Mahmoud et al. *Boundary Value Problems* (2024) 2024:49
https://doi.org/10.1186/s13661-024-01849-z

Boundary Value Problems
a SpringerOpen Journal

RESEARCH — Open Access

Asymptotic behaviour and boundedness of solutions for third-order stochastic differential equation with multi-delay

A.M. Mahmoud[1], D.A. Eisa[1], R.O.A. Taie[2] and D.A.M. Bakhit[1*]

*Correspondence: doaa_math90@yahoo.com
[1] Department of Mathematics, Faculty of Science, New Valley University, El-Khargah 72511, Egypt
Full list of author information is available at the end of the article

Abstract
In the present paper, we study stochastic stability and stochastic boundedness for the stochastic differential equation (SDE) with multi-delay of third order. The derived results extend and improve some earlier results in the relevant literature, which are related to the qualitative properties of solutions to third-order delay differential equations (DDEs) and SDEs with multi-delay. Two examples are given to illustrate the results.

Mathematics Subject Classification: 34D20; 35B5; 34K50

Keywords: Delay differential equations (DDEs); Stochastic delay differential equations (SDDEs); Multi-delay; Stochastic stability; Stochastic boundedness

1 Introduction

During the past several years, the DDEs and the differential equations (DEs) with multiple delays have received more attention because of their widely applied backgrounds, such as population ecology, heat exchanges, mechanics, and economics. Here, we can mention the books by Burton [15], Hale [17], Hale and Verduyn Lunel [18], Iannelli [19] and numerous researchers activities such that, Abdel-Razek et al. [1], Abou-El-Ela et al. [2], Ademola and Arawomo [10], Ademola et al [11, 13], Mahmoud and Bakhit [24] Omeike [35, 36] Remili and Beldjerd [37], Remili and Oudjedi [38–40], Remili [41], Tunç [43–48], and the references therein.

Moreover, another kind of the DEs is the stochastic delay differential equations (SDDEs), where relevant parameters are modeled as suitable stochastic processes; see the book by Gikhman and Skorokhod [16]. The SDDE is a DE whose coefficients are random numbers or random functions of the independent variable (or variables). It is the appropriate tool for describing systems with external noise. The models of SDDEs play an important role in a range of application areas, including biology, chemistry, epidemiology, mechanics, microelectronics, economics, and finance. For example, in biology, we see that recently, Fathalla A. Rihan [42] studied the SDDEs for the spread of Coronavirus Infection COVID-19.

Furthermore, SDDEs are crucial in ecology, epidemiology, and many other fields; see, for example, Arnold [14], Mao [29–33], Øksendal [34], and references therein.

In the last few decades, the theory of SDDEs has attracted much attention, and numerous papers have been published. Here, we can mention the works by Abou-El-Ela et al. [3–7], Ademola [8], Ademola et al. [9, 12], Liu [21], Liu and Raffoul [22], Luo [23], Mahmoud and Tunç [26–28], Tunç [49], Zhi and Liping [20], and the references therein. Recently, Mahmoud and Bakhit [25] established the properties of solutions for nonautonomous third-order stochastic differential equation with a constant delay

$$\begin{aligned} &x'''(t) + a(t)f\big(x(t),x'(t)\big)x''(t) + b(t)\phi\big(x(t)\big)x'(t) + c(t)\psi\big(x(t-r)\big) \\ &\quad + g(t,x)w'(t) = p\big(t,x(t),x'(t),x''(t)\big). \end{aligned}$$

The main purpose of this note is to establish new criteria for the uniformly stochastic asymptotical stability (USAS) and uniformly stochastic boundedness (USB) for solutions of the following more general third-order SDE with multi-delay as the form

$$\begin{aligned} &\big[h\big(x'(t)\big)x''(t)\big]' + \big(\psi_1\big(x,x'\big)x'\big)' + \sum_{i=1}^{n} Q_i\big(x\big(t-r_i(t)\big),x'\big(t-r_i(t)\big)\big) \\ &\quad + \sum_{i=1}^{n} f_i\big(x\big(t-r_i(t)\big)\big) + \alpha x\big(t-l(t)\big)w'(t) = \varepsilon\big(x,x',x''\big), \end{aligned} \tag{1.1}$$

where $r_i(t)$ is continuously differentiable functions with $0 \le r_i(t) \le \gamma_i$, $(i = 1,2,\dots,n)$, $\gamma_i > 0$ are constants, ψ_1, Q_i, f_i and ε are continuous functions in their respective arguments, with $Q_i(x,0) = Q(0,y) = 0$ and $f_i(0) = 0$. In addition, $l(t)$ is a continuous function and defined from $[0,\infty)$ to $[0,l_1]$. $w(t) \in \mathbb{R}^n$ is a standard Brownian motion.

Consider the following notations

$$\begin{aligned} &\Phi_1(t) = \frac{\partial \psi_1}{\partial x}\frac{dx}{dt} + \frac{\partial \psi_1}{\partial y}\frac{dy}{dt}, \qquad h\big(x'(t)\big) = H(t), \\ &\Phi_2(t) = \frac{H'(t)}{H^2(t)} \quad \text{and} \quad \Phi_3(t) = \left(\frac{\psi_1(x,y)}{H(t)}\right)'. \end{aligned}$$

Therefore, equivalent system of (1.1) can be written as

$$\begin{aligned} x' &= y, \\ y' &= \frac{z}{H(t)}, \\ z' &= -\psi_1(x,y)\frac{z}{H(t)} - y\Phi_1(t) - \sum_{i=1}^{n} Q_i(x,y) \\ &\quad + \sum_{i=1}^{n}\left\{\int_{t-r_i(t)}^{t} \frac{\partial Q_i(x(s),y(s))}{\partial x} y(s)\,ds + \int_{t-r_i(t)}^{t} \frac{\partial Q_i(x(s),y(s))}{\partial y}\frac{z(s)}{H(s)}\,ds\right\} \\ &\quad - \sum_{i=1}^{n} f_i(x) + \sum_{i=1}^{n}\int_{t-r_i(t)}^{t} f_i'\big(x(s)\big)y(s)\,ds - \alpha x\big(t-l(t)\big)w'(t) + \varepsilon\left(x,y,\frac{z}{H(t)}\right). \end{aligned} \tag{1.2}$$

Remarks

(1) Whenever $\alpha x(t-l(t))w'(t) = 0$, and we consider the case that $i = 1$, then equation (1.1) reduces to a DDE of third order discussed in [39].
(2) Suppose that $\alpha = 0$, $h(x'(t)) = g(x''(t))$, $\psi_1(x,x') = h(x'(t))$, and with $i = 1$ if we let $Q(x(t-r(t)), x'(t-r(t))) = (\varphi(x(t))x(t))'$, then (1.1) can be reduced to the equation studied in [41].
(3) In the case $i = 1$, $\alpha = 0$ and if $h(x'(t)) = 1$, $(\psi_1(x,x')x')' = f(x,x')x''$, then equation (1.1) specialises to that considered in [2]. Our results generalize all the previous results.
(4) Whenever, $h(x'(t)) = 1$, $(\psi_1(x,x')x')' = a(t)f(x(t),x'(t))x''(t)$, and when $i = 1$, $Q_i(x(t-r_i(t)), x'(t-r_i(t))) = b(t)\phi(x(t))x'(t)$, $f_i(x(t-r_i(t))) = c(t)\psi(x(t-r))$, and $\alpha x(t-l(t)) = g(t,x)$, then (1.1) reduces to the studied equation in [25]. Thus, equation (1.1) generalizes the results obtained in [25]. Hence, our results include and extend all the previous results.

2 Stability results

Let $B(t) = (B_1(t),\dots,B_m(t))$ be an m-dimensional Brownian motion defined on the probability space. Consider an n-dimensional SDDE

$$dx(t) = N_1(t,x_t)\,dt + N_2(t,x_t)\,dB(t), \quad x_t(\theta) = x(t+\theta) - r \le \theta \le 0, t \ge t_0, \tag{2.1}$$

with initial value $x(0) = x_0 \in \mathcal{C}([-r,0];\mathbb{R}^n)$. Suppose that $N_1 : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $N_2 : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ satisfy the local Lipschitz and the linear growth conditions. Hence, for any given initial value $x(0) = x_0 \in \mathbb{R}^n$, it is known that equation (2.1) has a unique continuous solution on $t \ge 0$, which is known as $x(t;x_0)$ in this section. Suppose that $N_1(t,0) = 0$ and $N_2(t,0) = 0$, for all $t \ge 0$. Hence, the SDDE admits the zero solution $x(t;0) \equiv 0$.

Consider a functional $W(t,\varphi)$ that can be represented in the form $W(t,\varphi) = W(t,\varphi(0), \varphi(s))$, $s < 0$, for $\varphi = x_t$, put

$$W_\varphi(t,\varphi) = W(t,\varphi) = W(t,x_t) = W\bigl(t,x,x(t+s)\bigr), \quad x = \varphi(0) = x(t), s < 0,$$

and suppose that the function $W_\varphi(t,x)$ has a continuous derivative with respect to t and two continuous derivatives with respect to x.

Let $C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n;\mathbb{R}^+)$ denote the family of nonnegative functionals $W(t,x_t)$ defined on $\mathbb{R}^+ \times \mathbb{R}^n$, which are once continuously differentiable in t and twice continuously differentiable in x.

By the Itô formula, we have

$$dW(t,x_t) = \mathcal{L}W(t,x_t)\,dt + W_x(t,x_t)N_2(t,x_t)\,dB(t),$$

where

$$\begin{aligned} LW(t,x_t) &= W_t(t,x_t) + W_x(t,x_t)N_1(t,x_t) \\ &\quad + \frac{1}{2}\operatorname{trace}\bigl[N_2^T(t,x_t)W_{xx}(t,x_t)N_2(t,x_t)\bigr], \end{aligned} \tag{2.2}$$

such that

$$W_x = (W_{x_1}, \dots, W_{x_n}), \qquad W_t(t,x_t) = \frac{\partial W(t,x_t)}{\partial t},$$
$$W_x(t,x_t) = \left(\frac{\partial W(t,x_t)}{\partial x_1}, \dots, \frac{\partial W(t,x_t)}{\partial x_n}\right).$$

Furthermore,

$$W_{xx} = (W_{x_i x_j})_{n\times n} = \left(\frac{\partial^2 W(t,x_t)}{\partial x_i x_j}\right)_{n\times n}, \quad i,j = 1,2,3,\dots,n.$$

Now, we will give some definitions

Definition 2.1 [32] The zero solution of (2.1) is said to be stochastically stable or stable in probability if for every pair of $\varepsilon \in (0,1)$ and $r > 0$, there exists a $\delta = \delta(\varepsilon, r) > 0$ such that

$$P\bigl\{\bigl|x(t;x_0)\bigr| < r \text{ for all } t \geq 0\bigr\} \geq 1 - \varepsilon,$$

whenever $|x_0| < \delta$. Otherwise, it is said to be stochastically unstable.

Definition 2.2 [32] The zero solution of (2.1) is said to be stochastically asymptotically stable if it is stochastically stable, and, moreover, for every $\varepsilon \in (0,1)$, there exists a $\delta_0 = \delta_0(\varepsilon) > 0$, such that

$$P\Bigl\{\lim_{t\to\infty} x(t;x_0) = 0\Bigr\} \geq 1 - \varepsilon,$$

whenever $|x_0| < \delta_0$.

Definition 2.3 [22] (Stochastic boundedness) A solution $x(t;t_0,x_0)$ of (2.1) is said to be stochastically bounded, or bounded in probability, if it satisfies

$$E^{x_0}\bigl\|x(t;t_0,x_0)\bigr\| \leq C\bigl(\|x_0\|, t_0\bigr), \quad \text{for all } t \geq t_0,$$

where E^{x_0} denotes the expectation operator with respect to the probability law associated with x_0, and $C : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a constant depending on t_0 and x_0. We say that solutions of (2.1) are uniformly stochastically bounded if C is independent of t_0.

Hypotheses Suppose that there exist positive constants a_0, a, μ, D, C, b_i, c_i, d_i, L_i, M_i, N_i, A_i, B_i, C_i, D_i, γ_i, H_1, H_2 and l_1, such that

- (h_1) $1 < a \leq \psi_1(x,y) \leq a_0$, $y\frac{\partial \psi_1}{\partial x} \leq 0$ for all $x, y \in \mathbb{R}$.
- (h_2) $\frac{Q_i(x,y)}{y} \geq b_i > 0$, $y \neq 0$; $f_i(x) \geq d_i x$ with $\sup\{f_i'(x)\} = \frac{c_i}{2}$, $f_i(x)\operatorname{sgn} x > 0$ for $x \neq 0$ and $|f_i'(x)| \leq L_i$.
- (h_3) $H_1 \leq H(t) \leq H_2 \leq 1$, $H_1(a-1) \geq 2\mu$.
- (h_4) $|\frac{\partial Q_i}{\partial x}| \leq M_i$, $|\frac{\partial Q_i}{\partial y}| \leq N_i$ and $r_i(t) \leq \gamma_i$, $r_i'(t) \leq \beta_i$, $0 < \beta_i \leq 1$.
- (h_5) $ab_i - c_i > 2M_i + 2b_i + 6$.
- (h_6) $0 < l(t) \leq l_1$, $|l'(t)| \leq \frac{1}{2}$.

(h_7) $2\alpha^2 \leq 2H_1 d_i - H_1(a + b_i + 2)$.
(h_8) $\int_{-\infty}^{\infty} |\frac{\partial \psi_1(u,v)}{\partial u}| \, du + \int_{-\infty}^{\infty} |\frac{\partial \psi_1(u,v)}{\partial v}| \, dv \leq D < \infty$, $\int_{-\infty}^{\infty} |h'(u)| \, du \leq C < \infty$.

Theorem 2.1 *Assuming that the hypotheses* (h_1)–(h_8) *hold true provided that*

$$\gamma_i \leq \min\Bigg[\bigg\{\frac{2H_1 d_i - H_1(b_i + a + 1) - 2\alpha^2}{2A_i}\bigg\}, \bigg\{\frac{B_i(ab_i - c_i - 2M_i - 2b_i - 6)}{4(\mu B_i N_i + D_i)}\bigg\}, \bigg\{\frac{B_i H_1(aH_1 + 2H_1\mu + H_1)}{2(B_i A_i + C_i)}\bigg\}\Bigg],$$

where

$$\begin{aligned} A_i &= H_1(M_i + L_i) + N_i, \\ B_i &= 1 - \beta_i, \\ C_i &= N_i\big((\mu + 1)H_1 + 1\big), \\ D_i &= (M_i + L_i)\big\{H_1(\mu + 1) + \mu H_1(1 - \beta_i) + 1\big\}, \end{aligned} \tag{2.3}$$

with

$$\mu = \sum_{i=1}^{n} \frac{ab_i + c_i}{4b_i}.$$

Then, the zero solution of (1.1) *is USAS.*

Proof The main tool of the stability results is the continuously differentiable functional $W_1 = W_1(x_t, y_t, z_t)$, defined as

$$W_1 = \exp\bigg(-\frac{\omega(t)}{\mu_1}\bigg)U_1,$$

where

$$\omega(t) = \int_0^t \eta_1(s)\, ds, \quad \text{such that } \eta_1(t) = \big|\Phi_1(t)\big| + \big|\Phi_2(t)\big|.$$

Considering $\varepsilon \equiv 0$, we can observe that the Lyapunov functional $U_1 = U_1(x_t, y_t, z_t)$, where $x_t = x(t + s)$, $s \leq 0$, can be written as follows

$$\begin{aligned} U_1 &= \mu \sum_{i=1}^{n} \int_0^x f_i(\xi)\, d\xi + y \sum_{i=1}^{n} f_i(x) + \mu \int_0^y \psi_1(x, \eta)\eta d\eta \\ &\quad + \sum_{i=0}^{n} \int_0^y Q_i(x, \eta)\, d\eta + \mu yz + \frac{1}{2H(t)} z^2 + x^2 + xz \\ &\quad + \frac{\alpha^2}{H_1} \int_{t-l(t)}^{t} x^2(s)\, ds + \sum_{i=1}^{n} \lambda_i \int_{-r_i(t)}^{0} \int_{t+s}^{t} y^2(\theta)\, d\theta\, ds \\ &\quad + \sum_{i=1}^{n} \delta_i \int_{-r_i(t)}^{0} \int_{t+s}^{t} z^2(\theta)\, d\theta\, ds. \end{aligned} \tag{2.4}$$

Since the integrals $\int_{t-l(t)}^{t} x^2(s)\,ds$, $\int_{-r_i(t)}^{0}\int_{t+s}^{t} y^2(\theta)\,d\theta\,ds$ and $\int_{-r_i(t)}^{0}\int_{t+s}^{t} z^2(\theta)\,d\theta\,ds$ are positive, from the conditions (h_1)–(h_3), we conclude

$$U_1 \geq \mu\sum_{i=1}^{n}\int_0^x f_i(\xi)\,d\xi + y\sum_{i=1}^{n} f_i(x) + \frac{1}{2}\mu a y^2 + \frac{1}{2}y^2\sum_{i=1}^{n} b_i + \mu yz + \frac{1}{2}z^2 + x^2 + xz.$$

Therefore, we get

$$U_1 \geq \sum_{i=1}^{n}\frac{1}{2b_i}\big(b_i y + f_i(x)\big)^2 + \left(\mu y + \frac{z}{2}\right)^2 + \left(x + \frac{z}{2}\right)^2 + \frac{1}{2}\mu(a-2\mu)y^2 + \sum_{i=1}^{n}\frac{1}{2b_i y^2}\left[4\int_0^x f_i(\xi)\left\{\int_0^y \big(\mu b_i - f_i'(\xi)\big)\eta\,d\eta\right\}d\xi\right].$$

Since $\mu = \sum_{i=1}^{n}\frac{ab_i+c_i}{4b_i}$ and $\sup\{f'(x)\} = \frac{c_i}{2}$, it follows that

$$a - 2\mu = \sum_{i=1}^{n}\frac{ab_i - c_i}{2b_i} > 0,$$

and

$$\sum_{i=1}^{n}\big(\mu b_i - f_i'(x)\big) \geq \sum_{i=1}^{n}\frac{ab_i - c_i}{4} > 0.$$

Then, we get

$$\sum_{i=1}^{n}\frac{1}{2b_i y^2}\left[4\int_0^x f_i(\xi)\left\{\int_0^y \big(\mu b_i - f_i'(\xi)\big)\eta\,d\eta\right\}d\xi\right] \geq \sum_{i=1}^{n}\frac{ab_i - c_i}{4b_i}\int_0^x f_i(\xi)\,d\xi,$$

which tends to the following

$$U_1 \geq \sum_{i=1}^{n}\frac{1}{2b_i}\big(b_i y + f_i(x)\big)^2 + \left(\mu y + \frac{z}{2}\right)^2 + \left(x + \frac{z}{2}\right)^2 + \frac{1}{2}\mu\sum_{i=1}^{n}\left(\frac{ab_i - c_i}{2b_i}\right)y^2 + \sum_{i=1}^{n}\frac{ab_i - c_i}{4b_i}\int_0^x f_i(\xi)\,d\xi. \tag{2.5}$$

Hence, there exists a positive constant E_1, such that

$$U_1 \geq E_1\big(x^2 + y^2 + z^2\big). \tag{2.6}$$

In view of the hypotheses (h_1)–(h_4) and the following inequalities

$$\int_{-r_i(t)}^{0}\int_{t+s}^{t} y^2(\theta)\,d\theta\,ds \leq \|y\|^2\int_{t-r_i(t)}^{t}\big(\theta - t + r_i(t)\big)\,d\theta,$$

$$\int_{-r_i(t)}^{0}\int_{t+s}^{t} z^2(\theta)\,d\theta\,ds \le \|z\|^2 \int_{t-r_i(t)}^{t}\big(\theta - t + r_i(t)\big)\,d\theta,$$

and

$$\int_{t-l(t)}^{t} x^2(s)\,ds \le l(t)\|x\|^2.$$

Therefore, we can write (2.4) as

$$\begin{aligned} U_1 \le{}& \frac{1}{2}\mu x^2\sum_{i=1}^{n} L_i + xy\sum_{i=1}^{n} L_i + \frac{1}{2}\mu a_0 y^2 + \frac{1}{2}y^2\sum_{i=0}^{n} M_i + \mu yz + \frac{1}{2H_1}z^2 \\ &+ x^2 + xz + \frac{\alpha^2}{H_1}l(t)\|x\|^2 + \sum_{i=1}^{n}\big(\lambda_i\|y\|^2 + \delta_i\|z\|^2\big)\int_{t-r_i(t)}^{t}\big(\theta - t + r_i(t)\big)\,d\theta. \end{aligned}$$

Since $r_i(t) \le \gamma_i$ and $l(t) \le l_1$, with applying the estimate $2pq \le (p^2 + q^2)$, we find

$$\begin{aligned} U_1 \le{}& \sum_{i=1}^{n}\left\{\frac{L_i(\mu+1)+3}{2} + \frac{\alpha^2}{H_1}l_1\right\}\|x\|^2 \\ &+ \sum_{i=1}^{n}\left\{\frac{\mu(a+a_0+1)+(L_i+M_i)+\gamma_i^2\lambda_i}{2}\right\}\|y\|^2 \\ &+ \sum_{i=1}^{n}\left\{\frac{1+H_1(\mu+1+\gamma_i^2\delta_i)}{2H_1}\right\}\|z\|^2. \end{aligned} \tag{2.7}$$

Then, there exists a positive constant E_2, such that

$$U_1 \le E_2\big(x^2 + y^2 + z^2\big). \tag{2.8}$$

Now, using the equivalent system (1.2) with $\varepsilon = 0$ and the Itô formula (2.2), the derivative of the Lyapunov functional U_1 is given by

$$\begin{aligned} LU_1 ={}& y^2\sum_{i=1}^{n} f_i'(x) + \mu\int_0^y y\frac{\partial\psi_1(x,\eta)}{\partial x}\eta\,d\eta + \sum_{i=1}^{n}\int_0^y \frac{\partial Q_i(x,\eta)}{\partial x}y\,d\eta + \frac{\mu}{H(t)}z^2 \\ &- \left(\mu y^2 + \frac{1}{H(t)}yz + xy\right)\Phi_1(t) - \mu y\sum_{i=1}^{n} Q_i(x,y) - \frac{\psi_1(x,y)}{(H(t))^2}z^2 \\ &- \frac{\psi_1(x,y)}{H(t)}xz - x\sum_{i=1}^{n} Q_i(x,y) - x\sum_{i=1}^{n} f_i(x) + 2xy + yz \\ &- \frac{H'(t)}{2(H(t))^2}z^2 + \frac{\alpha^2}{H_1}x^2(t) - \frac{\alpha^2}{H_1}x^2\big(t-l(t)\big)\big(1-l'(t)\big) \\ &+ \frac{\alpha^2}{2H_1}x^2\big(t-l(t)\big) + y^2\sum_{i=1}^{n}\lambda_i\gamma_i + z^2\sum_{i=1}^{n}\delta_i\gamma_i \\ &- \sum_{i=1}^{n}\lambda_i(1-\beta_i)\int_{t-r_i(t)}^{t} y^2(\theta)\,d\theta - \sum_{i=1}^{n}\delta_i(1-\beta_i)\int_{t-r_i(t)}^{t} z^2(\theta)\,d\theta \end{aligned}$$

$$+\left(x+\mu y+\frac{1}{H(t)}z\right)\left[\sum_{i=1}^{n}\int_{t-r_i(t)}^{t}\left\{\frac{\partial Q_i(x(s),y(s))}{\partial x}y(s)\right.\right.$$
$$\left.\left.+\frac{\partial Q_i(x(s),y(s))}{\partial y}\frac{z(s)}{H(s)}+f'\big(x(s)\big)y(s)\,ds\right\}\right].$$

Therefore, using the definition of $\Phi_2(t)$ and considering the conditions (h_1)–(h_4) of Theorem 2.1, we have

$$\begin{aligned}
LU_1 \le{}& \frac{1}{2}y^2\sum_{i=1}^{n}c_i+\frac{\mu}{H_1}z^2-\left(\mu y^2+\frac{1}{H(t)}yz+xy\right)\Phi_1(t)+\frac{y^2}{2}\sum_{i=1}^{n}M_i-\frac{1}{2}\Phi_2(t)z^2\\
&-az^2-axz-xy\sum_{i=1}^{n}b_i-x^2\sum_{i=1}^{n}d_i+2xy+yz+\frac{\alpha^2}{H_1}x^2(t)+y^2\sum_{i=1}^{n}\lambda_i\gamma_i\\
&+z^2\sum_{i=1}^{n}\delta_i\gamma_i-\sum_{i=1}^{n}\lambda_i(1-\beta_i)\int_{t-r_i(t)}^{t}y^2(\theta)\,d\theta\\
&-\sum_{i=1}^{n}\delta_i(1-\beta_i)\int_{t-r_i(t)}^{t}z^2(\theta)\,d\theta\\
&-\mu y^2\sum_{i=1}^{n}b_i+\left(x+\mu y+\frac{1}{H_1}z\right)\left[\sum_{i=1}^{n}\int_{t-r_i(t)}^{t}\left\{\left|\frac{\partial Q_i(x(s),y(s))}{\partial x}\right|y(s)\right.\right.\\
&\left.\left.+\left|\frac{\partial Q_i(x(s),y(s))}{\partial y}\right|\frac{z(s)}{H(s)}+f'\big(x(s)\big)y(s)\,ds\right\}\right].
\end{aligned}\tag{2.9}$$

Suppose that

$$\Phi(t)=-\left(\mu y^2+\frac{1}{H(t)}yz+xy\right)\Phi_1(t)-\frac{1}{2}\Phi_2(t)z^2.$$

Using the Schwarz inequality $|pq|\le\frac{1}{2}(p^2+q^2)$ and (h_3), we can write the above equation as

$$\Phi(t)\le\left(\mu y^2+\frac{1}{2H_1}y^2+\frac{1}{2H_1}z^2+\frac{1}{2}\big(x^2+y^2\big)\right)\big|\Phi_1(t)\big|+\frac{1}{2}z^2\big|\Phi_2(t)\big|.$$

Therefore, we get

$$\begin{aligned}
\Phi(t)&\le\left\{\left(\mu+\frac{1}{H_1}+1\right)\big|\Phi_1(t)\big|+\frac{1}{2}\big|\Phi_2(t)\big|\right\}\big(x^2+y^2+z^2\big)\\
&\le\left\{\left(\mu+\frac{1}{H_1}+1\right)+\frac{1}{2}\right\}\big(\big|\Phi_1(t)\big|+\big|\Phi_2(t)\big|\big)\big(x^2+y^2+z^2\big).
\end{aligned}$$

For the positive constant E_3, the last inequality becomes

$$\Phi(t)\le E_3\eta_1(t)\big(x^2+y^2+z^2\big),$$

where

$$E_3=\max\left\{\frac{1}{2},\mu+\frac{1}{H_1}+1\right\}.$$

It follows from (2.6) that

$$\Phi(t) \le \frac{E_3}{E_1}\eta_1(t)U_1. \tag{2.10}$$

Thus, by (2.9), (2.10) and the fact that $2pq \le (p^2 + q^2)$, we obtain the following estimate

$$\begin{aligned} LU_1 \le & -\frac{1}{2H_1}\sum_{i=1}^{n}\left\{2H_1 d_i - H_1(a + b_i + 2) - 2\alpha^2 - \bigl(H_1(M_i + L_i) + N_i\bigr)\gamma_i\right\}x^2 \\ & - \sum_{i=1}^{n}\left\{\mu b_i - \frac{c_i}{2} - \frac{M_i}{2} - \frac{b_i}{2} - \frac{3}{2} - \frac{\mu(H_1(M_i + L_i) + N_i)}{2H_1}\gamma_i - \lambda_i\gamma_i\right\}y^2 \\ & - \sum_{i=1}^{n}\left\{\frac{H_1(a-1) - 2\mu}{2H_1} - \frac{\mu(H_1(M_i + L_i) + N_i)}{2H_1^2}\gamma_i - \delta_i\gamma_i\right\}z^2 + \frac{E_3}{E_1}\eta_1(t)U_1 \\ & + \sum_{i=1}^{n}\left\{\frac{1}{2H_1}(M_i + L_i)\bigl(H_1(\mu + 1) + 1\bigr) - (1 - \beta_i)\lambda_i\right\}\int_{t-r_i(t)}^{t} y^2(\theta)\,d\theta \\ & + \sum_{i=1}^{n}\left\{\frac{N_i}{2H_1^2}\bigl(H_1(\mu + 1) + 1\bigr) - (1 - \beta_i)\delta_i\right\}\int_{t-r_i(t)}^{t} z^2(\theta)\,d\theta. \end{aligned} \tag{2.11}$$

If we let

$$\lambda_i = \frac{(M_i + L_i)\{H_1(\mu + 1) + 1\}}{2H_1(1 - \beta_i)},$$

and

$$\delta_i = \frac{N_i\{H_1(\mu + 1) + 1\}}{2H_1^2(1 - \beta_i)}.$$

We also have $\mu b_i - \frac{c_i}{2} = \frac{ab_i - c_i}{4} > 0$ and $H_1(a - 1) \ge 2\mu$; therefore, (2.11) becomes

$$\begin{aligned} LU_1 \le & -\frac{1}{2H_1}\sum_{i=1}^{n}\left\{2H_1 d_i - H_1(a + b_i + 2) - 2\alpha^2 - \bigl(H_1(M_i + L_i) + N_i\bigr)\gamma_i\right\}x^2 \\ & - \sum_{i=1}^{n}\Biggl[\frac{ab_i - c_i - 2M_i - 2b_i - 6}{4} \\ & - \frac{(M_i + L_i)\{H_1(\mu + 1) + \mu H_1(1 - \beta_i) + 1\} + \mu N_i(1 - \beta_i)}{2H_1(1 - \beta_i)}\gamma_i\Biggr]y^2 \\ & - \sum_{i=1}^{n}\left\{\frac{H_1(a-1) - 2\mu}{2H_1} - \frac{(1 - \beta_i)(H_1(M_i + L_i) + N_i) + N_i((\mu + 1)H_1 + 1)}{2(1 - \beta_i)H_1^2}\gamma_i\right\}z^2 \\ & + \frac{E_3}{E_1}\eta_1(t)U_1. \end{aligned}$$

Now, in view of (2.3), the last inequality becomes

$$LU_1 \le -\frac{1}{2H_1}\sum_{i=1}^{n}\left\{2H_1 d_i - H_1(a + b_i + 2) - 2\alpha^2 - A_i\gamma_i\right\}x^2$$

$$
\begin{aligned}
&-\sum_{i=1}^{n}\left\{\frac{ab_i-c_i-2M_i-2b_i-6}{4}-\frac{D_i+\mu N_iA_i}{2H_1B_i}\gamma_i\right\}y^2\\
&-\sum_{i=1}^{n}\left\{\frac{H_1(a-1)-2\mu}{2H_1}-\frac{B_iA_i+C_i}{2B_iH_1^2}\gamma_i\right\}z^2\\
&+\frac{E_3}{E_1}\eta_1(t)U_1.
\end{aligned}
$$

Hence, for the positive constant $E_4>0$, we obtain

$$LU_1\le -E_4\left(x^2+y^2+z^2\right)+\frac{E_3}{E_1}\eta_1(t)U_1.$$

Now, if we let

$$
\begin{aligned}
&\sigma_1(t)=\min\left\{x'(0),x'(t)\right\}, &&\sigma_2(t)=\max\left\{x'(0),x'(t)\right\}\\
&\sigma_3(t)=\min\left\{x''(0),x''(t)\right\}, &&\sigma_4(t)=\max\left\{x''(0),x''(t)\right\},
\end{aligned}
$$

then we get

$$
\begin{aligned}
\omega(t)&=\int_0^t\eta_1(s)\,ds\\
&=\int_0^t\left\{\left|\Phi_1(s)\right|+\left|\Phi_2(s)\right|\right\}ds\\
&\le\int_{\sigma_1(t)}^{\sigma_2(t)}\left|\frac{\partial\psi_1(u,v)}{\partial u}\right|du+\int_{\sigma_3(t)}^{\sigma_4(t)}\left|\frac{\partial\psi_1(u,v)}{\partial v}\right|dv+\int_{\sigma_1(t)}^{\sigma_2(t)}\left|\frac{H'(u)}{(H(u))^2}\right|du\\
&\le\int_{-\infty}^{\infty}\left|\frac{\partial\psi_1(u,v)}{\partial u}\right|du+\int_{-\infty}^{\infty}\left|\frac{\partial\psi_1(u,v)}{\partial v}\right|dv+\frac{1}{H_1^2}\int_{-\infty}^{\infty}\left|h'(u)\right|du.
\end{aligned}
$$

From the condition (h_6), it follows that

$$\omega(t)\le D+\frac{C}{H_1^2}<\infty. \tag{2.12}$$

Because of

$$W_1=\exp\left(-\frac{\omega(t)}{\mu_1}\right)U_1,\qquad \mu_1=\frac{E_1}{E_3}.$$

The stochastic derivative of the above equation is

$$LW_1\le\exp\left(-\frac{\omega(t)}{\mu_1}\right)\left(LU_1-\frac{E_3}{E_1}\eta_1(t)U_1\right).$$

Therefore, for the positive constant D_1, we conclude that

$$LW_1\le -D_1\left(x^2+y^2+z^2\right). \tag{2.13}$$

Hence, from the results (2.6), (2.8), and (2.13), all conditions of the Lemma of the stability in [8, 14] are satisfied. Therefore, the proof of Theorem 2.1 is now complete. □

3 Uniformly stochastically boundedness results

Theorem 3.1 *Assume that the hypotheses (h_1)–(h_8) hold true and suppose that there exist positive constants F_i, K_i, and m such that*

$$F_i = a(a_0 + 1) + H_1(\mu + 1) + 1)(M_i + L_i)(a + 1) + (ab_i - c_i)L_i, \tag{3.1}$$

and

$$K_i = H_1(\mu + 1) + 1 + a(a_0 + 1) + H_1(ab_i - c_i)N_i. \tag{3.2}$$

Furthermore, we assume that

$$\alpha^2 < \sum_{i=1}^{n} \frac{H_1\{(ab_i - c_i + 1)d_i - (a + b_i + 2)\}}{a + 1},$$

and

$$\left|\varepsilon\left(x, x', x''\right)\right| \leq m.$$

Provided that the positive constant γ_i satisfies the following

$$\begin{aligned}\gamma_i \leq \min\Bigg[&\left\{\frac{H_1(H_1(ab_i - c_i + 1)d_i - H_1(b_i + a + H_1) - (a + 1)\alpha^2)}{(ab_i - c_i + H_1)A_i}\right\},\\ &\left\{\frac{H_1^2 B_i(ab_i - c_i + 2ac_i - 2M_i - 2b_i - 6)}{4(\mu B_i N_i(\mu H_1 + aa_0) + F_i H_1)}\right\},\\ &\left\{\frac{B_i H_1(H_1(a + 1) - 2\mu)}{2(B_i A_i(a_0 + 1) + K_i)}\right\}\Bigg].\end{aligned}$$

Then, all solutions of (1.1) *are USB.*

Proof Here, consider $\varepsilon \neq 0$ and define the Lyapounov functional as follows

$$U(x_t, y_t, z_t) = U_1(x_t, y_t, z_t) + U_2(x_t, y_t, z_t),$$

where U_1 is defined in (2.4), and we define U_2 as

$$\begin{aligned}U_2 = {}& a\frac{\psi_1(x, y)}{H(t)} \sum_{i=1}^{n} \int_0^x f_i(\xi)\, d\xi + aa_0 \int_0^y \psi_1(x, \eta)\eta d\eta + a \sum_{i=1}^{n} f_i(x)y\\ &+ \frac{1}{2H_1} \sum_{i=1}^{n} b_i(ab_i - c_i)x^2 + \frac{1}{2} \sum_{i=1}^{n} c_i y^2 + \sum_{i=1}^{n} (ab_i - c_i)\left(\frac{z}{H_1} + a_0 y\right)x\\ &+ \frac{a_0}{2H_1}z^2 + \frac{aa_0}{H_1}yz + \frac{a}{H_1}\alpha^2 \int_{t-l(t)}^{t} x^2(s)\, ds.\end{aligned} \tag{3.3}$$

Since $\int_{t-l(t)}^{t} x^2(s)\, ds$ is nonnegative, recall the hypotheses (h_1)–(h_4), and then U_2 becomes

$$U_2 \geq \frac{a^2}{H_1} \sum_{i=1}^{n} \int_0^x f_i(\xi)\, d\xi + \frac{1}{2}a^2 a_0 y^2 + a \sum_{i=1}^{n} f_i(x)y + \frac{a_0}{2H_1}z^2 + \frac{aa_0}{H_1}yz$$

$$+\frac{1}{2H_1}\sum_{i=1}^{n} b_i(ab_i-c_i)x^2+\frac{1}{2}\sum_{i=1}^{n} c_i y^2+\sum_{i=1}^{n}(ab_i-c_i)\left(\frac{z}{H_1}+a_0y\right)x.$$

Furthermore, we have $H_2 \le H(t) \le H_1 \le 1$, $\frac{1}{H_1} \ge 1$, so the above inequality leads to the following

$$\begin{aligned}U_2 \ge{}& \sum_{i=1}^{n}\frac{a^2}{2c_i y^2}\left[4\int_0^x f_i(\xi)\,d\xi\left\{\int_0^y\left(c_i-f_i'(\xi)\right)\right\}\eta\,d\eta\right]+\sum_{i=1}^{n}\frac{1}{2c_i}\left(c_i y+af_i(x)\right)^2\\&+\sum_{i=1}^{n}\frac{c_i}{2b_i}(z+ay)^2+\sum_{i=1}^{n}\frac{(ab_i-c_i)}{2b_i}\left\{b_i x+(z+ay)\right\}^2.\end{aligned} \tag{3.4}$$

Therefore, from (h_2), we find

$$\begin{aligned}U_2 \ge{}& \frac{a^2}{2}\sum_{i=1}^{n}\int_0^x f_i(\xi)\,d\xi+\sum_{i=1}^{n}\frac{1}{2c_i}\left(c_i y+af_i(x)\right)^2\\&+\sum_{i=1}^{n}\frac{c_i}{2b_i}(z+ay)^2+\sum_{i=1}^{n}\frac{(ab_i-c_i)}{2b_i}\left\{b_i x+(z+ay)\right\}^2.\end{aligned}$$

We can find a positive constant φ_1 such that the last inequality gives

$$U_2 \ge \varphi_1\left(x^2+y^2+z^2\right). \tag{3.5}$$

Thus, from (2.5) and (3.4), we conclude

$$\begin{aligned}U \ge{}& \sum_{i=1}^{n}\left(\frac{a^2}{2}+\frac{ab_i-c_i}{4b_i}\right)\int_0^x f_i(\xi)\,d\xi+\sum_{i=1}^{n}\frac{1}{2c_i}\left(c_i y+af_i(x)\right)^2\\&+\sum_{i=1}^{n}\frac{c_i}{2b_i}(z+ay)^2+\sum_{i=1}^{n}\frac{(ab_i-c_i)}{2b_i}\left\{b_i x+(z+ay)\right\}^2\\&+\sum_{i=1}^{n}\frac{1}{2b_i}\left(b_i y+f_i(x)\right)^2+\left(\mu y+\frac{z}{2}\right)^2+\left(x+\frac{z}{2}\right)^2+\frac{1}{2}\mu\left(\frac{ab_i-c_i}{2b_i}\right)y^2.\end{aligned}$$

Hence, for the positive constant φ_2, we get

$$U(x_t,y_t,z_t) \ge \varphi_2\left(x^2+y^2+z^2\right). \tag{3.6}$$

Since $|\frac{\partial Q_i}{\partial x}| \le M_i$, $|f_i'(x)| \le L_i$, $a \le \psi_1 \le a_0$ and $H_1 \le H(t) \le H_2 \le 1$, we can rewrite (3.3) in the following form

$$\begin{aligned}U_2 \le{}& \frac{aa_0}{H_1}\sum_{i=1}^{n} L_i x^2+\frac{aa_0}{2}y^2+\frac{a}{2}\sum_{i=1}^{n} L_i xy+\frac{1}{H_1}\sum_{i=1}^{n} b_i(ab_i-c_i)x^2\\&+\frac{1}{2}\sum_{i=1}^{n} c_i y^2+\left(\frac{z}{H_1}+a_0y\right)x\sum_{i=1}^{n} b_i(ab_i-c_i)\\&+\frac{a_0}{2H_1}z^2+\frac{aa_0}{H_1}yz+\frac{a}{H_1}\alpha^2 l(t)\|x\|^2.\end{aligned}$$

Applying the inequality $2pq \leq (p^2 + q^2)$ and using the condition $0 < l(t) \leq l_1$, it tends to

$$\begin{aligned} U_2 \leq{}& \frac{1}{2H_1}\sum_{i=1}^{n}\left\{aL_i(1+a_0)+(ab_i-c_i)(b_i+1+a_0H_1)+2a\alpha^2 l_1\right\}\|x\|^2 \\ &+\frac{1}{2H_1}\sum_{i=1}^{n}\left\{aa_0(H_1+1)+H_1\bigl(aL_i+c_i+a_0(ab_i-c_i)\bigr)\right\}\|y\|^2 \\ &+\frac{1}{2H_1}\sum_{i=1}^{n}\left\{a_0(a+1)+ab_i-c_i\right\}\|z\|^2. \end{aligned} \tag{3.7}$$

Then, with $\varphi_3 > 0$, we have

$$U_2(x_t,y_t,z_t) \leq \varphi_3\bigl(x^2+y^2+z^2\bigr). \tag{3.8}$$

Combining the inequality (2.7) with (3.7), we conclude

$$\begin{aligned} U \leq{}& \frac{1}{2H_1}\sum_{i=1}^{n}\left\{L_i\bigl(H_1(\mu+1)+a(1+a_0)\bigr)+(ab_i-c_i)(b_i+1+a_0H_1)\right. \\ &\left.+3H_1+\alpha^2 l_1(2a+1)\right\}\|x\|^2+\frac{1}{2H_1}\sum_{i=1}^{n}\left\{\mu aa_0(H_1+1)\right. \\ &\left.+a(a_0+1)+(L_i+M_i)+H_1\bigl(aL_i+c_i+a_0(ab_i-c_i)\bigr)+\gamma_i^2\lambda_i\right\}\|y\|^2 \\ &+\frac{1}{2H_1}\sum_{i=1}^{n}\left\{H_1(\mu+1)+1+a_0(a+1)+ab_i-c_i+\gamma_i^2\delta_i\right\}\|z\|^2. \end{aligned}$$

Hence, for the positive constant φ_4, the last inequality gives

$$U(x_t,y_t,z_t) \leq \varphi_4\bigl(x^2+y^2+z^2\bigr). \tag{3.9}$$

In view of the hypothesis of Theorem 3.1 and the Itô formula, the derivative of the Lyapunov functional (3.3) with respect to the system (1.2) becomes

$$\begin{aligned} LU_2 \leq{}& a\left(\frac{\psi_1(x,y)}{H(t)}\right)'\sum_{i=1}^{n}\int_0^x f_i(\xi)\,d\xi-\frac{1}{H_1}\Phi_1(t)\sum_{i=1}^{n}(ab_i-c_i)xy-\frac{ayz}{H_1}\Phi_1(t) \\ &-\frac{aa_0}{H_1}\Phi_1(t)y^2+aa_0\int_0^y y\frac{\partial\psi_1(x,\eta)}{\partial x}\eta\,d\eta-\frac{1}{H_1}\sum_{i=1}^{n}(ab_i-c_i)f_i(x)x \\ &-\frac{a}{2}\sum_{i=1}^{n}c_i y^2+\frac{a}{2H_1}\alpha^2x^2\bigl(t-l(t)\bigr)-\frac{a}{H_1}\alpha^2x^2\bigl(t-l(t)\bigr)\bigl(1-l'(t)\bigr) \\ &+\frac{a}{H_1}\alpha^2x^2(t)+\frac{m}{H_1}\sum_{i=1}^{n}(ab_i-c_i)|x|+\frac{aa_0m}{H_1}|y|+\frac{a_0m}{H_1}|z| \\ &+\left\{\frac{1}{H_1}\sum_{i=1}^{n}(ab_i-c_i)x+\frac{aa_0}{H_1}y+\frac{a_0}{H_1}z\right\}\left[\sum_{i=1}^{n}\int_{t-r_i(t)}^{t}\left\{\frac{\partial Q_i(x(s),y(s))}{\partial x}y(s)\right.\right. \\ &\left.\left.+\frac{\partial Q_i(x(s),y(s))}{\partial y}\frac{z(s)}{H(s)}+f'\bigl(x(s)\bigr)y(s)\,ds\right\}\right]. \end{aligned}$$

Now, we choose

$$\Phi_4(t) = a\Phi_3 \sum_{i=1}^{n} \int_0^x f_i(\xi)\, d\xi - \frac{1}{H_1}\Phi_1(t) \sum_{i=1}^{n} (ab_i - c_i)xy - \frac{ayz}{H_1}\Phi_1(t) - \frac{aa_0}{H_1}\Phi_1(t)y^2.$$

Since $|f_i'(x)| \leq L_i$, we obtain

$$\begin{aligned}\left|\Phi_4(t)\right| \leq{}& \frac{a}{2}\left|\Phi_3(t)\right| \sum_{i=1}^{n} L_i x^2 + \frac{1}{H_1}\left|\Phi_1(t)\right| \sum_{i=1}^{n} (ab_i - c_i)|x||y| \\ &+ \frac{a|y||z|}{H_1}\left|\Phi_1(t)\right| + \frac{aa_0}{H_1}\left|\Phi_1(t)\right|y^2.\end{aligned}$$

Using the fact that $2pq \leq (p^2 + q^2)$, we get

$$\left|\Phi_4(t)\right| \leq \sum_{i=1}^{n} \left\{\frac{1}{2}L_i + \frac{1}{H_1}(ab_i - c_i) + \frac{a}{H_1} + \frac{aa_0}{H_1}\right\}\left(\left|\Phi_1(t)\right| + \left|\Phi_3(t)\right|\right)\left(x^2 + y^2 + z^2\right).$$

If we let

$$\eta_2(t) = \left|\Phi_1(t)\right| + \left|\Phi_3(t)\right|,$$

then from (3.5), we conclude

$$\left|\Phi_4(t)\right| \leq \frac{\varphi_5}{\varphi_1} U_2 \eta_2(t), \tag{3.10}$$

where

$$\varphi_5 = \max\left\{\frac{1}{2}L_i, \frac{1}{H_1}(ab_i - c_i) + \frac{a}{H_1} + \frac{aa_0}{H_1}\right\}.$$

Considering the conditions $l(t) \leq \frac{1}{2}$, $y\frac{\partial \psi_1(x,y)}{\partial x} \leq 0$ and using equation (3.10), we find

$$\begin{aligned}LU_2 \leq{}& \frac{\varphi_5}{\varphi_1}\eta_2(t)U_2 - \frac{1}{H_1}\sum_{i=1}^{n}(ab_i - c_i)f_i(x)x - \frac{a}{2}\sum_{i=1}^{n} c_i y^2 \\ &+ \frac{a}{H_1}\alpha^2 x^2(t) + \frac{m}{H_1}\sum_{i=1}^{n}(ab_i - c_i)|x| + \frac{aa_0 m}{H_1}|y| + \frac{a_0 m}{H_1}|z| \\ &+ \left\{\frac{1}{H_1}\sum_{i=1}^{n}(ab_i - c_i)x + \frac{aa_0}{H_1}y + \frac{a_0}{H_1}z\right\}\left[\sum_{i=1}^{n}\int_{t-r_i(t)}^{t}\left\{\frac{\partial Q_i(x(s), y(s))}{\partial x}y(s)\right.\right. \\ &\left.\left.+ \frac{\partial Q_i(x(s), y(s))}{\partial y}\frac{z(s)}{H(s)} + f'\left(x(s)\right)y(s)\, ds\right\}\right].\end{aligned}$$

Now, from the hypotheses (h_2) and (h_4), we obtain

$$\begin{aligned}
LU_2 \leq{}& \frac{\varphi_5}{\varphi_1}\eta_2(t)U_2 + \left\{\frac{1}{H_1}\sum_{i=1}^{n}(ab_i - c_i)|x| + \frac{aa_0}{H_1}|y| + \frac{a_0}{H_1}|z|\right\}m \\
&+ \left\{\sum_{i=1}^{n}(ab_i - c_i)d_i - \frac{a\alpha^2}{H_1} - \sum_{i=1}^{n}\left(\frac{(ab_i - c_i)(H_1(M_i + L_i) + N_i)}{2H_1^2}\gamma_i\right)\right\}x^2 \\
&- \left\{\frac{a}{2}\sum_{i=1}^{n}c_i - \sum_{i=1}^{n}\left(\frac{aa_0(H_1(M_i + L_i) + N_i)}{2H_1^2}\gamma_i\right)\right\}y^2 \\
&+ \frac{a_0}{2H_1}\sum_{i=1}^{n}\left\{\left(H_1(M_i + L_i) + N_i\right)\gamma_i\right\}z^2 \\
&+ \frac{1}{2H_1}\sum_{i=1}^{n}\left\{a_0(M_i + L_i)(a + 1) + (ab_i - c_i)L_i\right\}\int_{t-r_i(t)}^{t} y^2(\theta)\,d\theta \\
&+ \frac{1}{2H_1^2}\sum_{i=1}^{n}\left\{a(a_0 + 1) + H_1(ab_i - c_i)\right\}\int_{t-r_i(t)}^{t} z^2(\theta)\,d\theta.
\end{aligned} \tag{3.11}$$

By compiling the above inequality with (2.11), from (3.1) and (3.2), we conclude

$$\begin{aligned}
LU \leq{}& \left\{\frac{1}{H_1}\sum_{i=1}^{n}(ab_i - c_i + 1)|x| + \left(\mu + \frac{aa_0}{H_1}\right)|y| + \frac{(a_0 + 1)}{H_1}|z|\right\}m + \frac{E_3}{E_1}\eta_1(t)U_1 \\
&+ \frac{\varphi_5}{\varphi_1}\eta_2(t)U_2 - \left\{\sum_{i=1}^{n}(ab_i - c_i + 1)d_i - \frac{1}{2}\sum_{i=1}^{n}(a + b_i + 2) - \frac{(a + 1)\alpha^2}{H_1}\right. \\
&\left. - \sum_{i=1}^{n}\left(\frac{(ab_i - c_i + H_1)A_i}{2H_1^2}\gamma_i\right)\right\}x^2 - \left\{\sum_{i=1}^{n}\left(\frac{ab_i - c_i + 2ac_i - 2M_i - 2b_i - 6}{4}\right)\right. \\
&\left. - \sum_{i=1}^{n}\left(\frac{(\mu H_1 + aa_0)A_i}{2H_1^2}\gamma_i - \lambda_i\gamma_i\right)\right\}y^2 \\
&- \frac{1}{2H_1}\left\{H_1(a + 1) - 2\mu - \frac{1}{H_1}(a_0 + 1)\sum_{i=1}^{n}A_i\gamma_i - \delta_i\gamma_i\right\}z^2 \\
&+ \sum_{i=1}^{n}\left\{\frac{1}{2H_1}F_i - (1 - \beta_i)\lambda_i\right\}\int_{t-r_i(t)}^{t} y^2(\theta)\,d\theta \\
&+ \sum_{i=1}^{n}\left\{\frac{1}{2H_1^2}K_i - (1 - \beta_i)\delta_i\right\}\int_{t-r_i(t)}^{t} z^2(\theta)\,d\theta.
\end{aligned}$$

We take

$$\lambda_i = \frac{F_i}{2H_1(1 - \beta_i)} \quad \text{and} \quad \delta_i = \frac{K_i}{2H_1^2(1 - \beta_i)}.$$

Therefore, from (2.3) and since $B_i = (1 - \beta_i)$, we obtain

$$LU \leq \left\{\frac{1}{H_1}\sum_{i=1}^{n}(ab_i - c_i + 1)|x| + \left(\mu + \frac{aa_0}{H_1}\right)|y| + \frac{(a_0 + 1)}{H_1}|z|\right\}m + \frac{E_3}{E_1}\eta_1(t)U_1$$

$$
\begin{aligned}
&+ \frac{\varphi_5}{\varphi_1}\eta_2(t)U_2 - \Biggl\{\sum_{i=1}^{n}(ab_i - c_i + 1)d_i - \sum_{i=1}^{n}(a + b_i + 1) - \frac{(a+1)\alpha^2}{H_1} \\
&- \sum_{i=1}^{n}\left(\frac{(ab_i - c_i + H_1)A_i}{2H_1^2}\gamma_i\right)\Biggr\}x^2 - \sum_{i=1}^{n}\Biggl\{\frac{ab_i - c_i + 2ac_i - 2M_i - 2b_i - 6}{4} \\
&- \frac{(\mu H_1 + aa_0)A_i}{2H_1^2}\gamma_i - \frac{F_i}{2H_1B_i}\gamma_i\Biggr\}y^2 \\
&- \Biggl\{\frac{1}{2H_1}\bigl(H_1(a+1) - 2\mu\bigr) - \frac{1}{2H_1^2}\sum_{i=1}^{n}\frac{(a_0+1)B_iA_i + K_i}{B_i}\gamma_i\Biggr\}z^2.
\end{aligned}
$$

Therefore, we can write the above inequality as follows

$$
\begin{aligned}
LU &\le \frac{E_3}{E_1}\eta_1(t)U_1 + \frac{\varphi_5}{\varphi_1}\eta_2(t)U_2 - \zeta\left(x^2 + y^2 + z^2\right) + \kappa\zeta\left(|x| + |y| + |z|\right) \\
&= \frac{E_3}{E_1}\eta_1(t)U_1 + \frac{\varphi_5}{\varphi_1}\eta_2(t)U_2 - \frac{\zeta}{2}\left(x^2 + y^2 + z^2\right) \\
&\quad - \frac{\zeta}{2}\left\{\left(|x| - \kappa\right)^2 + \left(|y| - \kappa\right)^2 + \left(|z| - \kappa\right)^2\right\} + \frac{3\zeta}{2}\kappa^2 \\
&\le \frac{E_3}{E_1}\eta_1(t)U_1 + \frac{\varphi_5}{\varphi_1}\eta_2(t)U_2 - \frac{\zeta}{2}\left(x^2 + y^2 + z^2\right) + \frac{3\zeta}{2}\kappa^2, \quad \text{for some } \kappa, \zeta > 0,
\end{aligned}
$$

where

$$
\kappa = m\max\left\{ab_i - c_i + 1, \mu + \frac{aa_0}{H_1}, \frac{a_0 + 1}{H_1}\right\}.
$$

From (2.8) and (3.8), we obtain the following estimate

$$
\begin{aligned}
\Phi_5(t) &= \frac{E_3}{E_1}\eta_1(t)U_1 + \frac{\varphi_5}{\varphi_1}\eta_2(t)U_2 \\
&\le \frac{E_3}{E_1}\eta_1(t)E_2\left(x^2 + y^2 + z^2\right) + \frac{\varphi_5}{\varphi_1}\eta_2(t)\varphi_3\left(x^2 + y^2 + z^2\right) \\
&\le \varphi_6\left(\left|\eta_1(t)\right| + \left|\eta_2(t)\right|\right)\left(x^2 + y^2 + z^2\right),
\end{aligned}
$$

where

$$
\varphi_6 = \max\left\{\frac{E_2E_3}{E_1}, \frac{\varphi_3\varphi_5}{\varphi_1}\right\}.
$$

According to inequality (3.6), we conclude

$$
\Phi_5(t) \le \frac{\varphi_6}{\varphi_2}\left(\left|\eta_1(t)\right| + \left|\eta_2(t)\right|\right)U.
$$

It follows that

$$
LU \le \frac{\varphi_6}{\varphi_2}\left(\left|\eta_1(t)\right| + \left|\eta_2(t)\right|\right)U - \frac{\zeta}{2}\left(x^2 + y^2 + z^2\right) + \frac{3\zeta}{2}\kappa^2. \tag{3.12}
$$

Define the Lyapunov functional $W_2(x_t, y_t, z_t)$ as follows

$$W_2 = \exp\left\{-\frac{\varphi_6}{\varphi_2}\eta_3(t)\right\}U(x_t, y_t, z_t),$$

where

$$\eta_3(t) = \int_0^t \left\{\eta_1(t) + \eta_2(t)\right\} d\eta.$$

Then, from the hypotheses h_1 and h_3 and (2.12), we conclude

$$\begin{aligned}
\eta_3(t) &= \int_0^t \big(\eta_1(s) + \eta_2(s)\big)\, ds \le D + \frac{C}{H_1^2} + \int_0^t \left\{\left|\Phi_1(s)\right| + \left|\Phi_3(s)\right|\right\} ds \\
&\le 2D + \frac{C}{H_1^2} + \frac{1}{H_1}\int_{\sigma_1(t)}^{\sigma_2(t)} \left|\frac{\partial\psi_1(u,v)}{\partial u}\right| du + \frac{1}{H_1}\int_{\sigma_3(t)}^{\sigma_4(t)} \left|\frac{\partial\psi_1(u,v)}{\partial v}\right| dv \\
&\quad + \frac{a_0}{H_1^2}\int_{\sigma_1(t)}^{\sigma_2(t)} \left|H'(u)\right| du \\
&\le 2D + \frac{C}{H_1^2} + \frac{1}{H_1}\int_{-\infty}^{\infty} \left|\frac{\partial\psi_1(u,v)}{\partial u}\right| du + \frac{1}{H_1}\int_{-\infty}^{\infty} \left|\frac{\partial\psi_1(u,v)}{\partial v}\right| dv \\
&\quad + \frac{a_0}{H_1^2}\int_{-\infty}^{\infty} \left|h'(u)\right| du.
\end{aligned}$$

It follows form (h_8) that

$$\eta_3(t) \le D\left(2 + \frac{1}{H_1}\right) + \frac{C}{H_1^2}(a_0 + 1) < \infty.$$

Then, the stochastically derivative of W_2 becomes

$$LW_2 = \exp\left\{-\frac{\varphi_6}{\varphi_2}\eta_3(t)\right\}\left\{LU - \frac{\varphi_6}{\varphi_2}\big(\left|\eta_1(t)\right| + \left|\eta_2(t)\right|\big)U\right\}.$$

Hence, from (3.12), we find

$$LW_2 \le M\left\{-\frac{\zeta}{2}\left(x^2 + y^2 + z^2\right) + \frac{3\zeta}{2}\kappa^2\right\}, \quad \text{for some } M > 0. \tag{3.13}$$

Thus, from inequalities (3.6) and (3.9) and by taking $\nu(t) = \zeta/2$, $\rho_4(t) = (3\zeta/2)\kappa^2$ and $n = 2$, we see that the conditions (i) and (ii) of Lemma 2.4 in [8, 14] are satisfied. As well as we can test that the condition (iii) is satisfied with $q_1 = q_2 = n = 2$ with $\rho_3 = 0$. Then, all conditions of Lemma 2.4 in [8, 14] are achieved.

So, with $\nu(t) = \zeta/2$, $\beta(t) = (3\zeta/2)\kappa^2$, $n = 2$, and $\rho_3 = 0$, we note that

$$\int_{t_0}^t \left\{\rho_3\nu(u) + \rho_4(u)\right\}e^{-\int_u^t \nu(s)\,ds}\, du = (3\zeta/2)\kappa^2\int_{t_0}^t e^{-\frac{\zeta}{2}\int_u^t ds}\, du \le 3\kappa^2,$$

for all $t \ge t_0 \ge 0$. Thus, condition (2.4) [8] holds. Now, since

$$g^T = \big(0 0 - \alpha x\big(t - l(t)\big)\big),$$

$$
\begin{aligned}
U_x &= (U_1)_x + (U_2)_x \\
&= \mu \sum_{i=1}^{n} f_i(x) + \sum_{i=1}^{n} f_i(x) + 2x + z + \frac{a\psi_i'(x,y)}{H(t)} \sum_{i=1}^{n} \int_{i=0}^{x} f_i(\xi)\,d\xi \\
&\quad + \frac{a\psi_i(x,y)}{H(t)} \sum_{i=1}^{n} f_i(\xi) + \sum_{i=1}^{n} \frac{b_i}{H_1}(ab_i - c_i)x + \sum_{i=1}^{n} (ab_i - c_i)\left(\frac{z}{H_1} + a_0 y\right), \\
U_y &= (U_1)_y + (U_2)_y \\
&= \sum_{i=1}^{n} f_i(x) + \mu\psi_i(x,y)y + \sum_{i=1}^{n} Q_i(x,y) + \mu z + a a_0 \psi_1(x,y)y \\
&\quad + a \sum_{i=1}^{n} f_i(x) + a_0 xy \sum_{i=1}^{n} c_i + \sum_{i=1}^{n} ab_i - c_i + \frac{a a_0}{H_1} z, \\
U_z &= (U_1)_z + (U_2)_z = \mu y + \frac{1}{H(t)} z + x + \frac{x}{H_1} \sum_{i=1}^{n} (ab_i - c_i)x + \frac{a}{H_1} z + \frac{a a_0}{H_1} y,
\end{aligned}
$$

we have

$$
\begin{aligned}
\left|U_{x_i}(t,x)G_{ik}(t,x)\right| \le \alpha \Bigg[&\left\{ \frac{H_1(\mu+1) + 3 + a(a_0+1)}{2H_1} \right\} x^2\bigl(t - l(t)\bigr) \\
&+ \left(\frac{H_1 + \sum_{i=1}^{n}(ab_i - c_i)}{2H_1} \right) x^2 + \left(\frac{\mu + a a_0}{2} \right) y^2 \\
&+ \left(\frac{a+1}{2H_1} \right) z^2 \Bigg] := \chi(t).
\end{aligned}
$$

Thus, condition (2.3) in [8, 14] is satisfied. Using Lemma 2.4 in [8, 14], we find that all solutions of (1.1) are USB, and we can also conclude

$$
E^{x_0}\left\|x(t,t_0,x_0)\right\| \le \left\{ \mathcal{C}x_0^2 + 3\kappa^2 \right\}^{\frac{1}{2}}, \quad \text{for all } t \ge t_0 \ge 0.
$$

Hence, the proof of Theorem 3.1 is now complete. □

4 Examples and discussion

Example 4.1 In a particular case $n = 1$, consider the following third-order SDDE

$$
\begin{aligned}
&\left\{ \left(\frac{3}{4} + \frac{1}{4} e^{-4x'} \right) x'' \right\}' + \left\{ \left(19 + \frac{\pi}{2} + \arctan\bigl(xx'\bigr) \right) x' \right\}' + 9x\bigl(t - r(t)\bigr)x'\bigl(t - r(t)\bigr) \\
&\quad + \sin\Bigl(x\bigl(t - r(t)\bigr)x'\bigl(t - r(t)\bigr)\Bigr) + 25x\bigl(t - r(t)\bigr) \\
&\quad + \frac{x(t - r(t))}{1 + x(t - r(t))} + \frac{1}{4}\sin\left(x\left(t - \frac{1}{2}e^{-t} \right) \right) = 0.
\end{aligned}
\tag{4.1}
$$

The equivalent system of (4.1) is

$$
\begin{aligned}
x' &= y, \\
y' &= \frac{z}{\frac{3}{4} + \frac{1}{4} e^{-4y}},
\end{aligned}
$$

$$\begin{aligned}
z' = &-\left(19 + \frac{\pi}{2} + \arctan(xy)\right)\frac{z}{\frac{3}{4}+\frac{1}{4}e^{-4y}} \\
&- y\left(\frac{-y^2}{1+x^2y^2} - \frac{x}{1+x^2y^2}\frac{z}{\frac{3}{4}+\frac{1}{4}e^{-4y}}\right) - \big(9xy + \sin(xy)\big) \\
&+ \int_{t-r(t)}^{t} \big(9y(s) + y(s)\cos\big(x(s)y(s)\big)\big)y(s)\,ds \\
&+ \int_{t-r(t)}^{t} \big(9y(s) + y(s)\cos\big(x(s)y(s)\big)\big)\frac{z(s)}{\frac{3}{4}+\frac{1}{4}e^{-4y(s)}}\,ds - \left\{25x + \frac{x}{1+x^4}\right\} \\
&+ \int_{t-r(t)}^{t} \left\{25x(s) + \frac{1-2x^4(s)}{1+x^4(s)}\right\}ds - \frac{1}{4}\sin\left(x\left(t - \frac{1}{4}e^t\right)\right).
\end{aligned} \tag{4.2}$$

Comparing equation (1.2) with (4.2), we have

$$h(y) = \frac{3}{4} + \frac{1}{4}e^{-4y}, \quad \text{then } \frac{3}{4} \le h(y) \le 1.$$

Therefore, we get

$$H_1 = \frac{3}{4}, \qquad H_2 = 1.$$

The derivative of $h(y)$ is

$$h'(y) = -e^{-4y}.$$

Then, we find

$$\int_{-\infty}^{\infty} \left|-e^{-4v}\right| dv = 2\int_0^{\infty} \left|e^{-4v}\right| dv = \frac{1}{2} = C < \infty. \tag{4.3}$$

We can see that Fig. 1 illustrates the behavior of $h(y)$ in the interval $x \in [0, 50]$.

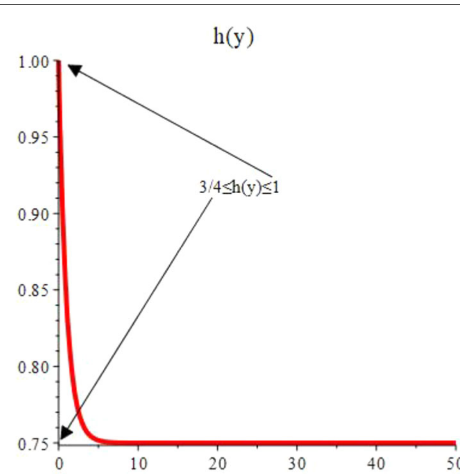

Figure 1 Trajectory of $h(y)$

We also have the function

$$\psi_1(x,y) = 19 + \frac{\pi}{2} - \arctan(xy), \quad \text{so } 19 \le \psi_1(x,y) \le 19 + \frac{\pi}{2},$$

then, we get $a = 19$ and $a_0 = 19 + \frac{\pi}{2}$.

We also obtain

$$\frac{\partial \psi_1(x,y)}{\partial x} = \frac{-y}{1 + x^2y^2}, \quad \text{so } y\frac{\partial \psi_1(x,y)}{\partial x} = \frac{-y^2}{1 + x^2y^2} \le 0,$$

and

$$\frac{\partial \psi_1(x,y)}{\partial y} = \frac{-x}{1 + x^2y^2}.$$

Therefore, we can conclude

$$\int_{-\infty}^{\infty} \left| \frac{-v}{1 + u^2v^2} \right| du + \int_{-\infty}^{\infty} \left| \frac{-u}{1 + u^2v^2} \right| dv = 2\pi = D < \infty. \tag{4.4}$$

Figure 2 shows the behavior of the function $\psi_1(x,y)$ through the interval $x \in [-4,4]$, $y \in [-4,4]$, and also it shows that $y\frac{\partial \psi_1}{\partial x} < 0$, for all x, y.

The function

$$Q(x,y) = 9xy + \sin(xy), \qquad Q(0,y) = Q(x,0) = 0$$

fulfills

$$\frac{Q(x,y)}{y} = 9x + \frac{\sin(xy)}{y} \ge 8 = b.$$

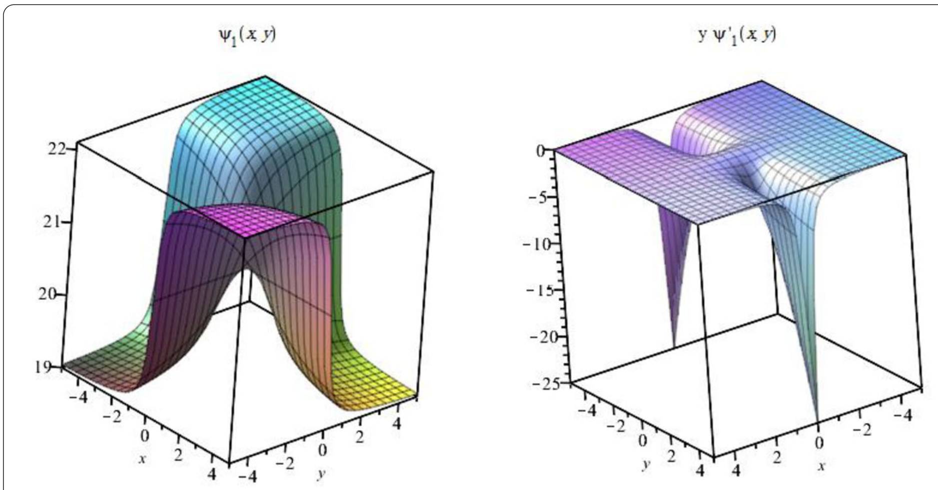

Figure 2 Trajectory of $\psi_1(x,y)$, $y\frac{\partial \psi_1}{\partial x}$

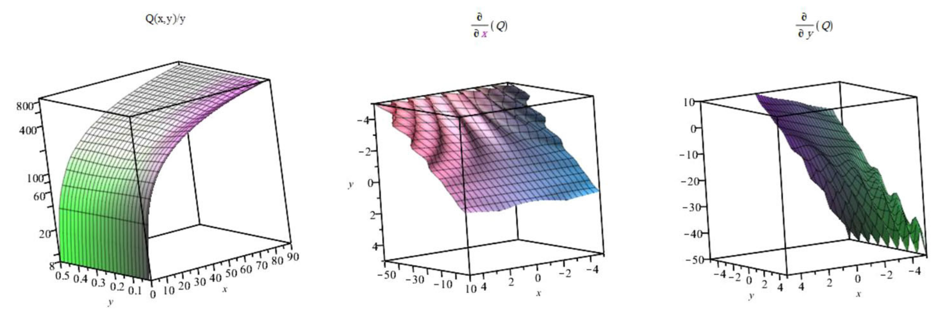

Figure 3 Trajectory of $\frac{\partial Q(x,y)}{\partial y}$, $\frac{\partial Q(x,y)}{\partial x}$ and $\frac{Q(x,y)}{y}$

The derivatives of $Q(x,y)$ are defined as follows

$$\frac{\partial Q(x,y)}{\partial x} = 9y + y\cos(xy), \quad \text{so} \left|\frac{\partial Q(x,y)}{\partial x}\right| \leq 10 = M,$$
$$\frac{\partial Q(x,y)}{\partial y} = 9x + x\cos(xy), \quad \text{so} \left|\frac{\partial Q(x,y)}{\partial y}\right| \leq 10 = N.$$

For the behavior of the functions $\frac{\partial Q(x,y)}{\partial y}$, $\frac{\partial Q(x,y)}{\partial x}$, and $\frac{Q(x,y)}{y}$, see Fig. 3.
Now, the function

$$f(x) = 25x + \frac{x}{1+x^4}, \quad \text{then } \frac{f(x)}{x} = 25 + \frac{1}{1+x^4} \geq 25 = d.$$

It follows that

$$f'(x) = 25 + \frac{1-2x^4}{(1+x^4)^2}, \qquad \left|f'(x)\right| \leq 26 = L.$$

Therefore, we find

$$\sup\left\{f'(x)\right\} = 26 = \frac{c}{2}.$$

Figure 4 gives the path of $\frac{f(x)}{x}, f'(x)$.
Finally, we obtain

$$\alpha x\bigl(t - l(t)\bigr) = \frac{1}{4}\sin\left(x\left(t - \frac{1}{2}e^{-t}\right)\right),$$
$$\text{so } \alpha = \frac{1}{4} \quad \text{and} \quad l(t) = \frac{1}{2}e^{-t}, \quad \text{then } \left|l'(t)\right| = \frac{1}{2}e^{-t} \leq \frac{1}{2}.$$

Figure 5 shows the behavior of the stochastic term $\frac{1}{4}\sin(x(t-\frac{1}{2}e^{t}))$, and it also shows that $|l'(t)| < \frac{1}{2}$ on the interval $[0,30]$.
Now, we have

$$\mu = \frac{ab+c}{4b} = 6.38, \quad \text{then } ab - c = 100 > 0,$$

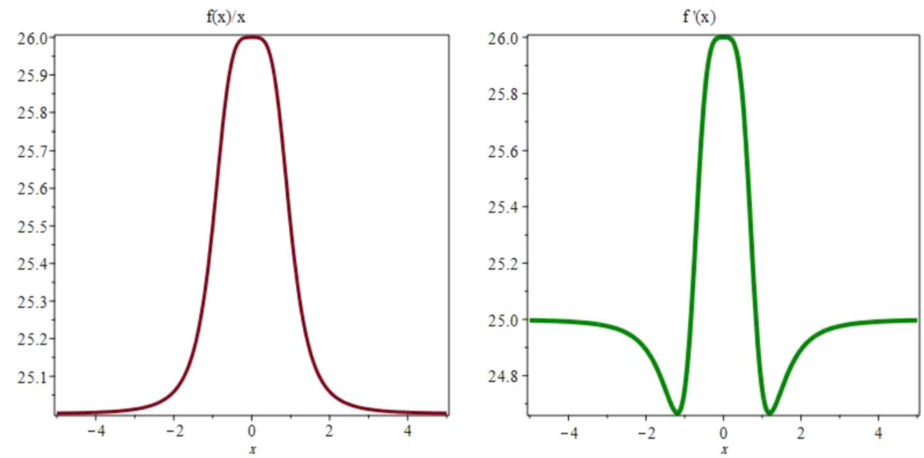

Figure 4 Path of $\frac{f(x)}{x}$, $f'(x)$

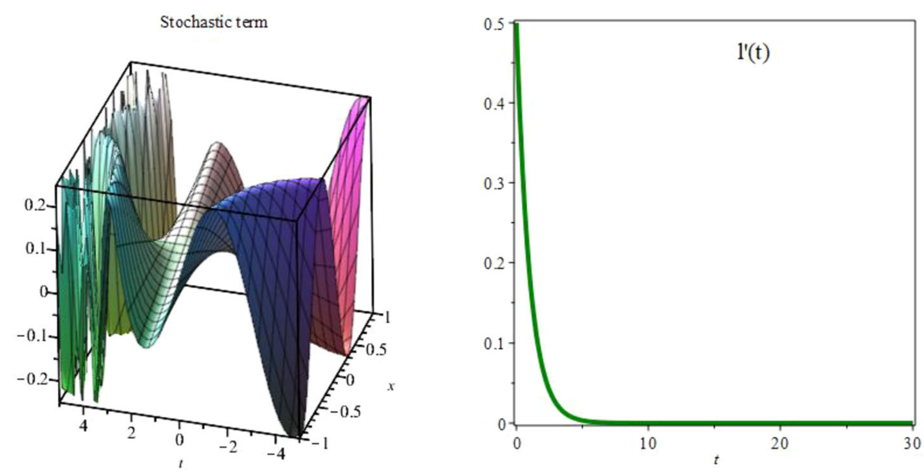

Figure 5 Trajectory of the stochastic term

and

$$2M + 2b + 6 = 42, \quad \text{so } ab - c > 2M + 2b + 6.$$

Since $\alpha^2 = \frac{1}{16}$, we have

$$2\alpha^2 = \frac{1}{8} < 2H_1 d - H_1(a + b + 2) = 15.75,$$

and

$$H_1(a - 1) = 13.5 > 2\mu.$$

Suppose that $\beta = \frac{1}{2}$, then we conclude

$$LU_1 \leq (10.42 - 24.67\gamma)x^2 - (14.5 - 585.89\gamma)y^2 - (0.49 - 125.06\gamma)z^2 + \frac{E_3}{E_1}\eta_1(t)U_1.$$

Therefore, we get

$$\gamma \leq \min(0.42, 0.025, 0.004).$$

Hence all hypotheses of Theorem 2.1 are achieved, then the zero solution of (4.1) is USAS.

Example 4.2 Consider the following SDDE

$$\begin{aligned} &\left\{\left(\frac{3}{4}+\frac{1}{4}e^{-4x'}\right)x''\right\}' + \left\{\left(19+\frac{\pi}{2}+\arctan\left(xx'\right)\right)\right\}' \\ &\quad + 9x\big(t-r(t)\big)x'\big(t-r(t)\big) \\ &\quad + \sin\left(x\big(t-r(t)\big)x'\big(t-r(t)\big)\right) + 25x\big(t-r(t)\big) \\ &\quad + \frac{x(t-r(t))}{1+x(t-r(t))} + \frac{1}{4}\sin\left(x\left(t-\frac{1}{2}e^{t}\right)\right) = \varepsilon\left(x,x',x''\right). \end{aligned} \tag{4.5}$$

Using the estimates in Example 4.1, we get

$$\begin{aligned} &H_1 = \frac{3}{4}, \qquad H_2 = 1, \qquad a = 19, \qquad a_0 = 19+\frac{\pi}{2}, \\ &b = 8, \qquad M = N = 10, \\ &c = 52, \qquad d = 25, \qquad L = 26, \qquad \mu = 6.38 \quad \text{and} \quad \alpha = \frac{1}{4}. \end{aligned}$$

Since

$$\frac{H_1\{(ab-c+1)d-(a+b+2)\}}{a+1} = 97.425,$$

then we get

$$\alpha^2 < \frac{H_1\{(ab-c+1)d-(a+b+2)\}}{a+1}.$$

Let $m = 0.01$, so we obtain

$$\begin{aligned} LU \leq &-(84.83-3313.56\gamma)x^2 - (508.5-27499.62\gamma)y^2 \\ &-(1.49-1395.63\gamma)z^2 + 1.35|x| + 4.48|y| + 0.25|z|, \end{aligned}$$

provided that

$$\gamma < \min(0.025, 0.019, 0.0001).$$

If we take $\zeta = 0.2$ and $m = 0.01$, then we find

$$\kappa = 0.01\max\{134.67, 447.92, 24.57\} \cong 4.48.$$

Now, we can satisfy the condition (ii) of Theorem 2.2 in [28] by taking

$$\nu = 0.1 \quad \text{and} \quad \rho_4(t) = \left(\frac{3\zeta}{2}\right)\kappa^2 = 6.04, \quad \text{with } n = 2.$$

Then, since $q_1 = q_2 = n = 2$, we get all assumptions of Theorem 2.2 [28] are satisfied. It follows from the above estimates, the following inequality holds

$$\int_{t_0}^{t} \{\rho_3 \nu(u) + \rho_4(u)\} e^{\int_{t_0}^{u} \nu(s)\, ds}\, du \le 3\kappa^2 = 60.2, \quad \text{for all } t \ge t_0 \ge 0.$$

And we also get

$$\begin{aligned} \left|U_{x_i}(t,x)G_{ik}(t,x)\right| &\le \frac{1}{4}\left(239.13x^2\big(t - l(t)\big) + 67.17x^2 + +168.77y^2 + 13.33z^2\right) \\ &:= \chi(t). \end{aligned}$$

Hence, Lemma 2.4 in [28] implies that the zero solution of (4.5) is USB.

Now, in view of Figs. 6 and 7, we find that the behavior for the solutions of (4.2) and (4.5) are asymptotically stable, such that the Figs. 6 and 7 illustrate the behavior of the solution, when $\alpha = 0.25$ and $\alpha = 1$, respectively. We note that, when α is increased, the stochasticity

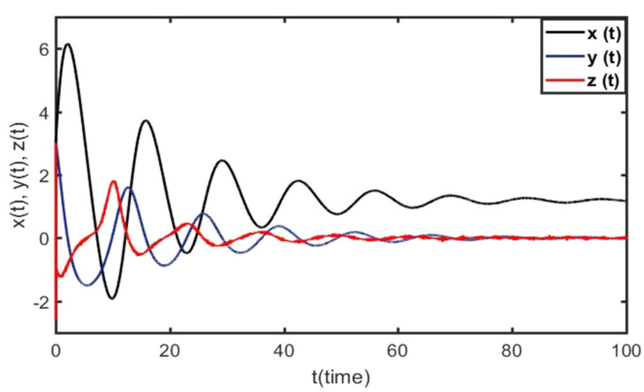

Figure 6 The behavior of the solutions with $\alpha = 0.25$

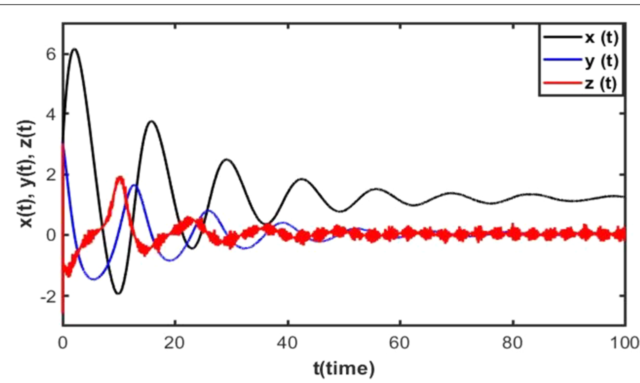

Figure 7 The behavior of the solutions with $\alpha = 1$

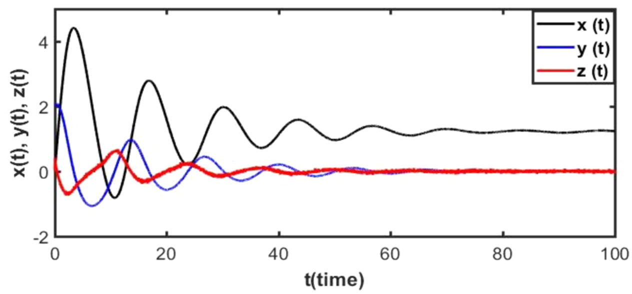

Figure 8 The behavior of the solutions with $\sin(x(t-\frac{1}{2}e^{-t}))=1$ and $\alpha=0.25$

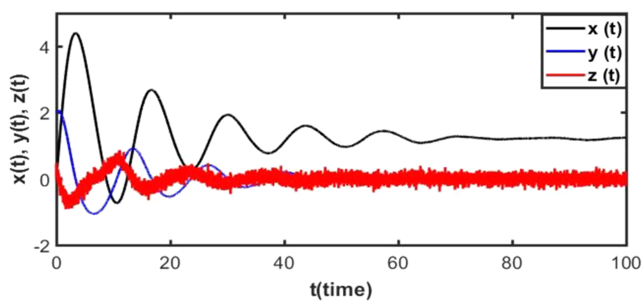

Figure 9 The behavior of the solutions with $\sin(x(t-\frac{1}{2}e^{-t}))=1$ and $\alpha=1$

becomes more pronounced. On the other hand, if we take the function $\sin(x(t-\frac{1}{2}e^{-t}))=1$, then we get Figs. 8 and 9, with $\alpha=0.25$ and $\alpha=1$, respectively.

Author contributions
All authors reviewed the manuscript A. M. Mahmoud D. A. Eisa, R. O. A. Taie and D. A. M. Bakhit1*

Funding
Open access funding provided by The Science, Technology & Innovation Funding Authority (STDF) in cooperation with The Egyptian Knowledge Bank (EKB).

Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable, because this article does not contain any studies with human or animal subjects.

Competing interests
The authors declare no competing interests.

Author details
[1]Department of Mathematics, Faculty of Science, New Valley University, El-Khargah 72511, Egypt. [2]Department of Mathematics, Faculty of Science, Assiut University, Assiut, Egypt.

Received: 27 December 2023 Accepted: 18 March 2024 Published online: 02 April 2024

References

1. Abdel-Razek, M.A., Mahmoud, A.M., Bakhit, D.A.M.: On stability and boundedness of solutions to a certain non-autonomous third-order functional differential equation with multiple deviating arguments. In: Differential Equations and Control Processes, vol. 1, pp. 84–103 (2019)
2. Abou-El-Ela, A.M.A., Sadek, A.I., Mahmoud, A.M.: Stability and boundedness of solutions of certain third-order nonlinear delay differential equation. Int. J. Autom. Control Syst. Eng. **9**(1), 9–15 (2009)

3. Abou-El-Ela, A.M.A., Sadek, A.I., Mahmoud, A.M.: On the stability of solutions for second-order stochastic delay differential equations. In: Differential Equations and Control Processes, vol. 2, pp. 1–13 (2015)
4. Abou-El-Ela, A.M.A., Sadek, A.I., Mahmoud, A.M., Farghaly, E.S.: New stability and boundedness results for solutions of a certain third-order nonlinear stochastic differential equation. Asian J. Math. Comput. Res. **5**(1), 60–70 (2015)
5. Abou-El-Ela, A.M.A., Sadek, A.I., Mahmoud, A.M., Farghaly, E.S.: Stability of solutions for certain third-order nonlinear stochastic delay differential equations. Ann. Appl. Math. **31**(3), 253–261 (2015)
6. Abou-El-Ela, A.M.A., Sadek, A.I., Mahmoud, A.M., Farghaly, E.S.: Asymptotic stability of solutions for a certain non-autonomous second-order stochastic delay differential equation. Turk. J. Math. **41**(2), 576–584 (2017)
7. Abou-El-Ela, A.M.A., Sadek, A.I., Mahmoud, A.M., Taie, R.O.A.: On the stochastic stability and boundedness of solutions for stochastic delay differential equation of the second-order. Chin. J. Math. **2015**, 1–8 (2015)
8. Ademola, A.T.: Stability, boundedness and uniqueness of solutions to certain third-order stochastic delay differential equations. In: Differential Equations and Control Processes, vol. 2, pp. 24–50 (2017)
9. Ademola, A.T., Akindeinde, S.O., Ogundare, B.S., Ogundiran, M.O., Adesina, O.A.: On the stability and boundedness of solutions to certain second-order nonlinear stochastic delay differential equations. J. Niger. Math. Soc. **38**(2), 185–209 (2019)
10. Ademola, A.T., Arawomo, P.O.: Uniform stability and boundedness of solutions of nonlinear delay differential equations of third-order. Math. J. Okayama Univ. **55**, 157–166 (2013)
11. Ademola, A.T., Arawomo, P.O., Ogunlaran, O.M., Oyekan, E.A.: Uniform stability, boundedness and asymptotic behaviour of solutions of some third-order nonlinear delay differential equations. Differ. Equ. Control Process. **4**, 43–66 (2013)
12. Ademola, A.T., Moyo, S., Ogundare, B.S., Ogundiran, M.O., Adesina, O.A.: Stability and boundedness of solutions to a certain second-order nonautonomous stochastic differential equation. Int. J. Anal. **2016**, 1–11 (2016)
13. Ademola, A.T., Ogundare, B.S., Adesina, O.A.: Stability, boundedness, and existence of periodic solutions to certain third-order delay differential equations with multiple deviating arguments. Int. J. Differ. Equ. **2015**, Article ID 213935 (2015)
14. Arnold, L.: Stochastic Differential Equations: Theory and Applications. Wiley, New York (1974)
15. Burton, T.A.: Stabitity and Periodic Solutions of Ordinary and Functional Differential Equations. Academic Press, San Diego (1985)
16. Gikhman, I.I., Skorokhod, A.V.: Stochastic Differential Equations. Springer, Berlin (1972)
17. Hale, J.K.: Theory of Functional Differential Equations. Springer, New York (1977)
18. Hale, J.K., Verduyn Lunel, S.M.: Introduction to Functional Differential Equations. Springer, New York (1993)
19. Iannelli, M.: Mathematics of Biology. Springer, Berlin (2010)
20. Li, Z., Xu, L.: Exponential stability in mean square of stochastic functional differential equations with infinite delay. Acta Appl. Math. **174**(8), 1–18 (2021)
21. Liu, K.: Stability of Infinite Dimensional Stochastic Differential Equation with Applications. Chapman & Hall, Boca Raton (2006)
22. Liu, R., Raffoul, Y.: Boundedness and exponential stability of highly nonlinear stochastic differential equations. Electron. J. Differ. Equ. **2009**, 143 (2009)
23. Luo, J.: Stochastically bounded solutions of a nonlinear stochastic differential equations. J. Comput. Appl. Math. **196**, 87–93 (2006)
24. Mahmoud, A.M., Bakhit, D.A.M.: Study of the stability behaviour and the boundedness of solutions to a certain third-order differential equation with a retarded argument. Ann. Appl. Math. **35**(1), 99–110 (2019)
25. Mahmoud, A.M., Bakhit, D.A.M.: On the properties of solutions for nonautonomous third-order stochastic differential equation with a constant delay. Turk. J. Math. **47**(1), 135–158 (2023)
26. Mahmoud, A.M., Tunç, C.: A new result on the stability of a stochastic differential equation of third-order with a time-lag. J. Egypt. Math. Soc. **26**(1), 202–210 (2018)
27. Mahmoud, A.M., Tunç, C.: Asymptotic stability of solutions for a kind of third-order stochastic differential equations with delays. Miskolc Math. Notes **20**(1), 381–393 (2019)
28. Mahmoud, A.M., Tunç, C.: Boundedness and exponential stability for a certain third-order stochastic delay differential equation. Dyn. Syst. Appl. **29**(2), 288–302 (2020)
29. Mao, X.: Exponential stability for stochastic differential equations with respect to semimartingales. In: Stochastic Processes and Their Applications, vol. 35, pp. 267–277 (1990)
30. Mao, X.: Stability of Stochastic Differential Equations with Respect to Semimartingales. Longman, Harlow (1991)
31. Mao, X.: Exponential Stability of Stochastic Differential Equations. Dekker, New York (1994)
32. Mao, X.: Stochastic Differential Equations and Applications. Horwood, Chichester (2007)
33. Mao, X., Yuan, C., Zou, J.: Stochastic differential delay equations of population dynamics. J. Math. Anal. Appl. **304**(1), 296–320 (2005)
34. Øksendal, B.: Stochastic Differential Equations, an Introduction with Applications. Springer, Berlin (2000)
35. Omeike, M.O.: Uniform ultimate boundedness of solutions of third-order nonlinear delay differential equations. Ann. Alexandru Ioan Cuza Univers. Math. **2**, 363–372 (2010)
36. Omeike, M.O.: Stability and boundedness of solutions of a certain system of third-order nonlinear delay differential equations. Acta Univ. Palacki. Olomuc., Fac. Rerum Nat., Math. **54**(1), 109–119 (2015)
37. Remili, M., Beldjerd, D.: On the asymptotic behavior of the solutions of third-order delay differential equations. Rend. Circ. Mat. Palermo **63**(3), 447–455 (2014)
38. Remili, M., Oudjedi, L.D.: Stability and boundedness of the solutions of non-autonomous third-order differential equations with delay. Acta Univ. Palacki. Olomuc., Fac. Rerum Nat., Math. **53**(2), 139–147 (2014)
39. Remili, M., Oudjedi, L.D.: Uniform ultimate boundedness and asymptotic behaviour of third-order nonlinear delay differential equation. Afr. Math. **27**, 1227–1237 (2016)
40. Remili, M., Oudjedi, L.D.: Boundedness and stability in third-order nonlinear differential equations with multiple deviating arguments. Arch. Math. **52**(2), 79–90 (2016)
41. Remili, M., Oudjedi, L.D., Beldjerd, D.: Stability and ultimate boundedness of solutions of some third-order differential equations with delay. J. Assoc. Arab Univ. Basic Appl. Sci. **23**, 90–95 (2017)

42. Rihan, F.A.: Delay Differential Equations and Applications to Biology. Springer, Berlin (2021)
43. Tunç, C.: On the stability and boundedness of solutions to third-order nonlinear differential equations with retarded argument. Nonlinear Dyn. **57**, 97–106 (2009)
44. Tunç, C.: Bound of solutions to third-order nonlinear differential equations with bounded delay. J. Franklin Inst. **347**(2), 415–425 (2010)
45. Tunç, C.: On some qualitative behaviors of solutions to a kind of third-order nonlinear delay differential equations. Electron. J. Qual. Theory Differ. Equ. **2010**, 12 (2010)
46. Tunç, C.: Stability and boundedness of the nonlinear differential equations with multiple deviating arguments. Afr. Math. **24**, 381–390 (2013)
47. Tunç, C.: Stability and boundedness in differential systems of third-order with variable delay. Proyecciones J. Math. **35**(3), 317–338 (2016)
48. Tunç, C., Gözen, M.: Stability and uniform boundedness in multidelay functional differential equations of third order. Abstr. Appl. Anal. **2013**, Article ID 248717 (2013)
49. Tunç, O., Tunç, C.: On the asymptotic stability of solutions of stochastic differential delay equations of second-order. J. Taibah Univ. Sci. **13**(1), 875–882 (2019)

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.