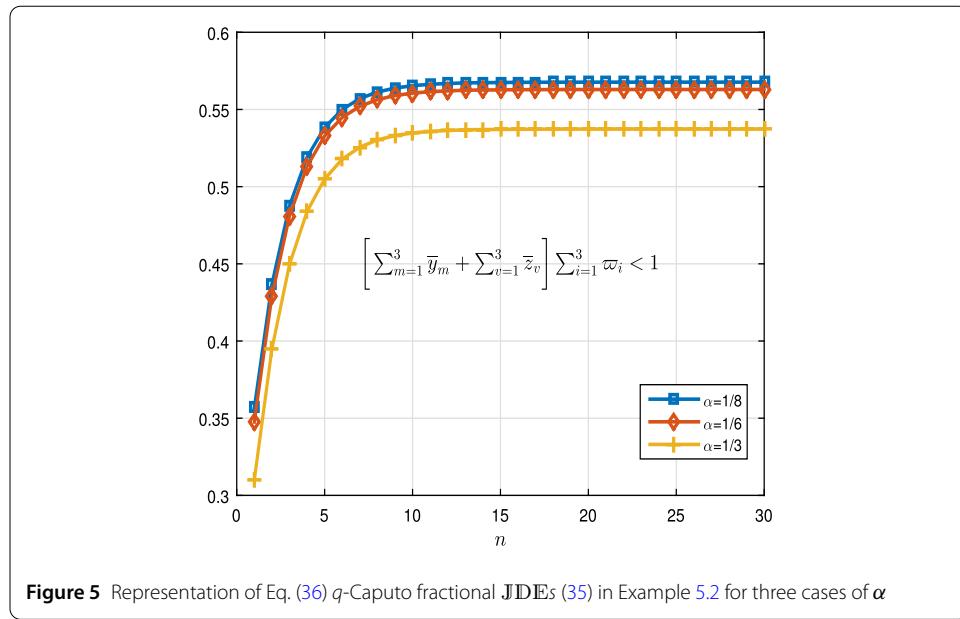


Figure 5 Representation of Eq. (36) q -Caputo fractional JDEs (35) in Example 5.2 for three cases of α

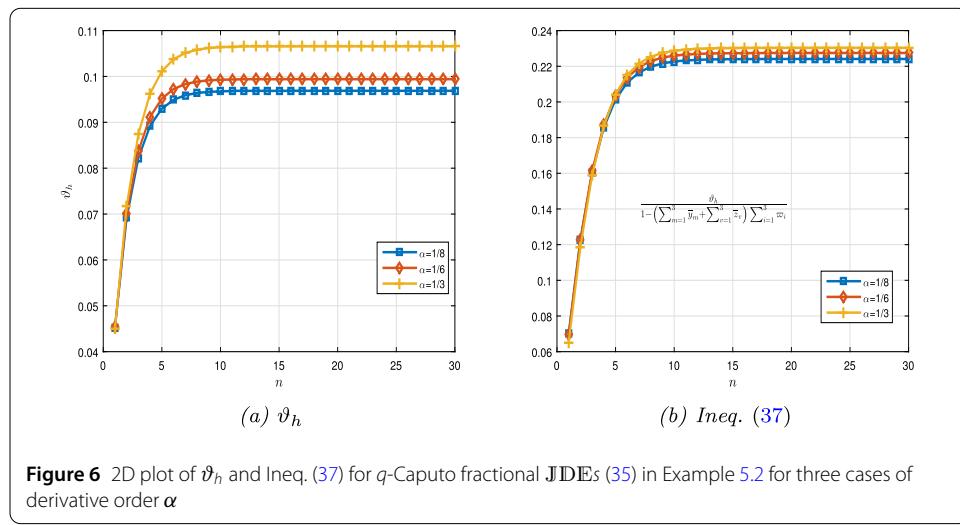


Figure 6 2D plot of ϑ_h and Ineq. (37) for q -Caputo fractional JDEs (35) in Example 5.2 for three cases of derivative order α

$$\approx \begin{cases} 0.097\chi^{\frac{\ln(3)}{5}}, & \alpha = \frac{1}{8}, \\ 0.099\chi^{\frac{\ln(3)}{5}}, & \alpha = \frac{1}{6}, \\ 0.107\chi^{\frac{\ln(3)}{5}}, & \alpha = \frac{1}{3}, \end{cases} = \vartheta_h h(\chi).$$

Table 5 shows these results. In addition, the curves drawn in Figs. 6a and 6b confirm the existence of ϑ_h and Ineq. (37) variables. Therefore, condition (H₇) is fulfilled with $h(\chi) = \chi^{\frac{\ln(3)}{5}}$ and $\vartheta_h = 0.097, 0.099, 0.107$ whenever $\alpha = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}$, respectively. Theorem 4.2 indicates that the q -FJP is \mathcal{UHR} (35) stable s.t.

$$\begin{aligned} \|v - \hat{v}\|_{\mathcal{F}} &\leq \frac{\vartheta_h}{1 - (\sum_{m=1}^3 \bar{y}_m + \sum_{v=1}^3 \bar{z}_v) \sum_{i=1}^3 \bar{w}_i} \bar{x} h(\chi) \\ &\approx \begin{cases} 0.224, & \alpha = \frac{1}{8}, \\ 0.227, & \alpha = \frac{1}{6}, \\ 0.230, & \alpha = \frac{1}{3}, \end{cases} \times \bar{x} h(\chi), \quad \bar{x} > 0, \chi \in \Lambda. \end{aligned} \tag{37}$$

6 Conclusion

We analyzed the q -FJIP, involving both ABCs and q -fractional CDs. Our main focus was on establishing certain conditions that guaranteed the EU of solution. For the validity of the suggested system, given in (1), we employed the Banach fixed point theorem and Leray-Schauder alternative. Additionally, we also explored the US outcomes and examined the resolution of our model (1) in specific circumstances. Our primary theoretical findings are demonstrated by means of a few examples.

Supplement

Algorithm 1 MATLAB lines for calculation q -factorial function

```

1  function p = qfactorialfunction (q,a,l)
2  s=1;
3  if l==0
4    p=s;
5  else
6    for i=0:l-1
7      s=s*(1-a*q^( i));
8    end;
9    p=s;
10 end;
11 end

```

Algorithm 2 MATLAB lines for q -gamma function

```

1  function p = qGammanew(q,x,n)
2  s=1;
3  for k=0:n-1
4    s=s*(1-q^(k+1))/(1-q^(x+k));
5  end;
6  p = s*(1-q)^(1-x);
7  end

```

Algorithm 3 MATLAB lines to calculate q -integral

```

1  function g= Iq_sigma(q,sigma,t,n,fun)
2  p=0;
3  for k=0:n
4    s=1;
5    for i=0:n
6      s=s*(1-q^(sigma+i-1))*(1-q^(k+i))/((1-q^( i+1)) ...
7      *(1-q^(sigma+k+i-1)));
8    end
9    p=p+s*q^k*eval(subs(fun ,t*q^k));
10 end;
11 g=round(p*(t^sigma)*(1-q)^sigma,6);
12 end

```

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Data Availability

No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate

Not applicable.

Consent for publication

Not applicable.

Competing interests

The authors declare no competing interests.

Author contributions

KHK: Actualization, methodology, formal analysis, validation, investigation and initial draft. AZ: Formal analysis, methodology, validation, investigation and initial draft. ILP: Methodology, formal analysis, validation and investigation. MES: Methodology, formal analysis, validation, actualization, investigation, software, simulation, initial draft and was a major contributor in writing the manuscript. All authors read and approved the final manuscript.

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