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Optimal decay-in-time rates of solutions to the Cauchy problem of 3D compressible magneto-micropolar fluids

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Abstract

This paper focuses on the long time behavior of the solutions to the Cauchy problem of the three-dimensional compressible magneto-micropolar fluids. More precisely, we aim to establish the optimal rates of temporal decay for the highest-order spatial derivatives of the global strong solutions by the method of decomposing frequency. Our result can be regarded as the further investigation of the one in (Wei, Guo and Li in J. Differ. Equ. 263:2457–2480, 2017), in which the authors only provided the optimal rates of temporal decay for the lower-order spatial derivatives of the perturbations of both the velocity and the micro-rotational velocity.

Keywords: Compressible magneto-micropolar fluids; Optimal rates of temporal decay; Highest-order spatial derivatives; Frequency splitting

1 Introduction

The three-dimensional (3D) motion of compressible magneto-micropolar fluids can be described as the following system [19]:

$$
\begin{cases}
\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0, \\
(\rho\mathbf{u})_t + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P(\rho) \\
\quad = (\mu+\nu)\Delta u + (\mu+\lambda-\nu)\nabla\operatorname{div}\mathbf{u} + 2\nu\nabla\times\boldsymbol{\omega} + (\nabla\times\mathbf{B})\times\mathbf{B}, \\
(\rho\boldsymbol{\omega})_t + \nabla\cdot(\rho\mathbf{u}\otimes\boldsymbol{\omega}) + 4\nu\boldsymbol{\omega} = \mu'\Delta\boldsymbol{\omega} + (\mu'+\lambda')\nabla\operatorname{div}\boldsymbol{\omega} + 2\nu\nabla\times\mathbf{u}, \\
\mathbf{B}_t - \nabla\times(\mathbf{u}\times\mathbf{B}) = -\nabla\times(\sigma\nabla\times\mathbf{B}), \\
\nabla\cdot\mathbf{B} = 0.
\end{cases}
\tag{1.1}
$$

Here the unknowns $\rho(x,t) > 0$, $\mathbf{u}(x,t)$, $\boldsymbol{\omega}(x,t)$, and $\mathbf{B}(x,t)$ mean the density, the velocity, the micro-rotational velocity, and the magnetic field, respectively. The pressure $P(\rho)$ is a smooth and strictly increasing scalar function. The positive constant ν is the so-called dynamics micro-rotation viscosity. Both the shear and bulk viscosity coefficients of fluids are represented by the parameters μ and λ satisfying that $\mu > 0$ and $2\mu + 3\lambda - 4\nu \geq 0$. Besides, the angular viscosity coefficients μ' and λ' fulfill the conditions $\mu' > 0$ and $2\mu' + 3\lambda' \geq 0$. The magnetic diffusivity coefficient is denoted by σ.

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While the magnetic field is absent, i.e., $\mathbf{B} = 0$, (1.1) reduces to the micropolar system. Huang, Kong, and Lian [9, 10] obtained the exponential stability of the generalized spherically symmetric solutions. For the Cauchy problem of the 3D compressible viscous micropolar system, Liu and Zhang [13] derived the optimal rates of decay-in-time of the global strong solutions with the smallness of initial perturbation in $H^N(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ with $N \geq 4$. Furthermore, Tong, Pan, and Tan [18] used the method of spectrum analyzing to establish both the lower and upper decay-in-time rates for the solutions, which explicitly shows that the obtained convergence rates are optimal. Recently, Qin and Zhang [16] investigated the optimal decay rates for higher-order derivatives of solutions of the 3D compressible micropolar fluids system. In particular, the authors showed that the highest-order spatial derivatives of both the perturbation density and the perturbation velocity converge to zero with the decay rate $(1 + t)^{-(\frac{3}{4}+\frac{N}{2})}$ for the $L^2(\mathbb{R}^3)$-norm.

Due to the strong nonlinearity and interactions among the physical quantities, it becomes more difficult to analyze the compressible magneto-micropolar system, i.e., $\mathbf{B} \neq 0$ in system (1.1). Amirat and Hamdache [1] extended the results in [5, 12] by establishing the global existence of weak solutions with finite energy for multi-dimensional compressible magneto-micropolar equations. The blow-up criterion of strong solutions to system (1.1) with initial vacuum can be referred to [22]. By employing L^p–L^q estimates for the linearized equations and the Fourier splitting method, Wei, Guo, and Li [19] first obtained the global-in-time existence and optimal temporal decay rates of the strong solutions to the Cauchy problem of system (1.1), in which the results can be read as follows: Supposing the initial data $(\rho_0, \mathbf{u}_0, \boldsymbol{\omega}_0, \mathbf{B}_0) \in H^N(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ satisfy

$$(\rho, \mathbf{u}, \boldsymbol{\omega}, \mathbf{B})|_{t=0} = (\rho_0, \mathbf{u}_0, \boldsymbol{\omega}, \mathbf{B}_0)(x,t) \to (\bar{\rho}, 0, 0, 0) \quad \text{as } |x| \to \infty \tag{1.2}$$

and

$$\left\|(\rho_0 - \bar{\rho}, \mathbf{u}_0, \boldsymbol{\omega}_0, \mathbf{B}_0)\right\|_{H^N(\mathbb{R}^3)} \leq \delta,$$

where the integer $N \geq 3$ and δ is small enough, then Cauchy problem (1.1)–(1.2) admits unique global-in-time strong solutions such that

$$\left\|\nabla^l(\rho - \bar{\rho}, \mathbf{u})(t)\right\|_{H^{N-l}(\mathbb{R}^3)} \leq C(1 + t)^{-\frac{3+2l}{4}}, \tag{1.3}$$

$$\left\|\nabla^l\boldsymbol{\omega}(t)\right\|_{H^{N-l}(\mathbb{R}^3)} \leq C(1 + t)^{-\frac{3+2(l+1)}{4}}, \tag{1.4}$$

$$\left\|\nabla^k\mathbf{B}(t)\right\|_{H^{N-k}(\mathbb{R}^3)} \leq C(1 + t)^{-\frac{3+2k}{4}} \tag{1.5}$$

for $l = 0, 1, \ldots, N - 1$ and $k = 0, 1, \ldots, N$. Without the L^1-integrability of the initial data, Jia, Tan, and Zhou [11] recently derived the temporal decay estimates of the solution in the homogeneous Sobolev and Besov spaces. For other mathematical issues of system (1.1), the interested readers can refer to [2, 4, 6, 7, 9, 10, 17, 20–24].

However, there is no available result about the long time behavior of the highest-order spatial derivatives of the perturbation $(\rho - \bar{\rho}, \mathbf{u}, \boldsymbol{\omega})$. In this paper, inspired by the new method of decomposing the frequency in [15], we establish the optimal temporal decay rates of the highest-order (i.e., Nth order) spatial derivatives of the global strong solution for the Cauchy problem (1.1)–(1.2).

Notation Before stating the main results, we shall introduce some basic notations used frequently in the sequel. The norm of Sobolev space $H^k(\mathbb{R}^3)$ is denoted by $\|\cdot\|_{H^k}$. We use $\langle f,g\rangle$ to denote the L^2-inner product between the functions f and g. L^p represents the usual Lebesgue space $L^p(\mathbb{R}^3)$ with the norm $\|\cdot\|_{L^p}$, where $1\le p\le\infty$. Enabling the cut-off function $\phi\in C_0^\infty(\mathbb{R}^3_\xi)$ to satisfy $\phi(\xi)=1$ when $|\xi|\le 1$ and $\phi(\xi)=0$ when $|\xi|\ge 2$, we define both the low-frequency and the high-frequency parts of f as follows:

$$f^l=\mathfrak{F}^{-1}\big[\phi(\xi)\widehat{f}\big] \quad \text{and} \quad f^h=\mathfrak{F}^{-1}\big[\big(1-\phi(\xi)\big)\widehat{f}\big],$$

where $\mathfrak{F}(f)$ or $\widehat{f}$ denotes the Fourier transform of f, and $\mathfrak{F}^{-1}$ is its inverse. The notation $f\lesssim g$ signifies that $f\le Cg$ with $C>0$ being a common constant that may vary from one line to another. $f\approx g$ means that $f\lesssim g$ and $g\lesssim f$. For simplicity, we denote $\|A\|_X+\|B\|_X$ by $\|(A,B)\|_X$.

Now we are in a position to state our main theorem.

Theorem 1.1 *Suppose* $(\rho_0-\bar{\rho},\mathbf{u}_0,\boldsymbol{\omega}_0,\mathbf{B}_0)\in H^N(\mathbb{R}^3)$ *with any given integer* $N\ge 3$. *There exists a constant* $\delta>0$ *such that if*

$$\big\|(\rho_0-\bar{\rho},\mathbf{u_0},\boldsymbol{\omega}_0,\mathbf{B}_0)\big\|_{H^N}\le\delta,$$

then Cauchy problem (1.1)–(1.2) *admits a unique global-in-time solution* $(\rho,\mathbf{u},\boldsymbol{\omega},\mathbf{B})$ *satisfying*

$$\begin{aligned}&\big\|(\rho-\bar{\rho},\mathbf{u},\boldsymbol{\omega},\mathbf{B})(t)\big\|_{H^N}^2+\int_0^t\Big(\big\|\nabla\rho(\tau)\big\|_{H^{N-1}}^2+\big\|\nabla(\mathbf{u},\boldsymbol{\omega},\mathbf{B})(\tau)\big\|_{H^N}^2\Big)\,\mathrm{d}\tau\\&\quad\lesssim\big\|(\rho_0-\bar{\rho},\mathbf{u}_0,\ \boldsymbol{\omega}_0,\mathbf{B}_0)\big\|_{H^N}^2.\end{aligned}\tag{1.6}$$

Moreover, if $(\rho_0-\bar{\rho},\mathbf{u}_0,\boldsymbol{\omega}_0,\mathbf{B}_0)\in L^1(\mathbb{R}^3)$, *then the following temporal decay estimates hold*:

$$\big\|\nabla^N(\rho-\bar{\rho},\mathbf{u},\mathbf{B})(t)\big\|_{L^2}\lesssim(1+t)^{-(\frac{3}{4}+\frac{N}{2})}\tag{1.7}$$

and

$$\big\|\nabla^N(\rho-\bar{\rho})^h,\mathbf{u}^h,\mathbf{B}^h)(t)\big\|_{L^2}+\big\|\nabla^N\boldsymbol{\omega}(t)\big\|_{L^2}\lesssim(1+t)^{-(\frac{5}{4}+\frac{N}{2})}.\tag{1.8}$$

Remark 1.1 The temporal decay rate in (1.7) is optimal in the sense that it is the same as the one of linear solutions shown in Lemma 2.1. In addition, the estimate (1.8) implies that the L^2-norms of any Nth order derivatives of the micro-rotational velocity approach zero along with a rate of decay-in-time $(1+t)^{(\frac{5}{4}+\frac{N}{2})}$. This convergence rate is quicker than the ones of both the density and the velocity.

Since the global-in-time existence and the *a priori* energy estimate (1.6) of the solution have been proved in [19], it suffices to establish both the temporal decay estimates (1.7) and (1.8). The remarkable thing is that, thanks to the formula (A.10), the high-frequency parts of solutions exhibit exponential decay-in-time by using an energy method, see the proof of Lemma 2.5. However, such property can not be expected for the lower frequency components. Therefore, we will first derive the decay-in-time estimates of the low-frequency parts in Sect. 2.1, and then the decay estimates of the high-frequency parts in Sect. 2.2. Based to the decay estimates as mentioned earlier and the formula (A.8), we further arrive at (1.7) (see (2.61) for the detailed derivation). Finally, we derive the faster decay-in-time (1.8) for the high-frequency parts and the micro-rotational velocity by a finer energy method in Sect. 3.

2 Decay-in-time of highest-order spatial derivatives

This section is served to establish the optimal temporal decay rate for the Nth order spatial derivatives of the solution $(\rho, \mathbf{u}, \mathbf{B})$ to Cauchy problem (1.1)–(1.2). Making use of the formulas

$$\begin{aligned}
&(\operatorname{curl}\mathbf{B}) \times \mathbf{B} = (\mathbf{B}\cdot\nabla)\mathbf{B} - \frac{1}{2}\nabla\big(|\mathbf{B}|^2\big),\\
&\Delta\mathbf{B} = \nabla\operatorname{div}\mathbf{B} - \operatorname{curl}\operatorname{curl}\mathbf{B},\\
&\operatorname{curl}(\mathbf{u}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{B} + \mathbf{u}\operatorname{div}\mathbf{B} - \mathbf{B}\operatorname{div}\mathbf{u},
\end{aligned}$$

and then letting $n = \rho - 1$, we can rewrite system (1.1)–(1.2) into the perturbation form:

$$\begin{cases}
n_t + \operatorname{div}\mathbf{u} = S_1,\\
\mathbf{u}_t + \gamma\nabla n - (\mu+\nu)\Delta\mathbf{u} - (\mu+\lambda-\nu)\nabla\operatorname{div}\mathbf{u} - 2\nu\nabla\times\boldsymbol{\omega} = S_2,\\
\boldsymbol{\omega}_t + 4\nu\boldsymbol{\omega} - \mu'\Delta\boldsymbol{\omega} - (\mu'+\lambda')\nabla\operatorname{div}\boldsymbol{\omega} - 2\nu\nabla\times\mathbf{u} = S_3,\\
\mathbf{B}_t - \sigma\Delta\mathbf{B} = S_4,\\
\nabla\cdot\mathbf{B} = 0\\
(n, \mathbf{u}, \omega, \mathbf{B})(x,0) = (n_0, \mathbf{u}_0, \omega_0, \mathbf{B}_0)(x,0) \to (0,0,0) \quad \text{as } |x| \to \infty.
\end{cases} \tag{2.1}$$

Here the positive constant $\gamma = P'(1)$, and the nonhomogeneous source terms S_i ($i = 1, 2, 3, 4$) are defined by

$$\begin{cases}
S_1 = -n\operatorname{div}\mathbf{u} - \mathbf{u}\cdot\nabla n,\\
S_2 = -\mathbf{u}\cdot\nabla\mathbf{u} - f(n)[(\mu+\nu)\Delta\mathbf{u} + (\mu+\lambda-\nu)\nabla\operatorname{div}\mathbf{u} + 2\nu\nabla\times\boldsymbol{\omega}] - h(n)\nabla n\\
\qquad + g(n)[\mathbf{B}\cdot\nabla\mathbf{B} - \frac{1}{2}\nabla(|\mathbf{B}|^2)],\\
S_3 = -\mathbf{u}\cdot\nabla\boldsymbol{\omega} - f(n)[\mu'\Delta\boldsymbol{\omega} + (\mu'+\lambda')\nabla\operatorname{div}\boldsymbol{\omega} - 4\nu\boldsymbol{\omega} + 2\nu\nabla\times\mathbf{u}],\\
S_4 = (\mathbf{B}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{B} - \mathbf{B}(\operatorname{div}\mathbf{u})
\end{cases} \tag{2.2}$$

with the nonlinear functions

$$f(n) = \frac{n}{n+1}, \qquad h(n) = \frac{P'(n+1)}{n+1} - P'(1), \quad \text{and} \quad g(n) = \frac{1}{n+1}.$$

By the *a priori* assumption

$$\big\|(n, \mathbf{u}, \boldsymbol{\omega}, \mathbf{B})(t)\big\|_{H^3} \le \delta \tag{2.3}$$

with some sufficiently small constant δ and Sobolev's inequality of $H^2 \hookrightarrow L^\infty$, we obtain

$$\frac{1}{2} \le n + 1 \le \frac{3}{2}.$$

This implies that

$$\left|f(n)\right|, \left|h(n)\right| \lesssim |n|, \tag{2.4}$$

and for any given $k \geq 1$ and $l \geq 0$,

$$\left|f^{(k)}(n)\right|, \left|h^{(k)}(n)\right|, \left|g^{(l)}(n)\right| \lesssim 1. \tag{2.5}$$

Define an increasing energy functional

$$M(t) = \sup_{0\leq\tau\leq t} (1+\tau)^{\frac{3+2N}{2}} \left\|\nabla^N(n, \mathbf{u}, \boldsymbol{\omega}, \mathbf{B})\right\|_{L^2}^2. \tag{2.6}$$

Then the following proposition leads us to estimate (1.7) in Theorem 1.1.

Proposition 2.1 *Under the assumptions in Theorem 1.1, it holds that*

$$M(t) \lesssim C_0 + \|\mathbf{U}_0\|_{L^1}^2, \tag{2.7}$$

where $\mathbf{U}_0 = (n_0, \mathbf{u}_0, \boldsymbol{\omega}_0, \mathbf{B}_0)$.

Afterwards, our dedication turns towards demonstrating the validity of Proposition 2.1.

2.1 Decay estimates on the low-frequency part

Before establishing the temporal decay estimates on the low-frequency part of the solutions $(n, \mathbf{u}, \boldsymbol{\omega}, \mathbf{B})$, we shall recall some decay results for the linearized system of (2.1).

Lemma 2.1 ([19]) *Let* $\widetilde{\mathbf{U}} = (\widetilde{n}, \widetilde{\mathbf{u}}, \widetilde{\boldsymbol{\omega}}, \widetilde{\mathbf{B}})$ *be the global solution to the Cauchy problem of the linearized system of* (2.1). *Then the following time decay properties hold for any integer* $m \geq 0$:

$$\left\|\nabla^m(\widetilde{n}, \widetilde{\mathbf{u}}, \widetilde{\mathbf{B}})\right\|_{L^q} \lesssim (1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{m}{2}} \left\|(\widetilde{n}_0, \widetilde{\mathbf{u}}_0, \widetilde{\mathbf{B}}_0)\right\|_{L^p} \tag{2.8}$$

and

$$\left\|\nabla^m \widetilde{\boldsymbol{\omega}}\right\|_{L^q} \lesssim (1+t)^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{m+1}{2}} \left\|(\widetilde{\mathbf{u}}_0, \widetilde{\boldsymbol{\omega}}_0)\right\|_{L^p}, \tag{2.9}$$

where $1 \leq p \leq 2 \leq q \leq \infty$.

Based on Lemma 2.1, we now present the time decay rate of the L^2-norm for the low-frequency part of the Nth order derivatives of the solution to the nonlinear system (2.1) as follows.

Lemma 2.2 *Under the assumptions in Theorem 1.1, the solution* $\mathbf{U} := (n, \mathbf{u}, \boldsymbol{\omega}, \mathbf{B})$ *to the Cauchy problem of nonlinear system* (2.1) *satisfies the following decay estimate*:

$$\left\|\nabla^N\left(n^l, \mathbf{u}^l, \boldsymbol{\omega}^l, \mathbf{B}^l\right)\right\|_{L^2} \lesssim \left(\|\mathbf{U}_0\|_{L^1} + \delta\sqrt{M(t)}\right)(1+t)^{-(\frac{3}{4}+\frac{N}{2})}. \tag{2.10}$$

Proof From Lemma 2.1, Duhamel's principle, Plancherel's theorem, and Hausdorff–Young's inequality, it holds that

$$\begin{aligned}\left\|\nabla^N\left(n^l,\mathbf{u}^l,\boldsymbol{\omega}^l,\mathbf{B}^l\right)\right\|_{L^2} \lesssim{}& (1+t)^{-(\frac{3}{4}+\frac{N}{2})}\|\mathbf{U}_0\|_{L^1} + \int_0^{\frac{t}{2}}(1+t-\tau)^{-(\frac{3}{4}+\frac{N}{2})}\left\|\mathbf{S}(\tau)\right\|_{L^1}\,\mathrm{d}\tau \\ &+ \int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{5}{4}}\left\|\nabla^{N-1}\mathbf{S}^l(\tau)\right\|_{L^1}\,\mathrm{d}\tau,\end{aligned} \tag{2.11}$$

where $\mathbf{S} = (S_1, S_2, S_3, S_4)^T$. By the method of decomposing the frequency, the last two terms on the right-hand side of (2.11) can be treated as follows. With the help of estimates (1.3)–(1.5), (2.4), (2.5), Lemmas A.1–A.2, and Lemma A.4, we can deduce that

$$\begin{aligned}\left\|\mathbf{S}(\tau)\right\|_{L^1} \lesssim{}& \left\|\operatorname{div}(n\mathbf{u})(\tau)\right\|_{L^1} + \left\|\mathbf{u}\cdot\nabla\mathbf{u}(\tau)\right\|_{L^1} + \left\|f(n)[2\nu\nabla\times\boldsymbol{\omega}](\tau)\right\|_{L^1} + \left\|\mathbf{u}\cdot\nabla\boldsymbol{\omega}(\tau)\right\|_{L^1} \\ &+ \left\|f(n)\left[(\mu+\nu)\Delta\mathbf{u} + (\mu+\lambda-\nu)\nabla\operatorname{div}\mathbf{u}\right](\tau)\right\|_{L^1} + \left\|f(n)\boldsymbol{\omega}(\tau)\right\|_{L^1} \\ &+ \left\|f(n)\left[\mu'\Delta\boldsymbol{\omega} + \left(\mu'+\lambda'\right)\nabla\operatorname{div}\boldsymbol{\omega}\right](\tau)\right\|_{L^1} + \left\|f(n)[\nabla\times\mathbf{u}]\right\|_{L^1} + \left\|h(n)\nabla n\right\|_{L^1} \\ &+ \left\|(\mathbf{B}\cdot\nabla)\mathbf{u}(\tau)\right\|_{L^1} + \left\|(\mathbf{u}\cdot\nabla)\mathbf{B}(\tau)\right\|_{L^1} + \left\|B(\operatorname{div}\mathbf{u})(\tau)\right\|_{L^1} \\ &+ \left\|g(n)\left[\mathbf{B}\cdot\nabla\mathbf{B} - \nabla\left(|\mathbf{B}|^2\right)/2\right]\right\|_{L^1} \\ \lesssim{}& \left\|(n,\mathbf{u},\mathbf{B})(\tau)\right\|_{L^2}\left\|\nabla(n,\mathbf{u},\boldsymbol{\omega},\mathbf{B})(\tau)\right\|_{L^2} + \left\|n(\tau)\right\|_{L^2}\left\|\Delta(\mathbf{u},\boldsymbol{\omega})(\tau)\right\|_{L^2} \\ &+ \left\|n(\tau)\right\|_{L^2}\left\|\boldsymbol{\omega}(\tau)\right\|_{L^2} \\ \lesssim{}& (1+\tau)^{-2}.\end{aligned} \tag{2.12}$$

Similarly, using estimates (1.3)–(1.5) and definition (2.6) of $M(t)$, we get

$$\begin{aligned}&\left\|\nabla^{N-1}\mathbf{S}^l(\tau)\right\|_{L^1} \\ &\quad\lesssim \left\|\nabla^{N-1}\operatorname{div}(n\mathbf{u})(\tau)\right\|_{L^1} + \left\|\nabla^{N-1}(\mathbf{u}\cdot\nabla\mathbf{u})(\tau)\right\|_{L^1} + \left\|\nabla^{N-1}\left[f(n)(2\nu\nabla\times\boldsymbol{\omega})\right](\tau)\right\|_{L^1} \\ &\qquad+ \left\|\nabla^{N-2}\left\{f(n)\left[(\mu+\nu)\Delta\mathbf{u} + (\mu+\lambda-\nu)\nabla\operatorname{div}\mathbf{u}\right]\right\}(\tau)\right\|_{L^1} + \left\|\nabla^{N-1}\left[(\mathbf{B}\cdot\nabla)\mathbf{u}\right](\tau)\right\|_{L^1} \\ &\qquad+ \left\|\nabla^{N-1}(\mathbf{u}\cdot\nabla\boldsymbol{\omega})(\tau)\right\|_{L^1} + \left\|\nabla^{N-1}\left\{f(n)\left[\mu'\Delta\boldsymbol{\omega} + \left(\mu'+\lambda'\right)\nabla\operatorname{div}\boldsymbol{\omega}\right]\right\}(\tau)\right\|_{L^1} \\ &\qquad+ \left\|\nabla^{N-1}f(n)(4\nu\boldsymbol{\omega})(\tau)\right\|_{L^1} + \left\|\nabla^{N-1}\left[f(n)(2\nu\nabla\times\mathbf{u})\right]\right\|_{L^1} + \left\|\nabla^{N-1}\left[(\mathbf{u}\cdot\nabla)\mathbf{B}(\tau)\right]\right\|_{L^1} \\ &\qquad+ \left\|\nabla^{N-1}\left\{g(n)\left[\mathbf{B}\cdot\nabla\mathbf{B} - \frac{1}{2}\nabla\left(|\mathbf{B}|^2\right)\right]\right\}(\tau)\right\|_{L^1} + \left\|\nabla^{N-1}\left[\mathbf{B}(\operatorname{div}\mathbf{u})\right](\tau)\right\|_{L^1} \\ &\qquad+ \left\|\nabla^{N-1}\left[h(n)\nabla n\right](\tau)\right\|_{L^1} \\ &\quad\lesssim \left\|(n,\mathbf{u},\mathbf{B})(\tau)\right\|_{L^2}\left\|\nabla^N(n,\mathbf{u},\boldsymbol{\omega},\mathbf{B})(\tau)\right\|_{L^2} + \left\|\nabla^{N-1}(n,\mathbf{u},\mathbf{B})(\tau)\right\|_{L^2}\left\|\nabla(n,\mathbf{u},\boldsymbol{\omega},\mathbf{B})(\tau)\right\|_{L^2} \\ &\qquad+ \left\|\nabla^{N-2}n(\tau)\right\|_{L^2}\left\|\nabla^2(\mathbf{u},\boldsymbol{\omega})(\tau)\right\|_{L^2} + \left\|n(\tau)\right\|_{L^2}\left\|\nabla^{N-2}\boldsymbol{\omega}(\tau)\right\|_{L^2} \\ &\qquad+ \left\|\boldsymbol{\omega}(\tau)\right\|_{L^2}\left\|\nabla^{N-2}n(\tau)\right\|_{L^2} \\ &\quad\lesssim \delta(1+\tau)^{-\frac{3}{4}-\frac{N}{2}}\sqrt{M(\tau)} + (1+\tau)^{-1-\frac{N}{2}}.\end{aligned} \tag{2.13}$$

Inserting (2.12) and (2.13) into (2.11) gives estimate (2.10). □

2.2 Decay estimates on the high-frequency part

To prove Proposition 2.1, we also need to establish the temporal decay estimates on the higher-frequency parts of the highest-order derivatives of the global solutions. The following lemma first provides the energy dissipation for $\nabla^{N+1}(\mathbf{u}^h,\boldsymbol{\omega}^h,\mathbf{B}^h)$.

Lemma 2.3 *With the assumptions in Theorem 1.1, it holds that*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\gamma\big\|\nabla^N n^h\big\|_{L^2}^2+\big\|\nabla^N\mathbf{u}^h\big\|_{L^2}^2+\big\|\nabla^N\boldsymbol{\omega}^h\big\|_{L^2}^2+\big\|\nabla^N\mathbf{B}^h\big\|_{L^2}^2\big)\\&\qquad+(2\mu+\lambda-\nu)\big\|\nabla^{N+1}\mathbf{u}^h\big\|_{L^2}^2+(2\mu'+\lambda'\big\|\nabla^{N+1}\boldsymbol{\omega}^h\big\|_{L^2}^2+\sigma\big\|\nabla^{N+1}\mathbf{B}^h\big\|_{L^2}^2\\&\quad\lesssim\delta\big\|\big(\nabla^N n,\nabla^N\mathbf{u},\nabla^{N+1}\boldsymbol{\omega},\nabla^{N+1}\mathbf{u},\nabla^{N+1}\mathbf{B}\big)\big\|_{L^2}^2.\end{aligned}\tag{2.14}$$

Proof Performing the operator $\mathfrak{F}^{-1}[(1-\phi(\xi))\mathfrak{F}(\cdot)]$ onto system (2.1) yields

$$\begin{cases}\partial_t n^h+\operatorname{div}\mathbf{u}^h=S_1^h,\\ \partial_t\mathbf{u}^h+\gamma\nabla n^h-(\mu+\upsilon)\Delta\mathbf{u}^h-(\mu+\lambda-\upsilon)\nabla\operatorname{div}\mathbf{u}^h-2\upsilon\nabla\times\boldsymbol{\omega}^h=S_2^h,\\ \partial_t\boldsymbol{\omega}^h+4\upsilon\boldsymbol{\omega}^h-\mu'\Delta\boldsymbol{\omega}^h-(\mu'+\lambda')\nabla\operatorname{div}\boldsymbol{\omega}^h-2\nu\nabla\times\mathbf{u}^h=S_3^h,\\ \partial_t\mathbf{B}^h-\sigma\Delta\mathbf{B}^h=S_4^h.\end{cases}\tag{2.15}$$

By taking the L^2 inner product of $\nabla^N(2.15)_1$–$\nabla^N(2.15)_4$ with $\nabla^N n^h$, $\nabla^N\mathbf{u}^h$, $\nabla^N\boldsymbol{\omega}^h$, and $\nabla^N\mathbf{B}^h$, respectively, we further obtain

$$\begin{cases}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^N n^h\|_{L^2}^2+\langle\nabla^N n^h,\nabla^N\operatorname{div}\mathbf{u}^h\rangle=\langle\nabla^N S_1^h,\nabla^N n^h\rangle,\\ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^N\mathbf{u}^h\|_{L^2}^2+\gamma\langle\nabla^N\mathbf{u}^h,\nabla^N\nabla n^h\rangle-(\mu+\nu)\langle\nabla^N\mathbf{u}^h,\nabla^N\Delta\mathbf{u}^h\rangle\\ \qquad-(\mu+\lambda-\nu)\langle\nabla^N\mathbf{u}^h,\nabla^N\nabla\operatorname{div}\mathbf{u}^h\rangle-2\nu\langle\nabla^N\mathbf{u}^h,\nabla^N\nabla\times\boldsymbol{\omega}^h\rangle\\ \quad=\langle\nabla^N\mathbf{u}^h,\nabla^N S_2^h\rangle,\\ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^N\boldsymbol{\omega}^h\|_{L^2}^2+4\nu\langle\nabla^N\boldsymbol{\omega}^h,\nabla^N\boldsymbol{\omega}^h\rangle-\mu'\langle\nabla^N\boldsymbol{\omega}^h,\nabla^N\Delta\boldsymbol{\omega}^h\rangle\\ \qquad-(\mu'+\lambda')\langle\nabla^N\boldsymbol{\omega}^h,\nabla^N\nabla\operatorname{div}\boldsymbol{\omega}^h\rangle-2\nu\langle\nabla^N\boldsymbol{\omega}^h,\nabla^N\nabla\times\mathbf{u}^h\rangle\\ \quad=\langle\nabla^N\boldsymbol{\omega}^h,\nabla^N S_3^h\rangle,\\ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^N\mathbf{B}^h\|_{L^2}^2-\sigma\langle\nabla^N\mathbf{B}^h,\nabla^N\Delta\mathbf{B}^h\rangle=\langle\nabla^N\mathbf{B}^h,\nabla^N S_4^h\rangle.\end{cases}\tag{2.16}$$

Summing up the identities $\gamma\times(2.16)_1$, $(2.16)_2$, $(2.16)_3$, and $(2.16)_4$ leads to

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\gamma\big\|\nabla^N n^h\big\|_{L^2}^2+\big\|\nabla^N\mathbf{u}^h\big\|_{L^2}^2+\big\|\nabla^N\boldsymbol{\omega}^h\big\|_{L^2}^2+\big\|\nabla^N\mathbf{B}^h\big\|_{L^2}^2\big)\\&\qquad+(\mu+\nu)\big\|\nabla^{N+1}\mathbf{u}^h\big\|_{L^2}^2+(\mu+\lambda-\nu)\big\|\nabla^N\operatorname{div}\mathbf{u}^h\big\|_{L^2}^2+4\nu\big\|\nabla^N\boldsymbol{\omega}^h\big\|_{L^2}^2\\&\qquad+\mu'\big\|\nabla^{N+1}\boldsymbol{\omega}^h\big\|_{L^2}^2+\big(\mu'+\lambda'\big)\big\|\nabla^N\operatorname{div}\boldsymbol{\omega}^h\big\|_{L^2}^2+\sigma\big\|\nabla^{N+1}\mathbf{B}^h\big\|_{L^2}^2\\&\quad=4\nu\big\langle\nabla^N\boldsymbol{\omega}^h,\nabla^N\nabla\times\mathbf{u}^h\big\rangle+\gamma\big\langle\nabla^N n^h,\nabla^N S_1^h\big\rangle+\big\langle\nabla^N\mathbf{u}^h,\nabla^N S_2^h\big\rangle\\&\qquad+\big\langle\nabla^N\boldsymbol{\omega}^h,\nabla^N S_3^h\big\rangle+\big\langle\nabla^N\mathbf{B}^h,\nabla^N S_4^h\big\rangle\\&\quad=\sum_{i=1}^{5}J_i.\end{aligned}\tag{2.17}$$

We next deal with J_i $(i=1,2,\dots,5)$ terms by terms.

(1) *Term J_1.* Following Hölder's inequality and Young's inequality, it holds that

$$J_1 = 4\nu\langle\nabla^N\boldsymbol{\omega}^h, \nabla^N\nabla\times\mathbf{u}^h\rangle \le 4\nu\|\nabla^N\boldsymbol{\omega}^h\|_{L^2}^2 + \nu\|\nabla^N\nabla\times\mathbf{u}^h\|_{L^2}^2. \tag{2.18}$$

(2) *Term J_2.* To estimate the second term J_2, we reformulate it as follows:

$$\begin{aligned} J_2 &= \langle\nabla^N S_1^h, \nabla^N n^h\rangle \\ &= -\langle\nabla^N(n\operatorname{div}\mathbf{u} + \mathbf{u}\cdot\nabla n), \nabla^N n^h\rangle \\ &= -\langle\nabla^N(n\operatorname{div}\mathbf{u})^h, \nabla^N n^h\rangle - \langle\nabla^N(\mathbf{u}\cdot\nabla n)^h, \nabla^N n^h\rangle \\ &= \mathcal{K}_1 + \mathcal{K}_2. \end{aligned} \tag{2.19}$$

By Lemmas A.2, A.5, Hölder's inequality, and Young's inequality, one has

$$\begin{aligned} |\mathcal{K}_1| &\lesssim \|\nabla^N(n\operatorname{div}\mathbf{u})^h\|_{L^2}\|\nabla^N n^h\|_{L^2} \\ &\lesssim \|\nabla^N(n\operatorname{div}\mathbf{u})\|_{L^2}\|\nabla^N n\|_{L^2} \\ &\lesssim \left(\|n\|_{L^\infty}\|\nabla^N\operatorname{div}\mathbf{u}\|_{L^2} + \|\nabla^N n\|_{L^2}\|\operatorname{div}\mathbf{u}\|_{L^\infty}\right)\|\nabla^N n\|_{L^2} \\ &\lesssim \left(\|\nabla n\|_{H^1}\|\nabla^N\operatorname{div}\mathbf{u}\|_{L^2} + \|\nabla^N n\|_{L^2}\|\nabla\operatorname{div}\mathbf{u}\|_{H^1}\right)\|\nabla^N n\|_{L^2} \\ &\lesssim \delta\left(\|\nabla^N n\|_{L^2}^2 + \|\nabla^{N+1}\mathbf{u}\|_{L^2}^2\right). \end{aligned} \tag{2.20}$$

In addition, from the formula $f = f^h + f^l$, it holds that

$$\begin{aligned} \mathcal{K}_2 &= -\langle\nabla^N(\mathbf{u}\cdot\nabla n)^h, \nabla^N n^h\rangle \\ &= -\langle\nabla^N(\mathbf{u}\cdot\nabla n) - \nabla^N(\mathbf{u}\cdot\nabla n)^l, \nabla^N n^h\rangle \\ &= -\langle\nabla^N\left(\mathbf{u}\cdot\nabla n^h\right) + \nabla^N\left(\mathbf{u}\cdot\nabla n^l\right) - \nabla^N(\mathbf{u}\cdot\nabla n)^l, \nabla^N n^h\rangle \\ &= -\langle\nabla^N\left(\mathbf{u}\cdot\nabla n^h\right), \nabla^N n^h\rangle - \langle\nabla^N\left(\mathbf{u}\cdot\nabla n^l\right), \nabla^N n^h\rangle + \langle\nabla^N(\mathbf{u}\cdot\nabla n)^l, \nabla^N n^h\rangle \\ &= \mathcal{K}_{2,1} + \mathcal{K}_{2,2} + \mathcal{K}_{2,3}. \end{aligned} \tag{2.21}$$

Employing Lemma A.3 yields

$$\begin{aligned} |\mathcal{K}_{2,1}| &\lesssim |\langle\mathbf{u}\cdot\nabla\nabla^N n^h, \nabla^N n^h\rangle| + |\langle[\nabla^N, \mathbf{u}]\nabla n^h, \nabla^N n^h\rangle| \\ &\lesssim \frac{1}{2}|\langle\operatorname{div}\mathbf{u}, |\nabla^N n^h|^2\rangle| + \|[\nabla^N, \mathbf{u}]\nabla n^h\|_{L^2}\|\nabla^N n^h\|_{L^2} \\ &\lesssim \|\nabla\mathbf{u}\|_{L^\infty}\|\nabla^N n^h\|_{L^2}^2 \\ &\quad + \left(\|\nabla\mathbf{u}\|_{L^\infty}\|\nabla^N n^h\|_{L^2} + \|\nabla^N\mathbf{u}\|_{L^6}\|\nabla n^h\|_{L^3}\right)\|\nabla^N n^h\|_{L^2} \\ &\lesssim \|\nabla\mathbf{u}\|_{L^\infty}\|\nabla^N n^h\|_{L^2}^2 \\ &\quad + \left(\|\nabla^2\mathbf{u}\|_{H^1}\|\nabla^N n^h\|_{L^2} + \|\nabla^{N+1}\mathbf{u}\|_{L^2}\|\nabla n^h\|_{H^1}\right)\|\nabla^N n^h\|_{L^2} \\ &\lesssim \delta\left(\|\nabla^N n\|_{L^2}^2 + \|\nabla^{N+1}\mathbf{u}\|_{L^2}^2\right). \end{aligned} \tag{2.22}$$

With the help of Lemma A.6, we get

$$
\begin{aligned}
|\mathcal{K}_{2,2}| &\lesssim \big\|\nabla^N\big(\mathbf{u}\cdot\nabla n^l\big)\big\|_{L^2}\big\|\nabla^N n^h\big\|_{L^2} \\
&\lesssim \big(\|\mathbf{u}\|_{L^\infty}\big\|\nabla^{N+1}n^l\big\|_{L^2}+\big\|\nabla^N\mathbf{u}\big\|_{L^6}\big\|\nabla n^l\big\|_{L^3}\big)\big\|\nabla^N n^h\big\|_{L^2} \\
&\lesssim \big(\|\nabla\mathbf{u}\|_{H^1}\big\|\nabla^N n\big\|_{L^2}+\big\|\nabla^{N+1}\mathbf{u}\big\|_{L^2}\|\nabla n\|_{H^1}\big)\big\|\nabla^N n\big\|_{L^2} \\
&\lesssim \delta\big(\big\|\nabla^N n\big\|_{L^2}^2+\big\|\nabla^{N+1}\mathbf{u}\big\|_{L^2}^2\big).
\end{aligned} \tag{2.23}
$$

It can be obtained in a similar way that

$$
\begin{aligned}
|\mathcal{K}_{2,3}| &\lesssim \big\|\nabla^N(\mathbf{u}\cdot\nabla n)^l\big\|_{L^2}\big\|\nabla^N n^h\big\|_{L^2} \\
&\lesssim \big\|\nabla^{N-1}(\mathbf{u}\cdot\nabla n)\big\|_{L^2}\big\|\nabla^N n\big\|_{L^2} \\
&\lesssim \big(\|\mathbf{u}\|_{L^\infty}\big\|\nabla^N n\big\|_{L^2}+\big\|\nabla^{N-1}\mathbf{u}\big\|_{L^6}\|\nabla n\|_{L^3}\big)\big\|\nabla^N n\big\|_{L^2} \\
&\lesssim \big(\|\nabla\mathbf{u}\|_{H^1}\big\|\nabla^N n\big\|_{L^2}+\big\|\nabla^N\mathbf{u}\big\|_{L^2}\|\nabla n\|_{H^1}\big)\big\|\nabla^N n\big\|_{L^2} \\
&\lesssim \delta\big(\big\|\nabla^N n\big\|_{L^2}^2+\big\|\nabla^N\mathbf{u}\big\|_{L^2}^2\big).
\end{aligned} \tag{2.24}
$$

Gathering estimates (2.22)–(2.24), we can derive from (2.21) that

$$
|\mathcal{K}_2| \lesssim \delta\big\|\big(\nabla^N n,\nabla^N\mathbf{u},\nabla^{N+1}\mathbf{u}\big)\big\|_{L^2}^2. \tag{2.25}
$$

Furthermore, inserting (2.20) and (2.25) into (2.19), we arrive at

$$
|J_2| \lesssim \delta\big\|\big(\nabla^N n,\nabla^N\mathbf{u},\nabla^{N+1}\mathbf{u}\big)\big\|_{L^2}^2. \tag{2.26}
$$

(3) *Terms J_3 and J_4.* Following similar lines as in (2.19), the third term J_3 can be rewritten as follows:

$$
\begin{aligned}
J_3 = &-\big\langle\nabla^N(\mathbf{u}\cdot\nabla\mathbf{u})^h,\nabla^N\mathbf{u}^h\big\rangle \\
&+\big\langle\nabla^{N-1}\big\{f(n)\big[(\mu+\nu)\Delta\mathbf{u}+(\mu+\lambda-\nu)\nabla\operatorname{div}\mathbf{u}\big]\big\}^h,\nabla^N\operatorname{div}\mathbf{u}^h\big\rangle \\
&-2\nu\big\langle\nabla^N\big[f(n)(\nabla\times\boldsymbol{\omega})\big]^h,\nabla^N\mathbf{u}^h\big\rangle+\big\langle\nabla^{N-1}\big[h(n)\nabla n\big]^h,\nabla^N\operatorname{div}\mathbf{u}^h\big\rangle \\
&-\left\langle\nabla^N\left\{g(n)\left[\mathbf{B}\cdot\nabla\mathbf{B}-\frac{1}{2}\nabla\big(|B|^2\big)\right]\right\}^h,\nabla^N\mathbf{u}^h\right\rangle \\
:= &\ \mathcal{K}_3+\mathcal{K}_4+\mathcal{K}_5+\mathcal{K}_6+\mathcal{K}_7.
\end{aligned} \tag{2.27}
$$

From Lemmas A.2 and A.6, it holds for $\mathcal{K}_3$ that

$$
\begin{aligned}
|\mathcal{K}_3| &\lesssim \big\|\nabla^N(\mathbf{u}\cdot\nabla\mathbf{u})^h\big\|_{L^2}\big\|\nabla^N\mathbf{u}^h\big\|_{L^2} \\
&\lesssim \big\|\nabla^N(\mathbf{u}\cdot\nabla\mathbf{u})\big\|_{L^2}\big\|\nabla^{N+1}\mathbf{u}\big\|_{L^2} \\
&\lesssim \big(\|\mathbf{u}\|_{L^\infty}\big\|\nabla^{N+1}\mathbf{u}\big\|_{L^2}+\big\|\nabla^N\mathbf{u}\big\|_{L^6}\|\nabla\mathbf{u}\|_{L^3}\big)\big\|\nabla^{N+1}\mathbf{u}\big\|_{L^2} \\
&\lesssim \big(\|\nabla\mathbf{u}\|_{H^1}\big\|\nabla^{N+1}\mathbf{u}\big\|_{L^2}+\big\|\nabla^{N+1}\mathbf{u}\big\|_{L^2}\|\nabla\mathbf{u}\|_{H^1}\big)\big\|\nabla^{N+1}\mathbf{u}\big\|_{L^2} \\
&\lesssim \delta\big\|\nabla^{N+1}\mathbf{u}\big\|_{L^2}^2.
\end{aligned} \tag{2.28}
$$

Using Lemmas A.1, A.2, A.6 and Hausdorff–Young's inequality, we infer that

$$\begin{aligned}
|\mathcal{K}_4| &\lesssim \left\|\nabla^{N-1}\left\{f(n)\left[(\mu+\nu)\Delta\mathbf{u}+(\mu+\lambda-\nu)\nabla\operatorname{div}\mathbf{u}\right]\right\}^h\right\|_{L^2}\left\|\nabla^N\operatorname{div}\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \left\|\nabla^{N-1}\left\{f(n)\left[(\mu+\nu)\Delta\mathbf{u}+(\mu+\lambda-\nu)\nabla\operatorname{div}\mathbf{u}\right]\right\}\right\|_{L^2}\left\|\nabla^{N+1}\mathbf{u}\right\|_{L^2}\\
&\lesssim \left(\|n\|_{L^\infty}\left\|\nabla^{N+1}\mathbf{u}\right\|_{L^2}+\left\|\nabla^{N-1}n\right\|_{L^6}\|\Delta\mathbf{u}\|_{L^3}\right)\left\|\nabla^{N+1}\mathbf{u}\right\|_{L^2}\\
&\lesssim \left(\|\nabla n\|_{H^1}\left\|\nabla^{N+1}\mathbf{u}\right\|_{L^2}+\left\|\nabla^{N}n\right\|_{L^2}\|\Delta\mathbf{u}\|_{H^1}\right)\left\|\nabla^{N+1}\mathbf{u}\right\|_{L^2}\\
&\lesssim \delta\left(\left\|\nabla^{N+1}\mathbf{u}\right\|_{L^2}^2+\left\|\nabla^N n\right\|_{L^2}^2\right).
\end{aligned}\tag{2.29}$$

Applying a similar argument used for $\mathcal{K}_3$, we have

$$\begin{aligned}
|\mathcal{K}_5| &\lesssim \left\|\nabla^N\left[f(n)\nabla\times\boldsymbol{\omega}\right]^h\right\|_{L^2}\left\|\nabla^N\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \left\|\nabla^N\left[f(n)\nabla\times\boldsymbol{\omega}\right]\right\|_{L^2}\left\|\nabla^N\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \left(\|n\|_{L^\infty}\left\|\nabla^{N+1}\boldsymbol{\omega}\right\|_{L^2}+\left\|\nabla^N n\right\|_{L^2}\|\nabla\boldsymbol{\omega}\|_{L^\infty}\right)\left\|\nabla^{N+1}\mathbf{u}\right\|_{L^2}\\
&\lesssim \left(\|\nabla n\|_{H^1}\left\|\nabla^{N+1}\boldsymbol{\omega}\right\|_{L^2}+\left\|\nabla^N n\right\|_{L^2}\left\|\nabla^2\boldsymbol{\omega}\right\|_{H^1}\right)\left\|\nabla^{N+1}\mathbf{u}\right\|_{L^2}\\
&\lesssim \delta\left\|\left(\nabla^{N+1}\mathbf{u},\nabla^N n,\nabla^{N+1}\boldsymbol{\omega}\right)\right\|_{L^2}^2.
\end{aligned}\tag{2.30}$$

Besides, the following hold:

$$\begin{aligned}
|\mathcal{K}_6| &\lesssim \left\|\nabla^{N-1}\left[h(n)\nabla n\right]^h\right\|_{L^2}\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \left\|\nabla^{N-1}\left[h(n)\nabla n\right]\right\|_{L^2}\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \left(\|n\|_{L^\infty}\left\|\nabla^N n\right\|_{L^2}+\left\|\nabla^{N-1}n\right\|_{L^6}\|\nabla n\|_{L^3}\right)\left\|\nabla^{N+1}\mathbf{u}\right\|_{L^2}\\
&\lesssim \left(\|\nabla n\|_{H^1}\left\|\nabla^N n\right\|_{L^2}+\left\|\nabla^N n\right\|_{L^2}\|\nabla n\|_{H^1}\right)\left\|\nabla^{N+1}\mathbf{u}\right\|_{L^2}\\
&\lesssim \delta\left(\left\|\nabla^{N+1}\mathbf{u}\right\|_{L^2}^2+\left\|\nabla^N n\right\|_{L^2}^2\right)
\end{aligned}\tag{2.31}$$

and

$$\begin{aligned}
|\mathcal{K}_7| &\lesssim \left\|\nabla^N\left\{g(n)\left[\mathbf{B}\cdot\nabla\mathbf{B}-\frac{1}{2}\nabla\left(|\mathbf{B}|^2\right)\right]\right\}^h\right\|_{L^2}\left\|\nabla^N\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \left\|\nabla^N\left\{g(n)\left[\mathbf{B}\cdot\nabla\mathbf{B}-\frac{1}{2}\nabla\left(|\mathbf{B}|^2\right)\right]\right\}\right\|_{L^2}\left\|\nabla^N\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \left\|\nabla^N\left[\mathbf{B}\cdot\nabla\mathbf{B}-\frac{1}{2}\nabla\left(|\mathbf{B}|^2\right)\right]\right\|_{L^2}\left\|\nabla^N\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \left(\|\mathbf{B}\|_{L^\infty}\left\|\nabla^{N+1}\mathbf{B}\right\|_{L^2}+\left\|\nabla^N\mathbf{B}\right\|_{L^6}\|\nabla\mathbf{B}\|_{L^3}\right)\left\|\nabla^N\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \delta\left\|\left(\nabla^{N+1}\mathbf{B},\nabla^{N+1}\mathbf{u}\right)\right\|_{L^2}^2.
\end{aligned}\tag{2.32}$$

Inserting (2.28)–(2.32) into (2.27) gets that

$$|J_3|\lesssim \delta\left\|\left(\nabla^N n,\nabla^{N+1}\boldsymbol{\omega},\nabla^{N+1}\mathbf{u},\nabla^{N+1}\mathbf{B}\right)\right\|_{L^2}^2.\tag{2.33}$$

The estimate on term J_4 can be obtained from a similar argument used for J_3, and it is presented as follows:

$$|J_4| \lesssim \delta \left\| \left(\nabla^N n, \nabla^{N+1}\mathbf{u}, \nabla^{N+1}\boldsymbol{\omega}\right)\right\|_{L^2}^2. \tag{2.34}$$

(4) *Term J_5.* Recall that

$$\begin{aligned} J_5 &= \left\langle \nabla^N \left[(\mathbf{B}\cdot\nabla)\mathbf{u}\right]^h, \nabla^N \mathbf{B}^h\right\rangle - \left\langle \nabla^N\left[(\mathbf{u}\cdot\nabla)\mathbf{B}\right]^h, \nabla^N\mathbf{B}^h\right\rangle \\ &\quad - \left\langle \nabla^N\left[\mathbf{B}(\operatorname{div}\mathbf{u})\right]^h, \nabla^N \mathbf{B}^h\right\rangle \\ &:= \mathcal{K}_8 + \mathcal{K}_9 + \mathcal{K}_{10}. \end{aligned} \tag{2.35}$$

By Lemmas A.2 and A.6, we have

$$\begin{aligned} |\mathcal{K}_8| &\lesssim \left\|\nabla^N\left[(\mathbf{B}\cdot\nabla)\mathbf{u}\right]^h\right\|_{L^2}\left\|\nabla^N\mathbf{B}^h\right\|_{L^2} \\ &\lesssim \left\|\nabla^N\left[(\mathbf{B}\cdot\nabla)\mathbf{u}\right]\right\|_{L^2}\left\|\nabla^{N+1}\mathbf{B}\right\|_{L^2} \\ &\lesssim \left(\|\mathbf{B}\|_{L^\infty}\left\|\nabla^{N+1}\mathbf{u}\right\|_{L^2} + \left\|\nabla^N\mathbf{B}\right\|_{L^6}\|\nabla\mathbf{u}\|_{L^3}\right)\left\|\nabla^{N+1}\mathbf{B}\right\|_{L^2} \\ &\lesssim \delta\left(\left\|\nabla^{N+1}\mathbf{B}\right\|_{L^2}^2 + \left\|\nabla^{N+1}\mathbf{u}\right\|_{L^2}^2\right) \end{aligned} \tag{2.36}$$

and

$$\begin{aligned} |\mathcal{K}_9| &\lesssim \left\|\nabla^N\left[(\mathbf{u}\cdot\nabla)\mathbf{B}\right]^h\right\|_{L^2}\left\|\nabla^N\mathbf{B}^h\right\|_{L^2} \\ &\lesssim \left\|\nabla^N\left[(\mathbf{u}\cdot\nabla)\mathbf{B}\right]\right\|_{L^2}\left\|\nabla^{N+1}\mathbf{B}\right\|_{L^2} \\ &\lesssim \left(\|\mathbf{u}\|_{L^\infty}\left\|\nabla^{N+1}\mathbf{B}\right\|_{L^2} + \left\|\nabla^N\mathbf{u}\right\|_{L^6}\|\nabla\mathbf{B}\|_{L^3}\right)\left\|\nabla^{N+1}\mathbf{B}\right\|_{L^2} \\ &\lesssim \delta\left(\left\|\nabla^{N+1}\mathbf{B}\right\|_{L^2}^2 + \left\|\nabla^{N+1}\mathbf{u}\right\|_{L^2}^2\right). \end{aligned} \tag{2.37}$$

It holds for term $\mathcal{K}_{10}$ that

$$\begin{aligned} |\mathcal{K}_{10}| &\lesssim \left\|\nabla^N\left[\mathbf{B}(\operatorname{div}\mathbf{u})\right]^h\right\|_{L^2}\left\|\nabla^N\mathbf{B}^h\right\|_{L^2} \\ &\lesssim \left\|\nabla^N[\mathbf{B}\operatorname{div}\mathbf{u}]\right\|_{L^2}\left\|\nabla^{N+1}\mathbf{B}\right\|_{L^2} \\ &\lesssim \left(\|\mathbf{B}\|_{L^\infty}\left\|\nabla^{N+1}\mathbf{u}\right\|_{L^2} + \left\|\nabla^N\mathbf{B}\right\|_{L^6}\|\nabla\mathbf{u}\|_{L^3}\right)\left\|\nabla^{N+1}\mathbf{B}\right\|_{L^2} \\ &\lesssim \delta\left(\left\|\nabla^{N+1}\mathbf{B}\right\|_{L^2}^2 + \left\|\nabla^{N+1}\mathbf{u}\right\|_{L^2}^2\right). \end{aligned} \tag{2.38}$$

Inserting (2.36)–(2.38) into (2.35) yields that

$$|J_5| \lesssim \delta\left\|\left(\nabla^{N+1}\mathbf{u}, \nabla^{N+1}\mathbf{B}\right)\right\|_{L^2}^2. \tag{2.39}$$

Inserting estimates (2.18), (2.26), (2.33), (2.34), and (2.39) into (2.17), we can obtain the desired estimate (2.14). □

Next, we turn to provide the dissipation estimate for n^h.

Lemma 2.4 *With the assumptions in Theorem 1.1, it holds that*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\nabla^{N-1}\mathbf{u}^h\nabla^N n^h\,\mathrm{d}x+\frac{\gamma}{2}\left\|\nabla^N n^h\right\|_{L^2}^2\\&\quad\lesssim\delta\left\|\left(\nabla^N n,\nabla^N\mathbf{u},\nabla^{N+1}\mathbf{u},\nabla^{N+1}\mathbf{B},\nabla^{N+1}\boldsymbol{\omega}\right)\right\|_{L^2}^2+\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}^2+\left\|\nabla^{N+1}\boldsymbol{\omega}^h\right\|_{L^2}^2.\end{aligned}\tag{2.40}$$

Proof Performing the operator $\mathfrak{F}^{-1}[(1-\phi(\xi))\mathfrak{F}(\cdot)]$ to $\nabla^{N-1}(2.1)_2$ and taking the L^2 inner product of both the resulting equation and $\nabla^N n^h$, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\nabla^{N-1}\mathbf{u}^h\nabla^N n^h\,\mathrm{d}x+\gamma\left\|\nabla^N n^h\right\|_{L^2}^2\\&\quad=\left\|\nabla^N\mathbf{u}^h\right\|_{L^2}^2+(\mu+\nu)\left\langle\nabla^{N+1}\mathbf{u}^h,\nabla^N n^h\right\rangle+(\mu+\lambda-\nu)\left\langle\nabla^N\operatorname{div}\mathbf{u}^h,\nabla^N n^h\right\rangle\\&\qquad+2\nu\left\langle\nabla^{N-1}\nabla\times\boldsymbol{\omega}^h,\nabla^N n^h\right\rangle+\left\langle\nabla^N S_1^h,\nabla^{N-1}\mathbf{u}^h\right\rangle+\left\langle\nabla^{N-1}S_2^h,\nabla^N n^h\right\rangle.\\&\quad=\left\|\nabla^N\mathbf{u}^h\right\|_{L^2}^2+\sum_{i=6}^{10}J_i.\end{aligned}\tag{2.41}$$

(1) *Terms J_6, J_7, and J_8.* Using Hölder's inequality and Hausdorff–Young's inequality, we can estimate for term J_i ($6\le i\le 8$) as follows:

$$|J_6|\le\frac{3(\mu+\nu)^2}{2\gamma}\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}^2+\frac{\gamma}{6}\left\|\nabla^N n^h\right\|_{L^2}^2,\tag{2.42}$$

$$|J_7|\le\frac{3(\mu+\lambda-\nu)^2}{2\gamma}\left\|\nabla^N\operatorname{div}\mathbf{u}^h\right\|_{L^2}^2+\frac{\gamma}{6}\left\|\nabla^N n^h\right\|_{L^2}^2,\tag{2.43}$$

and

$$|J_8|\le\frac{6\nu^2}{\gamma}\left\|\nabla^N\boldsymbol{\omega}^h\right\|_{L^2}^2+\frac{\gamma}{6}\left\|\nabla^N n^h\right\|_{L^2}^2.\tag{2.44}$$

(2) *Term J_9.* For term J_9, we get

$$\begin{aligned}J_9&=-\left\langle\nabla^N(n\operatorname{div}\mathbf{u})^h,\nabla^{N-1}\mathbf{u}^h\right\rangle-\left\langle\nabla^N(\mathbf{u}\cdot\nabla n)^h,\nabla^{N-1}\mathbf{u}^h\right\rangle\\&=:\mathcal{K}_{11}+\mathcal{K}_{12}.\end{aligned}\tag{2.45}$$

By using Hölder's inequality, Young's inequality, Lemmas A.1, A.2, and A.6, we arrive at

$$\begin{aligned}|\mathcal{K}_{11}|&\lesssim\left\|\nabla^N(n\operatorname{div}\mathbf{u})^h\right\|_{L^2}\left\|\nabla^{N-1}\mathbf{u}^h\right\|_{L^2}\\&\lesssim\left\|\nabla^N(n\operatorname{div}\mathbf{u})\right\|_{L^2}\left\|\nabla^N\mathbf{u}^h\right\|_{L^2}\\&\lesssim\left(\|n\|_{L^\infty}\left\|\nabla^{N+1}\mathbf{u}\right\|_{L^2}+\left\|\nabla^N n\right\|_{L^2}\|\nabla\mathbf{u}\|_{L^\infty}\right)\left\|\nabla^{N+1}\mathbf{u}\right\|_{L^2}\\&\lesssim\delta\left(\left\|\nabla^N n\right\|_{L^2}^2+\left\|\nabla^{N+1}\mathbf{u}\right\|_{L^2}^2\right).\end{aligned}\tag{2.46}$$

In the same way, we can get

$$\begin{aligned}|\mathcal{K}_{12}|&\lesssim\left\|\nabla^{N-1}(\mathbf{u}\cdot\nabla n)^h\right\|_{L^2}\left\|\nabla^N\mathbf{u}^h\right\|_{L^2}\\&\lesssim\left\|\nabla^{N-1}(\mathbf{u}\cdot\nabla n)\right\|_{L^2}\left\|\nabla^N\mathbf{u}\right\|_{L^2}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left(\|\mathbf{u}\|_{L^\infty}\left\|\nabla^N n\right\|_{L^2}+\left\|\nabla^{N-1}\mathbf{u}\right\|_{L^6}\|\nabla n\|_{L^3}\right)\left\|\nabla^N\mathbf{u}\right\|_{L^2}\\
&\lesssim \delta\left(\left\|\nabla^N n\right\|_{L^2}^2+\left\|\nabla^N\mathbf{u}\right\|_{L^2}^2\right).
\end{aligned}
\tag{2.47}
$$

Substituting (2.46) and (2.47) into (2.45) leads to

$$
|J_9|\lesssim \delta\left\|\left(\nabla^N n,\nabla^N\mathbf{u},\nabla^{N+1}\mathbf{u}\right)\right\|_{L^2}^2.
\tag{2.48}
$$

(3) *Term* J_{10}. For term J_{10}, we directly have

$$
\begin{aligned}
J_{10}&=-\left\langle\nabla^{N-1}(\mathbf{u}\cdot\nabla\mathbf{u})^h,\nabla^N n^h\right\rangle\\
&\quad-\left\langle\nabla^{N-1}\left\{f(n)\left[(\mu+\nu)\Delta\mathbf{u}+(\mu+\lambda-\nu)\nabla\operatorname{div}\mathbf{u}\right]\right\}^h,\nabla^N n^h\right\rangle\\
&\quad-2\nu\left\langle\nabla^{N-1}\left[f(n)(\nabla\times\boldsymbol{\omega})\right]^h,\nabla^N n^h\right\rangle-\left\langle\nabla^{N-1}\left[h(n)\nabla n\right]^h,\nabla^N n^h\right\rangle\\
&\quad-\left\langle\nabla^{N-1}\left\{g(n)\left[\mathbf{B}\cdot\nabla\mathbf{B}-\frac{1}{2}\nabla\left(|B|^2\right)\right]\right\}^h,\nabla^N n^h\right\rangle\\
&:=\mathcal{K}_{13}+\mathcal{K}_{14}+\mathcal{K}_{15}+\mathcal{K}_{16}+\mathcal{K}_{17}.
\end{aligned}
\tag{2.49}
$$

With the help of Young's inequality and Hölder's inequality, we obtain

$$
\begin{aligned}
|\mathcal{K}_{13}|&\lesssim\left\|\nabla^{N-1}(\mathbf{u}\cdot\nabla\mathbf{u})^h\right\|_{L^2}\left\|\nabla^N n^h\right\|_{L^2}\\
&\lesssim\left\|\nabla^{N}(\mathbf{u}\cdot\nabla\mathbf{u})\right\|_{L^2}\left\|\nabla^N n^h\right\|_{L^2}\\
&\lesssim\left(\|\mathbf{u}\|_{L^\infty}\left\|\nabla^{N+1}\mathbf{u}\right\|_{L^2}+\left\|\nabla^N\mathbf{u}\right\|_{L^6}\|\nabla\mathbf{u}\|_{L^3}\right)\left\|\nabla^N n\right\|_{L^2}\\
&\lesssim\delta\left(\left\|\nabla^N n\right\|_{L^2}^2+\left\|\nabla^{N+1}\mathbf{u}\right\|_{L^2}^2\right)
\end{aligned}
\tag{2.50}
$$

and

$$
\begin{aligned}
|\mathcal{K}_{14}|&\lesssim\left\|\nabla^{N-1}\left\{f(n)\left[(\mu+\nu)\Delta\mathbf{u}+(\mu+\lambda-\nu)\nabla\operatorname{div}\mathbf{u}\right]\right\}^h\right\|_{L^2}\left\|\nabla^N n^h\right\|_{L^2}\\
&\lesssim\left(\|n\|_{L^\infty}\left\|\nabla^{N+1}\mathbf{u}\right\|_{L^2}+\left\|\nabla^{N-1}n\right\|_{L^6}\|\Delta\mathbf{u}\|_{L^3}\right)\left\|\nabla^N n\right\|_{L^2}\\
&\lesssim\delta\left(\left\|\nabla^N n\right\|_{L^2}^2+\left\|\nabla^{N+1}\mathbf{u}\right\|_{L^2}^2\right).
\end{aligned}
\tag{2.51}
$$

Similarly, we get

$$
\begin{aligned}
|\mathcal{K}_{15}|&\lesssim\left\|\nabla^{N-1}\left[f(n)(\nabla\times\boldsymbol{\omega})\right]^h\right\|_{L^2}\left\|\nabla^N n^h\right\|_{L^2}\\
&\lesssim\left\|\nabla^{N-1}\left[f(n)(\nabla\times\boldsymbol{\omega})\right]\right\|_{L^2}\left\|\nabla^N n^h\right\|_{L^2}\\
&\lesssim\left(\|n\|_{L^3}\left\|\nabla^N\boldsymbol{\omega}\right\|_{L^6}+\left\|\nabla^{N-1}n\right\|_{L^6}\|\nabla\boldsymbol{\omega}\|_{L^3}\right)\left\|\nabla^N n\right\|_{L^2}\\
&\lesssim\delta\left(\left\|\nabla^N n\right\|_{L^2}^2+\left\|\nabla^{N+1}\boldsymbol{\omega}\right\|_{L^2}^2\right)
\end{aligned}
\tag{2.52}
$$

and

$$
\begin{aligned}
|\mathcal{K}_{16}|&\lesssim\left\|\nabla^{N-1}\left[h(n)\nabla n\right]^h\right\|_{L^2}\left\|\nabla^N n^h\right\|_{L^2}\\
&\lesssim\left\|\nabla^{N}\left[h(n)\nabla n\right]\right\|_{L^2}\left\|\nabla^N n^h\right\|_{L^2}
\end{aligned}
$$

$$\begin{aligned}&\lesssim \left(\|n\|_{L^\infty}\left\|\nabla^N n\right\|_{L^2}+\left\|\nabla^{N-1}n\right\|_{L^6}\|\nabla n\|_{L^3}\right)\left\|\nabla^N n\right\|_{L^2}\\&\lesssim \delta\left\|\nabla^N n\right\|_{L^2}^2.\end{aligned}\tag{2.53}$$

For term $\mathcal{K}_{17}$, we have

$$\begin{aligned}|\mathcal{K}_{17}|&\lesssim\left\|\nabla^{N-1}\left\{g(n)\left[\mathbf{B}\cdot\nabla\mathbf{B}-\frac{1}{2}\nabla\left(|\mathbf{B}|^2\right)\right]\right\}^h\right\|_{L^2}\left\|\nabla^N n^h\right\|_{L^2}\\&\lesssim\left\|\nabla^{N}\left\{g(n)\left[\mathbf{B}\cdot\nabla\mathbf{B}-\frac{1}{2}\nabla\left(|\mathbf{B}|^2\right)\right]\right\}\right\|_{L^2}\left\|\nabla^N n^h\right\|_{L^2}\\&\lesssim\left(\|\mathbf{B}\|_{L^\infty}\left\|\nabla^{N+1}\mathbf{B}\right\|_{L^2}+\left\|\nabla^N\mathbf{B}\right\|_{L^6}\|\nabla\mathbf{B}\|_{L^3}\right)\left\|\nabla^N n\right\|_{L^2}\\&\lesssim\delta\left\|\left(\nabla^N n,\nabla^{N+1}\mathbf{B}\right)\right\|_{L^2}^2.\end{aligned}\tag{2.54}$$

Inserting (2.50)–(2.54) into (2.49) gets

$$|J_{10}|\lesssim\left\|\left(\nabla^N n,\nabla^{N+1}\mathbf{u},\nabla^{N+1}\boldsymbol{\omega},\nabla^{N+1}\mathbf{B}\right)\right\|_{L^2}^2.\tag{2.55}$$

Gathering (2.42)–(2.44), (2.48), and (2.55), we derive estimate (2.40) from (2.41). □

Based on the above two lemmas, we now present the temporal decay rate for the high-frequency of the Nth order derivatives of the solution.

Lemma 2.5 *With the assumptions in Theorem 1.1, it holds that*

$$\left\|\nabla^N\left(n^h,\mathbf{u}^h,\boldsymbol{\omega}^h,\mathbf{B}^h\right)(t)\right\|_{L^2}^2\lesssim\left(C_0+\delta\sqrt{M(t)}\right)(1+t)^{-(\frac{3}{4}+\frac{N}{2})}.\tag{2.56}$$

Proof By estimates (2.14), (2.40), Lemma A.6, and the smallness of δ, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{C}_1(t)+\left\|\nabla^N n^h\right\|_{L^2}^2+\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}^2+\left\|\nabla^{N+1}\boldsymbol{\omega}^h\right\|_{L^2}^2+\left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2}^2\\&\quad\lesssim\left\|\nabla^N\left(n^l,\mathbf{u}^l,\boldsymbol{\omega}^l,\mathbf{B}^l\right)\right\|_{L^2}^2,\end{aligned}\tag{2.57}$$

where

$$\mathfrak{C}_1(t)=D_1\left\|\nabla^N\left(n^h,\mathbf{u}^h,\boldsymbol{\omega}^h,\mathbf{B}^h\right)\right\|_{L^2}^2+\int_{\mathbb{R}^3}\nabla^{N-1}\mathbf{u}^h\nabla^N n^h\,\mathrm{d}x\tag{2.58}$$

with some large enough positive constant D_1. Recalling

$$C_1\mathfrak{C}_1(t)\lesssim\left\|\nabla^N n^h\right\|_{L^2}^2+\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}^2+\left\|\nabla^{N+1}\boldsymbol{\omega}^h\right\|_{L^2}^2+\left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2}^2,\tag{2.59}$$

thus, by the formula (A.10), Gronwall's inequality, Lemmas 2.2 and A.7, we can get

$$\begin{aligned}\mathfrak{C}_1(t)&\le\mathfrak{C}_1(0)e^{-C_1t}+\int_0^t e^{-C_1(t-\tau)}\left\|\nabla^N\left(n^l,\mathbf{u}^l,\boldsymbol{\omega}^l,\mathbf{B}^l\right)\right\|_{L^2}^2\mathrm{d}\tau\\&\le\mathfrak{C}_1(0)e^{-C_1t}+C\int_0^t e^{-C_1(t-\tau)}\left(\|\mathbf{U}_0\|_{L^1}^2+\delta^2M(\tau)\right)(1+\tau)^{-\frac{3+2N}{2}}\mathrm{d}\tau\end{aligned}$$

$$\lesssim (1+t)^{-\frac{3+2N}{2}}\left(\mathfrak{E}_1(0)+\|\mathbf{U}_0\|_{L^1}^2+\delta^2 M(\tau)\right). \tag{2.60}$$

Due to the relation

$$\mathfrak{E}_1(t)\approx \left\|\nabla^N\left(n^h,\mathbf{u}^h,\boldsymbol{\omega}^h,\mathbf{B}^h\right)\right\|_{L^2}^2,$$

estimate (2.56) can be easily obtained from (2.60). □

Proof of Proposition 2.1 Now we are in a position to prove Proposition 2.1. With the help of Lemmas 2.2, 2.5 and the frequency decompositions, we deduce that

$$\begin{aligned}\left\|\nabla^N(n,\mathbf{u},\boldsymbol{\omega},\mathbf{B})\right\|_{L^2}^2 &\le \left\|\nabla^N\left(n^h,\mathbf{u}^h,\boldsymbol{\omega}^h,\mathbf{B}^h\right)\right\|_{L^2}^2+\left\|\nabla^N\left(n^l,\mathbf{u}^l,\boldsymbol{\omega}^l,\mathbf{B}^l\right)\right\|_{L^2}^2\\ &\lesssim \left(\mathfrak{E}_1(0)+\|U_0\|_{L^1}^2+\delta^2M(t)\right)^2(1+t)^{-\frac{3+2N}{2}}\\ &\lesssim (1+t)^{-\frac{3+2N}{2}}\left(\mathfrak{E}_1(0)+\|\mathbf{U}_0\|_{L^1}^2+\delta^2M(t)\right).\end{aligned} \tag{2.61}$$

From the definition of $M(t)$, it holds that

$$M(t)\lesssim \mathfrak{E}_1(0)+\|\mathbf{U}_0\|_{L^1}^2+\delta^2M(t).$$

By the smallness of δ, we further obtain

$$M(t)\lesssim \mathfrak{E}_1(0)+\|\mathbf{U}_0\|_{L^1}^2.$$

Thus, we complete the proof of Proposition 2.1, which immediately implies the decay rate (1.7) in Theorem 1.1. □

3 Derivation of the decay-in-time of high-frequency parts

In this section, we further derive the optimal temporal rate of decay-in-time for the Nth order spatial derivatives of the solution $\boldsymbol{\omega}$. To this end, we shall first establish the decay estimate on $\|\nabla^N\boldsymbol{\omega}^l\|_{L^2}$, which is presented as follows.

Lemma 3.1 *With the assumptions in Theorem* 1.1, *the lower frequency part of solution* $\boldsymbol{\omega}$ *to the Cauchy problem of system* (2.1) *satisfies that*

$$\left\|\nabla^N\boldsymbol{\omega}^l(t)\right\|_{L^2}\lesssim (1+t)^{-(\frac{5}{4}+\frac{N}{2})}. \tag{3.1}$$

Proof Using estimate (2.9), Duhamel's principle, Plancherel theorem, and Hausdorff–Young's inequality, we have

$$\begin{aligned}\left\|\nabla^N\boldsymbol{\omega}^l(t)\right\|_{L^2}\lesssim{}& (1+t)^{-\frac{5}{4}-\frac{N}{2}}\left\|\boldsymbol{\omega}(0)\right\|_{L^1}+\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{5}{4}-\frac{N}{2}}\left\|S_3(\tau)\right\|_{L^1}\,\mathrm{d}\tau\\ &+\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{7}{4}}\left\|\nabla^{N-1}S_3^l(\tau)\right\|_{L^1}\,\mathrm{d}\tau.\end{aligned} \tag{3.2}$$

Employing a similar argument used for (2.12), we easily have

$$\left\|S_3(\tau)\right\|_{L^1}\lesssim (1+\tau)^{-2}. \tag{3.3}$$

On the other hand, from the decay rates (1.3)–(1.5), it holds for $\|\nabla^{N-1}S_3^l(\tau)\|_{L^1}$ that

$$\begin{aligned}
&\big\|\nabla^{N-1}S_3^l(\tau)\big\|_{L^1}\\
&\quad\lesssim \big\|\nabla^{N-1}(\mathbf{u}\cdot\nabla\boldsymbol{\omega})(\tau)\big\|_{L^1}+\big\|\nabla^{N-2}\big\{f(n)\big[\mu'\Delta\boldsymbol{\omega}+\big(\mu'+\lambda'\big)\nabla\operatorname{div}\boldsymbol{\omega}(\tau)\big]\big\}\big\|_{L^1}\\
&\qquad+\big\|\nabla^{N-1}f(n)(4\nu\boldsymbol{\omega})(\tau)\big\|_{L^1}+\big\|\nabla^{N-1}\big[f(n)(2\nu\nabla\times\mathbf{u})\big](\tau)\big\|_{L^1}\\
&\quad\lesssim \big\|\nabla^{N-1}(n,\mathbf{u})(\tau)\big\|_{L^2}\big\|\nabla(n,\mathbf{u},\boldsymbol{\omega})(\tau)\big\|_{L^2}+\big\|(n,\mathbf{u})(\tau)\big\|_{L^2}\big\|\nabla^{N}(n,\mathbf{u},\boldsymbol{\omega})(\tau)\big\|_{L^2}\\
&\qquad+\big\|\nabla^{N-2}n(\tau)\big\|_{L^2}\big\|\Delta\boldsymbol{\omega}(\tau)\big\|_{L^2}+\big\|n(\tau)\big\|_{L^2}\big\|\nabla^{N-1}\boldsymbol{\omega}(\tau)\big\|_{L^2}+\big\|\boldsymbol{\omega}(\tau)\big\|_{L^2}\big\|\nabla^{N-1}n(\tau)\big\|_{L^2}\\
&\quad\lesssim (1+\tau)^{-\frac34-\frac34-\frac N2}+(1+\tau)^{-\frac34-\frac{N-2}{2}-\frac94}+(1+\tau)^{-\frac34-\frac54-\frac{N-1}{2}}\\
&\quad\lesssim (1+\tau)^{-(\frac54+\frac N2)}.
\end{aligned}\tag{3.4}$$

Inserting (3.3) and (3.4) into (3.2), we arrive at the decay rate (3.1). □

We next establish the temporal decay estimates on $\|\nabla^N(n^h,\mathbf{u}^h,\boldsymbol{\omega}^h,\mathbf{B}^h)\|_{L^2}$.

Lemma 3.2 *With the assumptions in Theorem 1.1, it holds that*

$$\begin{aligned}
&\frac12\frac{\mathrm d}{\mathrm dt}\big(\gamma\big\|\nabla^N n^h\big\|_{L^2}^2+\big\|\nabla^N\mathbf{u}^h\big\|_{L^2}^2+\big\|\nabla^N\boldsymbol{\omega}^h\big\|_{L^2}^2+\big\|\nabla^N\mathbf{B}^h\big\|_{L^2}^2\big)\\
&\qquad+(2\mu+\lambda-\nu)\big\|\nabla^{N+1}\mathbf{u}^h\big\|_{L^2}^2+\big(2\mu'+\lambda'\big)\big\|\nabla^N\boldsymbol{\omega}^h\big\|_{L^2}^2+\sigma\big\|\nabla^N\mathbf{B}^h\big\|_{L^2}^2\\
&\quad\lesssim (1+t)^{-\frac{5+2N}{2}}+\big(\delta+(1+t)^{-\frac32}\big)\big(\big\|\nabla^N n^h\big\|_{L^2}^2+\big\|\nabla^{N+1}\mathbf{u}^h\big\|_{L^2}^2+\big\|\nabla^{N+1}\boldsymbol{\omega}^h\big\|_{L^2}^2\\
&\qquad+\big\|\nabla^{N+1}\mathbf{B}^h\big\|_{L^2}^2\big).
\end{aligned}\tag{3.5}$$

Proof Due to the decay rate (1.7) and the fine structures of the decomposition of both low and high frequencies, we next deal with the nonlinear terms J_i $(2\le i\le 5)$ on the right-hand side of (2.17) by a manner that is different from the one used in the proof of Lemma 2.5.

(1) *Term* J_2. For term $\mathcal{K}_1$, one has

$$\begin{aligned}
|\mathcal{K}_1|&\lesssim \big\|\nabla^N(n\operatorname{div}\mathbf{u})^h\big\|_{L^2}\big\|\nabla^N n^h\big\|_{L^2}\\
&\lesssim \big\|\nabla^N(n\operatorname{div}\mathbf{u})\big\|_{L^2}\big\|\nabla^N n^h\big\|_{L^2}\\
&\lesssim \big(\|n\|_{L^\infty}\big\|\nabla^N\operatorname{div}\big(\mathbf{u}^l+\mathbf{u}^h\big)\big\|_{L^2}+\big\|\nabla^N n\big\|_{L^2}\|\operatorname{div}\mathbf{u}\|_{L^\infty}\big)\big\|\nabla^N n^h\big\|_{L^2}\\
&\lesssim \big(\|\nabla n\|_{H^1}\big\|\nabla^N\operatorname{div}\mathbf{u}^l\big\|_{L^2}+\|\nabla n\|_{H^1}\big\|\nabla^N\operatorname{div}\mathbf{u}^h\big\|_{L^2}\\
&\qquad+\big\|\nabla^N n\big\|_{L^2}\|\nabla\operatorname{div}\mathbf{u}\|_{H^1}\big)\big\|\nabla^N n^h\big\|_{L^2}\\
&\lesssim \big((1+t)^{-\frac54-\frac34-\frac N2}+\delta\big\|\nabla^N\operatorname{div}\mathbf{u}^h\big\|_{L^2}\big)\big\|\nabla^N n^h\big\|_{L^2}\big)\\
&\lesssim (1+t)^{-\frac{5+2N}{2}}+\big(\delta+(1+t)^{-\frac32}\big)\big(\big\|\nabla^N\operatorname{div}\mathbf{u}^h\big\|_{L^2}^2+\big\|\nabla^N n^h\big\|_{L^2}^2\big).
\end{aligned}\tag{3.6}$$

For term $\mathcal{K}_{2,1}$, we can derive that

$$\begin{aligned}
|\mathcal{K}_{2,1}|&\lesssim \big|\big\langle\mathbf{u}\cdot\nabla\nabla^N n^h,\nabla^N n^h\big\rangle\big|+\big|\big\langle\big[\nabla^N,\mathbf{u}\big]\nabla n^h,\nabla^N n^h\big\rangle\big|\\
&\lesssim \frac12\big|\big\langle\operatorname{div}\mathbf{u},\big|\nabla^N n^h\big|^2\big\rangle\big|+\big\|\big[\nabla^N,\mathbf{u}\big]\nabla n^h\big\|_{L^2}\big\|\nabla^N n^h\big\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|\nabla \mathbf{u}\|_{L^\infty} \left\|\nabla^N n^h\right\|_{L^2}^2 \\
&\quad + \left(\|\nabla \mathbf{u}\|_{L^\infty} \left\|\nabla^N n^h\right\|_{L^2} + \left\|\nabla^N \mathbf{u}\right\|_{L^2} \left\|\nabla n^h\right\|_{L^\infty}\right) \left\|\nabla^N n^h\right\|_{L^2} \\
&\lesssim \delta \left\|\nabla^N n^h\right\|_{L^2}^2 + (1+t)^{-\frac{7}{4}-\frac{3}{4}-\frac{N}{2}} \left\|\nabla^N n^h\right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{5+2N}{2}} + \left((1+t)^{-\frac{5}{2}} + \delta\right) \left\|\nabla^N n^h\right\|_{L^2}^2. \qquad (3.7)
\end{aligned}
$$

Similarly, we also have

$$
\begin{aligned}
|\mathcal{K}_{2,2}| &\lesssim \left\|\nabla^N \left(\mathbf{u} \cdot \nabla n^l\right)\right\|_{L^2} \left\|\nabla^N n^h\right\|_{L^2} \\
&\lesssim \left(\|\mathbf{u}\|_{L^\infty} \left\|\nabla^{N+1} n^l\right\|_{L^2} + \left\|\nabla^N \mathbf{u}\right\|_{L^2} \left\|\nabla n^l\right\|_{L^\infty}\right) \left\|\nabla^N n^h\right\|_{L^2} \\
&\lesssim \left(\|\nabla \mathbf{u}\|_{H^1} \left\|\nabla^N n\right\|_{L^2} + \left\|\nabla^N \mathbf{u}\right\|_{L^2} \left\|\nabla^2 n\right\|_{H^1}\right) \left\|\nabla^N n^h\right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{5}{4}-\frac{3}{4}-\frac{N}{2}} \left\|\nabla^N n^h\right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{5+2N}{2}} + (1+t)^{-\frac{3}{2}} \left\|\nabla^N n^h\right\|_{L^2}^2 \qquad (3.8)
\end{aligned}
$$

and

$$
\begin{aligned}
|\mathcal{K}_{2,3}| &\lesssim \left\|\nabla^N (\mathbf{u} \cdot \nabla n)^l\right\|_{L^2} \left\|\nabla^N n^h\right\|_{L^2} \\
&\lesssim \left\|\nabla^{N-1} (\mathbf{u} \cdot \nabla n)\right\|_{L^2} \left\|\nabla^N n^h\right\|_{L^2} \\
&\lesssim \left(\|\mathbf{u}\|_{L^\infty} \left\|\nabla^N n\right\|_{L^2} + \left\|\nabla^{N-1} \mathbf{u}\right\|_{L^6} \|\nabla n\|_{L^3}\right) \left\|\nabla^N n^h\right\|_{L^2} \\
&\lesssim \left(\|\nabla \mathbf{u}\|_{H^1} \left\|\nabla^N n\right\|_{L^2} + \left\|\nabla^N \mathbf{u}\right\|_{L^2} \|\nabla n\|_{H^1}\right) \left\|\nabla^N n^h\right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{5}{4}-\frac{3}{4}-\frac{N}{2}} \left\|\nabla^N n^h\right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{5+2N}{2}} + (1+t)^{-\frac{3}{2}} \left\|\nabla^N n^h\right\|_{L^2}^2. \qquad (3.9)
\end{aligned}
$$

From estimates (3.7)–(3.9), it holds that

$$
|\mathcal{K}_2| \lesssim (1+t)^{-\frac{5+2N}{2}} + \left((1+t)^{-\frac{3}{2}} + \delta\right) \left\|\nabla^N n^h\right\|_{L^2}^2. \qquad (3.10)
$$

Inserting (3.6) and (3.10) into (2.19) yields that

$$
\begin{aligned}
|J_2| &\lesssim |\mathcal{K}_1| + |\mathcal{K}_2| \\
&\lesssim (1+t)^{-\frac{5+2N}{2}} + \left((1+t)^{-\frac{3}{2}} + \delta\right) \left(\left\|\nabla^N \operatorname{div} \mathbf{u}^h\right\|_{L^2}^2 + \left\|\nabla^N n^h\right\|_{L^2}^2\right). \qquad (3.11)
\end{aligned}
$$

(2) *Terms J_3 and J_4.* For term J_3, integrating by parts, we get

$$
\begin{aligned}
J_3 &= \left\langle \nabla^{N-1} (\mathbf{u} \cdot \nabla \mathbf{u})^h, \nabla^N \operatorname{div} \mathbf{u}^h \right\rangle \\
&\quad + \left\langle \nabla^{N-1} \left\{ f(n) \left[(\mu+\nu) \Delta \mathbf{u} + (\mu+\lambda-\nu) \nabla \operatorname{div} \mathbf{u} \right] \right\}^h, \nabla^N \operatorname{div} \mathbf{u}^h \right\rangle \\
&\quad + 2\nu \left\langle \nabla^{N-1} \left[f(n) (\nabla \times \boldsymbol{\omega}) \right]^h, \nabla^N \operatorname{div} \mathbf{u}^h \right\rangle + \left\langle \nabla^{N-1} \left[h(n) \nabla n \right]^h, \nabla^N \operatorname{div} \mathbf{u}^h \right\rangle \\
&\quad + \left\langle \nabla^{N-1} \left\{ g(n) \left[B \cdot \nabla B - \frac{1}{2} \nabla \left(|B|^2\right) \right] \right\}^h, \nabla^N \operatorname{div} \mathbf{u}^h \right\rangle \\
&:= \mathcal{K}_3' + \mathcal{K}_4' + \mathcal{K}_5' + \mathcal{K}_6' + \mathcal{K}_7'. \qquad (3.12)
\end{aligned}
$$

It holds for $\mathcal{K}_3'$ that

$$\begin{aligned}
\left|\mathcal{K}_3'\right| &\lesssim \left\|\nabla^{N-1}(\mathbf{u}\cdot\nabla\mathbf{u})^h\right\|_{L^2}\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \left\|\nabla^{N-1}(\mathbf{u}\cdot\nabla\mathbf{u})\right\|_{L^2}\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \left(\|\mathbf{u}\|_{L^\infty}\left\|\nabla^{N}\mathbf{u}\right\|_{L^2}+\left\|\nabla^{N-1}\mathbf{u}\right\|_{L^6}\|\nabla\mathbf{u}\|_{L^3}\right)\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \left(\|\nabla\mathbf{u}\|_{H^1}\left\|\nabla^{N}\mathbf{u}\right\|_{L^2}+\left\|\nabla^{N}\mathbf{u}\right\|_{L^2}\|\nabla\mathbf{u}\|_{H^1}\right)\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\\
&\lesssim (1+t)^{-\frac{5}{4}-\frac{3}{4}-\frac{N}{2}}\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\\
&\lesssim (1+t)^{-\frac{5+2N}{2}}+(1+t)^{-\frac{3}{2}}\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}^2.
\end{aligned}\tag{3.13}$$

For $\mathcal{K}_4'$, we have

$$\begin{aligned}
\left|\mathcal{K}_4'\right| &\lesssim \left\|\nabla^{N-1}\left\{f(n)\left[(\mu+\nu)\Delta\mathbf{u}+(\mu+\lambda-\nu)\nabla\operatorname{div}\mathbf{u}\right]\right\}^h\right\|_{L^2}\left\|\nabla^{N}\operatorname{div}\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \left\|\nabla^{N-1}\left\{f(n)\left[(\mu+\nu)\Delta\mathbf{u}+(\mu+\lambda-\nu)\nabla\operatorname{div}\mathbf{u}\right]\right\}\right\|_{L^2}\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \left(\|n\|_{L^\infty}\left\|\nabla^{N+1}\left(\mathbf{u}^l+\mathbf{u}^h\right)\right\|_{L^2}+\left\|\nabla^{N-1}n\right\|_{L^6}\|\Delta\mathbf{u}\|_{L^3}\right)\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \left(\|\nabla n\|_{H^1}\left\|\nabla^{N+1}\mathbf{u}^l\right\|_{L^2}+\|\nabla n\|_{H^1}\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}+\left\|\nabla^{N}n\right\|_{L^2}\|\Delta\mathbf{u}\|_{H^1}\right)\\
&\quad\times\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \left((1+t)^{-\frac{5}{4}-\frac{3}{4}-\frac{N}{2}}+\delta\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\right)\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\\
&\lesssim (1+t)^{-\frac{5+2N}{2}}+\left(\delta+(1+t)^{-\frac{3}{2}}\right)\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}^2.
\end{aligned}\tag{3.14}$$

Similarly, we deduce

$$\begin{aligned}
\left|\mathcal{K}_5'\right| &\lesssim \left\|\nabla^{N-1}\left[f(n)\nabla\times\boldsymbol{\omega}\right]^h\right\|_{L^2}\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \left\|\nabla^{N-1}\left[f(n)\nabla\times\boldsymbol{\omega}\right]\right\|_{L^2}\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \left(\|n\|_{L^\infty}\left\|\nabla^{N}\boldsymbol{\omega}\right\|_{L^2}+\left\|\nabla^{N-1}n\right\|_{L^6}\|\nabla\boldsymbol{\omega}\|_{L^3}\right)\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \left(\|\nabla n\|_{H^1}\left\|\nabla^{N}\boldsymbol{\omega}\right\|_{L^2}+\left\|\nabla^{N}n\right\|_{L^2}\|\nabla\boldsymbol{\omega}\|_{H^1}\right)\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\\
&\lesssim (1+t)^{-\frac{5}{4}-\frac{3}{4}-\frac{N}{2}}\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\\
&\lesssim (1+t)^{-\frac{5+2N}{2}}+(1+t)^{-\frac{3}{2}}\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}^2
\end{aligned}\tag{3.15}$$

and

$$\begin{aligned}
\left|\mathcal{K}_6'\right| &\lesssim \left\|\nabla^{N-1}\left[h(n)\nabla n\right]^h\right\|_{L^2}\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \left\|\nabla^{N-1}\left[h(n)\nabla n\right]\right\|_{L^2}\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \left(\|n\|_{L^\infty}\left\|\nabla^{N}n\right\|_{L^2}+\left\|\nabla^{N-1}n\right\|_{L^6}\|\nabla n\|_{L^3}\right)\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \left(\|\nabla n\|_{H^1}\left\|\nabla^{N}n\right\|_{L^2}+\left\|\nabla^{N}n\right\|_{L^2}\|\nabla n\|_{H^1}\right)\left\|\nabla^{N+1}\mathbf{u}\right\|_{L^2}\\
&\lesssim (1+t)^{-\frac{5}{4}-\frac{3}{4}-\frac{N}{2}}\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\\
&\lesssim (1+t)^{-\frac{5+2N}{2}}+(1+t)^{-\frac{3}{2}}\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}^2.
\end{aligned}\tag{3.16}$$

For term $\mathcal{K}_7'$, we have

$$\begin{aligned}
|\mathcal{K}_7'| &\lesssim \left\| \nabla^N \left\{ g(n)\left[\mathbf{B}\cdot\nabla\mathbf{B} - \frac{1}{2}\nabla\big(|\mathbf{B}|^2\big)\right]\right\}^h \right\|_{L^2} \left\|\nabla^N \mathbf{u}^h\right\|_{L^2} \\
&\lesssim \left\| \nabla^N \left\{ g(n)\left[\mathbf{B}\cdot\nabla\mathbf{B} - \frac{1}{2}\nabla\big(|\mathbf{B}|^2\big)\right]\right\} \right\|_{L^2} \left\|\nabla^N \mathbf{u}^h\right\|_{L^2} \\
&\lesssim \left\| \nabla^N \left[\mathbf{B}\cdot\nabla\mathbf{B} - \frac{1}{2}\nabla\big(|\mathbf{B}|^2\big)\right] \right\|_{L^2} \left\|\nabla^N \mathbf{u}^h\right\|_{L^2} \\
&\lesssim \big(\|\mathbf{B}\|_{L^\infty} \left\|\nabla^{N+1}\mathbf{B}\right\|_{L^2} + \left\|\nabla^N \mathbf{B}\right\|_{L^6} \|\nabla \mathbf{B}\|_{L^3}\big) \left\|\nabla^N \mathbf{u}^h\right\|_{L^2} \\
&\lesssim \big(\|\nabla\mathbf{B}\|_{H^1} \left\|\nabla^{N+1}\big(\mathbf{B}^l + \mathbf{B}^h\big)\right\|_{L^2} + \left\|\nabla^{N+1}\big(\mathbf{B}^l + \mathbf{B}^h\big)\right\|_{L^2} \|\nabla \mathbf{B}\|_{H^1}\big) \left\|\nabla^N \mathbf{u}^h\right\|_{L^2} \\
&\lesssim \big((1+t)^{-\frac{5}{4}-\frac{3}{4}-\frac{N}{2}} + \delta \left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2}\big) \left\|\nabla^{N+1} \mathbf{u}^h\right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{5+2N}{2}} + \big(\delta + (1+t)^{-\frac{3}{2}}\big)\big(\left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2}^2 + \left\|\nabla^N \mathbf{u}^h\right\|_{L^2}^2 \big).
\end{aligned} \tag{3.17}$$

Substituting (3.13)–(3.17) into (3.12), we can arrive at

$$|J_3| \lesssim (1+t)^{-\frac{5+2N}{4}} + \big(\delta + (1+t)^{-\frac{3}{2}}\big)\big(\left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2}^2 + \left\|\nabla^N \mathbf{u}^h\right\|_{L^2}^2 \big). \tag{3.18}$$

Employing a similar argument used for J_3, we have

$$|J_4| \lesssim (1+t)^{-\frac{5+2N}{4}} + \big(\delta + (1+t)^{-\frac{3}{2}}\big) \left\|\nabla^{N+1}\boldsymbol{\omega}^h\right\|_{L^2}^2. \tag{3.19}$$

(3) *Term J_5.* For the last term J_5, one has

$$\begin{aligned}
J_5 &= -\big\langle \nabla^{N-1}\big[(\mathbf{B}\cdot\nabla)\mathbf{u}\big]^h, \nabla^N \operatorname{div}\mathbf{B}^h\big\rangle + \big\langle \nabla^{N-1}\big[(\mathbf{u}\cdot\nabla)\mathbf{B}\big]^h, \nabla^N \operatorname{div}\mathbf{B}^h\big\rangle \\
&\quad + \big\langle \nabla^{N-1}\big[\mathbf{B}(\operatorname{div}\mathbf{u})\big]^h, \nabla^N \operatorname{div}\mathbf{B}^h\big\rangle \\
&=: \mathcal{K}_8' + \mathcal{K}_9' + \mathcal{K}_{10}'.
\end{aligned} \tag{3.20}$$

It holds for term $\mathcal{K}_8'$ that

$$\begin{aligned}
|\mathcal{K}_8'| &\lesssim \left\|\nabla^{N-1}\big[(\mathbf{B}\cdot\nabla)\mathbf{u}\big]^h\right\|_{L^2} \left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2} \\
&\lesssim \left\|\nabla^{N-1}\big[(\mathbf{B}\cdot\nabla)\mathbf{u}\big]\right\|_{L^2} \left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2} \\
&\lesssim \big(\|\mathbf{B}\|_{L^\infty} \left\|\nabla^N \mathbf{u}\right\|_{L^2} + \left\|\nabla^{N-1}\mathbf{B}\right\|_{L^6} \|\nabla\mathbf{u}\|_{L^3}\big) \left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2} \\
&\lesssim \big(\|\nabla\mathbf{B}\|_{H^1} \left\|\nabla^N \mathbf{u}\right\|_{L^2} + \left\|\nabla^{N}\mathbf{B}\right\|_{L^2} \|\nabla\mathbf{u}\|_{H^1}\big) \left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{5}{4}-\frac{3}{4}-\frac{N}{2}} \left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{5+2N}{2}} + (1+t)^{-\frac{3}{2}} \left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2}^2.
\end{aligned} \tag{3.21}$$

Similarly, we have

$$\begin{aligned}
|\mathcal{K}_9'| &\lesssim \left\|\nabla^{N-1}\big[(\mathbf{u}\cdot\nabla)\mathbf{B}\big]^h\right\|_{L^2} \left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2} \\
&\lesssim \left\|\nabla^{N-1}\big[(\mathbf{u}\cdot\nabla)\mathbf{B}\big]\right\|_{L^2} \left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left(\|\mathbf{u}\|_{L^\infty}\left\|\nabla^N\mathbf{B}\right\|_{L^2}+\left\|\nabla^{N-1}\mathbf{u}\right\|_{L^6}\|\nabla\mathbf{B}\|_{L^3}\right)\left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2}\\
&\lesssim \left(\|\nabla\mathbf{u}\|_{H^1}\left\|\nabla^N\mathbf{B}\right\|_{L^2}+\left\|\nabla^{N}\mathbf{u}\right\|_{L^2}\|\nabla\mathbf{B}\|_{H^1}\right)\left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2}\\
&\lesssim (1+t)^{-\frac{5}{4}-\frac{3}{4}-\frac{N}{2}}\left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2}\\
&\lesssim (1+t)^{-\frac{5+2N}{2}}+(1+t)^{-\frac{3}{2}}\left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2}^2
\end{aligned}
\tag{3.22}
$$

and

$$
\begin{aligned}
\left|\mathcal{K}_{10}'\right| &\lesssim \left\|\nabla^{N-1}\left[\mathbf{B}(\operatorname{div}\mathbf{u})\right]^h\right\|_{L^2}\left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2}\\
&\lesssim \left\|\nabla^{N-1}[\mathbf{B}\operatorname{div}\mathbf{u}]\right\|_{L^2}\left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2}\\
&\lesssim \left(\|\mathbf{B}\|_{L^\infty}\left\|\nabla^N\mathbf{u}\right\|_{L^2}+\left\|\nabla^{N-1}\mathbf{B}\right\|_{L^6}\|\nabla\mathbf{u}\|_{L^3}\right)\left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2}\\
&\lesssim \left(\|\nabla\mathbf{B}\|_{H^1}\left\|\nabla^N\mathbf{u}\right\|_{L^2}+\left\|\nabla^{N}\mathbf{B}\right\|_{L^2}\|\nabla\mathbf{u}\|_{H^1}\right)\left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2}\\
&\lesssim (1+t)^{-\frac{5}{4}-\frac{3}{4}-\frac{N}{2}}\left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2}\\
&\lesssim (1+t)^{-\frac{5+2N}{2}}+(1+t)^{-\frac{3}{2}}\left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2}^2.
\end{aligned}
\tag{3.23}
$$

Inserting (3.21)–(3.23) into (3.20) leads to

$$
|J_5| \lesssim (1+t)^{-\frac{5+2N}{4}}+(1+t)^{-\frac{3}{2}}\left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2}^2. \tag{3.24}
$$

Then we complete the proof of Lemma 3.2 by substituting estimates (3.11), (3.18), (3.19), and (3.24) into (2.17). □

To enclose the energy estimate, it is necessary to establish the dissipation estimate for $\nabla^N n^h$ in a different way.

Lemma 3.3 *With the assumptions in Theorem 1.1, it holds that*

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\nabla^{N-1}\mathbf{u}^h\nabla^N n^h\,\mathrm{d}x+\frac{\gamma}{2}\left\|\nabla^N n^h\right\|_{L^2}^2\\
&\quad\lesssim (1+t)^{-\frac{5+2N}{2}}+\left(\delta+(1+t)^{-\frac{3}{2}}\right)\left(\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}^2+\left\|\nabla^N n^h\right\|_{L^2}^2\right)\\
&\qquad+\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}^2+\left\|\nabla^{N+1}\boldsymbol{\omega}^h\right\|_{L^2}^2.
\end{aligned}
\tag{3.25}
$$

Proof Now, we aim to present the estimates on the last two terms on the right-hand side of (2.41).

(1) *Term J_9.* Integrating by parts, one has

$$
\begin{aligned}
|\mathcal{K}_{11}| &\lesssim \left\|\nabla^{N-1}(n\operatorname{div}\mathbf{u})^h\right\|_{L^2}\left\|\nabla^N\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \left\|\nabla^{N-1}(n\operatorname{div}\mathbf{u})\right\|_{L^2}\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \left(\|n\|_{L^\infty}\left\|\nabla^N\mathbf{u}\right\|_{L^2}+\left\|\nabla^{N-1}n\right\|_{L^6}\|\nabla\mathbf{u}\|_{L^3}\right)\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\\
&\lesssim \left(\|\nabla n\|_{H^1}\left\|\nabla^N\mathbf{u}\right\|_{L^2}+\left\|\nabla^{N}n\right\|_{L^2}\|\nabla\mathbf{u}\|_{H^1}\right)\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim (1+t)^{-\frac{5}{4}-\frac{3}{4}-\frac{N}{2}} \left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{5+2N}{2}} + (1+t)^{-\frac{3}{2}} \left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}^2
\end{aligned}
\tag{3.26}
$$

and

$$
\begin{aligned}
|\mathcal{K}_{12}| &\lesssim \left\|\nabla^{N-1}(\mathbf{u}\cdot\nabla n)^h\right\|_{L^2} \left\|\nabla^{N}\mathbf{u}^h\right\|_{L^2} \\
&\lesssim \left\|\nabla^{N-1}(\mathbf{u}\cdot\nabla n)\right\|_{L^2} \left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2} \\
&\lesssim \left(\|\mathbf{u}\|_{L^\infty}\left\|\nabla^{N} n\right\|_{L^2} + \left\|\nabla^{N-1}\mathbf{u}\right\|_{L^6}\|\nabla n\|_{L^3}\right)\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2} \\
&\lesssim \left(\|\nabla\mathbf{u}\|_{H^1}\left\|\nabla^{N} n\right\|_{L^2} + \left\|\nabla^{N}\mathbf{u}\right\|_{L^2}\|\nabla n\|_{H^1}\right)\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{5}{4}-\frac{3}{4}-\frac{N}{2}} \left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{5+2N}{2}} + (1+t)^{-\frac{3}{2}} \left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}^2.
\end{aligned}
\tag{3.27}
$$

Plugging (3.26) and (3.27) into (2.45), we have

$$
|J_9| \lesssim (1+t)^{-\frac{5+2N}{2}} + (1+t)^{-\frac{3}{2}} \left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}^2. \tag{3.28}
$$

(2) *Term* J_{10}. In the same way, we deduce

$$
\begin{aligned}
|\mathcal{K}_{13}| &\lesssim \left\|\nabla^{N-1}(\mathbf{u}\cdot\nabla \mathbf{u})^h\right\|_{L^2} \left\|\nabla^{N} n^h\right\|_{L^2} \\
&\lesssim \left\|\nabla^{N-1}(\mathbf{u}\cdot\nabla \mathbf{u})\right\|_{L^2} \left\|\nabla^{N} n^h\right\|_{L^2} \\
&\lesssim \left(\|\mathbf{u}\|_{L^\infty}\left\|\nabla^{N} \mathbf{u}\right\|_{L^2} + \left\|\nabla^{N-1}\mathbf{u}\right\|_{L^6}\|\nabla \mathbf{u}\|_{L^3}\right)\left\|\nabla^{N} n^h\right\|_{L^2} \\
&\lesssim \left(\|\nabla\mathbf{u}\|_{H^1}\left\|\nabla^{N} \mathbf{u}\right\|_{L^2} + \left\|\nabla^{N}\mathbf{u}\right\|_{L^2}\|\nabla \mathbf{u}\|_{H^1}\right)\left\|\nabla^{N} n^h\right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{5}{4}-\frac{3}{4}-\frac{N}{2}} \left\|\nabla^{N} n^h\right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{5+2N}{2}} + (1+t)^{-\frac{3}{2}} \left\|\nabla^{N} n^h\right\|_{L^2}^2
\end{aligned}
\tag{3.29}
$$

and

$$
\begin{aligned}
|\mathcal{K}_{14}| &\lesssim \left\|\nabla^{N-1}\left\{f(n)\left[(\mu+\nu)\Delta\boldsymbol{u} + (\mu+\lambda-\nu)\nabla\operatorname{div}\boldsymbol{u}\right]\right\}^h\right\|_{L^2} \left\|\nabla^{N} n^h\right\|_{L^2} \\
&\lesssim \left(\|n\|_{L^\infty}\left\|\nabla^{N+1}\left(\mathbf{u}^l+\mathbf{u}^h\right)\right\|_{L^2} + \left\|\nabla^{N-1} n\right\|_{L^6}\|\Delta\mathbf{u}\|_{L^3}\right)\left\|\nabla^{N} n^h\right\|_{L^2} \\
&\lesssim \left(\|\nabla n\|_{H^1}\left\|\nabla^{N+1}\mathbf{u}^l\right\|_{L^2} + \|\nabla n\|_{H^1}\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\right. \\
&\quad \left. + \left\|\nabla^{N} n\right\|_{L^2}\|\Delta\mathbf{u}\|_{H^1}\right)\left\|\nabla^{N} n^h\right\|_{L^2} \\
&\lesssim \left((1+t)^{-\frac{5}{4}-\frac{3}{4}-\frac{N}{2}} + \delta\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}\right)\left\|\nabla^{N} n^h\right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{5+2N}{2}} + \left(\delta + (1+t)^{-\frac{3}{2}}\right)\left(\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}^2 + \left\|\nabla^{N} n^h\right\|_{L^2}^2\right).
\end{aligned}
\tag{3.30}
$$

By Lemmas A.2 and A.5, we get

$$
\begin{aligned}
|\mathcal{K}_{15}| &\lesssim \left\|\nabla^{N-1}\left[f(n)(\nabla\times\boldsymbol{\omega})\right]^h\right\|_{L^2} \left\|\nabla^{N} n^h\right\|_{L^2} \\
&\lesssim \left\|\nabla^{N}\left[f(n)(\nabla\times\boldsymbol{\omega})\right]\right\|_{L^2} \left\|\nabla^{N} n^h\right\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \left(\|n\|_{L^\infty}\left\|\nabla^N \boldsymbol{\omega}\right\|_{L^2}+\left\|\nabla^{N-1} n\right\|_{L^6}\|\nabla \boldsymbol{\omega}\|_{L^3}\right)\left\|\nabla^N n^h\right\|_{L^2} \\
&\lesssim \left(\|\nabla n\|_{H^1}\left\|\nabla^N \boldsymbol{\omega}\right\|_{L^2}+\left\|\nabla^N n\right\|_{L^2}\|\nabla \boldsymbol{\omega}\|_{H^1}\right)\left\|\nabla^N n^h\right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{5}{4}-\frac{3}{4}-\frac{N}{2}}\left\|\nabla^N n^h\right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{5+2N}{2}}+(1+t)^{-\frac{3}{2}}\left\|\nabla^N n^h\right\|_{L^2}^2
\end{aligned}
\tag{3.31}
$$

and

$$
\begin{aligned}
|\mathcal{K}_{16}| &\lesssim \left\|\nabla^{N-1}\left[h(n)\nabla n\right]^h\right\|_{L^2}\left\|\nabla^N n^h\right\|_{L^2} \\
&\lesssim \left\|\nabla^{N-1}\left[h(n)\nabla n\right]\right\|_{L^2}\left\|\nabla^N n^h\right\|_{L^2} \\
&\lesssim \left(\|n\|_{L^\infty}\left\|\nabla^N n\right\|_{L^2}+\left\|\nabla^{N-1} n\right\|_{L^6}\|\nabla n\|_{L^3}\right)\left\|\nabla^N n\right\|_{L^2} \\
&\lesssim \left(\|\nabla n\|_{H^1}\left\|\nabla^N n\right\|_{L^2}+\left\|\nabla^N n\right\|_{L^2}\|\nabla n\|_{H^1}\right)\left\|\nabla^N n^h\right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{5}{4}-\frac{3}{4}-\frac{N}{2}}\left\|\nabla^N n^h\right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{5+2N}{2}}+(1+t)^{-\frac{3}{2}}\left\|\nabla^N n^h\right\|_{L^2}^2.
\end{aligned}
\tag{3.32}
$$

For term $\mathcal{K}_{17}$, we have

$$
\begin{aligned}
|\mathcal{K}_{17}| &\lesssim \left\|\nabla^{N-1}\left\{g(n)\left[\mathbf{B}\cdot\nabla\mathbf{B}-\frac{1}{2}\nabla\left(|\mathbf{B}|^2\right)\right]\right\}^h\right\|_{L^2}\left\|\nabla^N n^h\right\|_{L^2} \\
&\lesssim \left\|\nabla^{N-1}\left\{g(n)(\mathbf{B}\cdot\nabla\mathbf{B})-\left[g(n)\frac{1}{2}\nabla\left(|\mathbf{B}|^2\right)\right]^h\right\}\right\|_{L^2}\left\|\nabla^N n^h\right\|_{L^2} \\
&\lesssim \left(\|\mathbf{B}\|_{L^\infty}\left\|\nabla^N \mathbf{B}\right\|_{L^2}+\left\|\nabla^{N-1}\mathbf{B}\right\|_{L^6}\|\nabla\mathbf{B}\|_{L^3}\right)\left\|\nabla^N n^h\right\|_{L^2} \\
&\lesssim \left(\|\nabla\mathbf{B}\|_{H^1}\left\|\nabla^N \mathbf{B}\right\|_{L^2}+\left\|\nabla^N\mathbf{B}\right\|_{L^2}\|\mathbf{B}\|_{H^1}\right)\left\|\nabla^N n^h\right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{5}{4}-\frac{3}{4}-\frac{N}{2}}\left\|\nabla^N n^h\right\|_{L^2} \\
&\lesssim (1+t)^{-\frac{5+2N}{2}}+(1+t)^{-\frac{3}{2}}\left\|\nabla^N n^h\right\|_{L^2}^2.
\end{aligned}
\tag{3.33}
$$

Putting estimates (3.29)–(3.33) together, we can derive from (2.49) that

$$
|J_{10}| \lesssim (1+t)^{-\frac{5+2N}{2}}+\left(\delta+(1+t)^{-\frac{3}{2}}\right)\left(\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}^2+\left\|\nabla^N n^h\right\|_{L^2}^2\right).
\tag{3.34}
$$

Plugging (2.42)–(2.44), (3.28), and (3.34) into (2.41), we can derive estimate (3.25). □

From Lemmas 3.2, 3.3 and the smallness of δ, we can obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{E}_2(t)+\left\|\nabla^N n^h\right\|_{L^2}^2+\left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}^2+\left\|\nabla^{N+1}\boldsymbol{\omega}^h\right\|_{L^2}^2+\left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2}^2 \\
&\quad\lesssim (1+t)^{-\frac{5+2N}{2}},
\end{aligned}
\tag{3.35}
$$

where

$$
\mathfrak{E}_2(t)=D_2\left\|\nabla^N\left(n^h,\mathbf{u}^h,\boldsymbol{\omega}^h,\mathbf{B}^h\right)\right\|_{L^2}^2+\int_{\mathbb{R}^3}\nabla^{N-1}\mathbf{u}^h\nabla^N n^h\,\mathrm{d}x
\tag{3.36}
$$

with some large enough positive constant D_2. Besides, it can be easily obtained that

$$\left\|\nabla^N n^h\right\|_{L^2}^2 + \left\|\nabla^{N+1}\mathbf{u}^h\right\|_{L^2}^2 + \left\|\nabla^{N+1}\boldsymbol{\omega}^h\right\|_{L^2}^2 + \left\|\nabla^{N+1}\mathbf{B}^h\right\|_{L^2}^2 \geq C_2\mathfrak{C}_2(t). \tag{3.37}$$

Making use of estimates (3.35), (3.37) and Gronwall's inequality yields

$$\mathfrak{C}_2 \lesssim (1+t)^{-\frac{5+2N}{2}}. \tag{3.38}$$

Thanks to the relations

$$\mathfrak{C}_2 \approx \left\|\nabla^N\left(n^h, \mathbf{u}^h, \boldsymbol{\omega}^h, \mathbf{B}^h\right)\right\|_{L^2}^2$$

and $f = f^l + f^h$, the decay rate (1.8) can be derived from (3.1) and (3.38). Thus we complete the proof of Theorem 1.1.

Appendix

This appendix is devoted to providing some basic mathematical tools used frequently in the previous sections. The detailed proof of the following Gagliardo–Nirenberg inequality can be referred to [3, 14].

Lemma A.1 *Let $0 \leq i, j \leq k$, it holds that*

$$\left\|\nabla^i(f)\right\|_{L^p} \lesssim \left\|\nabla^j f\right\|_{L^q}^{1-\alpha}\left\|\nabla^k f\right\|_{L^r}^{\alpha}, \tag{A.1}$$

where $\alpha \in [\frac{i}{k}, 1]$ and satisfies

$$\frac{i}{3} - \frac{1}{p} = \left(\frac{i}{3} - \frac{1}{q}\right)(1-\alpha) + \frac{k}{3} - \frac{1}{r}\alpha. \tag{A.2}$$

Especially, while $p = q = r = 2$, we have

$$\left\|\nabla^i(f)\right\|_{L^2} \lesssim \left\|\nabla^j f\right\|_{L^2}^{\frac{k-i}{k-j}}\left\|\nabla^k f\right\|_{L^r}^{\frac{i-j}{k-j}}. \tag{A.3}$$

Lemma A.2 *It holds that for $k \geq 0$,*

$$\left\|\nabla^k(fg)\right\|_{L^p} \lesssim \|f\|_{L^{p_1}}\left\|\nabla^k g\right\|_{L^{p_2}} + \left\|\nabla^k f\right\|_{L^{p_3}}\|g\|_{L^{p_4}}, \tag{A.4}$$

where $p_1, p_2, p_3 \in (1, +\infty)$ and

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}. \tag{A.5}$$

We can further deduce the following commutator estimate from Lemma A.2.

Lemma A.3 *Let f and g be smooth functions belonging to $H^k \cap L^\infty$ for any integer $k \geq 1$, and then we define the commutator*

$$\left[\nabla^k, f\right]g = \nabla^k(fg) - f\nabla^k g. \tag{A.6}$$

It holds that

$$\left\|\left[\nabla^k, f\right] g\right\|_{L^p} \lesssim \|\nabla f\|_{L^{p_1}}\left\|\nabla^{k-1} g\right\|_{L^{p_2}}+\left\|\nabla^k f\right\|_{L^{p_3}}\|g\|_{L^{p_4}}. \tag{A.7}$$

Here p, p_1, p_2, p_3 are defined as in Lemma A.2.

Lemma A.4 *Assume that $\|f\|_{L^\infty} \leq 1$. Let $F(f)$ be a smooth function of f with bounded derivatives of any order. Then, for any given integer $k \geq 1$ and any given real number $1 \leq p \leq \infty$, we have*

$$\left\|\nabla^k[F(f)]\right\|_{L^p} \lesssim \left\|\nabla^k f\right\|_{L^p}.$$

Lemma A.5 *We have, for any function $f \in H^2(\mathbb{R}^3)$,*

(i) $\|f\|_{L^r} \lesssim \|f\|_{H^1}$, $2 \leq r \leq 6$,
(ii) $\|f\|_{L^\infty} \lesssim \|\nabla f\|_{H^1}$.

Lemma A.6 *If $f \in H^k(\mathbb{R}^3)$, then we have*

$$\left\|\nabla^k f\right\|_{L^2} \leq \left\|\nabla^k f^h\right\|_{L^2}+\left\|\nabla^k f^l\right\|_{L^2}, \quad k \geq 0, \tag{A.8}$$

$$\left\|\nabla^k f^l\right\|_{L^2} \lesssim \left\|\nabla^{k-1} f^l\right\|_{L^2}, \quad k \geq 1, \tag{A.9}$$

$$\left\|\nabla^k f^h\right\|_{L^2} \lesssim \left\|\nabla^{k+1} f^h\right\|_{L^2}, \quad k \geq 1. \tag{A.10}$$

Lemma A.7 ([8]) *Suppose that $b_1, b_2 \in \mathbb{R}^3$ and $b_1 > 0$, $b_2 > 0$, it holds that for $t \in \mathbb{R}_+$,*

$$\int_0^t (1+\tau)^{-b_1} e^{-b_2(t-\tau)}\, \mathrm{d}\tau \lesssim C(b_1, b_2)(1+t)^{-b_1}, \tag{A.11}$$

where $C(b_1, b_2)$ is the constant that only depends on b_1, b_2.

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The authors declare no competing interests.

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