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Positive solutions for a semipositone anisotropic p-Laplacian problem

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Abstract

In this paper, a semipositone anisotropic p-Laplacian problem

$$-\Delta_{\overrightarrow{p}} u = \lambda f(u),$$

on a bounded domain with the Dirchlet boundary condition is considered, where $A(u^q - 1) \leq f(u) \leq B(u^q - 1)$ for $u > 0$, $f(0) < 0$ and $f(u) = 0$ for $u \leq -1$. It is proved that there exists $\lambda^* > 0$ such that if $\lambda \in (0, \lambda^*)$, then the problem has a positive weak solution $u_\lambda \in L^\infty(\overline{\Omega})$ via combining Mountain-Pass arguments, comparison principles, and regularity principles.

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Keywords: Mountain-Pass Theorem; Semipositone problem; Anisotropic p-Laplacian; Positive solutions

1 Introduction

Mathematically, a positione is a particular kind of eigenvalue problem involving a nonlinear function on the reals that is continuous, positive, and monotone. A semipositone is an eigenvalue problem that would be a positone eigenvalue problem except that the nonlinear function is not positive when its argument is zero.

Semipositone problems naturally arise in various studies. For example, consider the Rozenwig–McArthur equations in the analysis of competing species where "harvesting" takes place. The study of positive solutions to these problems, unlike the positone case, turns into a nontrivial question as 0 is not a subsolution, making the method of sub-supersolutions difficult to apply. Semipositone problems, again unlike positone problems, give rise to the interesting phenomenon of symmetry breaking (see [8]).

Consider the nonlinear eigenvalue problems of the form

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

When f is positive and monotone, it is referred to in the literature as a positone problem. The case where f satisfies, $f(0) < 0$, f is monotone and eventually positive, is referred to in the literature as a semipositone problem. The study of positive solutions to semipostone problems is considerably more challenging, since the range of a solution must include regions where f is negative as well as where f is positive. The study of semipositone problems was first formally introduced by Castro et al. in 1988 (see [7]) in the case of Dirichlet boundary conditions, where several challenging differences were noted in their study when compared to the study of positone problems.

Perera et al. [16] consider the p-superlinear semipositone p-Laplacian problem

$$\begin{cases} -\Delta_p u = u^{q-1} - \mu & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and proved the the existence of ground-state positive solutions (see [4–6, 9] for other cases).

Alves et al. [2] prove the existence of a solution for the class of the semipositone problem

$$\begin{cases} -\Delta u = h(x)(f(u) - a) & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \end{cases}$$

via the variational method together with estimates that involve the Riesz potential (see also [1, 10, 11, 21]).

Fu et al. [14] prove the existence of positive solutions for a class of semipositone problems with singular Trudinger–Moser nonlinearities. The proof is based on compactness and regularity arguments.

Castro et al. [6] study the existence of positive weak solutions to the problem (1.1). Here, we refer to [6] and study the existence of positive weak solutions to the problem

$$\begin{cases} -\Delta_{\overrightarrow{p}} u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $-\Delta_{\overrightarrow{p}}$ is the anisotropic p-Laplace operator, Ω is an open smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$ and the function $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f(0) < 0$ (semipositone), which implies that $u = 0$ is not a subsolution to (1.2), making the finding of positive solutions rather challenging (see [15]).

We set $\overrightarrow{p} := (p_1, \dots, p_N)$, where

$$1 < p_1, p_2, \dots, p_N, \quad \sum_{i=1}^{N} \frac{1}{p_i} > 1,$$

$p_+ := \max\{p_i : i = 1, \dots, N\}$ and $p_- := \min\{p_i : i = 1, \dots, N\}$.

Let $\overline{p}$ denote the harmonic means $\overline{p} = N/(\sum_{i=1}^N \frac{1}{p_i})$, and define

$$p^\star := \frac{N}{(\sum_{i=1}^N \frac{1}{p_i}) - 1} = \frac{N\overline{p}}{N - \overline{p}} \quad \text{and} \quad p_\infty := \max\{p_+, p^\star\}.$$

Here and after, we assume $p_+ < p^\star$. Thus, $p_\infty = p^\star$:

(H_1) Suppose there exist $q \in (p_+ - 1, p^\star - 1)$, $A > 0$, $B > 0$ such that

$$\begin{cases} A(u^q - 1) \leq f(u) \leq B(u^q - 1) & \text{for } u > 0, \\ f(u) = 0 & \text{for } u \leq -1. \end{cases} \tag{1.3}$$

(H_2) Assume an Ambrosetti–Rabinowitz-type condition, i.e., that there exist $\theta > p_+$ and $M \in \mathbb{R}$ such that

$$uf(u) \geq \theta F(u) + M, \tag{1.4}$$

where

$$F(u) = \int_0^u f(s)\, ds.$$

Remark 1.1 Equation (1.3) implies that there exist positive real numbers A_1, B_1 such that

$$F(u) \leq B_1\big(|u|^{q+1} + 1\big) \quad \text{for all } u \in \mathbb{R} \tag{1.5}$$

and

$$F(u) \geq A_1\big(|u|^{q+1} + 1\big) \quad \text{for all } u \in \mathbb{R}. \tag{1.6}$$

With respect to the above, the main result of this paper is Theorem 1.2. Our result extends the result of [5, Theorem 1.1] and [6, Theorem 1.1].

Theorem 1.2 *There exists $\lambda^* > 0$ such that if $\lambda \in (0, \lambda^*)$, then the problem (1.2) has a positive weak solution $u_\lambda \in L^\infty(\overline{\Omega})$.*

The rest of the paper is organized as follows. In Sect. 2, the suitable function space that is the anisotropic Sobolev space is recalled and necessary facts are also recalled. In Sect. 3, we study the Mountain-Pass Theorem and Palais–Smale condition for the problem. In Sect. 4, we present the proof of the main result, Theorem 1.2, which shows the existence of a positive solution of the problem (1.2).

2 Function spaces

Here, we define the anisotropic Sobolev spaces (see [18–20] and references therein), to which the solutions for our problems naturally belong, by

$$\begin{cases} W^{1,\overrightarrow{p}}(\Omega) := \{u \in W^{1,1}(\Omega) : \int_\Omega |\frac{\partial u}{\partial x_i}|^{p_i} < \infty, i = 1, \dots, N\}, \\ W_0^{1,\overrightarrow{p}}(\Omega) = W^{1,\overrightarrow{p}}(\Omega) \cap W_0^{1,1}(\Omega) \end{cases} \tag{2.1}$$

with the norm

$$\|u\|_{W^{1,\overrightarrow{p}}(\Omega)} := \int_\Omega \big|u(x)\big|\, dx + \sum_{i=1}^N \bigg(\int_\Omega \bigg|\frac{\partial u}{\partial x_i}\bigg|^{p_i} dx\bigg)^{\frac{1}{p_i}}.$$

We consider $W_0^{1,\overrightarrow{p}}(\Omega)$ endowed with the norm

$$\|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)} := \sum_{i=1}^{N}\left(\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx\right)^{\frac{1}{p_i}} = \sum_{i=1}^{N}\|u\|_{W_0^{1,p_i}(\Omega)}.$$

We recall the following theorem [13, Theorem 1].

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with a Lipschitz boundary. If*

$$p_i > 1, \quad \text{for all } i = 1,\dots,N, \qquad \sum_{i=1}^{N}\frac{1}{p_i} > 1,$$

then for all $r \in [1,p^*]$, there is a continuous embedding $W_0^{1,\overrightarrow{p}}(\Omega) \subset L^r(\Omega)$. For $r < p^*$, the embedding is compact.

Definition 2.2 An element $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ is called a weak solution to (1.2), if

$$\sum_{i=1}^{N}\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i}\,dx = \lambda\int_\Omega f(u)\phi\,dx \tag{2.2}$$

for all $\phi \in W_0^{1,\overrightarrow{p}}(\Omega)$.

Associated to (1.2) we have the functional $J_\lambda : W_0^{1,\overrightarrow{p}}(\Omega) \to \mathbb{R}$ defined by

$$J_\lambda(u) := \sum_{i=1}^{N}\frac{1}{p_i}\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx - \lambda\int_\Omega F\big(u(x)\big)\,dx. \tag{2.3}$$

Remark 2.3 J_λ is a functional of class C^1 and the critical points of the functional J_λ are the weak solutions of (1.2) (see [17] for a similar argument).

By the Mountain-Pass Theorem we can prove the existence of one solution of (1.2) and then we show for the proper value of λ that the solution is positive.

3 Mountain-Pass Theorem and Palais–Smale condition

The next two lemmas prove that J_λ satisfies the geometric hypotheses of the Mountain-Pass Theorem.

Lemma 3.1 *Assume $\phi \in W_0^{1,\overrightarrow{p}}(\Omega)$ denotes a positive differentiable function with $\|\phi\|_{W_0^{1,\overrightarrow{p}}(\Omega)} = 1$. There exists $\lambda_1 > 0$ such that if $\lambda \in (0,\lambda_1)$, then $J_\lambda(c\lambda^{-r}\phi) \leq 0$, where $r = \frac{1}{q+1-p_+} > 0$, $c = ((N+1)p_-^{-1}A_1^{-1}\|\phi\|_{q+1}^{-q-1})^r$ and A_1 is given by (1.6).*

Proof Since

$$\|\phi\|_{W_0^{1,\vec{p}}(\Omega)} = \sum_{i=1}^N \left(\int_\Omega \left|\frac{\partial \phi}{\partial x_i}\right|^{p_i} dx\right)^{\frac{1}{p_i}} = 1,$$

then $\int_\Omega |\frac{\partial \phi}{\partial x_i}|^{p_i} dx \leq 1$ for all $i = 1,\dots,N$. Also, $p_i > 1$, therefore

$$\int_\Omega \left|\frac{\partial \phi}{\partial x_i}\right|^{p_i} dx \leq \left(\int_\Omega \left|\frac{\partial \phi}{\partial x_i}\right|^{p_i} dx\right)^{\frac{1}{p_i}},$$

hence,

$$\sum_{i=1}^N \left(\int_\Omega \left|\frac{\partial \phi}{\partial x_i}\right|^{p_i} dx\right) \leq \sum_{i=1}^N \left(\int_\Omega \left|\frac{\partial \phi}{\partial x_i}\right|^{p_i} dx\right)^{\frac{1}{p_i}}.$$

Let $s = c\lambda^{-r}$, then by (1.6), we have

$$\begin{aligned}
J_\lambda(s\phi) &= \sum_{i=1}^N \int_\Omega \frac{|\frac{\partial(s\phi)}{\partial x_i}|^{p_i}}{p_i} dx - \lambda \int_\Omega F(s\phi)\, dx \\
&= \sum_{i=1}^N \frac{s^{p_i}}{p_i} \int_\Omega \left|\frac{\partial \phi}{\partial x_i}\right|^{p_i} dx - \lambda \int_\Omega F(s\phi)\, dx \\
&\leq \frac{\sum_{i=1}^N s^{p_i}}{p_-} - \lambda \int_\Omega F(s\phi)\, dx \\
&\leq \frac{\sum_{i=1}^N s^{p_i}}{p_-} - \lambda A_1 \int_\Omega \left(s^{q+1}\phi^{q+1} - 1\right) dx \\
&\leq \frac{Ns^{p_+}}{p_-} - \lambda A_1 \int_\Omega \left(s^{q+1}\phi^{q+1} - 1\right) dx \\
&= \frac{Ns^{p_+}}{p_-} - A_1 s^{q+1} \|\phi\|_{q+1}^{q+1} \lambda + \lambda A_1 |\Omega| \\
&= \left\{\frac{Nc^{p_+}\lambda^{-rp_+}}{p_-} - A_1 c^{q+1} \lambda^{-r(q+1)+1} \|\phi\|_{q+1}^{q+1}\right\} + \lambda A_1 |\Omega| \\
&\leq c^{p_+}\left\{\frac{N\lambda^{-rp_+}}{p_-} - A_1 c^{q+1-p_+} \lambda^{-r(q+1)+1} \|\phi\|_{q+1}^{q+1}\right\} + \lambda A_1 |\Omega|.
\end{aligned} \tag{3.1}$$

Thus,

$$\begin{aligned}
J_\lambda(s\phi) &\leq c^{p_+}\left\{\frac{N\lambda^{-rp_+}}{p_-} - \frac{N+1}{p_-}\lambda^{-r(q+1)+1}\right\} + \lambda A_1 |\Omega| \\
&= c^{p_+}\lambda^{-rp_+}\left\{\frac{N}{p_-} - \frac{N+1}{p_-}\lambda^{-r(q+1)+1+rp_+}\right\} + \lambda A_1 |\Omega| \\
&= -\frac{c^{p_+}\lambda^{-rp_+}}{p_-} + \lambda A_1 |\Omega|.
\end{aligned} \tag{3.2}$$

Taking $\lambda_1 < \min\{1, (p_- A_1 |\Omega| c^{-p_+})^{\frac{-1}{(1+rp_+)}}\}$, the lemma is proven. □

Lemma 3.2 *Assume* $r = \frac{1}{q+1-p_+} > 0$. *There exists* $\tau > 0$, $c_1 > 0$ *and* $\lambda_2 \in (0,1)$ *such that if* $\|u\|_{W_0^{1,p_+}(\Omega)} = \tau\lambda^{-\tau}$, *then* $J_\lambda(u) \geq c_1(\tau\lambda^{-r})^{p_+}$ *for all* $\lambda \in (0,\lambda_2)$.

Proof By the Sobolev embedding Theorem 2.1, there exists $K_1 > 0$ such that if $u \in W_0^{1,p_+}(\Omega)$, then $\|u\|_{q+1} \leq K_1\|u\|_{W_0^{1,p_+}(\Omega)}$. Assume

$$\tau = \min\left\{\left(2p_+K_1^{q+1}B_1\right)^{-r}, c\|u\|_{W_0^{1,p_+}(\Omega)}\right\}. \tag{3.3}$$

If $\|u\|_{W_0^{1,p_+}(\Omega)} = \tau\lambda^{-r}$, then

$$\begin{aligned}
J_\lambda(u) &= \sum_{i=1}^{N}\frac{1}{p_i}\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx - \lambda\int_\Omega F(u)\,dx \\
&\geq \frac{1}{p_+}\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_+}dx - \lambda\int_\Omega F(u)\,dx \\
&= \frac{\tau\lambda^{-r}}{p_+} - \lambda\int_\Omega F(u)\,dx \\
&\geq \frac{(\tau\lambda^{-r})^{p_+}}{p_+} - \lambda\int_\Omega B_1|u|^{q+1}\,dx - \lambda|\Omega|B_1 \\
&\geq \frac{(\tau\lambda^{-r})^{p_+}}{p_+} - \lambda B_1K_1^{q+1}\|u\|^{q+1}_{W_0^{1,p_+}(\Omega)} - \lambda|\Omega|B_1 \\
&= \frac{(\tau\lambda^{-r})^{p_+}}{p_+} - \lambda B_1K_1^{q+1}\tau^{(q+1)}\lambda^{-r(q+1)} - \lambda|\Omega|B_1 \\
&= \lambda^{-rp_+}\left\{\frac{\tau^{p_+}}{2p_+} - \lambda^{1+rp_+}|\Omega|B_1\right\} \\
&\geq \lambda^{-rp_+}\frac{\tau^{p_+}}{4p_+},
\end{aligned} \tag{3.4}$$

where we have used that $\tau \leq (2p_+K_1^{q+1}B_1)^{-r}$ (see (3.3)). Taking $c_1 = \frac{\tau^{p_+}}{4p_+}$ and $\lambda_2 = \tau^{\frac{p_+}{1+rp_+}}(4p_+B_1|\Omega|)^{-\frac{1}{1+rp_+}}$, the lemma is proven. □

Next, using the Mountain-Pass Theorem we prove that (1.2) has a solution $u_\lambda \in W_0^{1,\vec{p}}(\Omega)$.

Lemma 3.3 *Let* $\lambda_3 = \min\{\lambda_1,\lambda_2\}$. *There exists* $c_2 > 0$ *such that, for each* $\lambda \in (0,\lambda_3)$, *the functional* J_λ *has a critical point* u_λ *of mountain-pass type that satisfies* $J_\lambda(u_\lambda) \leq c_2\lambda^{-p_+r}$.

Proof First, we show that J_λ satisfies the Palais–Smale condition.

Assume that $\{u_n\}_n$ is a sequence in $W_0^{1,\vec{p}}(\Omega)$ such that $\{J_\lambda(u_n)\}_n$ is bounded and $J_\lambda'(u_n) \to 0$. Hence, there exists $\nu > 0$ such that

$$\langle J_\lambda'(u_n), u_n\rangle \leq \|u_n\|_{W_0^{1,\vec{p}}(\Omega)}$$

for $n \geq \nu$. Thus,

$$-\sum_{i=1}^{N}\int_\Omega\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i}dx - \sum_{i=1}^{N}\left(\int_\Omega\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i}\right)^{\frac{1}{p_i}}dx \leq -\lambda\int_\Omega f(u_n)u_n\,dx \quad \text{for } n \geq \nu.$$

Let K be a constant such that $|J_\lambda(u_n)| \le K$ for all $n = 1, 2, \dots$. From (1.4), we obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i} dx - \frac{\lambda}{\theta} \int_\Omega f(u_n) u_n \, dx + \frac{\lambda}{\theta} M|\Omega| \\
&\quad \le \sum_{i=1}^{N} \frac{1}{p_i} \int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i} dx - \lambda \int_\Omega F(u_n)\, dx \\
&\quad \le K.
\end{aligned}$$

From the last two inequalities we have

$$\sum_{i=1}^{N} \left(\frac{1}{p_i} - \frac{1}{\theta}\right) \int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i} dx - \sum_{i=1}^{N} \frac{1}{\theta} \left(\int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i}\right)^{\frac{1}{p_i}} dx \le K - \frac{\lambda}{\theta} M|\Omega|. \tag{3.5}$$

Now, we consider two cases. Case (i): If $(\int_\Omega |\frac{\partial u_n}{\partial x_i}|^{p_i})^{\frac{1}{p_i}} \le 1$, for $i = 1, \dots, N$, then $\{u_n\}$ is a bounded sequence. Case (ii): If there exists $1 \le j \le N$ such that $(\int_\Omega |\frac{\partial u_n}{\partial x_j}|^{p_j})^{\frac{1}{p_j}} > 1$, then

$$\left(\int_\Omega \left|\frac{\partial u_n}{\partial x_j}\right|^{p_j} dx\right)^{\frac{1}{p_j}} \le \int_\Omega \left|\frac{\partial u_n}{\partial x_j}\right|^{p_j} dx.$$

This shows (3.5) can be written as

$$\left(\frac{1}{p_+} - \frac{1}{\theta}\right) \left(\int_\Omega \left|\frac{\partial u_n}{\partial x_j}\right|^{p_j} dx\right)^{\frac{1}{p_j}} - \sum_{i=1}^{N} \frac{1}{\theta} \left(\int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i} dx\right)^{\frac{1}{p_i}} \le K - \frac{\lambda}{\theta} M|\Omega|.$$

This proves that $\{u_n\}$ is a bounded sequence. Thus, without loss of generality, we may assume that $\{u_n\}$ converges weakly. Let $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ be its weak limit. Since $q < \frac{Np_+}{(N-p_+)}$, by the Sobolev embedding theorem we may assume that $\{u_n\}$ converges to u in $L^q(\Omega)$. These assumptions and Hölder's inequality imply

$$\int_\Omega \lambda f(u_n)(u_n - u) \to 0. \tag{3.6}$$

From (3.6) and $\lim_{n\to+\infty} J'_\lambda(u_n) = 0$, we have

$$\lim_{n\to+\infty} \sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2} \frac{\partial u_n}{\partial x_i} \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right) dx = 0. \tag{3.7}$$

Using again that u is the weak limit of $\{u_n\}$ in $W_0^{1,\overrightarrow{p}}(\Omega)$ we also have

$$\lim_{n\to+\infty} \sum_{i=1}^{N} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i} \left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right) dx = 0. \tag{3.8}$$

By Hölder's inequality,

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega\left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2}\frac{\partial u_n}{\partial x_i}-\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)\left(\frac{\partial u_n}{\partial x_i}-\frac{\partial u}{\partial x_i}\right)dx\\
&\geq\sum_{i=1}^{N}\int_\Omega\left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i}-\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-1}\left|\frac{\partial u}{\partial x_i}\right|-\left|\frac{\partial u}{\partial x_i}\right|^{p_i-1}\left|\frac{\partial u_n}{\partial x_i}\right|+\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\right)dx\\
&=\sum_{i=1}^{N}\left(\int_\Omega\left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i}+\left|\frac{\partial u}{\partial x_i}\right|^{p_i}\right)dx-\int_\Omega\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-1}\left|\frac{\partial u}{\partial x_i}\right|dx\right.\\
&\quad\left.-\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i-1}\left|\frac{\partial u_n}{\partial x_i}\right|dx\right)\\
&\geq\sum_{i=1}^{N}\left[\int_\Omega\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i}dx-\left(\int_\Omega\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i}dx\right)^{\frac{p_i-1}{p_i}}\left(\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx\right)^{\frac{1}{p_i}}\right.\\
&\quad\left.-\left(\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx\right)^{\frac{p_i-1}{p_i}}\left(\int_\Omega\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i}dx\right)^{\frac{1}{p_i}}+\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx\right]\\
&=\sum_{i=1}^{N}\left\{\left[\left(\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx\right)^{\frac{p_i-1}{p_i}}-\left(\int_\Omega\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i}dx\right)^{\frac{p_i-1}{p_i}}\right]\right.\\
&\quad\left.\times\left[\left(\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx\right)^{\frac{1}{p_i}}-\left(\int_\Omega\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i}dx\right)^{\frac{1}{p_i}}\right]\right\}\\
&\geq 0.
\end{aligned}\tag{3.9}$$

The relations (3.7)–(3.9) imply that

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{N}\left\{\left[\left(\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx\right)^{\frac{p_i-1}{p_i}}-\left(\int_\Omega\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i}dx\right)^{\frac{p_i-1}{p_i}}\right]\right.\\
&\quad\left.\times\left[\left(\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx\right)^{\frac{1}{p_i}}-\left(\int_\Omega\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i}dx\right)^{\frac{1}{p_i}}\right]\right\}=0.
\end{aligned}$$

This shows that for each $i=1,\dots,N$

$$\lim_{n\to\infty}\left(\int_\Omega\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i}dx\right)^{\frac{1}{p_i}}=\left(\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx\right)^{\frac{1}{p_i}},$$

which implies that $\lim_{n\to\infty}\|u_n\|_{W_0^{1,\vec{p}}(\Omega)}=\|u\|_{W_0^{1,\vec{p}}(\Omega)}$. Since $u_n\rightharpoonup u$, $u_n\to u$ in $W_0^{1,\vec{p}}(\Omega)$. This proves that J_λ satisfies the Palais–Smale condition.

From (3.1) we obtain

$$\begin{aligned}
\max\{J_\lambda(s\phi):s\geq 0\}&\leq\left(\frac{Np_+}{p_-}\right)^{r(q+1)}\frac{C^{1+p_+r}((q+1)^{r(q-p_+)+r}-p_+)}{D^{p_+r}p_+(q+1)^{r(q+1)}}\lambda^{-p_+r}+\lambda A_1|\Omega|\\
&:=c_2'\lambda^{-p_+r}+\lambda A_1|\Omega|\\
&\leq c_2'\lambda^{-p_+r}+\lambda^{-p_+r}A_1|\Omega|\\
&:=c_2\lambda^{-p_+r},
\end{aligned}\tag{3.10}$$

where $C = \max\{\int_\Omega |\frac{\partial u}{\partial x_i}|^{p_i}\,dx : \text{for } 1 \leq i \leq N\}$ and $D = A_1\|\phi\|_{q+1}^{q+1}$. With this estimate and Lemma 3.2, the existence of $u_\lambda \in W_0^{1,\overrightarrow{p}}(\Omega)$ such that $\nabla J_\lambda(u_\lambda) = 0$ and

$$c_1\left(\tau\lambda^{-r}\right)^{p_+} \leq J_\lambda(u_\lambda) \leq c_2\lambda^{-p_+r} \tag{3.11}$$

follows by the Mountain-Pass Theorem. □

Remark 3.4 The solution $u_\lambda \in W_0^{1,\overrightarrow{p}}(\Omega)$ is indeed in $L^\infty(\Omega)$ (see [12, Lemma 2.4]) and [3, Sect. 4]).

Lemma 3.5 *Let u_λ be as in Lemma 3.3. Then, there is a positive constant M_0 such that*

$$M_0\lambda^{-r} \leq \|u_\lambda\|_\infty. \tag{3.12}$$

Proof Note that there exists $c_1 > 0$ such that $J(u_\lambda) \geq c_1\lambda^{-rp_+}$. On the other hand, $F(s) \geq \min F > -\infty$ and $f(s)s \leq B_1(|s|^{q+1} + |s|)$ for all $s \in \mathbb{R}$. Then, there is a constant $C_1 > 0$ such that

$$\begin{aligned}
\lambda\int_\Omega f(u_\lambda)u_\lambda\,dx &= \sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx \\
&\geq p_- \sum_{i=1}^N \frac{1}{p_i}\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx \\
&\geq p_- J(u_\lambda) + p_-\lambda\int_\Omega F(u_\lambda)\,dx \\
&\geq p_- C_1\lambda^{-rp_+} + p_-|\Omega|\lambda\min F \\
&\geq c_1\lambda^{-rp_+}.
\end{aligned}$$

Thus, $\lim_{\lambda\to 0}\|u_\lambda\|_\infty = +\infty$. On the other hand, by (1.5),

$$\begin{aligned}
\lambda\int_\Omega f(u_\lambda)u_\lambda\,dx &\leq B_1\lambda\int_\Omega \left(|u_\lambda|^{q+1} + |u_\lambda|\right)dx \\
&\leq B_1\lambda\int_\Omega \left(\|u_\lambda\|_\infty^{q+1} + \|u_\lambda\|_\infty\right)dx \\
&\leq 2B_1|\Omega|\lambda\|u_\lambda\|_\infty^{q+1},
\end{aligned}$$

where we have used the fact that $0 < \lambda < 1$. Finally, taking $M_0 = \frac{C_1}{2B_1|\Omega|}$, the lemma is proven. □

Lemma 3.6 *Let u_λ be as in Lemma 3.3. Then, there exists $c_3 > 0$ such that*

$$\sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i} dx \leq c_3\lambda^{-rp_+} \tag{3.13}$$

for all $\lambda \in (0, \lambda_3)$.

Proof By (1.4) and the definition of u_λ,

$$\begin{aligned}\lambda\int_\Omega \frac{\theta-p_+}{\theta}u_\lambda f(u_\lambda)\,dx &\le \lambda\int_\Omega \big(u_\lambda f(u_\lambda)-p_+F(u_\lambda)\big)\,dx-\frac{\lambda p_+M|\Omega|}{\theta}\\ &=\sum_{i=1}^N\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx-p_+\int_\Omega F(u_\lambda)\,dx-\frac{\lambda p_+M|\Omega|}{\theta}\\ &\le p_+\left(\sum_{i=1}^N\frac{1}{p_i}\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx-\int_\Omega F(u_\lambda)\,dx\right)-\frac{p_+\lambda M|\Omega|}{\theta}\\ &\le c_2\lambda^{-rp_+}+\frac{\lambda p_+M|\Omega|}{\theta}\\ &\le 2c_2\lambda^{-rp_+},\end{aligned}\tag{3.14}$$

where we have used $0<\lambda<1$. Now, the result follows from (3.14) and the fact that u_λ is a weak solution of (1.2). □

4 Existence of a positive solution

Now, we can prove Theorem 1.2 as follows.

Proof Suppose there exists a sequence $\{\lambda_j\}_j, 1>\lambda_j>0$ for all j, converging to 0 such that the measure $m(\{x\in\Omega; u_{\lambda_j}(x)\le 0\})>0$.

Letting $w_j=\frac{u_{\lambda_j}}{\|u_{\lambda_j}\|_\infty}$, we see that

$$-\sum_{i=1}^N\|u_{\lambda_j}\|_\infty^{p_i-1}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial w_j}{\partial x_i}\right|^{p_i-2}\frac{\partial w_j}{\partial x_i}\right)=\lambda_j f(u_{\lambda_j}).\tag{4.1}$$

From Lemmas 3.5 and 3.6 there is a constant C_3 such that

$$\begin{aligned}\sum_{i=1}^N\int_\Omega\left|\frac{\partial w_j}{\partial x_i}\right|^{p_i}dx &=\sum_{i=1}^N\left(\frac{1}{\|u_{\lambda_j}\|_\infty}\right)^{p_i}\int_\Omega\left|\frac{\partial u_{\lambda_j}}{\partial x_i}\right|^{p_i}dx\\ &\le\sum_{i=1}^N\frac{1}{(M_0\lambda^{-r})^{p_i}}\int_\Omega\left|\frac{\partial u_{\lambda_j}}{\partial x_i}\right|^{p_i}dx\\ &\le M_1\frac{1}{\lambda^{-rp_+}}\sum_{i=1}^N\int_\Omega\left|\frac{\partial u_{\lambda_j}}{\partial x_i}\right|^{p_i}dx\\ &\le C_3.\end{aligned}\tag{4.2}$$

This shows that for each $i=1,\dots,N$

$$\int_\Omega\left|\frac{\partial w_j}{\partial x_i}\right|^{p_i}dx\le c_3\tag{4.3}$$

and therefore

$$\|w_j\|_{W_0^{1,\vec{p}}(\Omega)}=\sum_{i=1}^N\left(\int_\Omega\left|\frac{\partial w_j}{\partial x_i}\right|^{p_i}dx\right)^{\frac{1}{p_i}}\le D_3.\tag{4.4}$$

By [3, Proposition 4.1] (or [13, Theorem 2]) the sequence w_j is uniformly bounded in $L^\infty(\Omega)$. Therefore, one may denote its limit by ω.

Next, using comparison principles [12, Lemma 2.5], we prove that $w(x) \geq 0$.

Let $v_0 \in W_0^{1,p_+}(\Omega)$ be the solution of

$$\begin{cases} -\Delta_{p_+} v_0 = 1 & \text{in } \Omega, \\ v_0 = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.5}$$

Let $K_j := \lambda_j \min\{f(t); t \in \mathbb{R}\} \|u_{\lambda_j}\|_\infty^{1-p_+}$. The solution v_j of the equation

$$\begin{cases} -\Delta_{p_+} v_j = K_j & \text{in } \Omega, \\ v_0 = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.6}$$

is given by $v_j = (-K_j)^{\frac{1}{p_+-1}} v_0$.

Since $\lambda_j f(u_{\lambda_j}) \|u_{\lambda_j}\|_\infty^{1-p_+} \geq K_j$, it follows by the comparison principle in [12, Lemma 2.5] that $w_j \geq v_j$. Then, the fact that $v_j(x) \to 0$ as $j \to 0$ implies that $w(x) \geq 0$ for all $x \in \Omega$.

Since, by hypothesis, $q > p_+ - 1$, we have $s = \frac{Np_+ r}{(N-p_+)} > 1$. This result, together with the Sobolev embedding Theorem, (1.3) and Lemma 3.6, gives

$$\begin{aligned} \int_\Omega \left|f(u_{\lambda_j})\right|^s \|u_{\lambda_j}\|_\infty^{s(1-p_+)}\, dx &\leq B^s 2^{s-1} \int_\Omega \left(|u_{\lambda_j}|^{(q+1-p_+)s} + 1\right) dx \\ &\leq C\left(\|u_{\lambda_j}\|_{W_0^{1,p_+}(\Omega)}^{\frac{Np_+}{N-p_+}} + 1\right) \\ &\leq C\left(c_3 \lambda_j^{-r\frac{Np_+}{N-p_+}} + 1\right), \end{aligned} \tag{4.7}$$

where $C > 0$ is a constant independent of j and, without loss of generality, we have assumed $\|u_{\lambda_j}\|_\infty \geq 1$. From (4.7) and the fact that $\frac{rNp_+}{(sN-sp_+)} = 1$ we see that $\{\lambda_j f(u_{\lambda_j}) \|u_{\lambda_j}\|_\infty^{(1-p_+)}\}$ is bounded in $L^s(\Omega)$, so we may assume that it converges weakly. Let $z \in L^s(\Omega)$ be the weak limit of such a sequence. Since $\lambda_j \|u_{\lambda_j}\|_\infty^{(1-p_+)} \to 0$ as $j \to +\infty$ and f is bounded from below, $z \geq 0$. Now, if $\phi \in C_0^\infty(\Omega)$, then

$$\begin{aligned} \sum_{i=1}^N \int_\Omega \left|\frac{\partial w}{\partial x_i}\right|^{p_i-2} \left\langle \frac{\partial w}{\partial x_i}, \frac{\partial \phi}{\partial x_i} \right\rangle dx &= \lim_{j\to\infty} \sum_{i=1}^N \int_\Omega \left|\frac{\partial w_j}{\partial x_i}\right|^{p_i-2} \left\langle \frac{\partial w_j}{\partial x_i}, \frac{\partial \phi}{\partial x_i} \right\rangle dx \\ &= \lim_{j\to\infty} \sum_{i=1}^N \int_\Omega \|u_{\lambda_j}\|_\infty^{1-p_i} \left|\frac{\partial u_{\lambda_j}}{\partial x_i}\right|^{p_i-2} \left\langle \frac{\partial u_{\lambda_j}}{\partial x_i}, \frac{\partial \phi}{\partial x_i} \right\rangle dx \\ &\geq \lim_{j\to\infty} \|u_{\lambda_j}\|_\infty^{1-p_+} \sum_{i=1}^N \int_\Omega \left|\frac{\partial u_{\lambda_j}}{\partial x_i}\right|^{p_i-2} \left\langle \frac{\partial u_{\lambda_j}}{\partial x_i}, \frac{\partial \phi}{\partial x_i} \right\rangle dx \\ &= \lim_{j\to\infty} \int_\Omega \|u_{\lambda_j}\|_\infty^{1-p_+} \lambda_j f(u_{\lambda_j}) \phi\, dx \\ &= \int_\Omega z\phi\, dx. \end{aligned} \tag{4.8}$$

Therefore, $-\Delta_{\overrightarrow{p}} w \geq z$. Since $\|w_j\|_\infty = 1$, $w \neq 0$. By [12, Lemma 2.5], $w > 0$ in Ω.

Therefore, since $\{w_j\}_j$ converges w in $L^\infty(\Omega)$, for sufficiently large j, $w_j(x) > 0$ for all $x \in \Omega$. Hence, $u_{\lambda_j}(x) > 0$ for all $x \in \Omega$, which contradicts the assumption that

$$m\left(\left\{x; u_{\lambda_j}(x) < 0\right\}\right) > 0.$$

This contradiction proves Theorem 1.2. □

Data availability
Not applicable.

Declarations

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Not applicable.

Competing interests
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