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Quasilinear Schrödinger equations with superlinear terms describing the Heisenberg ferromagnetic spin chain

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Abstract

In this paper, we consider a model problem arising from a classical planar Heisenberg ferromagnetic spin chain:

$$-\Delta u + V(x)u - \frac{u}{\sqrt{1-u^2}}\Delta\sqrt{1-u^2} = c|u|^{p-2}u, \quad x \in \mathbb{R}^N,$$

where $2 < p < 2^*$, $c > 0$ and $N \geq 3$. By the cutoff technique, the change of variables and the L^∞ estimate, we prove that there exists $c_0 > 0$, such that for any $c > c_0$ this problem admits a positive solution. Here, in contrast to the Morse iteration method, we construct the L^∞ estimate of the solution. In particular, we give the specific expression of c_0.

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Keywords: Schrödinger equations; L^∞ estimate; Heisenberg ferromagnet

1 Introduction

This paper is concerned with the existence of standing-wave solutions for quasilinear Schrödinger equations of the form

$$iz_t = -\Delta z + W(x)z - \rho\big(|z|^2\big)z - \kappa\,\Delta l\big(|z|^2\big)l'\big(|z|^2\big)z, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $W(x)$ is a given potential, κ is a real constant, and ρ, l are real functions of essentially pure power forms. Quasilinear equations of the form (1.1) appear more naturally in mathematical physics and have been derived as models of several physical phenomena corresponding to various types of l. For instance, the case of $l(s) = s$ is used for the superfluid film equation in plasma physics [1]. In the case $l(s) = (1+s)^{\frac{1}{2}}$, (1.1) models the self-channeling of a high-power ultrashort-wavelength laser in matter [2]. If $l(s) = (1-s)^{\frac{1}{2}}$, (1.1) also appears in the theory of the Heisenberg ferromagnetic spin chain. We refer to [3–6] and their references for more details on this subject.

Here, our special interest is in the existence of standing-wave solutions, that is, solutions of type $\phi(x,t) = \exp(iFt)u(x)$, where $F \in \mathbb{R}$ and $u > 0$ is a real function. It is well known that

ϕ satisfies (1.1) if and only if the function u solves the following equation of the elliptic type:

$$-\Delta u + V(x)u - \kappa \Delta l\big(u^2\big)l'\big(u^2\big)u = \rho\big(u^2\big)u, \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $V(x) = W(x) + F$ is the new potential function. If we let $l(s) = (1-s)^{\frac{1}{2}}$, $\rho(s) = \varepsilon'(1-s)^{-\frac{1}{2}}$ and $V(x) = \lambda + \varepsilon'$, we obtain the equation

$$-\Delta u + \lambda u - \frac{\kappa u}{\sqrt{1-u^2}}\Delta\sqrt{1-u^2} = \varepsilon'\frac{u}{\sqrt{1-u^2}} - \varepsilon' u, \quad x \in \mathbb{R}^N, \tag{1.3}$$

which originally appears in the Heisenberg ferromagnetic spin chain. In the mathematical literature, few results are known on (1.3). In a one-dimensional space, Brüll et al. [7] studied the ground states u for (1.3) with $\lim_{|x|\to\infty} u(x) = 0$. For a higher-dimensional space, in [4], Takeno and Homma constructed the expression of the solution to boundary value problems for second-order nonlinear ordinary differential equations.

More recently, Wang in [8] considered the following quasilinear Schrödinger equation:

$$-\Delta u + \lambda u - \frac{u}{\sqrt{1-u^2}}\Delta\sqrt{1-u^2} = \varepsilon'\frac{u}{\sqrt{1-u^2}} - \varepsilon' u, \quad x \in \mathbb{R}^3. \tag{1.4}$$

He generalized the result given in [7] to a three-dimensional space.

The main objective of the present paper is to study the following quasilinear Schrödinger equation

$$-\Delta u + V(x)u - \frac{u}{\sqrt{1-u^2}}\Delta\sqrt{1-u^2} = c|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.5}$$

that is, the case $l(s) = (1-s)^{\frac{1}{2}}$, $\rho(s) = cs^{\frac{p-2}{2}}$. To the best of our knowledge, up to now, there are no results for (1.5) on $\mathbb{R}^N$ for the superlinear case.

We observe that the critical point of the functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left[\left(1 - \frac{u^2}{1-u^2}\right)|\nabla u|^2 + V(x)u^2\right]\mathrm{d}x - \frac{c}{p}\int_{\mathbb{R}^N}|u|^p\,\mathrm{d}x \tag{1.6}$$

solves the Euler–Lagrange equation (1.5). From the variational point of view, there exist two difficulties to overcome for this functional (1.6). One is that the functional is not well defined in $H^1(\mathbb{R}^N)$. The other is how to guarantee the positiveness of the principle part. In order to overcome these two difficulties, we will focus on the following functional:

$$I_0(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left[\left(1 - \frac{\kappa u^2}{1-\kappa u^2}\right)|\nabla u|^2 + V(x)u^2\right]\mathrm{d}x - \frac{c\kappa^{\frac{p-2}{2}}}{p}\int_{\mathbb{R}^N}|u|^p\,\mathrm{d}x, \tag{1.7}$$

where $\kappa > 0$ is a constant. Obviously, if u_κ is a critical point of $I_0(u)$, then u_κ solves the equation

$$-\Delta u + V(x)u - \frac{u}{\sqrt{1-\kappa u^2}}\Delta\sqrt{1-\kappa u^2} = c\kappa^{\frac{p-2}{2}}|u|^{p-2}u, \quad x \in \mathbb{R}^N. \tag{1.8}$$

For the solution u_κ of (1.8), we rescale $u_\kappa = \kappa^{-\frac{1}{2}}u$. Then, u satisfies (1.5). Furthermore, according to [9], (1.8) can be reformulated as the following problems of the form:

$$-\operatorname{div}\bigl(g^2(u)\nabla u\bigr) + g(u)g'(u)|\nabla u|^2 + V(x)u = c\kappa^{\frac{p-2}{2}}|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.9}$$

where $g(t) = \sqrt{1 - \frac{\kappa t^2}{1-\kappa t^2}}$. It is obvious that $g(t)$ is a singular function. Now, to avoid the singularity, by using the cutoff technique introduced in [8], we continuously extend the domain of the function $g(t)$ to all of $[0, +\infty)$. More precisely, we consider the function

$$g_\kappa(t) = \begin{cases} \sqrt{1 - \frac{\kappa t^2}{1-\kappa t^2}} & \text{if } 0 \le t < \frac{1}{\sqrt{\theta\kappa}}; \\ \sqrt{\frac{2\theta}{(\theta-1)^2\sqrt{\theta\kappa}t} + \frac{\theta^2-5\theta+2}{(\theta-1)^2}} & \text{if } t \ge \frac{1}{\sqrt{\theta\kappa}}, \end{cases} \tag{1.10}$$

where $\theta > \frac{5+\sqrt{17}}{2}$. Clearly, $g_\kappa(t) \in C^1([0, +\infty), [0, +\infty))$ and $g_\kappa(t)$ decreases in $[0, +\infty)$. Substituting this form for $g(t)$ in (1.9), we obtain the following Schrödinger equation:

$$-\operatorname{div}\bigl(g_\kappa^2(u)\nabla u\bigr) + g_\kappa(u)g_\kappa'(u)|\nabla u|^2 + V(x)u = c\kappa^{\frac{p-2}{2}}|u|^{p-2}u, \quad x \in \mathbb{R}^N \tag{1.11}$$

and the critical point of the functional

$$I_\kappa(u) = \frac{1}{2}\int_{\mathbb{R}^N}\bigl[g_\kappa^2(u)|\nabla u|^2 + V(x)u^2\bigr]\,\mathrm{d}x - \frac{c\kappa^{\frac{p-2}{2}}}{p}\int_{\mathbb{R}^N}\bigl(u^+\bigr)^p\,\mathrm{d}x \tag{1.12}$$

satisfies the equation (1.11).

Here, the previously defined $g_\kappa(t)$ is obviously bounded satisfying $0 < a_1 \le g_\kappa(t) \le 1$, where $a_1 = \frac{\sqrt{\theta^2-5\theta+2}}{\theta-1}$. Hence, the functional $I_\kappa(u)$ is regular and nonsmooth. For the existence and the L^∞ estimate of the critical point of the functional (1.12), we follow the ideas shown in [9, 10] and make the change of variables:

$$v = G_\kappa(u) = \int_0^u g_\kappa(s)\,\mathrm{d}s, \quad u = G_\kappa^{-1}(v). \tag{1.13}$$

Thus, by using the change of variable (1.13), the nonsmooth functional $I_\kappa(u)$ can be transformed into a smooth functional

$$J_\kappa(v) = \frac{1}{2}\int_{\mathbb{R}^N}\bigl[|\nabla v|^2 + V(x)G_\kappa^{-1}(v)^2\bigr]\,\mathrm{d}x - \frac{c\kappa^{\frac{p-2}{2}}}{p}\int_{\mathbb{R}^N}\bigl(G_\kappa^{-1}(v)^+\bigr)^p\,\mathrm{d}x \tag{1.14}$$

and the quasilinear problem (1.11) is reformulated as a semilinear equation

$$-\Delta v + V(x)\frac{G_\kappa^{-1}(v)}{g_\kappa(G_\kappa^{-1}(v))} = c\kappa^{\frac{p-2}{2}}\frac{(G_\kappa^{-1}(v)^+)^{p-1}}{g_\kappa(G_\kappa^{-1}(v))}, \quad x \in \mathbb{R}^N. \tag{1.15}$$

Consequently, in order to find the nontrivial solutions of (1.11), it suffices to show the existence of the nontrivial solutions of (1.15). We also observe that if v_κ is a critical point of the functional $J_\kappa(v)$, then $u_\kappa = G_\kappa^{-1}(v_\kappa)$ is a solution of the problem (1.11). Hence, in this way we only need to discuss the existence of the critical point v_κ of the smooth functional $J_\kappa(v)$ by the critical-point theory. In what follows, we assume that

$$c\kappa^{\frac{p-2}{2}} = 1$$

only as a convenience. If we can prove that the critical point u_κ of the functional (1.12) satisfies

$$|u_\kappa|_\infty = \left|G_\kappa^{-1}(v_\kappa)\right|_\infty < \frac{1}{\sqrt{\theta\kappa}} = \theta^{-\frac{1}{2}} c^{\frac{1}{p-2}}, \tag{1.16}$$

then this function u_κ is good for what we want since $g_\kappa(u) = g(u) = \sqrt{1 - \frac{\kappa t^2}{1-\kappa t^2}}$ under this situation. That is, in this case, the functional (1.12) is exactly the functional (1.7) and thus u_κ is a weak solution of equations (1.8) and (1.9). Then, the function

$$u = \kappa^{\frac{1}{2}} u_\kappa = c^{-\frac{1}{p-2}} u_\kappa \tag{1.17}$$

is the solution of (1.5).

Based on the description in the previous paragraph, the key step is to construct the estimate of $|v_\kappa|_\infty$. Then, we can achieve the expression of c_0 by the inequality (1.16) such that, if $c > c_0$, (1.16) holds and so $u = c^{-\frac{1}{p-2}} u_\kappa$ solves the equation (1.5). To this aim, according to the arguments in [11], we first obtain the H^1 estimate of v_κ. Then, combining this H^1 estimate, we construct the L^∞ estimate $|v_\kappa|_\infty$. We must point out explicitly that, instead of the Morse iteration method used in [11], we use the method of converting integral inequalities into differential inequalities, which can be found in Lemma 5.1 on p. 71 in Ladyzhenskaya and Ural'tseva [12] and is used to study the L^∞ estimate of the nonlinear elliptic equations on bounded domains, to construct the estimate of $|v_\kappa|_\infty$. Moreover, all the constants in this estimate are well known.

Throughout this paper, we assume the potential $V(x) \in C^1(\mathbb{R}^N, \mathbb{R})$ satisfies

(V_1) $V(x) \geq V_0 > 0$;

(V_2) $V(x) \leq V_\infty$

and we make use of the following notations: Let X be the completion of the space $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \left[\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x\right]^{\frac{1}{2}}.$$

By (V_1) and (V_2), X is equivalent to $H^1(\mathbb{R}^N)$. The symbols $|u|_q$ and $|u|_\infty$ are used for the norm of the space $L^q(\mathbb{R}^N)$ with $2 \leq q < +\infty$ and $q = \infty$, respectively.

The corresponding result is as follows:

Theorem 1.1 *For all $\theta > \frac{5+\sqrt{17}}{2}$, let*

$$c_0 := \theta^{\frac{p-2}{2}} 2^b \theta_1^{-(p-2)(1+p(2^*-p))} C_N^{(p-2)(1+p(2^*-p))} \left(\frac{2p}{p-2} J_\infty(v_1)\right)^{(p-2)(1+(p-2)(2^*-p))}, \tag{1.18}$$

where $b = (p-2)(1 + 2^* - p + \frac{2}{2^*-p})$, $p > 2$, $\theta_1^2 = \frac{\theta^2-5\theta+2}{(\theta-1)^2}$, C_N is the best Sobolev constant and v_1 is the least energy solution of the functional

$$J_\infty(v) = \frac{1}{2}\int_{\mathbb{R}^N} \left[|\nabla v|^2 + V_\infty \theta_1^{-2} v^2\right] \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^N} |v|^p \,\mathrm{d}x.$$

Then, for $c > c_0$, the quasilinear problem (1.5) admits a solution u under the conditions (V_1) and (V_2).

Furthermore, we obtain a Pohozaev identity for this class of quasilinear equations, which is used to prove the nonexistence results of solution for (1.5), while we justify that $p = 2^*$ is the critical exponent for equation (1.5).

Theorem 1.2 *Suppose $p \geq 2^*$, $|u| \leq u_0 < 1$ for some u_0 and $V(x)$ satisfies $2V(x) + x \cdot \nabla V(x) \geq 0$ for all $x \in \mathbb{R}^N$. If $u \in C^2(\mathbb{R}^N)$ is a classical solution of (1.5), then $u \equiv 0$.*

2 The modified problem

In this section, we consider the following equation

$$-\Delta u + V(x)u - \frac{u}{\sqrt{1-\kappa u^2}}\Delta\sqrt{1-\kappa u^2} = c\kappa^{\frac{p-2}{2}}\left(u^+\right)^{p-1}, \quad x \in \mathbb{R}^N \text{ and } \kappa > 0. \tag{2.1}$$

If we rescale the solution of (2.1) $u_\kappa = \kappa^{-\frac{1}{2}}u$, then u solves

$$-\Delta u + V(x)u - \frac{u}{\sqrt{1-u^2}}\Delta\sqrt{1-u^2} = c\left(u^+\right)^{p-1}, \quad x \in \mathbb{R}^N. \tag{2.2}$$

In what follows, we will establish a positive solution of (2.1). To this aim, we first introduce $g_\kappa(t)$ defined in (1.10) and focus on the following Schrödinger equation:

$$-\operatorname{div}\left(g_\kappa^2(u)\nabla u\right) + g_\kappa(u)g_\kappa'(u)|\nabla u|^2 + V(x)u = c\kappa^{\frac{p-2}{2}}\left(u^+\right)^{p-1}, \quad x \in \mathbb{R}^N. \tag{2.3}$$

We will prove that there exists a positive solution u_κ for (2.3) with $|u_\kappa| \leq \frac{1}{\sqrt{\theta\kappa}}$. Direct calculation shows that u_κ is indeed a solution of (2.1) and thus $\sqrt{\kappa}u_\kappa$ is a solution of (2.2).

It is well known that (2.3) is the Euler–Lagrange equation associated with the energy functional

$$I_\kappa(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left[g_\kappa^2(u)|\nabla u|^2 + V(x)u^2\right]\mathrm{d}x - \frac{c\kappa^{\frac{p-2}{2}}}{p}\int_{\mathbb{R}^N}\left(u^+\right)^p\mathrm{d}x. \tag{2.4}$$

Thus, by using the change of variable (1.13) and recalling our assumption $c\kappa^{\frac{p-2}{2}} = 1$, the nonsmooth functional $I_\kappa(u)$ can be transformed into a smooth functional

$$J_\kappa(v) = \frac{1}{2}\int_{\mathbb{R}^N}\left[|\nabla v|^2 + V(x)G_\kappa^{-1}(v)^2\right]\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^N}\left(G_\kappa^{-1}(v)^+\right)^p\mathrm{d}x \tag{2.5}$$

and the quasilinear problem (2.3) is reformulated as a semilinear equation

$$-\Delta v + V(x)\frac{G_\kappa^{-1}(v)}{g_\kappa(G_\kappa^{-1}(v))} = \frac{(G_\kappa^{-1}(v)^+)^{p-1}}{g_\kappa(G_\kappa^{-1}(v))}, \quad x \in \mathbb{R}^N. \tag{2.6}$$

Therefore, in order to find the positive solution of (2.3), it suffices to study the solutions of (2.6) via the mountain-pass theorem. Thus, we need the following lemma to show some properties of the inverse function $G_\kappa^{-1}(t)$.

Lemma 2.1 *For any $\theta > \frac{5+\sqrt{17}}{2}$, we have*

(1). $\theta_1 := \frac{\sqrt{\theta^2-5\theta+2}}{\theta-1} < g_\kappa(t) \leq 1$ *for all* $t \geq 0$;

(2). $\lim_{t\to 0}\frac{G_\kappa^{-1}(t)}{t} = 1$;

(3). $\lim_{t\to\infty} \frac{G_\kappa^{-1}(t)}{t} = \frac{1}{\theta_1}$;
(4). $t \le G_\kappa^{-1}(t) \le \frac{1}{\theta_1} t$ *for all* $t \ge 0$;
(5). $-\frac{\theta}{(\theta-1)(\theta-2)} \le \frac{t}{g_\kappa(t)} g_\kappa'(t) \le 0$ *for all* $t \ge 0$.

Proof This lemma is mainly from [8], here the proof is provided to readers only as a convenience. By the definition of $g_\kappa(t)$ and L'Hospital's rule, properties (1)–(3) are obvious. By (1), for $t > 0$, we have $\theta_1 t \le G_\kappa(t) \le g_\kappa(0)t$, which implies (4). Now, we prove the property (5). If $t < \frac{1}{\sqrt{\theta\kappa}}$, we have

$$\frac{tg_\kappa'(t)}{g_\kappa(t)} = \frac{t(g_\kappa^2(t))'}{2g_\kappa^2(t)} = \frac{-\kappa t^2}{(1-\kappa t^2)(1-2\kappa t^2)} \ge -\frac{\theta}{(\theta-1)(\theta-2)}$$

by direct computation. If $t \ge \frac{1}{\sqrt{\theta\kappa}}$, we also have

$$\frac{tg_\kappa'(t)}{g_\kappa(t)} \ge -\frac{\theta}{(\theta-1)(\theta-2)}. \qquad \square$$

In the following lemma, we establish the geometric hypotheses of the mountain-pass theorem.

Lemma 2.2 *There exist* ρ_0, a_0, *such that* $J_\kappa(v) \ge a_0$ *for* $\|v\| = \rho_0$. *Moreover, there exists* $e \in H^1(\mathbb{R}^N)$ *such that* $J_\kappa(e) < 0$.

Proof By Lemma 2.1-(4) and Sobolev embedding, we have

$$\begin{aligned} J_\kappa(v) &= \frac{1}{2}\int_{\mathbb{R}^N} \left[|\nabla v|^2 + V(x)G_\kappa^{-1}(v)^2\right] \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^N} \left(G_\kappa^{-1}(v)^+\right)^p \mathrm{d}x \\ &\ge \frac{1}{2}\int_{\mathbb{R}^N} \left[|\nabla v|^2 + V(x)v^2\right] \mathrm{d}x - \frac{\theta_1^{-p}}{p}\int_{\mathbb{R}^N} \left(v^+\right)^p \mathrm{d}x \\ &\ge \frac{1}{2}\|v\|^2 - C\|v\|^p. \end{aligned}$$

Therefore, by choosing ρ_0 small, we know that

$$a_0 = \frac{1}{2}\rho_0^2 - C\rho_0^p > 0$$

and, hence,

$$J_\kappa(v) \ge a_0 \quad \text{for } \|v\| = \rho_0.$$

In order to prove the existence of $e \in \mathbb{R}^N$ such that $J_\kappa(e) < 0$, we fix $\psi \in C_0^\infty(\mathbb{R}^N, [0,1])$ with $\operatorname{supp}\phi = \bar{B}_1$. Thus, by Lemma 2.1-(4), we obtain

$$\begin{aligned} J_\kappa(t\phi) &= \frac{t^2}{2}\int_{\mathbb{R}^N} \left[|\nabla\phi|^2 + V(x)G_\kappa^{-1}(t\phi)^2\right] \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^N} \left(G_\kappa^{-1}(t\phi)^+\right)^p \mathrm{d}x \\ &\le \frac{t^2}{2}\int_{\mathbb{R}^N} \left[|\nabla\phi|^2 + \theta_1^{-2}V_\infty\phi^2\right] \mathrm{d}x - \frac{t^p}{p}\int_{\mathbb{R}^N} \phi^p \,\mathrm{d}x. \end{aligned}$$

Since $p > 2$, it follows that $J_\kappa(t\phi) \to -\infty$ as $t \to \infty$. Then, the result follows considering $e = t\phi$ for t large enough. □

In consequence of Lemma 2.2, we can apply the mountain-pass theorem without the (PS)-condition found in [13] to obtain a $(\mathrm{PS})_{c_\kappa}$ sequence $\{v_n\}$, where c_κ is the well-known mountain-pass level associated with the function J_κ, that is,

$$J_\kappa(v_n) \to c_\kappa \quad \text{and} \quad J'_\kappa(v_n) \to 0 \quad \text{as } n \to \infty.$$

Lemma 2.3 *The sequence $\{v_n\}$ is bounded.*

Proof As $\{v_n\} \subset H^1(\mathbb{R}^N)$ is a Palais–Smale sequence, we know that

$$\begin{aligned} J_\kappa(v_n) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(x)\left|G_\kappa^{-1}(v_n)\right|^2\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^N}\left(G_\kappa^{-1}(v_n)^+\right)^p\mathrm{d}x \\ &= c_\kappa + o(1). \end{aligned} \tag{2.7}$$

Moreover, for any $\phi \in H^1(\mathbb{R}^N)$, we have $J'_\kappa(v_n)\phi = o(1)\|\phi\|$, that is,

$$\int_{\mathbb{R}^N}\left[\nabla v_n \nabla\phi + V(x)\frac{G_\kappa^{-1}(v_n)}{g_\kappa(G_\kappa^{-1}(v_n))}\phi - \frac{(G_\kappa^{-1}(v_n)^+)^{p-1}}{g_\kappa(G_\kappa^{-1}(v_n))}\phi\right]\mathrm{d}x = o(1)\|\phi\|. \tag{2.8}$$

Now, fixing $\phi = G_\kappa^{-1}(v_n)g_\kappa(G_\kappa^{-1}(v_n))$, it follows from Lemma 2.1-(5) that

$$\left|\nabla\left(G_\kappa^{-1}(v_n)g_\kappa\left(G_\kappa^{-1}(v_n)\right)\right)\right| \le \left[1 + \frac{G_\kappa^{-1}(v_n)}{g_\kappa(G_\kappa^{-1}(v_n))}g'_\kappa\left(G_\kappa^{-1}(v_n)\right)\right]|\nabla v_n| \le |\nabla v_n|. \tag{2.9}$$

On the other hand, using Lemma 2.1-(1) and (4), we have

$$\left|G_\kappa^{-1}(v_n)g_\kappa\left(G_\kappa^{-1}(v_n)\right)\right| \le \theta_1^{-1}|v_n|. \tag{2.10}$$

Combining (2.9) and (2.10), we see that $\phi \in H^1(\mathbb{R}^N)$ with $\|\phi\| \le \theta_1^{-2}\|v_n\|$. Thus, using $\phi = G_\kappa^{-1}(v_n)g_\kappa(G_\kappa^{-1}(v_n))$ as a test function in (2.8), we derive that

$$\begin{aligned} o(1)\|v_n\| &= J'_\kappa(v_n)G_\kappa^{-1}(v_n)g_\kappa\left(G_\kappa^{-1}(v_n)\right) \\ &= \int_{\mathbb{R}^N}\left[\left(1 + \frac{G_\kappa^{-1}(v_n)}{g_\kappa(G_\kappa^{-1}(v_n))}g'_\kappa\left(G_\kappa^{-1}(v_n)\right)\right)|\nabla v_n|^2 + V(x)\left|G_\kappa^{-1}(v_n)\right|^2 \right. \\ &\quad \left. - \left(G_\kappa^{-1}(v_n)^+\right)^p\right]\mathrm{d}x \\ &\le \int_{\mathbb{R}^N}\left[|\nabla v_n|^2 + V(x)\left|G_\kappa^{-1}(v_n)\right|^2 - \left(G_\kappa^{-1}(v_n)^+\right)^p\right]\mathrm{d}x. \end{aligned} \tag{2.11}$$

Therefore, combining (2.7), (2.8), and (2.11), we infer the inequality

$$\begin{aligned} pc_\kappa + o(1) + o(1)\|v_n\| &= pJ_\kappa(v_n) - J'_\kappa(v_n)G_\kappa^{-1}(v_n)g_\kappa\left(G_\kappa^{-1}(v_n)\right) \\ &\ge \frac{p-2}{2}\int_{\mathbb{R}^N}\left[|\nabla v_n|^2 + V(x)\left|G_\kappa^{-1}(v_n)\right|^2\right]\mathrm{d}x \end{aligned}$$

$$\geq \frac{p-2}{2}\|v_n\|^2,$$

which shows the boundedness of $\{v_n\}$. □

Since $\{v_n\}$ is a bounded sequence in $H^1(\mathbb{R}^N)$, there exist $v_\kappa \in H^1(\mathbb{R}^N)$ and a subsequence of $\{v_n\}$, still denoted by itself, such that

$$v_n \rightharpoonup v_\kappa \quad \text{in } H^1\big(\mathbb{R}^N\big),$$
$$v_n \to v_\kappa \quad \text{in } L^q_{\mathrm{loc}}\big(\mathbb{R}^N\big) \text{ for } q \in [2, 2^*)$$

and

$$v_n(x) \to v_\kappa(x) \quad \text{a.e. in } \mathbb{R}^N.$$

Proposition 2.1 *The weak limit of v_κ of $\{v_n\}$ is a nontrivial critical point of J_κ and $J_\kappa(v_\kappa) \leq c_\kappa$.*

Proof To begin with, we first prove that v_κ is a weak solution. To this aim, we must prove that

$$J'_\kappa(v_\kappa)\phi = 0, \quad \forall \phi \in H^1\big(\mathbb{R}^N\big),$$

that is,

$$\int_{\mathbb{R}^N}\left[\nabla v_\kappa \nabla\phi + V(x)\frac{G_\kappa^{-1}(v_\kappa)}{g_\kappa(G_\kappa^{-1}(v_\kappa))}\phi - \frac{(G_\kappa^{-1}(v_\kappa)^+)^{p-1}}{g_\kappa(G_\kappa^{-1}(v_\kappa))}\phi\right]\mathrm{d}x = 0, \quad \forall \phi \in H^1\big(\mathbb{R}^N\big).$$

Since $C_0^\infty(\mathbb{R}^N)$ is dense in $\mathbb{R}^N$, we will show the last equality only for $\phi \in C_0^\infty(\mathbb{R}^N)$.

In what follows, for each $R > 0$, we consider $\phi_R \in C_0^\infty(\mathbb{R}^N)$ verifying

$$0 \leq \phi_R \leq 1 \quad \text{for } x \in \mathbb{R}^N,$$
$$\phi_R(x) = 1 \quad \text{for } x \in B_R(0)$$

and

$$\phi_R(x) = 0, \quad \forall x \in B_{2R}^c(0).$$

By [13], there is $z \in L^q(B_{2R}(0))$ such that

$$|v_n| \leq \big|z(x)\big| \quad \text{a.e. in } B_{2R}(0).$$

Consequently,

$$\frac{G_\kappa^{-1}(v_n)}{g_\kappa(G_\kappa^{-1}(v_n))}v_n \to \frac{G_\kappa^{-1}(v_\kappa)}{g_\kappa(G_\kappa^{-1}(v_\kappa))}v_\kappa \quad \text{a.e. in } B_{2R}(0)$$

and

$$\frac{(G_\kappa^{-1}(v_n)^+)^{p-1}}{g_\kappa(G_\kappa^{-1}(v_n))}v_n \to \frac{(G_\kappa^{-1}(v_\kappa)^+)^{p-1}}{g_\kappa(G_\kappa^{-1}(v_\kappa))}v_\kappa \quad \text{a.e. in } B_{2R}(0).$$

Moreover, by Lemma 2.1-(1) and (4),

$$\left|V(x)\frac{G_\kappa^{-1}(v_n)}{g_\kappa(G_\kappa^{-1}(v_n))}v_n\phi_R\right| \le \theta_1^{-2}V_\infty|v_n|^2|\phi_R| \le \theta_1^{-2}V_\infty\left|z(x)\right|^2|\phi_R|$$

and

$$\left|\frac{(G_\kappa^{-1}(v_n)^+)^{p-1}}{g_\kappa(G_\kappa^{-1}(v_n))}v_n\phi_R\right| \le \theta_1^{-(p-1)}|v_n|^p|\phi_R| \le \theta_1^{-(p-1)}\left|z(x)\right|^p|\phi_R|.$$

Hence, by the Lebesgue Dominate Theorem, we have

$$\int_{\mathbb{R}^N} V(x)\frac{G_\kappa^{-1}(v_n)}{g_\kappa(G_\kappa^{-1}(v_n))}v_n\phi_R\,\mathrm{d}x \to \int_{\mathbb{R}^N} V(x)\frac{G_\kappa^{-1}(v_\kappa)}{g_\kappa(G_\kappa^{-1}(v_\kappa))}v_\kappa\phi_R\,\mathrm{d}x \tag{2.12}$$

and

$$\int_{\mathbb{R}^N} \frac{(G_\kappa^{-1}(v_n)^+)^{p-1}}{g_\kappa(G_\kappa^{-1}(v_n))}v_n\phi_R\,\mathrm{d}x \to \int_{\mathbb{R}^N} \frac{(G_\kappa^{-1}(v_\kappa)^+)^{p-1}}{g_\kappa(G_\kappa^{-1}(v_\kappa))}v_\kappa\phi_R\,\mathrm{d}x. \tag{2.13}$$

The same type of arguments shows the limits below

$$\int_{\mathbb{R}^N} V(x)\frac{G_\kappa^{-1}(v_n)}{g_\kappa(G_\kappa^{-1}(v_n))}v_\kappa\phi_R\,\mathrm{d}x \to \int_{\mathbb{R}^N} V(x)\frac{G_\kappa^{-1}(v_\kappa)}{g_\kappa(G_\kappa^{-1}(v_\kappa))}v_\kappa\phi_R\,\mathrm{d}x \tag{2.14}$$

and

$$\int_{\mathbb{R}^N} \frac{(G_\kappa^{-1}(v_n)^+)^{p-1}}{g_\kappa(G_\kappa^{-1}(v_n))}v_\kappa\phi_R\,\mathrm{d}x \to \int_{\mathbb{R}^N} \frac{(G_\kappa^{-1}(v_\kappa)^+)^{p-1}}{g_\kappa(G_\kappa^{-1}(v_\kappa))}v_\kappa\phi_R\,\mathrm{d}x. \tag{2.15}$$

Now, the above limits combined with $J'_\kappa(v_n)(v_n\phi) = o_n(1)$ and $J'_\kappa(v_n)(v_\kappa\phi) = o_n(1)$ give

$$\int_{\mathbb{R}^N} |\nabla v_n - \nabla v_\kappa|^2\phi_R(x)\,\mathrm{d}x \to 0$$

and then it follows that

$$\int_{B_R(0)} |\nabla v_n - \nabla v_\kappa|^2\,\mathrm{d}x \to 0.$$

Recalling that R is arbitrary and $v_n \to v_\kappa$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, we are able to conclude the $v_n \to v_\kappa$ in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$. Thereby,

$$J'_\kappa(v_n)\phi \to J'_\kappa(v_\kappa)\phi, \quad \forall\phi \in C_0^\infty\left(\mathbb{R}^N\right).$$

Since $J'_\kappa(v_n)\phi = o_n(1)$, the last limit yields $J'_\kappa(v_\kappa)\phi = 0$ for all $\phi \in C_0^\infty(\mathbb{R}^N)$, that is, v_κ is a critical point for J_κ.

Now, we will show that $v_\kappa \not\equiv 0$. To this aim, we suppose that $v_\kappa = 0$ and claim that in this case $\{v_n\}$ is also a Palais–Smale sequence for functional $J_{\kappa,\infty} : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$J_{\kappa,\infty}(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x + \frac{1}{2}V_\infty \int_{\mathbb{R}^N} \left|G_\kappa^{-1}(v)\right|^2 \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^N} \left(G_\kappa^{-1}(v)^+\right)^p \mathrm{d}x. \tag{2.16}$$

On the other hand, we know that $V(x) \to V_\infty$ as $|x| \to \infty$, $|G_\kappa^{-1}(s)| \le \theta_1^{-1}|s|$ and $v_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, therefore

$$J_\kappa(v_n) - J_{\kappa,\infty}(v_n) = \frac{1}{2}\int_{\mathbb{R}^N} \left[V(x) - V_\infty\right]\left|G_\kappa^{-1}(v_n)\right|^2 \mathrm{d}x \to 0. \tag{2.17}$$

Moreover, as $\frac{|G^{-1}(s)|}{g(G^{-1}(s))} \le \theta_1^{-1}|s|$, it follows that

$$\sup_{\|\phi\|\le 1} \left|J'_\kappa(v_n)\phi - J'_{\kappa,\infty}(v_n)\phi\right| = \sup_{\|\phi\|\le 1} \left|\int_{\mathbb{R}^N} \left[V(x) - V_\infty\right] \frac{G_\kappa^{-1}(v_n)}{g_\kappa(G_\kappa^{-1}(v_\kappa))}\phi\,\mathrm{d}x\right| \to 0. \tag{2.18}$$

Next, we claim that for all $R > 0$, the vanishing

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |v_n|^2\,\mathrm{d}x = 0 \tag{2.19}$$

cannot occur. Suppose, by contradiction, that (2.19) occurs, then by Lions' compactness lemma [14], $v_n \to 0$ in $L^q(\mathbb{R}^N)$ for any $q \in (2, 2^*)$. Jointly with Lemma 2.1, we derive that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left(G_\kappa^{-1}(v_n)^+\right)^p \mathrm{d}x = 0$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{(G_\kappa^{-1}(v_n)^+)^{p-1}}{g_\kappa(G_\kappa^{-1}(v_n))} v_n\,\mathrm{d}x = 0.$$

Moreover, using the limits below

$$\lim_{s\to 0} \frac{1}{s^2}\left[\left|G_\kappa^{-1}(s)\right|^2 - \frac{G_\kappa^{-1}(s)}{g_\kappa(G_\kappa^{-1}(s)}s\right] = \lim_{s\to\infty} \frac{1}{|s|^p}\left[\left|G_\kappa^{-1}(s)\right|^2 - \frac{G_\kappa^{-1}(s)}{g_\kappa(G_\kappa^{-1}(s)}s\right] = 0,$$

we also have

$$\lim_{n\to\infty}\left[\left|G_\kappa^{-1}(v_n)\right|^2 - \frac{G_\kappa^{-1}(v_n)}{g_\kappa(G_\kappa^{-1}(v_n)}v_n\right]\mathrm{d}x = 0.$$

Therefore,

$$\begin{aligned} 2c_\kappa + o(1) &= 2J_\kappa(v_n) - J'_\kappa(v_n)v_n \\ &= \int_{\mathbb{R}^N}\left[\left|G_\kappa^{-1}(v_n)\right|^2 - \frac{G_\kappa^{-1}(v_n)}{g_\kappa(G_\kappa^{-1}(v_n))}v_n\right]\mathrm{d}x \\ &\quad - \frac{2}{p}\int_{\mathbb{R}^N}\left(G_\kappa^{-1}(v_n)^+\right)^p \mathrm{d}x + \int_{\mathbb{R}^N} \frac{(G_\kappa^{-1}(v_n)^+)^{p-1}}{g_\kappa(G_\kappa^{-1}(v_n))}v_n\,\mathrm{d}x \to 0, \end{aligned}$$

which is a contradiction, since $c_\kappa \ge a_0 > 0$.

Thus, $\{v_n\}$ does not vanish and there exist α, $R > 0$ and $\{y_n\} \subset \mathbb{R}^N$ verifying

$$\lim_{n\to\infty} \int_{B_R(y)} |v_n|^2 \, dx \geq \alpha > 0. \tag{2.20}$$

Setting $\tilde{v}_n = v_n(x + y_n)$ and using that $\{v_n\}$ is a Palais–Smale sequence for $J_{\kappa,\infty}$, we know that $\{\tilde{v}_n\}$ is also a Palais–Smale sequence for $J_{\kappa,\infty}$. Therefore, there is $\tilde{v}_\kappa \in H^1(\mathbb{R}^N)$ such that

$$\tilde{v}_n \to \tilde{v}_\kappa \quad \text{in } H^1_{\mathrm{loc}}(\mathbb{R}^N) \quad \text{and} \quad J'_{\kappa,\infty}(\tilde{v}_\kappa) = 0.$$

Furthermore, by (2.20), we also have $\tilde{v}_\kappa \neq 0$. Henceforward, without loss of generality, we assume that

$$\tilde{v}_n(x) \to \tilde{v}_\kappa(x) \quad \text{and} \quad \nabla\tilde{v}_n(x) \to \nabla\tilde{v}_\kappa(x) \quad \text{a.e. in } \mathbb{R}^N.$$

The last limit, together with Fatous' Lemma, lead to

$$\begin{aligned}
2c_\kappa &= \limsup_{n\to\infty}\bigl[2J_{\kappa,\infty}(\tilde{v}_n) - J'_{\kappa,\infty}(\tilde{v}_n)G_\kappa^{-1}(v_n)g_\kappa\bigl(G_\kappa^{-1}(\tilde{v}_n)\bigr)\bigr] \\
&= -\limsup_{n\to\infty}\biggl[\int_{\mathbb{R}^N} \frac{G_\kappa^{-1}(\tilde{v}_n)g'_\kappa(G_\kappa^{-1}(\tilde{v}_n))}{g_\kappa(G_\kappa^{-1}(\tilde{v}_n))}|\nabla\tilde{v}_n|^2 \, dx - \frac{2-p}{p}\int_{\mathbb{R}^N}\bigl(G_\kappa^{-1}(\tilde{v}_n)^+\bigr)^p \, dx\biggr] \\
&\geq -\int_{\mathbb{R}^N} \frac{G_\kappa^{-1}(\tilde{v}_\kappa)g'_\kappa(G_\kappa^{-1}(\tilde{v}_\kappa))}{g_\kappa(G_\kappa^{-1}(\tilde{v}_\kappa))}|\nabla\tilde{v}_\kappa|^2 \, dx - \frac{2-p}{p}\int_{\mathbb{R}^N}\bigl(G_\kappa^{-1}(\tilde{v}_\kappa)^+\bigr)^p \, dx \\
&= 2J_{\kappa,\infty}(\tilde{v}_\kappa) - J'_{\kappa,\infty}(\tilde{v}_\kappa)G_\kappa^{-1}(\tilde{v}_\kappa)g_\kappa\bigl(G_\kappa^{-1}(\tilde{v}_\kappa)\bigr) \\
&= 2J_{\kappa,\infty}(\tilde{v}_\kappa),
\end{aligned} \tag{2.21}$$

which shows that $J_{\kappa,\infty}(\tilde{v}_\kappa) \leq c_\kappa$. Now, following the arguments given in [15], if we define

$$\tilde{v}_{\kappa,t}(x) = \begin{cases} \tilde{v}_\kappa(x/t) & \text{if } t > 0; \\ 0 & \text{if } t = 0 \end{cases}$$

and $\gamma(t) = \tilde{v}_{\kappa,t}(x)$, we achieve

$$\max_{t\geq 0} J_{\kappa,\infty}\bigl(\gamma(t)\bigr) = J_{\kappa,\infty}(\tilde{v}_\kappa)$$

and $J_{\kappa,\infty}(\gamma(L)) < 0$ for sufficiently large $L > 1$. Then, by the definition of c_κ, there holds

$$c_\kappa \leq \max_{t\in[0,1]} J_\kappa\bigl(\hat{\gamma}(t)\bigr) := J_\kappa\bigl(\hat{\gamma}(\bar{t})\bigr) < J_{\kappa,\infty}\bigl(\hat{\gamma}(\bar{t})\bigr) \leq \max_{t\in[0,1]} J_{\kappa,\infty}\bigl(\gamma(t)\bigr) = J_{\kappa,\infty}(\tilde{v}_\kappa) \leq c_\kappa,$$

which is a contradiction. Thereby, v_κ is a nontrivial critical point for J_κ. Moreover, repeating the same type of arguments explored in (2.21), we have that $J_\kappa(v_\kappa) \leq c_\kappa$. □

3 L^∞ estimate

This section is mainly to show the L^∞ estimate of the function $v_\kappa = G_\kappa(u_\kappa)$ obtained in Proposition 2.1. To this aim, we need the following fact first to show the H^1 estimate of v_κ.

Lemma 3.1 *The solution v_κ satisfies $\|v_\kappa\|^2 \le \frac{2pc_\kappa}{p-2}$.*

Proof As v_κ is a critical point of J_κ, it follows that

$$\begin{aligned} pc_\kappa &= pJ_\kappa(v_\kappa) - J'(v_\kappa)G_\kappa^{-1}(v_\kappa)g_\kappa\big(G_\kappa^{-1}(v_\kappa)\big) \\ &\ge \frac{p-2}{2}\left[\int_{\mathbb{R}^N} |\nabla v_\kappa|^2\,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)\big|G_\kappa^{-1}(v_\kappa)\big|^2\,\mathrm{d}x\right]. \end{aligned}$$

Then, by Lemma 2.1-(4),

$$pc_\kappa \ge \frac{p-2}{2}\left[\int_{\mathbb{R}^N} |\nabla v_\kappa|^2\,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)|v_\kappa|^2\,\mathrm{d}x\right],$$

which implies that

$$\|v_\kappa\|^2 \le \frac{2pc_\kappa}{p-2}. \qquad \square$$

From now on, we consider the functional

$$J_\infty(v) = \frac{1}{2}\int_{\mathbb{R}^N}\big(|\nabla v|^2 + V_\infty\theta_1^{-2}v^2\big)\,\mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^N}\big(v^+\big)^p\,\mathrm{d}x$$

and we denote c_∞ the mountain-pass level associated with J_∞, which is independent of κ. Since $J_\kappa(v) \le J_\infty(v)$, we deduce that $c_\kappa \le c_\infty$. Consequently, by Lemma 3.1, the solution v_κ must satisfy the estimate

$$\|v_\kappa\|^2 \le \frac{2pc_\infty}{p-2}. \tag{3.1}$$

Now, we construct the estimate of $|v_\kappa|_\infty$ via the following two lemmas.

Lemma 3.2 *The solution v_κ of the semilinear equation (2.6) satisfies*

$$\int_{A_l} |v_\kappa - l|\,\mathrm{d}x \le 2C_N^2\theta_1^{-p}\alpha^{p-2}|A_l|^{1+a},$$

where $A_l = \{x \in \mathbb{R}^N : v_\kappa(x) > l\}$, $\alpha = |v_\kappa|_{2^*}$, $a = 1 - \frac{p}{2^*}$, and $|A_l|$ denotes the Lebesgue measure of the set A_l.

Proof For any $\phi \in H^1(\mathbb{R}^N)$, the solution v_κ of (2.6) satisfies

$$\int_{\mathbb{R}^N} \nabla v_\kappa \nabla\phi\,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)\frac{G_\kappa^{-1}(v_\kappa)\phi}{g_\kappa(G_\kappa^{-1}(v_\kappa))}\,\mathrm{d}x = \int_{\mathbb{R}^N} \frac{(G_\kappa^{-1}(v_\kappa)^+)^{p-1}\phi}{g_\kappa(G_\kappa^{-1}(v_\kappa))}\,\mathrm{d}x. \tag{3.2}$$

By taking $\phi = (v_\kappa - l)^+$ as a test function in (3.2) with $l > 0$, applying Lemma 2.1-(1) and (4), we have

$$\begin{aligned}\int_{A_l} |\nabla v_\kappa|^2 \,\mathrm{d}x &\le \int_{A_l} \frac{(G_\kappa^{-1}(v_\kappa)^+)^{p-1}(v_\kappa - l)^+}{g_\kappa(G_\kappa^{-1}(v_\kappa))}\,\mathrm{d}x \\ &\le \theta_1^{-1} \int_{A_l} \left|G_\kappa^{-1}(v_\kappa)\right|^{p-1}(v_\kappa - l)\,\mathrm{d}x \\ &\le \theta_1^{-1}\left(\int_{A_l} \left|G_\kappa^{-1}(v_\kappa)\right|^{2^*}\mathrm{d}x\right)^{\frac{p-1}{2^*}}\left(\int_{A_l} |v-l|^{2^*}\,\mathrm{d}x\right)^{\frac{1}{2^*}} |A_l|^{\frac{2^*-p}{2^*}} \\ &\le \theta_1^{-p}\alpha^{p-2}\left(\int_{A_l} |v_\kappa|^{2^*}\,\mathrm{d}x\right)^{\frac{1}{2^*}}\left(\int_{A_l} |v-l|^{2^*}\,\mathrm{d}x\right)^{\frac{1}{2^*}} |A_l|^{\frac{2^*-p}{2^*}}.\end{aligned} \tag{3.3}$$

Combining the Sobolev inequality

$$\left(\int_{A_l} |v_\kappa - l|^{2^*}\,\mathrm{d}x\right)^{\frac{1}{2^*}} \le C_N\left(\int_{A_l} |\nabla v_\kappa|^2\,\mathrm{d}x\right)^{\frac{1}{2}}$$

and the Minkowski inequality, we have

$$\begin{aligned}&\left(\int_{A_l} |v_\kappa - l|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}} \\ &\quad\le C_N^2\theta_1^{-p}\alpha^{p-2}\left[\left(\int_{A_l} |v_\kappa - l|^{2^*}\,\mathrm{d}x\right)^{\frac{1}{2^*}} + lA_l^{\frac{1}{2^*}}\right]\left(\int_{A_l} |v_\kappa - l|^{2^*}\,\mathrm{d}x\right)^{\frac{1}{2^*}} |A_l|^{\frac{2^*-p}{2^*}} \\ &\quad\le C_N^2\theta_1^{-p}\alpha^{p-2}\left[\left(\int_{A_l} |v_\kappa - l|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}} |A_l|^{\frac{2^*-p}{2^*}}\right. \\ &\qquad\left. + l|A_l|^{1-\frac{p-1}{2^*}}\left(\int_{A_l} |v_\kappa - l|^{2^*}\,\mathrm{d}x\right)^{\frac{1}{2^*}}\right].\end{aligned} \tag{3.4}$$

Moreover, by the Hölder inequality, we have

$$l|A_l| \le \int_{A_l} |v_\kappa|\,\mathrm{d}x \le \left(\int_{A_l} |v_\kappa|^{2^*}\,\mathrm{d}x\right)^{\frac{1}{2^*}} |A_l|^{1-\frac{1}{2^*}} \le \alpha|A_l|^{1-\frac{1}{2^*}},$$

that is,

$$|A_l| \le \left(\frac{\alpha}{l}\right)^{2^*}. \tag{3.5}$$

If we take $l_0 = \alpha(2C_N^2\theta_1^{-p}\alpha^{p-2})^{\frac{1}{2^*-p}}$, we have

$$C_N^2\theta_1^{-p}\alpha^{p-2}|A_{l_0}|^{\frac{2^*-p}{2^*}} \le C_N^2\theta_1^{-p}\alpha^{p-2}\left(\frac{\alpha}{l_0}\right)^{2^*-p} = \frac{1}{2}. \tag{3.6}$$

Consequently, combining (3.4) and (3.6), we conclude, if $l > l_0$, that

$$\left(\int_{A_l} |v_\kappa - l|^{2^*}\,\mathrm{d}x\right)^{\frac{1}{2^*}} \le 2C_N^2\theta_1^{-p}\alpha^{p-2}l|A_l|^{1-\frac{p-1}{2^*}}. \tag{3.7}$$

Thus, jointly with

$$\int_{A_l} |v_\kappa - l|\,\mathrm{d}x \le \left(\int_{A_l} |v_\kappa - l|^{2^*}\,\mathrm{d}x\right)^{\frac{1}{2^*}} |A_l|^{1-\frac{1}{2^*}},$$

we finally have

$$\int_{A_l} |v_\kappa - l|\,\mathrm{d}x \le 2C_N^2\theta_1^{-p}\alpha^{p-2}l|A_l|^{1+a}. \tag{3.8}$$

Lemma 3.3 *The solution v_κ of the semilinear equation* (2.6) *has the following estimate*:

$$|v_\kappa|_\infty \le 2^{1+\frac{1}{a}}\left(2\theta_1^{-p}C_N^2\right)^{2^*-p}\alpha^{1+(p-2)(2^*-p)}.$$

Proof Inspired by Lemma 5.1 of [12], we consider the function

$$f(l) = \int_{A_l} |v_\kappa - l|\,\mathrm{d}x.$$

For this function, we have $-f'(l) = |A_l|$. Therefore, (3.8) can be rewritten as

$$f(l) \le 2C_N^2\theta_1^{-p}\alpha^{p-2}l\left(-f'(l)\right)^{1+a}. \tag{3.9}$$

If we integrate this inequality with respect to l from l_0 to $l_{\max} := |v_\kappa|_\infty$, we obtain

$$l_{\max}^{\frac{a}{1+a}} \le l_0^{\frac{a}{1+a}} + \left(2C_N^2\theta_1^{-p}\alpha^{p-2}\right)^{\frac{1}{1+a}}\left(f(l_0)^{\frac{a}{1+a}} - f(l_{\max})^{\frac{a}{1+a}}\right).$$

Moreover, jointly with (3.5), recalling that $l_0 = \alpha(2C_N^2\theta_1^{-p}\alpha^{p-2})^{\frac{1}{2^*-p}}$, we infer that

$$|A_{l_0}|^a \le \left(\frac{\alpha}{l_0}\right)^{2^*a} = \left(2C_N^2\theta_1^{-p}\alpha^{p-2}\right)^{-1}$$

and then, by (3.8),

$$\left(f(l_0)\right)^{\frac{a}{1+a}}\left(2\theta_1^{-p}C_N\right)^{\frac{1}{1+a}} \le \left(2C_N^2\theta_1^{-p}\alpha^{p-2}l_0|A_{l_0}|^{1+a}\right)^{\frac{a}{1+a}}\left(2\theta_1^{-p}C_N\right)^{\frac{1}{1+a}} = l_0^{\frac{a}{1+a}}. \tag{3.10}$$

Therefore, we have

$$l_{\max}^{\frac{a}{1+a}} \le 2l_0^{\frac{a}{1+a}},$$

which implies the desired inequality

$$|v_\kappa|_\infty = l_{\max} \le 2^{1+\frac{1}{a}}\left(2\theta_1^{-p}C_N^2\right)^{2^*-p}\alpha^{1+(p-2)(2^*-p)}$$

$$= 2^{b_1}\left(\theta_1^{-p}C_N^2\right)^{2^*-p}\alpha^{1+(p-2)(2^*-p)},$$

where $b_1 = 1 + \frac{1}{a} + 2^* - p$. □

4 Proof of Theorem 1.1

Proof of Theorem 1.1 A direct consequence of Proposition 2.1 and Lemma 3.3 is that $v_\kappa = G_\kappa(u_\kappa)$ solves (1.15) and has the estimate

$$|v_\kappa|_\infty \le 2^{b_1}\left(\theta_1^{-p}C_N^2\right)^{2^*-p}\alpha^{1+(p-2)(2^*-p)}. \tag{4.1}$$

Combining Lemma 2.1-(4) and (3.1), we infer that

$$\begin{aligned} |u_\kappa|_\infty \le \theta_1^{-1}|v_\kappa|_\infty &\le \theta_1^{-1}2^{b_1}\left(\theta_1^{-p}C_N^2\right)^{2^*-p}\alpha^{1+(p-2)(2^*-p)} \\ &\le \theta_1^{-1}2^{b_1}\left(\theta_1^{-p}C_N^2\right)^{2^*-p}\left(C_N\|v_\kappa\|\right)^{1+(p-2)(2^*-p)} \\ &\le \theta_1^{-1}2^{b_1}\theta_1^{-p(2^*-p)}C_N^{1+p(2^*-p)}\left(\frac{2p}{p-2}J_\infty(v_1)\right)^{1+(p-2)(2^*-p)}. \end{aligned} \tag{4.2}$$

Now, to ensure that

$$|u_\kappa|_\infty < \frac{1}{\sqrt{\theta\kappa}} = \theta^{-\frac{1}{2}}c^{\frac{1}{p-2}}, \tag{4.3}$$

we select

$$c_0 = 2^{b_1(p-2)}\theta^{\frac{p-2}{2}}\theta_1^{-(p-2)(1+p(2^*-p))}C_N^{(p-2)(1+p(2^*-p))}\left(\frac{2p}{p-2}J_\infty(v_1)\right)^{(p-2)(1+(p-2)(2^*-p))}.$$

Thus, inequality (4.3) can be satisfied if only $c > c_0$. Obviously, equation (1.11) is indeed equation (1.8) under the situation of $|u_\kappa|_\infty < \frac{1}{\sqrt{\theta\kappa}}$. Hence, u_κ solves (1.8) and then $u = \kappa^{\frac{1}{2}}u_\kappa = c^{-\frac{1}{p-2}}u_\kappa$ is the solution of (1.5). Thus, we complete the proof. □

5 Proof of Theorem 1.2

In this section, we will prove the nonexistence results for equation (1.5). To this aim, we first show a Pohozaev identity and we justify that the critical exponent for this class of problems is 2^*.

Lemma 5.1 (Pohozaev identity). *Suppose $F(x,u,r) \in C^1(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ satisfies*

$$\operatorname{div} F_r(x,u,\nabla u) = F_u(x,u,\nabla u), \tag{5.1}$$

where

$$F_r(x,u,r) = \left(F_{r_1}(x,u,r), F_{r_2}(x,u,r), \dots, F_{r_N}(x,u,r),\right), \quad r = (r_1, r_2, \dots, r_N),$$

$$F_{r_i}(x,u,r) = \frac{\partial F(x,u,r)}{\partial r_i}, \quad i = 1,2,\dots,N$$

and

$$F_u(x,u,r) = \frac{\partial F(x,u,r)}{\partial u}.$$

Then, if $F(x,u,\nabla u)$, $x \cdot F_x(x,u,\nabla u)$, *and* $F_r(x,u,\nabla u)\cdot\nabla u \in L^1(\mathbb{R}^N)$, *there holds the following identity*

$$N\int_{\mathbb{R}^N} F(x,u,\nabla u)\,\mathrm{d}x + \int_{\mathbb{R}^N} x\cdot F_x(x,u,\nabla u)\,\mathrm{d}x - \int_{\mathbb{R}^N} F_r(x,u,\nabla u)\cdot\nabla u\,\mathrm{d}x = 0. \tag{5.2}$$

We omit the proof of this lemma, since it can be mainly found in [16].

To present the Pohozaev identity associated to (1.5), we rewrite equation (1.5) as

$$\operatorname{div}\left(\left(1-\frac{u^2}{1-u^2}\right)\nabla u\right) - \frac{u|\nabla u|^2}{(1-u^2)^2} + V(x)u = c|u|^{p-2}u. \tag{5.3}$$

Thus, the integrands in (5.2) can be expressed as

$$\begin{aligned} F(x,u,\nabla u) &= \frac{1}{2}\left(1-\frac{u^2}{1-u^2}\right)|\nabla u|^2 + \frac{V(x)}{2}u^2 - \frac{c|u|^p}{p}, \\ x\cdot F_x(x,u,\nabla u) &= \frac{1}{2}\big(x\cdot\nabla V(x)\big)u^2 \end{aligned}$$

and

$$F_r(x,u,\nabla u)\cdot\nabla u = \left(1-\frac{u^2}{1-u^2}\right)|\nabla u|^2.$$

Moreover, if $|u| \le u_0 < 1$, we have

$$\left|1-\frac{u^2}{1-u^2}\right| \le C.$$

Consequently, we achieve the following lemma based on Lemma 5.1 under the conditions $|\nabla u|^2$, $V(x)u^2$, $(x\cdot\nabla V(x))u^2$, and $u^p \in L^1(\mathbb{R}^N)$.

Lemma 5.2 *Suppose that* $u \in C^2(\mathbb{R}^N)$ *is a solution of* (1.5) *and* $|u| \le u_0 < 1$. *Then,*

$$\begin{aligned} &\frac{N-2}{2}\int_{\mathbb{R}^N}\left(1-\frac{u^2}{1-u^2}\right)|\nabla u|^2\,\mathrm{d}x + \int_{\mathbb{R}^N}\frac{NV(x)+(x\cdot\nabla V(x))}{2}u^2\,\mathrm{d}x \\ &\quad = \frac{cN}{p}\int_{\mathbb{R}^N}|u|^p\,\mathrm{d}x \end{aligned} \tag{5.4}$$

if $|\nabla u|^2$, $V(x)u^2$, $(x\cdot\nabla V(x))u^2$, *and* $u^p \in L^1(\mathbb{R}^N)$.

Now, we show the nonexistence result of the solution for (1.5).

Proof of Theorem 1.2 On the one hand, the Pohozaev identity associated to (1.5) is

$$\begin{aligned}&\int_{\mathbb{R}^N}\left(1-\frac{u^2}{1-u^2}\right)|\nabla u|^2\,\mathrm{d}x+\frac{1}{N-2}\int_{\mathbb{R}^N}\big(NV(x)+\big(x\cdot\nabla V(x)\big)\big)u^2\,\mathrm{d}x\\&\quad=\frac{\eta 2^*}{p}\int_{\mathbb{R}^N}|u|^p\,\mathrm{d}x.\end{aligned}\tag{5.5}$$

On the other hand, multiplying (5.3) by u and integrating it, we have

$$\int_{\mathbb{R}^N}\left(1-\frac{u^2}{1-u^2}\right)|\nabla u|^2\,\mathrm{d}x-\int_{\mathbb{R}^N}\frac{u^2|\nabla u|^2}{(1-u^2)^2}\,\mathrm{d}x+\int_{\mathbb{R}^N}V(x)u^2\,\mathrm{d}x=\eta\int_{\mathbb{R}^N}|u|^p\,\mathrm{d}x.\tag{5.6}$$

Combining (5.5) and (5.6), it follows that

$$\begin{aligned}&\int_{\mathbb{R}^N}\frac{u^2|\nabla u|^2}{(1-u^2)^2}\,\mathrm{d}x+\frac{1}{N-2}\int_{\mathbb{R}^N}\big(2V(x)+\big(x\cdot\nabla V(x)\big)\big)u^2\,\mathrm{d}x\\&\quad=\eta\left(\frac{2^*}{p}-1\right)\int_{\mathbb{R}^N}|u|^p\,\mathrm{d}x.\end{aligned}\tag{5.7}$$

Thus, if $p\geq 2^*$ and $2V(x)+(x\cdot\nabla V(x))\geq 0$, we conclude that

$$\int_{\mathbb{R}^N}\frac{u^2|\nabla u|^2}{(1-u^2)^2}\,\mathrm{d}x=0,$$

which implies that $u=0$ and we complete the proof of Theorem 1.2. □

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Declarations

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The authors declare no competing interests.

Author contributions
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