Shao et al. *Boundary Value Problems* (2024) 2024:35
https://doi.org/10.1186/s13661-024-01842-6

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Caputo fractional backward stochastic differential equations driven by fractional Brownian motion with delayed generator

Yunze Shao[1], Junjie Du[2*], Xiaofei Li[1], Yuru Tan[1] and Jia Song[1]

*Correspondence:
dujunjie0420@163.com
[2]School of Information Engineering, Jingzhou University, Jingzhou 434020, Hubei, China
Full list of author information is available at the end of the article

Abstract

Over the years, the research of backward stochastic differential equations (BSDEs) has come a long way. As a extension of the BSDEs, the BSDEs with time delay have played a major role in the stochastic optimal control, financial risk, insurance management, pricing, and hedging. In this paper, we study a class of BSDEs with time-delay generators driven by Caputo fractional derivatives. In contrast to conventional BSDEs, in this class of equations, the generator is also affected by the past values of solutions. Under the Lipschitz condition and some new assumptions, we present a theorem on the existence and uniqueness of solutions.

Keywords: Fractional backward stochastic differential equations; Caputo; Fractional Brownian motion; Time-delay generator

1 Introduction

The fractional derivative introduces a memory effect, enabling the system to exert a long-term influence on past inputs. Simultaneously, the delay is manifested as the system response persists over time, eventually fully reflecting changes in the input after a certain period has elapsed. Both these concepts effectively capture and describe the enduring memory correlation of a system. Because of their advantages in describing the time behavior of a system or process, backward stochastic differential equations (BSDEs) with fractional derivative or delay are widely used in many fields such as optimal control, finance, biology, and physics.

The theory of BSDEs bridges the gap between randomness and determinism and was first proposed by Bismut [2] in 1973 to explain a stochastic version of the process associated with Pontryagin's maximum principle, but subsequent work has been carried out in a linear context. In 1978, Bismut [3] introduced a class of nonlinear BSDEs (Riccati equations) and proved the existence and uniqueness of their solutions. In 1990, Peng and Pardoux [19] solved the existence and uniqueness problem of the equations under Lipschitz conditions and proposed a general form of BSDEs. In subsequent work, they established a theoretical system of BSDEs, focusing on solving the structure and diffusion part of the generator in the equation. This led to a series of results and studies (Bahlali et al. [1], Zhang et al. [24]), making this theory more perfect. Further, this theory itself has been developed into many different branches (Ma et al. [17], Zhao et al. [26], Pardoux and Răşcanu [20]).

The fractional Brownian motion $B^H = \{B_t^H, t \geq 0\}$ (fBm) with Hurst parameter H ($0 < H < 1$), proposed by Kolmogorov [15], is a central Gaussian process. Due to its favorable autocorrelation and long-memory properties, it can serve as a valuable supplement to the study of BSDEs. Therefore Hu and Peng [13] introduced fractional-order Brownian motion and studied general linear and nonlinear BSDEs driven by a fractional-order Brownian motion. Under some mild assumptions, they established the existence and uniqueness of understanding. In the case of nonlinear, BSDE is solved by obtaining an inequality leading to the fixed point theorem, and the existence and uniqueness of the solution is proved when the Hurst parameter $H \in (\frac{1}{2}, 1)$. Maticiuc and Nie [18] then obtained some general results for fractional-order BSDEs through a rigorous approach, extending the discussion to fractional BSVIs. Borkowska [14] studied generalized BSDEs driven by a fractional Brownian motion. Douissi et al. [9] proposed a new mean-field anticipation BSDE driven by a fractional Brownian motion. Wen and Shi [23] studied the existence and uniqueness of expected backward stochastic differential equations driven by a fractional Brownian motion under the Lipschitz condition and a sufficiently small time horizon T. Li et al. [16] derived a European option pricing formula based on the FSDE model with fractional derivative $\mathrm{d}^\alpha f = \Gamma(1+\alpha)[\mathrm{d}f - f'(x)\,\mathrm{d}x]$. Inspired by Li's method, Chen et al. [4] combined fractional derivatives and fractional differential equations and obtained a BSDE with Caputo fractional derivative of the form

$$\begin{aligned} Y_t = Y_T &+ \int_t^T \left(\frac{\alpha \cdot (\alpha - 1) \cdot (T-s)^{\alpha-2}}{\Gamma(1+\alpha)} F(s) - f(s) \right) \mathrm{d}s \\ &- \int_t^T \frac{z(s,T)}{\Gamma(1+\alpha)}\,\mathrm{d}B_s, \quad \alpha = 2H \in (1,2), \end{aligned}$$

where Γ is the gamma function, f is the generator related to the present time, and B is a standard Brownian motion. Chen et al. [4] constructed a new norm to obtain the existence and uniqueness of the equation.

The mathematical delay method plays an important role in many fields such as stochastic optimal control, financial risk, insurance management, pricing, and hedging. Delong and Imkeller [7, 8] studied the following class of BSDEs with time-delay generators:

$$Y(t) = \xi + \int_t^T f(s, Y_s, Z_s)\,ds - \int_t^T Z(s)\,dW(s), \quad t \in [0,T],$$

where $(Y_s, Z_s) = (Y(s+u), Z(s+u)), -T \leq u \leq 0$, and the generated elements can be time dependent on the past values of solutions and weighted by a time-delay function. Examples of BSDEs with time-delay generators, exhibiting either multiple solutions or no solution, are provided, and some properties of the solutions of these BSDEs with time-delay generators are discussed. Subsequently, Wen [22] extended such fractional-order equations. The existence and uniqueness of various BSDE models and their solutions have been proved due to their excellent mathematical properties, broad application prospects, and deep connection with partial differential equations (Zhang et al. [25], Zhuang [27], Delng et al. [6], Peng and Yang [21]).

As an extension of BSDEs, Caputo fractional backward stochastic differential equations with delay driven by a fractional Brownian motion play an important role in stochastic optimal control, financial risk, insurance management, pricing, and hedging. In particular, for pricing problems in finance, by introducing the Caputo fractional derivative and lag or delay term, the market reaction lag effect can be considered in the stock price model, and the change of stock price can be described and predicted more accurately. However, to the best of our knowledge, there have been no studies on such BSDEs. Therefore in this paper, we mainly study the existence and uniqueness of such BSDEs. Specifically, we investigate the following fractional-order BSDEs with a tiem-delay generator:

$$\begin{cases} Y_t = Y_T + \int_t^T \big(\frac{\alpha(\alpha-1)(T-s)^{\alpha-2}}{\Gamma(1+\alpha)} F(s,\eta_s,Y_s,Z_s,Y_{s-\delta_1(s)},Z_{s-\delta_2(s)}) \\ \qquad - f(s,\eta_s,Y_s,Z_s,Y_{s-\delta_3(s)},Z_{s-\delta_4(s)})\big)\,ds \\ \qquad - \int_t^T \frac{Z_s}{\Gamma(1+\alpha)}\,dB_s^H, \quad 0 \le t \le T, \\ Y_t = \varphi_t, \qquad Z_t = \psi_t, \quad -\delta \le t < 0. \end{cases} \tag{1.1}$$

This paper is organized as follows: In Sect. 2, we introduce some conditions, theorems, and hypotheses to facilitate the subsequent proof. In Sect. 3, we prove the existence and uniqueness of solutions to the equations. The comparison theorem of the equations is obtained in Sect. 4. In Sect. 5, we conclude our investigation.

2 Preliminaries

We first review some relevant definitions, lemma, and propositions and make some necessary assumptions. For more specific content, we refer to Decreusefond and Ustunel [5], Duncan et al. [10], and Hu [11].

2.1 Fractional Brownian motion

A fractional Brownian motion with Hurst parameter $H \in (0,1)$ is a zero-mean Gaussian process $B^H = \{B_t^H, t \ge 0\}$ with covariance

$$E\big(B_t^H B_s^H\big) = \frac{1}{2}\big(t^{2H} + s^{2H} + |t-s|^{2H}\big).$$

When $H = \frac{1}{2}$, this process becomes a standard Brownian motion. When $H \in (0,\frac{1}{2})$, time series show a negative long-term dependence, that is, there is little correlation between past and future values, and future changes are difficult to predict. Throughout the paper, we only consider $H \in (\frac{1}{2},1)$, and the time series shows a positive long-term dependence.

Let $(\Omega,\mathcal{F},\mathcal{P})$ be a complete probability space with filtration $\mathcal{F}$ is generated by an fBm B^H. For two continuous functions ξ and η on $[0,T]$, we define the Hilbert scalar product and norm

$$\langle \xi,\eta\rangle_t = \int_0^t \int_0^t \phi(u-v)\xi_u\eta_v\,\mathrm{d}u\,\mathrm{d}v \quad \text{and} \quad \|\xi\|_t = \sqrt{\langle \xi,\xi\rangle_t},$$

where $\phi(x) = 2H(2H-1)|x|^{2H-2}$ for $x \in R$. Under this Hilbert scalar product, we use Θ_t to represent the completion of a continuous function. Let $\mathcal{P}_T$ be the set of all fractional Brownian motion polynomials over $[0,T]$. The elements therein are represented as

$$F(\omega) = f\bigg(\int_0^T \xi_1(t)\,dB_t^H,\dots,\int_0^T \xi_n(t)\,dB_t^H\bigg),$$

where ξ_i ($i = 1,2,\dots,n$) are continuous functions on $[0,T]$, and f is a polynomial of n variables.

The Malliavin derivative D_s^H of a polynomial functional F is defined by

$$D_s^H F = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\left(\int_0^T \xi_1(t)\,dB_t^H,\dots,\int_0^T \xi_n(t)\,dB_t^H\right)\xi_j(s), \quad 0 \le s \le T.$$

The completion of $\mathcal{P}_T$ represented by $\mathbb{D}_{1,2}^H$ has the norm

$$\|F\|_{H,1,2}^2 = E\|F\|_T{}^2 + E\left\|D_s^H F\right\|_T^2.$$

We also have the derivative

$$\mathbb{D}_t^H F = \int_0^T \phi(t-u) D_v^H F\,dv.$$

Proposition 2.1 (*Hu* [11], *Proposition* 6.25) *Denote by* $\mathbb{L}_H^{1,2}$ *the space of all continuous processes* $F_s : (\Omega,\mathcal{F},P) \to \Theta_t$ *satisfying* $E[\|F\|_T^2 + \int_0^T\int_0^T |\mathbb{D}_s^H F_t|^2\,ds\,dt] \le \infty$. *If* $F_s \in \mathbb{L}_H^{1,2}$, *then the Itô-type stochastic integral* $\int_0^T F(t)\,dB_t^H$ *exists in* $L^2(\Omega,\mathcal{F},\mathcal{P})$, *and*

$$E\left(\int_0^T F(t)\,dB_t^H\right) = 0,$$
$$E\left(\int_0^T F(t)\,dB_t^H\right)^2 = E\left(\|F\|_{\Theta_t}^2\right) + E\int_0^T\int_0^T \mathbb{D}_s^H F_t \mathbb{D}_t^H F_s\,ds\,dt.$$

Proposition 2.2 (*Hu* [11], *Theorem* 10.3) *For* $i = 1,2$, *let* $g_i(s) \in \mathbb{D}_H^{1,2}$, *and l et* f_i *and* g_i *be two real-valued processes satisfying* $E\int_0^T (|f_i(s)|^2 + |g_i(s)|^2)\,ds < \infty$. *We assume that* $\mathbb{D}_t^H f_i(s)$ *and* $\mathbb{D}_t^H g_i(s)$ *are continuously differentiable for almost all* $(s,t) \in [0,T]^2$ *and* $\omega \in \Omega$. *Suppose that* $E[\int_0^T\int_0^T |\mathbb{D}_t^H g_i(s)|^2\,ds\,dt] < \infty$. *Denote*

$$G_i(t) = \int_0^T f_i(s)\,ds + \int_0^T g_i(s)\,dB_s^H, \quad 0 \le t \le T.$$

Then

$$\begin{aligned} G_1(t)G_2(t) = {}& \int_0^t G_1(s)f_2(s)\,ds + \int_0^t G_1(s)g_2(s)\,dB_s^H + \int_0^t G_2(s)f_1(s)\,ds \\ & + \int_0^t G_2(s)g_1(s)\,dB_s^H + \int_0^t \mathbb{D}_s^H G_1(s)g_2(s)\,ds + \int_0^t \mathbb{D}_s^H G_2(s)g_1(s)\,ds. \end{aligned}$$

Proposition 2.3 (*Hu and Peng* [13] *and Maticiuc and Nie* [18]) *Let* $(Y_t,Z_t) \in \widetilde{\mathcal{V}}_{[-\delta,T]} \times \widetilde{\mathcal{V}}_{[-\delta,T]}^H$ *be the solution to the fractional BSDE* (2.1), *we have the relation*

$$\mathbb{D}_t^H Y_t = \frac{\widehat{\sigma}(t)}{\sigma(t)} Z_t, \quad 0 \le t \le T.$$

Therefore there exists a constant $M > 0$ *such that*

$$\frac{t^{2H-1}}{M} Z_t \le \mathbb{D}_t^H Y_t = \frac{\hat{\sigma}_t}{\sigma_t} Z_t \le M t^{2H-1} Z_t, \quad 0 \le t \le T.$$

2.2 Caputo fractional derivative

In the studies of real financial markets, it appeared that the stochastic differential equation of fractional Brownian motion can only describe the noise memory but cannot be used to study the trend memory effect of the stock price. Subsequently, fractional derivatives were introduced to describe the trend memory process, which is another effective tool to describe the memory effect, for example, the Caputo fractional derivative [16] with Hurst index $H \in (\frac{1}{2}, 1)$:

$$d^{\alpha} f = \Gamma(1+\alpha)\big[df - f'(x)\,dx\big], \quad \alpha = 2H.$$

Some of results for this Caputo fractional derivative are given in [16, Lemma 5]:

$$\text{(1)} \quad \int_0^t f(x)\big(dx^{\alpha}\big) = \alpha \int_0^t (t-x)^{\alpha-1} f(x)\,dx, \quad 0<\alpha<1,$$
$$\text{(2)} \quad \int_0^t f(x)(dx)^{\alpha} = \alpha(\alpha-1) \int_0^t (t-x)^{\alpha-2} F(x)\,dx, \quad 1<\alpha<2, \qquad F(t) = \int_0^t f(x)\,dx.$$

Combined with a linear stochastic differential equation, stochastic differential equation with Caputo fractional derivative is derived:

$$\mathrm{d}Y_t = \frac{f(t,Y_t,Z_t)}{\Gamma(1+\alpha)}(\mathrm{d}t)^{\alpha} + \frac{Z_t}{\Gamma(1+\alpha)}\,\mathrm{d}B_t + f(t,Y_t,Z_t)\,\mathrm{d}t, \quad 1<\alpha<2, 0.5<H<1.$$

Integrating this equation in the interval $[t,T]$ and then by combining the derivation method of Caputo SDE and BSDE [4], we can get a Caputo fractional BSDE driven by a fractional Brownian motion with time delay:

$$\begin{cases} Y_t = Y_T + \int_t^T \big(\frac{\alpha(\alpha-1)(T-s)^{\alpha-2}}{\Gamma(1+\alpha)} F(s,\eta_s,Y_s,Z_s,Y_{s-\delta_1(s)},Z_{s-\delta_2(s)}) \\ \qquad - f(s,\eta_s,Y_s,Z_s,Y_{s-\delta_3(s)},Z_{s-\delta_4(s)})\big)\,ds \\ \qquad - \int_t^T \frac{Z_s}{\Gamma(1+\alpha)}\,dB_s^H, \quad 0 \le t \le T, \\ Y_t = \varphi_t, \qquad Z_t = \psi_t, \quad -\delta \le t < 0. \end{cases} \tag{2.1}$$

2.3 Assumptions and conditions

To prove our results, we make the following statement. First, we have

$$\eta_t = \eta_0 + \int_0^t b_s\,ds + \int_0^t \sigma_s\,dB_s^H, \quad 0 \le t \le T, \tag{2.2}$$

where η_0 is a constant, b_t and σ_t are two bounded deterministic continuously differentiable functions from $[0,T]$ to $\mathbb{R}$, and $\sigma_s > 0$. Then we have

$$\|\sigma\|_t^2 = \int_0^t \int_0^t \phi(u-v)\sigma_u\sigma_v\,du\,dv = H(2H-1)\int_0^t \int_0^t |u-v|^{2H-2}\sigma_u\sigma_v\,du\,dv.$$

Denote $\frac{d}{dt}(\|\sigma\|_t^2) = 2\hat{\sigma}_t\sigma_t$ and $\hat{\sigma}_u = \int_0^t \phi(u-v)\sigma_v\,dv$. In addition, consider the following sets:

$$L^2(\mathcal{F}_t;\mathbb{R}) := \big\{\varphi : \Omega \to \mathbb{R} | \varphi \text{ is } \mathcal{F}_t\text{- measurable}, E\big[|\varphi|^2\big] < \infty\big\};$$

$$L^2_{\mathcal{F}}(0,T;\mathbb{R}) := \bigg\{\varphi : [0,T]\times\Omega \to \mathbb{R} | \varphi \text{ is a progressively measurable process,}$$
$$E\bigg[\int_0^T \big|\varphi(t)\big|^2\bigg]\mathrm{d}t\bigg] < \infty\bigg\};$$

$$\mathcal{C}^{k,l}\big([0,T]\times\mathbb{R}\big)$$
$$:= \big\{\varphi(t,x) : [0,T]\times\mathbb{R} \to \mathbb{R}|\varphi \text{ is } k \text{ times differentiable with respect to } t \in [0,T]$$
$$\text{and } l \text{ times continuously differentiable with respect to } x \in \mathbb{R}\big\};$$

$$\mathcal{C}^{k,l}_{\text{pol}}\big([0,T]\times\mathbb{R}\big) := \big\{\varphi(t,x)|\varphi \in \mathcal{C}^{k,l}_{\text{pol}}\big([0,T]\times\mathbb{R}\big),$$
$$\text{and all derivatives of } \varphi \text{ are of polynomial growth}\big\};$$

$$\mathcal{V}_{[0,T]} := \bigg\{\varphi(t,x)|\varphi \in \mathcal{C}^{1,3}_{\text{pol}}\big([0,T]\times\mathbb{R}\big) \text{ with } \frac{\partial\varphi}{\partial t} \in \mathcal{C}^{0,1}_{\text{pol}}\big([0,T]\times\mathbb{R}\big)\bigg\}.$$

Denote the completions of $\mathcal{V}_{[t_1,t_2]}$ by $\tilde{\mathcal{V}}_{[t_1,t_2]}$ and $\tilde{\mathcal{V}}^H_{[t_1,t_2]}$ with the corresponding β-norms

$$\|\varphi\|_\beta = \bigg(E\int_{t_1}^{t_2} e^{\beta t}\big|\varphi(t)\big|^2\,dt\bigg)^{\frac{1}{2}}, \qquad \|\mu\|_\beta = \bigg(E\int_{t_1}^{t_2} e^{\beta t}t^{2H-1}\big|\mu(t)\big|^2\,dt\bigg)^{\frac{1}{2}},$$

where $\beta \geq 0$ is a constant; apparently, they are Banach spaces. In addition, we need to make the following assumptions.

Assumption 2.1 δ_i, $i = 1,2,3,4$, *are* R^+*-valued continuous functions on* $[0,T]$ *such that* $0 \leq s - \delta_i(s) \leq s$, *where* δ *is a constant.*

(A1) There exists $L \geq 0$ such that for all $t \in [0,T]$ and for all nonnegative integrable g,

$$\int_t^T g\big(s-\delta_i(s)\big)\,ds \leq L\int_{t-\delta}^T g(s)\,ds, \quad i = 1,2,3,4.$$

(A2) The generator (F,f): $(F,f)(t,x,y,z,y',z') : [0,T]\times\mathbb{R}^5 \to \mathbb{R}$ is a $\mathcal{C}^{0,1}_{\text{pol}}$-continuous function, there exist $C_1, C_2 \geq 0$ such that (F,f) satisfies the following conditions: for all $t \in [0,T]$ and $i = 1,2,3,4$, we have

$$\begin{aligned}
&\big|F(t,x,y,z,y_{t-\delta_i(\cdot)},z_{t-\delta_i(\cdot)}) - F\big(t,x,y',z',y'_{t-\delta_i(\cdot)},z'_{t-\delta_i(\cdot)}\big)\big|^2\\
&\quad\leq C_1\big(\big|y-y'\big|^2 + t^{2H-1}\big|z-z'\big|^2 + \big|y_{t-\delta(\cdot)} - y'_{t-\delta(\cdot)}\big|^2\\
&\qquad + (t-\delta)^{2H-1}\big|z_{t-\delta(\cdot)} - z'_{t-\delta(\cdot)}\big|^2\big)\\
&\big|f(t,x,y,z,y_{t-\delta_i(\cdot)},z_{t-\delta_i(\cdot)}) - f\big(t,x,y',z',y'_{t-\delta_i(\cdot)},z'_{t-\delta_i(\cdot)}\big)\big|^2\\
&\quad\leq C_2\big(\big|y-y'\big|^2 + t^{2H-1}\big|z-z'\big|^2 + \big|y_{t-\delta(\cdot)} - y'_{t-\delta(\cdot)}\big|^2\\
&\qquad + (t-\delta)^{2H-1}\big|z_{t-\delta(\cdot)} - z'_{t-\delta(\cdot)}\big|^2\big).
\end{aligned}$$

3 Existence and uniqueness

We first investigate properties of the solutions. If, under hypothetical conditions, there exists a pair of processes (Y_t, Z_t) satisfying equation (2.1), then we can derive that there exists a solution to equation (2.1).

Theorem 3.1 *Let* (F,f) *satisfy* (A1) *and* (A2). *Suppose* $\varphi_t \in \widetilde{\mathcal{V}}_{[-\delta,T]}$ *and* $\psi_t \in \widetilde{\mathcal{V}}_{[-\delta,T]}$. *Then BSDEs* (2.1) *admit a unique solution* $(Y_t, Z_t) \in \widetilde{\mathcal{V}}_{[-\delta,T]} \times \widetilde{\mathcal{V}}^H_{[-\delta,T]}$.

Proof For given pair $(y,z) \in \widetilde{\mathcal{V}}_{[-\delta,T]} \times \widetilde{\mathcal{V}}^H_{[-\delta,T]}$, we consider the following BSDEs:

$$\begin{cases} Y_t = Y_T + \int_t^T \frac{\alpha(\alpha-1)(T-s)^{\alpha-2}}{\Gamma(1+\alpha)} F(s,\eta_s,Y_s,Z_s,Y_{s-\delta_1(s)},Z_{s-\delta_2(s)}) \\ \qquad -f(s,\eta_s,Y_s,Z_s,Y_{s-\delta_3(s)},Z_{s-\delta_4(s)})\,ds \\ \qquad -\int_t^T \frac{Z_s}{\Gamma(1+\alpha)}\,dB^H_s, \quad 0 \le t \le T; \\ Y_t = \varphi(t), \qquad Z_t = \psi(t), \qquad Y_T = g(X_T), \quad -\delta \le t < 0. \end{cases} \tag{3.1}$$

We define a mapping $\Lambda: \widetilde{\mathcal{V}}_{[-\delta,T]} \times \widetilde{\mathcal{V}}^H_{[-\delta,T]} \to \widetilde{\mathcal{V}}_{[-\delta,T]} \times \widetilde{\mathcal{V}}^H_{[-\delta,T]}$ such that $\Lambda[(y_.,z_.)] = (Y_.,Z_.)$. If we can prove that Λ is a compressed map, then we get the desired result. So, let us complete the proof.

For two arbitrary elements $(y.,z.)$ and $(y'.,z'.)$ in $\widetilde{\mathcal{V}}_{[-\delta,T]} \times \widetilde{\mathcal{V}}^H_{[-\delta,T]}$, we have $\Lambda[(y.,z.)] = (Y.,Z.)$ and $\Lambda[(y'.,z'.)] = (Y'.,Z'.)$. Define

$$\begin{aligned} &\hat{y}_. \triangleq y_. - y'_., \qquad \hat{z}_. \triangleq z_. - z'_. \\ &\hat{Y}_. \triangleq Y_. - Y'_., \qquad \hat{Z}_. \triangleq Z_. - Z'_. \end{aligned}$$

Then applying Itô s formula to $e^{\beta t}|\hat{Y}_t|^2$, by Proposition 2.2 we obtain

$$\begin{aligned} d\left(e^{\beta t}|\hat{Y}_t|^2\right) &= \beta e^{\beta t}|\hat{Y}_t|^2\,dt + 2e^{\beta t}|\hat{Y}_t|d|\hat{Y}_t| + e^{\beta t}d|\hat{Y}_t|^2 \\ &= \beta e^{\beta t}|\hat{Y}_t|^2\,dt - 2e^{\beta t}|\hat{Y}_t|\left|\frac{\alpha(\alpha-1)(T-t)^{\alpha-2}}{(1+\alpha)}\right||\hat{F}_t|\,dt + 2e^{\beta t}|\hat{Y}_t||\hat{f}_t| \\ &\quad + \frac{2e^{\beta t}|\hat{Y}_t||\hat{Z}_t|}{(1+\alpha)}\,dB^H_t + \frac{2e^{\beta t}\mathbb{D}^H_t|\hat{Y}_t||\hat{Z}_t|}{\Gamma(1+\alpha)}\,dt. \end{aligned}$$

Integrating over interval $[t,T]$ and converting by deformation, we easily have

$$\begin{aligned} &e^{\beta t}|\hat{Y}_t|^2 + \beta\int_t^T e^{\beta s}|\hat{Y}_s|^2\,ds + \frac{2}{\Gamma(1+\alpha)}\int_t^T e^{\beta s}|\hat{Y}_s||\hat{Z}_s|\,dB^H_s \\ &\quad + \frac{2}{\Gamma(1+\alpha)}\int_t^T e^{\beta s}\mathbb{D}^H_s|\hat{Y}_s||\hat{Z}_s|\,ds \\ &= e^{\beta T}|\hat{Y}_T|^2 + 2\int_t^T e^{\beta s}\left|\frac{\alpha(\alpha-1)(T-s)^{\alpha-2}}{\Gamma(1+\alpha)}\right||\hat{Y}_s||\hat{F}_s|\,ds - 2\int_t^T e^{\beta s}|\hat{Y}_s||\hat{f}_s|\,ds. \end{aligned}$$

Here, because of the difficulty of integrating the term of the Malliavin derivative $\mathbb{D}^H_s|\hat{Y}_s|$, we need to transform it. According to the above equation and Proposition 2.3, we have

$$\begin{aligned} &e^{\beta t}|\hat{Y}_t|^2 + \beta\int_t^T e^{\beta s}|\hat{Y}_s|^2\,ds + \frac{2}{\Gamma(1+\alpha)}\int_t^T e^{\beta s}|\hat{Y}_s||\hat{Z}_s|\,dB^H_s \\ &\quad + \frac{2}{M\Gamma(1+\alpha)}\int_t^T e^{\beta s}s^{2H-1}|\hat{Z}_s|^2\,ds \\ &\le e^{\beta T}|\hat{Y}_T|^2 + 2\int_t^T e^{\beta s}\left|\frac{\alpha(\alpha-1)(T-s)^{\alpha-2}}{\Gamma(1+\alpha)}\right||\hat{Y}_s||\hat{F}_s|\,ds - 2\int_t^T e^{\beta s}|\hat{Y}_s||\hat{f}_s|\,ds. \end{aligned}$$

Taking the expectation on both sides and applying the inequality $2AB \leq A^2 + B^2$, we have

$$\begin{aligned}
&E\bigg[e^{\beta t}|\hat{Y}_t|^2 - e^{\beta T}|\hat{Y}_T|^2 + \beta\int_t^T e^{\beta s}|\hat{Y}_s|^2\,ds + \frac{2}{\Gamma(1+\alpha)}\int_t^T e^{\beta s}|\hat{Y}_s||\hat{Z}_s|\,dB_s^H \\
&\quad + \frac{2}{M\Gamma(1+\alpha)}\int_t^T e^{\beta s}s^{2H-1}|\hat{Z}_s|^2\,ds\bigg] \\
&\leq E\bigg[2\int_t^T e^{\beta s}\bigg|\frac{\alpha(\alpha-1)(T-s)^{\alpha-2}}{\Gamma(1+\alpha)}\bigg||\hat{Y}_s||\hat{F}_s|\,ds - 2\int_t^T e^{\beta s}|\hat{Y}_s||\hat{f}_s|\,ds\bigg] \\
&= E\bigg[2\int_t^T e^{\beta s}\bigg|\frac{\alpha(\alpha-1)(T-s)^{\alpha-2}}{\Gamma(1+\alpha)}\bigg||\hat{Y}_s|\big|F(s,\eta_s,y_s,z_s,y_{s-\delta_1(s)},z_{s-\delta_2(s)}) \\
&\quad - F\big(s,\eta_s,y'_s,z'_s,y'_{s-\delta_1(s)},z'_{s-\delta_2(s)}\big)\big|\,ds \\
&\quad - 2\int_t^T e^{\beta s}|\hat{Y}_s|\big|f(s,\eta_s,y_s,z_s,y_{s-\delta_3(s)},z_{s-\delta_4(s)}) - f\big(s,\eta_s,y'_s,z'_s,y'_{s-\delta_3(s)},z'_{s-\delta_4(s)}\big)\big|\,ds\bigg] \\
&\leq E\bigg[\frac{\beta}{4}\int_t^T e^{\beta s}|\hat{Y}_s|^2\,ds + \frac{4}{\beta}\int_t^T e^{\beta s}\bigg|\frac{\alpha(\alpha-1)(T-s)^{\alpha-2}}{\Gamma(1+\alpha)}\bigg|^2\big|F(s,\eta_s,y_s,z_s,y_{s-\delta_1(s)},z_{s-\delta_2(s)}) \\
&\quad - F\big(s,\eta_s,y'_s,z',y'_{s-\delta_1(s)},z'{}_{s-\delta_2(s)}\big)\big|^2 + \frac{\beta}{4}\int_t^T e^{\beta s}|\hat{Y}_s|^2\,ds \\
&\quad + \frac{4}{\beta}\int_t^T e^{\beta s}\big|f(s,\eta_s,y_s,z_s,y_{s-\delta_3(s)},z_{s-\delta_4(s)}) \\
&\quad - f\big(s,\eta_s,y'_s,z'_s,y'_{s-\delta_3(s)},z'_{s-\delta_4(s)}\big)\big|^2\,ds\bigg] \\
&= E\bigg[\frac{\beta}{2}\int_t^T e^{\beta s}|\hat{Y}_s|^2\,ds + \frac{4}{\beta}\int_t^T e^{\beta s}\bigg|\frac{\alpha(\alpha-1)(T-s)^{\alpha-2}}{\Gamma(1+\alpha)}\bigg|^2\big|F(s,\eta_s,y_s,z_s,y_{s-\delta_1(s)},z_{s-\delta_2(s)}) \\
&\quad - F\big(s,\eta_s,y'_s,z'_s,y'_{s-\delta_1(s)},z'{}_{s-\delta_2(s)}\big)\big|^2 + \frac{4}{\beta}\int_t^T e^{\beta s}\big|f(s,\eta_s,y_s,z_s,y_{s-\delta_3(s)},z_{s-\delta_4(s)}) \\
&\quad - f\big(s,\eta_s,y'_s,z'_s,y'_{s-\delta_3(s)},z'{}_{s-\delta_4(s)}\big)\big|^2\,ds\bigg].
\end{aligned}\tag{3.2}$$

Since $\int_t^T \frac{\alpha(\alpha-1)(T-s)^{\alpha-2}}{\Gamma(1+\alpha)}\,ds = \frac{\alpha(T-t)^{\alpha-1}}{\Gamma(1+\alpha)} - (T-t)$, it is a continuous bounded function on $[t,T]$. By assumptions (A1) and (A2) and Jensen's inequality there exists a constant $k > 0$ such that

$$\begin{aligned}
&E\bigg[e^{\beta t}|\hat{Y}_t|^2 - e^{\beta T}|\hat{Y}_T|^2 + \frac{\beta}{2}\int_t^T e^{\beta s}|\hat{Y}_s|^2\,ds + \frac{2}{\Gamma(1+\alpha)}\int_t^T e^{\beta s}|\hat{Y}_s||\hat{Z}_s|\,dB_s^H \\
&\quad + \frac{2}{M\Gamma(1+\alpha)}\int_t^T e^{\beta s}s^{2H-1}|\hat{Z}_s|^2\,ds\bigg] \\
&\leq E\bigg[\frac{4k}{\beta}\int_t^T e^{\beta s}\big|F(s,\eta_s,y_s,z_s,y_{s-\delta_1(s)},z_{s-\delta_2(s)}) - F\big(s,\eta_s,y'_s,z'_s,y'_{s-\delta_1(s)},z'_{s-\delta_2(s)}\big)\big|^2\,ds \\
&\quad + \frac{4}{\beta}\int_t^T e^{\beta s}\big|f(s,\eta_s,y_s,z_s,y_{s-\delta_3(s)},z_{s-\delta_4(s)}) - f\big(s,\eta_s,y'_s,z'_s,y'_{s-\delta_3(s)},z'_{s-\delta_4(s)}\big)\big|^2\,ds\bigg] \\
&\leq E\bigg[\bigg(\frac{4k}{\beta}C_1 + \frac{4}{\beta}C_2\bigg)\int_{t-\delta}^T e^{\beta s}\big(|\hat{y}_s|^2 + s^{2H-1}|\hat{z}_s|^2 + |\hat{y}_{s-\delta}|^2 + (s-\delta)^{2H-1}|\hat{z}_{s-\delta}|^2\big)\,ds\bigg]
\end{aligned}$$

$$\leq E\bigg[\bigg(\frac{4k}{\beta}C_1+\frac{4}{\beta}C_2\bigg)(L+1)\int_{t-\delta}^{T}e^{\beta s}\big(|\hat{y}_s|^2+s^{2H-1}|\hat{z}_s|^2\big)\,ds\bigg]. \tag{3.3}$$

Let $t=0$ and multiply both sides by $\frac{M\Gamma(1+\alpha)}{2}$; since $\Gamma(1+\alpha)\in(1,2)$, we get

$$\begin{aligned}
&E\bigg[\int_0^T e^{\beta s}\bigg(\frac{\beta M}{4}|\hat{Y}_s|^2+s^{2H-1}|\hat{Z}_s|^2\bigg)\,ds\bigg]\\
&\quad\leq E\bigg[\int_0^T e^{\beta s}\bigg(\frac{\beta M\Gamma(1+\alpha)}{4}|\hat{Y}_s|^2+s^{2H-1}|\hat{Z}_s|^2\bigg)\,ds\bigg]\\
&\quad\leq E\bigg[\bigg(\frac{4k}{\beta}C_1+\frac{4}{\beta}C_2\bigg)(L+1)M\int_{-\delta}^{T}e^{\beta s}\big(|\hat{y}|^2+s^{2H-1}\big|\hat{z}^2\big|\big)\,ds\bigg].
\end{aligned}$$

Denoting $\beta=(4kC_1+4C_2)(L+1)M+\frac{4}{M}$, we have

$$\begin{aligned}
&E\bigg[\int_0^T e^{\beta s}\big(|\hat{Y}_s|^2+s^{2H-1}|\hat{Z}_s|^2\big)\,ds\bigg]\\
&\quad\leq\frac{1}{2}E\bigg[\int_{-\delta}^{T}e^{\beta s}\big(|\hat{y}_s|^2+s^{2H-1}|\hat{z}_s|^2\big)\,ds\bigg],
\end{aligned}$$

that is,

$$\|\hat{Y},\hat{Z}\|_\beta\leq\frac{1}{\sqrt{2}}\|\hat{y},\hat{z}\|_\beta.$$

Therefore the map Λ is a strictly compressed mapping, which means that equation (3.1) has a unique solution. □

Remark Note that in this paper, we assume that $H\in(\frac{1}{2},1)$. However, in this paper the generators in fractional-order backward stochastic differential equations only consider present and past time and do not include future time. Next, we make a supplement to the scope of this study and expand the scope of the generator. Consider the following generator for equations that include not only present and past time, but also future time:

$$\begin{cases}
Y_t=\xi_t+\int_t^T\frac{\alpha(\alpha-1)(T-s)^{\alpha-2}}{\Gamma(1+\alpha)}F(s,Y_{s-\delta_1(s)},Z_{s-\delta_2(s)},Y_s,Z_s,Y_{s+\delta_3(s)},Z_{s+\delta_4(s)})\\
\qquad-f(s,Y_{s-\delta_5(s)},Z_{s-\delta_6(s)},Y_s,Z_s,Y_{s+\delta_7(s)},Z_{s+\delta_8(s)})\,ds\\
\qquad-\int_t^T\frac{Z_t}{\Gamma(1+\alpha)}\,dB_t^H,\quad 0\leq t\leq T;\\
Y(t)=\xi(t),\qquad Z(t)=\phi(t),\qquad T\leq t\leq T+\delta.
\end{cases} \tag{3.4}$$

Here we make the necessary assumptions:

(A3) δ_i, $i=1,2,\dots,8$, are R^+-valued continuous functions on $[0,T]$ such that $\delta_i(\cdot)\leq\delta$, where δ is a normal constant. There exist $L',L''\geq 0$ such that for all $t\in[0,T]$ and all nonnegative and integrable g,

$$\int_t^T g\big(s-\delta_i(s)\big)\,ds\leq L'\int_{t-\delta}^{T}g(s)\,ds,\quad i=1,2,\dots,8,$$

$$\int_t^T g\big(s+\delta_i(s)\big)\,ds\leq L''\int_t^{T+\delta}g(s)\,ds,\quad i=1,2,\dots,8.$$

Assume that the generator $(F,f)(\omega,t,a,b,x,y,c,d)$:
$\Omega \times [0,T] \times L^2(\mathcal{F}_{s'},\mathbb{R}) \times L^2(\mathcal{F}_s,\mathbb{R}) \times \mathbb{R}^2 \times L^2(\mathcal{F}_{r'},\mathbb{R}) \times L^2(\mathcal{F}_r,\mathbb{R}) \to L^2(\mathcal{F}_t,\mathbb{R}), s'$, $s \in [-\delta,T], r', r \in [0,T+\delta]$, satisfies the following conditions:

(A4) There exists a constant $D > 0$ such that for all $t \in [0,T]$,

$$\begin{aligned}
&\left|(F,f)(t,a,b,x,y,c,d) - (F,f)\left(t,a',b',x,y,c',d'\right)\right| \\
&\quad \le D\left(\left|a-a'\right| + t^{H-\frac{1}{2}}\left|b-b'\right| + \left|y-y'\right| + t^{H-\frac{1}{2}}\left|z-z'\right|\right. \\
&\qquad \left. + E^{\mathcal{F}_t}\left\{\left|c-c'\right| + t^{H-\frac{1}{2}}\left|d-d'\right|\right\}\right),
\end{aligned}$$

(A5) $E[\int_t^T |F(t,0,0,0,0,0,0)|^2\,ds] < \infty$, $E[\int_t^T |f(t,0,0,0,0,0,0)|^2\,ds] < \infty$.

With the help of the above method, we have the existence and uniqueness of the solution to equation (3.4).

4 Comparison theorem

In this section, we further study the properties of the solutions to the following Caputo-driven fractional backward stochastic differential equations:

$$\begin{cases}
-dY_t = ((\frac{\alpha(\alpha-1)(T-t)^{\alpha-2}}{\Gamma(1+\alpha)})F(t,\eta_t,Y_t,Z_t,Y_{t-\delta_1(t)}) - f(t,\eta_t,Y_t,Z_t,Y_{t-\delta_3(t)}))\,dt \\
\qquad\qquad - \frac{Z_t}{\Gamma(1+\alpha)}\,dB_t^H, \quad 0 \le t \le T; \\
Y_t = \varphi(t), \qquad Y_T = g(X_T), \quad -\delta \le t < 0.
\end{cases}$$

By Theorem 3.1 this equation has a unique solution in $\widetilde{\mathcal{V}}_{[-\delta,T]} \times \widetilde{\mathcal{V}}_{[-\delta,T]}^H$. Next, we give a comparison theorem of the equations studied in this paper based on Hu [12, Theorem 12.3].

Theorem 4.1 *For $i = 1,2$, let $g_i(X_T)$ and $\varphi_i(t)$ be continuously differentiable functions of polynomial growth together with their derivatives. Assume that $F_i(t,x,y,z,y_\delta)$, $\partial F_i(t,x,y,z,y_\delta)$, $f_i(t,x,y,z,y_\delta)$, and $\partial f_i(t,x,y,z,y_\delta)$ are uniformly Lipschitz continuous with respect to y and z and satisfy* (A1) *and* (A2) *in the interval* $[0,T] \times \mathbb{R}^4$. *Moreover, assume that F_2 is increasing with respect to y_δ and f_2 is decreasing about y_δ. If $\varphi_1(t) \le \varphi_2(t)$ for all $t \in [-\delta,0]$ and*

$$\begin{aligned}
&\left(\left(\frac{\alpha(\alpha-1)(T-t)^{\alpha-2}}{\Gamma(1+\alpha)}\right)F_1(t,\eta_t,Y_t,Z_t,Y_{t-\delta_1(t)}) - f_1(t,\eta_t,Y_t,Z_t,Y_{t-\delta_3(t)})\right) \\
&\quad \le \left(\left(\frac{\alpha(\alpha-1)(T-t)^{\alpha-2}}{\Gamma(1+\alpha)}\right)F_2(t,\eta_t,Y_t,Z_t,Y_{t-\delta_1(t)}) - f_2(t,\eta_t,Y_t,Z_t,Y_{t-\delta_3(t)})\right), \\
&y_{t-\delta_1(t)}, y_{t-\delta_3(t)} \in L^2_{\mathcal{F}}[-\delta,t],
\end{aligned}$$

then $Y_1(t) \le Y_2(t)$.

Proof First, consider the following BSDE with terminal condition:

$$\begin{cases} -dY_1(t) = ((\frac{\alpha(\alpha-1)(T-t)^{\alpha-2}}{\Gamma(1+\alpha)})F_1(t,\eta_t,Y_1(t),Z_1(t),Y_1(t-\delta_1(t)) \\ \qquad\qquad -f_1(t,\eta_t,Y_1(t),Z_1(t),Y_1(t-\delta_3(t))) \\ \qquad\qquad -\frac{Z_1(t)}{\Gamma(1+\alpha)}\,dB_t^H, \quad 0\le t\le T, \\ Y_1(t)=\varphi_1(t), \quad -\delta\le t\le 0. \end{cases}$$

From Theorem 3.1 we know that this equation has a unique solution $(Y_1,Z_1)\in\mathcal{V}_{[-\delta,T]}\times\mathcal{V}^H_{[0,T]}$. Next, consider another equation

$$\begin{cases} -dY_3(t) = ((\frac{\alpha(\alpha-1)(T-t)^{\alpha-2}}{\Gamma(1+\alpha)})F_2(t,\eta_t,Y_3(t),Z_3(t),Y_1(t-\delta_1(t)) \\ \qquad\qquad -f_2(t,\eta_t,Y_3(t),Z_3(t),Y_1(t-\delta_3(t))) \\ \qquad\qquad -\frac{Z_3(t)}{\Gamma(1+\alpha)}\,dB_t^H, \quad 0\le t\le T, \\ Y_3(t)=\varphi_2(t), \quad -\delta\le t\le 0. \end{cases}$$

Obviously, the only solution to this equation is (Y_3,Z_3). Meanwhile, we have the following assumptions: $\varphi_1(t)\le\varphi_2(t)$, and

$$\begin{aligned} &\left(\left(\frac{\alpha(\alpha-1)(T-t)^{\alpha-2}}{\Gamma(1+\alpha)}\right)F_1(t,\eta_t,Y_t,Z_t,Y_{t-\delta_1(t)})-f_1(t,\eta_t,Y_t,Z_t,Y_{t-\delta_3(t)})\right) \\ &\quad\le\left(\left(\frac{\alpha(\alpha-1)(T-t)^{\alpha-2}}{\Gamma(1+\alpha)}\right)F_2(t,\eta_t,Y_t,Z_t,Y_{t-\delta_1(t)})-f_2(t,\eta_t,Y_t,Z_t,Y_{t-\delta_3(t)})\right) \\ &y_{t-\delta_1(t)},y_{t-\delta_3(t)}\in L^2_{\mathcal{F}}[-\delta,t]. \end{aligned}$$

According to Hu et al. [12, Theorem 12.3], we have

$$Y_1(t)\le Y_3(t)\quad\text{a.e., a.s.}$$

In the same way, we set up the following equation:

$$\begin{cases} -dY_4(t) = ((\frac{\alpha(\alpha-1)(T-t)^{\alpha-2}}{\Gamma(1+\alpha)})F_2(t,\eta_t,Y_4(t),Z_4(t),Y_3(t-\delta_1(t)) \\ \qquad\qquad -f_2(t,\eta_t,Y_4(t),Z_4(t),Y_3(t-\delta_3(t))) \\ \qquad\qquad -\frac{Z_4(t)}{\Gamma(1+\alpha)}\,dB_t^H, \quad 0\le t\le T; \\ Y_4(t)=\varphi_2(t), \quad -\delta\le t\le 0. \end{cases}$$

The unique solution to this equation is represented by $(Y_4,Z_4)\in\tilde{\mathcal{V}}_{[-\delta,T]}\times\tilde{\mathcal{V}}^H_{[0,T]}$. As $F_2(t,x,y,z,y_\delta)$ is increasing with respect to y_δ, $f_2(t,x,y,z,y_\delta)$ is decreasing with respect to y_δ, and $Y_1(t)<Y_3(t)$, it follows that

$$\begin{aligned} &\left(\left(\frac{\alpha(\alpha-1)(T-t)^{\alpha-2}}{\Gamma(1+\alpha)}\right)F_2\big(t,\eta_t,Y(t),Z(t),Y_1\big(t-\delta_1(t)\big)\big) \\ &\qquad-f_2\big(t,\eta_t,Y(t),Z(t),Y_1\big(t-\delta_3(t)\big)\big)\right) \end{aligned}$$

$$\leq \left(\left(\frac{\alpha(\alpha-1)(T-t)^{\alpha-2}}{\Gamma(1+\alpha)}\right)F_2\big(t,\eta_t,Y(t),Z(t),Y_3\big(t-\delta_1(t)\big)\big)\right.$$
$$\left.-f_2\big(t,\eta_t,Y(t),Z(t),Y_3\big(t-\delta_3(t)\big)\big)\right)$$
$$y_{t-\delta_1(t)},y_{t-\delta_3(t)}\in L^2_{\mathcal{F}}[-\delta,t].$$

Similarly,

$$Y_3(t)\leq Y_4(t)\quad \text{a.e., a.s.}$$

Finally, consider the following BSDEs for $n=5,6,\ldots$:

$$\begin{cases}-dY_n(t)=((\frac{\alpha(\alpha-1)(T-t)^{\alpha-2}}{\Gamma(1+\alpha)})F_2(t,\eta_t,Y_n(t),Z_n(t),Y_{n-1}(t-\delta_1(t)) \\ \qquad\qquad -f_2(t,\eta_t,Y_n(t),Z_n(t),Y_{n-1}(t-\delta_3(t))) \\ \qquad\qquad -\frac{Z_n(t)}{\Gamma(1+\alpha)}\,dB_t^H,\quad 0\leq t\leq T, \\ Y_n(t)=\varphi_2(t),\quad -\delta\leq t\leq 0.\end{cases}$$

From the above we can conclude that

$$Y_4(t)\leq Y_5(t)\leq Y_6(t)\leq Y_7(t)\leq\cdots\leq Y_n(t)\leq\cdots\quad \text{a.e., a.s.}$$

Let us scale up the result so that we get a Cauchy sequence $\{Y_n,Z_n\}_{n\geq 3}$. Denote $\hat{Y}_n(t)=Y_n(t)-Y_{n-1}(t)$ and $\hat{Z}_n(t)=Z_n(t)-Z_{n-1}(t)$. From estimate (3.3), assumptions (A1) and (A2), and Jensen's inequality we have

$$E\Big[e^{\beta t}\big|\hat{Y}_n(t)\big|^2-e^{\beta T}\big|\hat{Y}_n(T)\big|^2+\frac{\beta}{2}\int_t^T e^{\beta s}\big|\hat{Y}_n(s)\big|^2\,ds$$
$$+\frac{2}{M\Gamma(1+\alpha)}\int_t^T e^{\beta s}s^{2H-1}\big|\hat{Z}_n(s)\big|^2\,ds\Big]$$
$$\leq E[\frac{4}{\beta}\int_t^T e^{\beta s}\left[\frac{\alpha(\alpha-1)(T-s)^{\alpha-2}}{\Gamma(1+\alpha)}\right]^2\big|F\big(s,\eta_s,\tilde{Y}_n(s),\tilde{Z}_n(s),\tilde{Y}_{n-1}\big(s-\delta_1(s)\big)\big)$$
$$-F\big(s,\eta_s,\tilde{Y}_{n-1}(s),\tilde{Z}_{n-1}(s),\tilde{Y}_{n-2}\big(s-\delta_1(s)\big)\big)\big|^2\,ds+\frac{4}{\beta}\int_t^T e^{\beta s}\big|f\big(s,\eta_s,\tilde{Y}_n(s),\tilde{Z}_n(s),$$
$$\tilde{Y}_{n-1}\big(s-\delta_1(s)\big)\big)-f\big(s,\eta_s,\tilde{Y}_{n-1}(s),\tilde{Z}_{n-1}(s),\tilde{Y}_{n-2}\big(s-\delta_1(s)\big)\big)\big|^2\,ds]$$
$$\leq\frac{4}{\beta}(kC_1+C_2)E\left[\int_t^T e^{\beta s}\big(\big|\hat{Y}_n(s)\big|^2+s^{2H-1}\big|\hat{Z}_n(s)\big|^2\big)\,ds+\int_{t-\delta}^T e^{\beta s}\big|\hat{Y}_{n-1}(s)\big|^2\,ds\right].$$

Then letting $t=0$, multiplying both sides by $\frac{M\Gamma(1+\alpha)}{2}$, and noting that $\Gamma(1+\alpha)\in(1,2)$, we have

$$E\left[\int_0^T e^{\beta s}\left(\frac{\beta M}{4}\big|\hat{Y}_n(s)\big|^2+s^{2H-1}\big|\hat{Z}_n(s)\big|^2\right)ds\right]$$
$$\leq\frac{4M\Gamma(1+\alpha)}{\beta}(kC_1+C_2)$$

$$\times E\left[\int_0^T e^{\beta s}\left(\left|\hat{Y}_n(s)\right|^2 + s^{2H-1}\left|\hat{Z}_n(s)\right|^2\right) ds + \int_{-\delta}^T e^{\beta s}\left|\hat{Y}_{n-1}(s)\right|^2 ds\right].$$

Denoting $\beta = 16M\Gamma(1+\alpha)(kC_1 + C_2) + \frac{4}{M}$, we have

$$\begin{aligned}
&E\left[\int_0^T e^{\beta s}\left(\frac{\beta M}{4}\left|\hat{Y}_n(s)\right|^2 + s^{2H-1}\left|\hat{Z}_n(s)\right|^2\right) ds\right] \\
&\quad \leq E\left[\int_0^T e^{\beta s}\left(\left|\hat{Y}_n(s)\right|^2 + s^{2H-1}\left|\hat{Z}_n(s)\right|^2\right) ds\right] + \frac{1}{4}E\left[\int_{-\delta}^T e^{\beta s}\left|\hat{Y}_{n-1}(s)\right|^2 ds\right].
\end{aligned}$$

Therefore

$$\begin{aligned}
&E\left[\int_0^T e^{\beta s}\left(\left|\hat{Y}_n(s)\right|^2 + s^{2H-1}\left|\hat{Z}_n(s)\right|^2\right) ds\right] \\
&\quad \leq \frac{1}{3}E\left[\int_{-\delta}^T e^{\beta s}\left|\hat{Y}_{n-1}(s)\right|^2 ds\right] \\
&\quad \leq \frac{1}{3}E\left[\int_{-\delta}^T e^{\beta s}\left|\hat{Y}_{n-1}(s)\right|^2 ds + \int_0^T e^{\beta s} s^{2H-1}\left|\hat{Z}_{n-1}(s)\right|^2\right].
\end{aligned}$$

Hence

$$\begin{aligned}
&E\left[\int_0^T e^{\beta s}\left(\left|\hat{Y}_n(s)\right|^2 + s^{2H-1}\left|\hat{Z}_n(s)\right|^2\right) ds\right] \\
&\quad \leq \left(\frac{1}{3}\right)^{n-4} E\left[\int_{-\delta}^T e^{\beta s}\left|\hat{Y}_4(s)\right|^2 ds + \int_0^T e^{\beta s} s^{2H-1}\left|\hat{Z}_4(s)\right|^2\right].
\end{aligned}$$

Obviously, $(\hat{Y}_n)_{n\geq 4}$ and $(\hat{Z}_n)_{n\geq 4}$ are Cauchy sequences in the Banach spaces $\tilde{\mathcal{V}}_{[-\delta,T]}$ and $\tilde{\mathcal{V}}^H_{[0,T]}$, respectively. Their limits are represented by (Y_t, Z_t) for all $0 \leq t \leq T$. Therefore (Y_t, Z_t) is a solution of the following fractional BSDE:

$$\begin{cases}
-dY_t = \left(\left(\frac{\alpha(\alpha-1)(T-t)^{\alpha-2}}{\Gamma(1+\alpha)}\right)F_2(t, \eta_t, Y_t, Z_t, Y_{t-\delta_1(t)}) - f_2(t, \eta_t, Y_t, Z_t, Y_{t-\delta_3(t)})\right) dt \\
\qquad\qquad - \frac{Z_t}{\Gamma(1+\alpha)}\, dB^H_t, \quad 0 \leq t \leq T, \\
Y_t = \varphi_2(t), \quad -\delta \leq t < 0.
\end{cases}$$

At the same time, by Theorem 3.1 on the uniqueness of the solution it is also the solution of the following equation:

$$\begin{cases}
-dY_2(t) = \left(\left(\frac{\alpha(\alpha-1)(T-t)^{\alpha-2}}{\Gamma(1+\alpha)}\right)F_2(t, \eta_t, Y_2(t), Z_2(t), Y_2(t-\delta_1(t))\right) \\
\qquad\qquad - f_2(t, \eta_t, Y_2(t), Z_2(t), Y_2(t-\delta_3(t)))\big) \\
\qquad\qquad - \frac{Z_2(t)}{\Gamma(1+\alpha)}\, dB^H_t, \quad 0 \leq t \leq T, \\
Y_2(t) = \varphi_2(t), \quad -\delta \leq t \leq 0.
\end{cases}$$

Hence

$$Y(t) = Y_2(t) \quad \text{a.s.}$$

Since

$$Y_1(t) \le Y_3(t) \le Y_4(t) \le Y(t),$$

we get $Y_1(t) \le Y_2(t)$, as desired, □

Remark Under assumptions (A3), (A4), and (A5), equation (3.4) is still applicable to the above equation, and the comparison theorem of the equation is also valid.

Example To illustrate the results, consider the equation

$$\begin{cases} Y_t = Y_T + \int_t^T \big(\frac{\alpha(\alpha-1)(T-s)^{\alpha-2}}{\Gamma(1+\alpha)} \int_0^s (Y(u) + Z(u) + Y(u-\delta_3(u)) + Z(u-\delta_4(u)))\,du \\ \quad - (Y(s) + Z(s) + Y(s-\delta_3(s)) \\ \quad + Z(s-\delta_4(s))\,ds - \int_t^T \frac{Z_s}{\Gamma(1+\alpha)}\,dB_s^H, \quad 0 \le t \le T, \\ Y_t = \varphi_t, \qquad Z_t = \psi_t, \quad -\delta \le t < 0. \end{cases}$$

where for $i = 1,2,3,4$, the generator satisfies (A1) and (A2). Suppose $\varphi_t \in \widetilde{\mathcal{V}}_{[-\delta,T]}$ and $\psi_t \in \widetilde{\mathcal{V}}_{[-\delta,T]}$. Then BSDE (2.1) admits a unique solution $(Y_t, Z_t) \in \widetilde{\mathcal{V}}_{[-\delta,T]} \times \widetilde{\mathcal{V}}^H_{[-\delta,T]}$.

5 Conclusions

In this paper, we introduced the fractional derivative and time-delay generator into backward stochastic differential equations. We obtained the existence and uniqueness of a solution and gave an example to illustrate our results. However, the analytical solution to BSDEs is not easy to get and becomes more difficult after the introduction of fractional derivatives and time delays. Our future work will focus on the numerical solution of this equation. At the same time, since fractional derivatives and time delay both provide effective mathematical tools for the study of complex systems and phenomena, it seems interesting to explore their applications in finance.

Acknowledgements
The authors would like to thank the editor for handling this manuscript.

Funding
This research is supported by the National Natural Science Foundation of China (62076039), Natural Science Foundation of Hubei Province (2022CFB023), Education Science Planning Project of Hubei Province (2022GA031), and college student innovation entrepreneurship project from Yangtze University (Yz2022293).

Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

Author contributions
All the authors contributed equally and significantly in writing this paper. All the authors read and approved the final manuscript.

Author details
[1]School of Information and Mathematics, Yangtze University, Jingzhou, 434023, Hubei, China. [2]School of Information Engineering, Jingzhou University, Jingzhou 434020, Hubei, China.

Received: 28 December 2023 Accepted: 22 February 2024 Published online: 07 March 2024

References

1. Bahlali, K., Essaky, E.H., Ouknine, Y.: Reflected backward stochastic differential equation with jumps and locally Lipschitz coefficient. Random Oper. Stoch. Equ. (2002)
2. Bismut, J.-M.: Conjugate convex functions in optimal stochastic control. J. Math. Anal. Appl. **44**(2), 384–404 (1973)
3. Bismut, J.-M.: An introductory approach to duality in optimal stochastic control. SIAM Rev. **20**(1), 62–78 (1978)
4. Chen, J., Ke, S., Li, X., Liu, W.: Existence, uniqueness and stability of solutions to fractional backward stochastic differential equations. App. Math. Sci. Eng. **30**(1), 811–829 (2022)
5. Decreusefond, L., Üstünel, A.S.: Stochastic analysis of the fractional Brownian motion. Potential Anal. **10**, 177–214 (1999)
6. Delong, L.: Applications of time-delayed backward stochastic differential equations to pricing, hedging and portfolio management (2010). arXiv preprint arXiv:1005.4417
7. Delong, L., Imkeller, P.: Backward stochastic differential equations with time delayed generators – results and counterexamples. Ann. Appl. Probab. **20**(4), 1512–1536 (2010)
8. Delong, Ł., Imkeller, P.: On Malliavin's differentiability of BSDEs with time delayed generators driven by Brownian motions and Poisson random measures. Stoch. Process. Appl. **120**(9), 1748–1775 (2010)
9. Douissi, S., Wen, J., Shi, Y.: Mean-field anticipated BSDEs driven by fractional Brownian motion and related stochastic control problem. Appl. Math. Comput. **355**, 282–298 (2019)
10. Duncan, T.E., Hu, Y., Pasik-Duncan, B.: Stochastic calculus for fractional Brownian motion I. Theory. SIAM J. Control Optim. **38**(2), 582–612 (2000)
11. Hu, Y.: Integral Transformations and Anticipative Calculus for Fractional Brownian Motions. Am. Math. Soc., Providence (2005)
12. Hu, Y., Ocone, D., Song, J.: Some results on backward stochastic differential equations driven by fractional Brownian motions. In: Stochastic Analysis and Applications to Finance: Essays in Honour of Jia-an Yan, pp. 225–242. World Scientific, Singapore (2012)
13. Hu, Y., Peng, S.: Backward stochastic differential equation driven by fractional Brownian motion. SIAM J. Control Optim. **48**(3), 1675–1700 (2009)
14. Jańczak-Borkowska, K.: Generalized BSDEs driven by fractional Brownian motion. Stat. Probab. Lett. **83**(3), 805–811 (2013)
15. Kolmogorov, A.N.: Wienersche Spiralen und einige andere interessante Kurven in Hilbertscen Raum. C. R. (Dokl.) Acad. Sci. URSS **26**, 115–118 (1940)
16. Li, Q., Zhou, Y., Zhao, X., Ge, X., et al.: Fractional order stochastic differential equation with application in european option pricing. Discrete Dyn. Nat. Soc. 2014 (2014)
17. Ma, J., Protter, P., San Martín, J., Torres, S.: Numerical method for backward stochastic differential equations. Ann. Appl. Probab., 302–316 (2002)
18. Maticiuc, L., Nie, T.: Fractional backward stochastic differential equations and fractional backward variational inequalities. J. Theor. Probab. **28**, 337–395 (2015)
19. Pardoux, E., Peng, S.: Adapted solution of a backward stochastic differential equation. Syst. Control Lett. **14**(1), 55–61 (1990)
20. Pardoux, E., Răşcanu, A., Pardoux, E., Răşcanu, A.: Backward stochastic differential equations. Stochastic Differential Equations, Backward SDEs, Partial Differential Equations, 353–515 (2014)
21. Peng, S., Yang, Z.: Anticipated backward stochastic differential equations. Ann. Probab. **37**(3), 877–902 (2009)
22. Wen, J.: Fractional backward stochastic differential equations with delayed generator (2022). arXiv preprint arXiv:2211.16826
23. Wen, J., Shi, Y.: Anticipative backward stochastic differential equations driven by fractional Brownian motion. Stat. Probab. Lett. **122**, 118–127 (2017)
24. Zhang, P., Ibrahim, A.I.N., Mohamed, N.A.: Mean-field and anticipated BSDEs with time-delayed generator. Mathematics **11**(4), 888 (2023)
25. Zhang, P., Mohamed, N.A., Ibrahim, A.I.N.: Mean-field and anticipated BSDEs with time-delayed generator. Mathematics **11**(4), 888 (2023)
26. Zhao, W., Zhang, W., Ju, L.: A numerical method and its error estimates for the decoupled forward–backward stochastic differential equations. Commun. Comput. Phys. **15**(3), 618–646 (2014)
27. Zhuang, Y.: Non-zero sum differential games of anticipated forward-backward stochastic differential delayed equations under partial information and application. Adv. Differ. Equ. **2017**(1), 383 (2017)

Publisher's Note