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On a new version of Hermite–Hadamard-type inequality based on proportional Caputo-hybrid operator

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Abstract

In mathematics and the applied sciences, as a very useful tool, fractional calculus is a basic concept. Furthermore, in many areas of mathematics, it is better to use a new hybrid fractional operator, which combines the proportional and Caputo operators. So we concentrate on the proportional Caputo-hybrid operator because of its numerous applications. In this research, we introduce a novel extension of the Hermite–Hadamard-type inequalities for proportional Caputo-hybrid operator and establish an identity. Then, taking into account this novel generalized identity, we develop some integral inequalities associated with the left-side of Hermite–Hadamard-type inequalities for proportional Caputo-hybrid operator. Moreover, to illustrate the newly established inequalities, we give some examples with the help of graphs.

Mathematics Subject Classification: 26D07; 26D10; 26D15; 26A33

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1 Introduction

Convex analysis is a field of study that has found its application in various domains of optimization theory, energy systems, engineering applications, and physics. This analysis holds a significant position in these areas of mathematics, in particular, in the study of inequalities. The most well-known inequality in convex theory is the Hermite–Hadamard inequality. This inequality provided by Hermite and Hadamard [11, 16] is expressed as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1}$$

where $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers, and $a, b \in I$ with $a < b$. If f is concave, then both inequalities in the statement hold in the reverse direction.

Upper and lower bounds for the average value of a convex function on a compact interval are provided by the Hermite–Hadamard inequality. Numerous disciplines, including integral calculus, probability theory, statistics, optimization, and number theory, use this inequality. It is a valuable tool for tackling physical problems that demand the computation of function averages. With the emergence of new problems, its applications continue to expand, making it as an effective tool for solving a wide range of mathematical problems. Moreover, the Hermite–Hadamard inequality is characterized by the trapezoidal and midpoint inequalities on its right and left parts. Researchers' studies have concentrated on these kinds of inequality. Trapezoid-type inequalities for the case of convex functions were first presented by Dragomir and Agarwal [10], whereas midpoint-type inequalities for the case of convex functions were first established by Kırmacı [22]. Following the establishment of these inequalities, there has been considerable research activity in this field [2, 7, 18].

Fractional calculus has a strong historical basis. The beginnings can be traced back to the correspondence between Leibniz and L'Hôpital. With the aid of this calculus, we can more precisely characterize the behavior of complex systems, particularly those displaying noninteger-order dynamics. It extends the notions of standard calculus due to the presence of fractional orders. In recent times, fractional calculus is a developing branch of mathematics, which plays a significant role in capturing the dynamics of intricate systems across diverse science because of the new fractional integral and derivative such as Caputo–Fabrizio [8], Atangana–Baleanu [4], tempered [27], etc.

A fundamental class of fractional integral operators is the Riemann–Liouville integral operators [21].

Definition 1 For $f \in L_1[a,b]$, the Riemann–Liouville integrals of order $\alpha > 0$ are given by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt, \quad x > a,$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,dt, \quad x < b,$$

where Γ is the gamma function, and $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$. Obviously, the Riemann–Liouville integrals will be equal to classical integrals for $\alpha = 1$.

Sarıkaya et al. [30] and Iqbal et al. [19] obtained many fractional midpoint- and trapezoid-type inequalities for convex functions, respectively. Next, Sarıkaya and Yıldırım [31] derived a distinct expression of the Hermite–Hadamard inequality with the help of fractional integrals in the following form.

Theorem 1 *Let $f \in L_1[a,b]$ be a convex function on $[a,b]$. Then we have the following inequalities for fractional integrals*:

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{(\frac{a+b}{2})+}^{\alpha} f(b) + J_{(\frac{a+b}{2})-}^{\alpha} f(a) \right] \le \frac{f(a)+f(b)}{2}$$

for $\alpha > 0$.

For other results on fractional integral inequalities, see [6, 12, 14, 24, 25, 33] and references therein.

Another important definition in fractional analysis is the following [28]:

Definition 2 Let $\alpha > 0$ and $\alpha \notin \{1, 2, \ldots\}$, $n = [\alpha] + 1$, and let $f \in AC^n[a, b]$ denote the space of functions having n absolutely continuous derivatives. The left-sided and right-sided Caputo fractional derivatives of order α are defined as follows:

$$ {}^{C}D^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\,dt, \quad x > a, $$

and

$$ {}^{C}D^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^b (t-x)^{n-\alpha-1} f^{(n)}(t)\,dt, \quad x < b. $$

If $\alpha = n \in \{1, 2, 3, \ldots\}$ and the usual nth-order derivative $f^{(n)}(x)$ exists, then the Caputo fractional derivative ${}^{C}D^{\alpha}_{a^+}f(x)$ coincides with $f^{(n)}(x)$, whereas ${}^{C}D^{\alpha}_{b^-}f(x)$ coincides with the latter up to a constant multiplier $(-1)^n$. For $n = 1$ and $\alpha = 0$, we have ${}^{C}D^{\alpha}_{a^+}f(x) = {}^{C}D^{\alpha}_{b^-}f(x) = f(x)$.

The Caputo derivative is defined as the application of a fractional integral to the standard derivative of a function, whereas the Riemann–Liouville fractional derivative is obtained by differentiating the fractional integral of a function concerning its independent variable of order n. The Caputo fractional derivative necessitates more suitable initial conditions in contrast to the conventional Riemann–Liouville fractional derivative considering fractional differential equations [9]. Besides, the operator of a proportional derivative denoted as ${}^{P}D_{\alpha}f(t)$ is given by [3]

$$ {}^{P}D_{\alpha}f(t) = K_1(\alpha, t)f(t) + K_0(\alpha, t)f'(t). $$

Under certain assumptions, K_1 and K_0 are functions of $\alpha \in [0, 1]$ and $t \in \mathbb{R}$; also, the function f is differentiable with respect to $t \in \mathbb{R}$. It is related to the wide and expanding area of conformable derivatives. Additionally, in the realm of control theory, the utilization of this operator is common. Thus, studying the Caputo and proportional derivatives has become much more important in recent years [1, 13, 15, 17, 20, 23, 26, 32].

The definition provided by Baleanu et al. [5] combines the notions of proportional and Caputo derivatives in a novel way to create a hybrid fractional operator that can be expressed as a linear combination of Riemann–Liouville and Caputo fractional derivatives.

Definition 3 Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I°, and let f, f' be locally in $L_1(I)$. Then the proportional Caputo-hybrid operator may be defined as follows:

$$ {}^{PC}_{0}D^{\alpha}_{t}f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \bigl[K_1(\alpha, \tau)f(\tau) + K_0(\alpha, \tau)f'(\tau)\bigr](t-\tau)^{-\alpha}\,d\tau, $$

where $\alpha \in [0,1]$, and K_1 and K_0 are functions satisfying the following conditions:

$$\lim_{\alpha\to 0^+} K_0(\alpha,\tau) = 0; \qquad \lim_{\alpha\to 1} K_0(\alpha,\tau) = 1; \quad K_0(\alpha,\tau) \neq 0, \alpha \in (0,1];$$
$$\lim_{\alpha\to 0} K_1(\alpha,\tau) = 0; \qquad \lim_{\alpha\to 1^-} K_1(\alpha,\tau) = 1; \quad K_1(\alpha,\tau) \neq 0, \alpha \in [0,1).$$

Sarıkaya [29] suggested a novel concept based on Definition 3 via different functions K_1 and K_0. They also provided the Hermite–Hadamard inequality using the following:

Definition 4 Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I° such that $f, f' \in L_1(I)$. The left- and right-sided proportional Caputo-hybrid operators of order α are defined, respectively, as follows:

$$ {}^{PC}_{a^+}D^{\alpha}_{b}f(b) = \frac{1}{\Gamma(1-\alpha)} \int_a^b \bigl[K_1(\alpha, b-\tau)f(\tau) + K_0(\alpha, b-\tau)f'(\tau)\bigr](b-\tau)^{-\alpha}\,d\tau $$

and

$$ {}^{PC}_{b^-}D^{\alpha}_{a}f(a) = \frac{1}{\Gamma(1-\alpha)} \int_a^b \bigl[K_1(\alpha, \tau-a)f(\tau) + K_0(\alpha, \tau-a)f'(\tau)\bigr](\tau-a)^{-\alpha}\,d\tau, $$

where $\alpha \in [0,1]$, and $K_0(\alpha,\tau) = (1-\alpha)^2\tau^{1-\alpha}$ and $K_1(\alpha,\tau) = \alpha^2\tau^{\alpha}$.

Theorem 2 *Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I°, the interior of the interval I, where $a, b \in I^\circ$ with $a < b$, and let f and f' be convex functions on I. Then we have the following inequalities*:

$$\begin{aligned}
&\alpha^2(b-a)^{\alpha} f\left(\frac{a+b}{2}\right) + \frac{1}{2}(1-\alpha)(b-a)^{1-\alpha} f'\left(\frac{a+b}{2}\right) \\
&\quad \le \frac{\Gamma(1-\alpha)}{2(b-a)^{1-\alpha}}\bigl[{}^{PC}_{a^+}D^{\alpha}_{b}f(b) + {}^{PC}_{b^-}D^{\alpha}_{a}f(a)\bigr] \\
&\quad \le \alpha^2(b-a)^{\alpha}\left[\frac{f(a)+f(b)}{2}\right] + (1-\alpha)(b-a)^{1-\alpha}\left[\frac{f'(a)+f'(b)}{4}\right].
\end{aligned}$$

This work aims to analyze analogs of the Hermite–Hadamard-type inequalities concerning Riemann integrals for the proportional Caputo-hybrid operator. For this, we first give a Hermite–Hadamard inequality in different presentation using the proportional Caputo-hybrid operator as defined by Sarıkaya [29]. Moreover, we present an identity for the first part of this inequality, called the midpoint side. To establish different midpoint-type inequalities, this identity is required. Furthermore, we provide several examples supported to demonstrate the established inequalities by graphical representations. Under appropriate assumptions on α, these conclusions expand and generalize the inequalities discovered in earlier works.

2 Main results

In this section, firstly, differently from the literature, we obtain a Hermite–Hadamard inequality for the proportional Caputo-hybrid operator as follows.

Theorem 3 *Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I^o, where $a, b \in I^o$ satisfy $a < b$, and let f and f' be convex functions on I. Then we have the following inequalities*:

$$\begin{aligned}
&\alpha^2(b-a)^\alpha 2^{-\alpha} f\left(\frac{a+b}{2}\right) + (1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2} f'\left(\frac{a+b}{2}\right) \\
&\quad \le \frac{\Gamma(1-\alpha)}{2^\alpha (b-a)^{-\alpha+1}}\left[{}^{PC}_{(\frac{a+b}{2})^+}D^\alpha_b f(b) + {}^{PC}_{(\frac{a+b}{2})^-}D^\alpha_a f(a)\right] \\
&\quad \le \alpha^2(b-a)^\alpha 2^{-\alpha}\left(\frac{f(a)+f(b)}{2}\right) + (1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}\left(\frac{f'(a)+f'(b)}{2}\right).
\end{aligned} \tag{2}$$

Proof Since f and f' are convex functions on $[a, b]$, we have

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2}\left[f\left(\frac{t}{2}a + \frac{2-t}{2}b\right) + f\left(\frac{2-t}{2}a + \frac{t}{2}b\right)\right]$$

and

$$f'\left(\frac{a+b}{2}\right) \le \frac{1}{2}\left[f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) + f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right)\right].$$

Then, multiplying these expressions by $\alpha^2(b-a)^\alpha 2^{-\alpha}$ and $(1-\alpha)^2(b-a)^{1-\alpha}2^{\alpha-1}t^{1-2\alpha}$, respectively, we obtain

$$\begin{aligned}
&\alpha^2(b-a)^\alpha 2^{-\alpha} f\left(\frac{a+b}{2}\right) \\
&\quad \le \frac{1}{2}\left[\alpha^2(b-a)^\alpha 2^{-\alpha} f\left(\frac{t}{2}a + \frac{2-t}{2}b\right) + \alpha^2(b-a)^\alpha 2^{-\alpha} f\left(\frac{2-t}{2}a + \frac{t}{2}b\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
&(1-\alpha)^2(b-a)^{1-\alpha}2^{\alpha-1}t^{1-2\alpha} f'\left(\frac{a+b}{2}\right) \\
&\quad \le \frac{1}{2}\left[(1-\alpha)^2(b-a)^{1-\alpha}2^{\alpha-1}t^{1-2\alpha} f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\right. \\
&\qquad \left. + (1-\alpha)^2(b-a)^{1-\alpha}2^{\alpha-1}t^{1-2\alpha} f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right)\right].
\end{aligned}$$

Placing these two statements side by side and summing them up yield

$$\begin{aligned}
&\alpha^2(b-a)^\alpha 2^{-\alpha} f\left(\frac{a+b}{2}\right) + (1-\alpha)^2(b-a)^{1-\alpha}2^{\alpha-1}t^{1-2\alpha} f'\left(\frac{a+b}{2}\right) \\
&\quad \le \frac{1}{2}\left[\alpha^2(b-a)^\alpha 2^{-\alpha} t^\alpha f\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\right. \\
&\qquad \left. + (1-\alpha)^2(b-a)^{1-\alpha}2^{\alpha-1}t^{1-\alpha} f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\right]t^{-\alpha} \\
&\qquad + \frac{1}{2}\left[\alpha^2(b-a)^\alpha 2^{-\alpha} t^\alpha f\left(\frac{2-t}{2}a + \frac{t}{2}b\right)\right. \\
&\qquad \left. + (1-\alpha)^2(b-a)^{1-\alpha}2^{\alpha-1}t^{1-\alpha} f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right)\right]t^{-\alpha}.
\end{aligned}$$

Integrating both sides of the inequality with respect to $t \in [0,1]$, we deduce that

$$\begin{aligned}
&\alpha^2(b-a)^\alpha 2^{-\alpha} f\left(\frac{a+b}{2}\right) + \frac{1}{2}(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-1}f'\left(\frac{a+b}{2}\right) \\
&\quad \le \frac{1}{2}\int_0^1 \left[\alpha^2(b-a)^\alpha 2^{-\alpha} t^\alpha f\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\right. \\
&\qquad \left. + (1-\alpha)^2(b-a)^{1-\alpha}2^{\alpha-1}t^{1-\alpha}f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right)\right] t^{-\alpha}\,dt \\
&\qquad + \frac{1}{2}\int_0^1 \left[\alpha^2(b-a)^\alpha 2^{-\alpha} t^\alpha f\left(\frac{2-t}{2}a + \frac{t}{2}b\right)\right. \\
&\qquad \left. + (1-\alpha)^2(b-a)^{1-\alpha}2^{\alpha-1}t^{1-\alpha}f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right)\right] t^{-\alpha}\,dt.
\end{aligned}$$

Using a variable substitution, we arrive at

$$\begin{aligned}
&\alpha^2(b-a)^\alpha 2^{-\alpha} f\left(\frac{a+b}{2}\right) + \frac{1}{2}(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-1}f'\left(\frac{a+b}{2}\right) \\
&\quad \le \frac{2^{-\alpha}}{(b-a)^{-\alpha+1}} \int_{\frac{a+b}{2}}^{b} \left[\alpha^2(b-\tau)^\alpha f(\tau) + (1-\alpha)^2(b-\tau)^{1-\alpha}f'(\tau)\right](b-\tau)^{-\alpha}\,d\tau \\
&\qquad + \frac{2^{-\alpha}}{(b-a)^{-\alpha+1}} \int_{a}^{\frac{a+b}{2}} \left[\alpha^2(\tau-a)^\alpha f(\tau) + (1-\alpha)^2(\tau-a)^{1-\alpha}f'(\tau)\right](\tau-a)^{-\alpha}\,d\tau \\
&\quad = \frac{2^{-\alpha}}{(b-a)^{-\alpha+1}} \int_{\frac{a+b}{2}}^{b} \left[K_1(\alpha,b-\tau)f(\tau) + K_0(\alpha,b-\tau)f'(\tau)\right](b-\tau)^{-\alpha}\,d\tau \\
&\qquad + \frac{2^{-\alpha}}{(b-a)^{-\alpha+1}} \int_{a}^{\frac{a+b}{2}} \left[K_1(\alpha,\tau-a)f(\tau) + K_0(\alpha,\tau-a)f'(\tau)\right](\tau-a)^{-\alpha}\,d\tau \\
&\quad = \frac{\Gamma(1-\alpha)}{2^\alpha(b-a)^{-\alpha+1}}\left[{}^{PC}_{(\frac{a+b}{2})^+}D^\alpha_b f(b) + {}^{PC}_{(\frac{a+b}{2})^-}D^\alpha_a f(a)\right].
\end{aligned}$$

Therefore the left side of inequality (2) is demonstrated. To verify the second side of (2), by the convexity of f and f' on $[a,b]$, we have

$$f\left(\frac{t}{2}a + \frac{2-t}{2}b\right) + f\left(\frac{2-t}{2}a + \frac{t}{2}b\right) \le f(a) + f(b)$$

and

$$f'\left(\frac{t}{2}a + \frac{2-t}{2}b\right) + f'\left(\frac{2-t}{2}a + \frac{t}{2}b\right) \le f'(a) + f'(b).$$

Multiplying these two expressions by $\alpha^2(b-a)^\alpha 2^{-\alpha}$ and $(1-\alpha)^2(b-a)^{1-\alpha}2^{\alpha-1}t^{1-2\alpha}$, we obtain

$$\begin{aligned}
&\alpha^2(b-a)^\alpha 2^{-\alpha} f\left(\frac{t}{2}a + \frac{2-t}{2}b\right) + \alpha^2(b-a)^\alpha 2^{-\alpha} f\left(\frac{2-t}{2}a + \frac{t}{2}b\right) \\
&\quad \le \alpha^2(b-a)^\alpha 2^{-\alpha}\left[f(a) + f(b)\right]
\end{aligned}$$

and

$$\begin{aligned}
&(1-\alpha)^2(b-a)^{1-\alpha}2^{\alpha-1}t^{1-2\alpha}f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\\
&\quad+(1-\alpha)^2(b-a)^{1-\alpha}2^{\alpha-1}t^{1-2\alpha}f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\\
&\leq(1-\alpha)^2(b-a)^{1-\alpha}2^{\alpha-1}t^{1-2\alpha}\bigl[f'(a)+f'(b)\bigr].
\end{aligned}$$

Adding these two inequalities, we get

$$\begin{aligned}
&\frac{1}{2}\left[\alpha^2(b-a)^{\alpha}2^{-\alpha}t^{\alpha}f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right.\\
&\quad\left.+(1-\alpha)^2(b-a)^{1-\alpha}2^{\alpha-1}t^{1-\alpha}f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]t^{-\alpha}\\
&\quad+\frac{1}{2}\left[\alpha^2(b-a)^{\alpha}2^{-\alpha}t^{\alpha}f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right.\\
&\quad\left.+(1-\alpha)^2(b-a)^{1-\alpha}2^{\alpha-1}t^{1-\alpha}f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right]t^{-\alpha}\\
&\leq\alpha^2(b-a)^{\alpha}2^{-\alpha}\left[\frac{f(a)+f(b)}{2}\right]+(1-\alpha)^2(b-a)^{1-\alpha}2^{\alpha-1}t^{1-2\alpha}\left[\frac{f'(a)+f'(b)}{2}\right].
\end{aligned}$$

Integrating both sides of the inequality over $t\in[0,1]$, we get

$$\begin{aligned}
&\frac{1}{2}\int_0^1\left[\alpha^2(b-a)^{\alpha}2^{-\alpha}t^{\alpha}f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right.\\
&\quad\left.+(1-\alpha)^2(b-a)^{1-\alpha}2^{\alpha-1}t^{1-\alpha}f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right]t^{-\alpha}\,dt\\
&\quad+\frac{1}{2}\int_0^1\left[\alpha^2(b-a)^{\alpha}2^{-\alpha}t^{\alpha}f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right.\\
&\quad\left.+(1-\alpha)^2(b-a)^{1-\alpha}2^{\alpha-1}t^{1-\alpha}f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right]t^{-\alpha}\,dt\\
&\leq\alpha^2(b-a)^{\alpha}2^{-\alpha}\left[\frac{f(a)+f(b)}{2}\right]+(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}\left[\frac{f'(a)+f'(b)}{2}\right].
\end{aligned}$$

Now a change of variable leads us to

$$\begin{aligned}
&\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{(\frac{a+b}{2})^+}D^{\alpha}_{b}f(b)+{}^{PC}_{(\frac{a+b}{2})^-}D^{\alpha}_{a}f(a)\right]\\
&\quad\leq\alpha^2(b-a)^{\alpha}2^{-\alpha}\left[\frac{f(a)+f(b)}{2}\right]+(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}\left[\frac{f'(a)+f'(b)}{2}\right].
\end{aligned}$$

Consequently, we achieve the required second side of inequality (2). □

Remark 1 In the particular case where α tends to 1 in Theorem 3, we get inequality (1). Therefore we see that inequality (2) is an extension of classical Hermite–Hadamard inequality.

Now we present an example illustrating our theorem.

Example 1 Let us consider the function $f:[1,2]\to\mathbb{R}$ given by $f(x)=x^3$. Then the left- and right-hand sides of (2) are

$$\alpha^2 2^{-\alpha-3}27+(1-\alpha)2^{\alpha-4}27:=\Psi_1 \quad \text{and} \quad \alpha^2 2^{-\alpha-1}9+(1-\alpha)2^{\alpha-3}15:=\Psi_2.$$

Furthermore, from the equalities

$$\begin{aligned}
&\int_1^{\frac{3}{2}}\left[\alpha^2(\tau-1)^\alpha\tau^3+(1-\alpha)^2(\tau-1)^{1-\alpha}3\tau^2\right](\tau-1)^{-\alpha}\,d\tau\\
&\quad=\alpha^2\int_1^{\frac{3}{2}}\tau^3\,d\tau+3(1-\alpha)^2\int_1^{\frac{3}{2}}(\tau-1)^{1-2\alpha}\tau^2\,d\tau\\
&\quad=\frac{65}{64}\alpha^2+\left(1-\alpha^2\right)\left(3\frac{2^{2\alpha-5}}{2-\alpha}+6\frac{2^{2\alpha-3}}{3-2\alpha}+3\frac{2^{2\alpha-3}}{1-\alpha}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{3}{2}}^{2}\left[\alpha^2(2-\tau)^\alpha\tau^3+(1-\alpha)^2(2-\tau)^{1-\alpha}3\tau^2\right](2-\tau)^{-\alpha}\,d\tau\\
&\quad=\alpha^2\int_{\frac{3}{2}}^{2}\tau^3\,d\tau+3(1-\alpha)^2\int_{\frac{3}{2}}^{2}(2-\tau)^{1-2\alpha}\tau^2\,d\tau\\
&\quad=\frac{175}{64}\alpha^2+\left(1-\alpha^2\right)\left(3\frac{2^{2\alpha-5}}{2-\alpha}-12\frac{2^{2\alpha-3}}{3-2\alpha}+12\frac{2^{2\alpha-3}}{1-\alpha}\right)
\end{aligned}$$

it follows that

$$\begin{aligned}
&\frac{\Gamma(1-\alpha)}{2^\alpha(b-a)^{-\alpha+1}}\left[{}^{PC}_{(\frac{a+b}{2})^+}D^\alpha_b f(b)+{}^{PC}_{(\frac{a+b}{2})^-}D^\alpha_a f(a)\right]\\
&\quad=\frac{15}{4}\frac{\alpha^2}{2^\alpha}+(1-\alpha)^2\left(6\frac{2^{\alpha-5}}{2-\alpha}-6\frac{2^{\alpha-3}}{3-2\alpha}+15\frac{2^{\alpha-3}}{1-\alpha}\right):=\Psi_3.
\end{aligned}$$

Thus, in view of (2), we obtain the inequality

$$\begin{aligned}
\alpha^2 2^{-\alpha-1}9+(1-\alpha)2^{\alpha-3}15 &\le \frac{15}{4}\frac{\alpha^2}{2^\alpha}+(1-\alpha)^2\left(6\frac{2^{\alpha-5}}{2-\alpha}-6\frac{2^{\alpha-3}}{3-2\alpha}+15\frac{2^{\alpha-3}}{1-\alpha}\right)\\
&\le \alpha^2 2^{-\alpha-3}27+(1-\alpha)2^{\alpha-4}27.
\end{aligned}\tag{3}$$

Inequality (3) is illustrated in Fig. 1.

The following lemma is essential for demonstrating other our main results.

Lemma 1 *Let $f:I\subset\mathbb{R}^+\to\mathbb{R}$ be a twice differentiable function on I^o, where $a,b\in I^o$ satisfy $a<b$, and let $f,f',f''\in L_1[a,b]$. Then we have the following identity*:

$$\alpha^2(b-a)^{\alpha+1}2^{-\alpha-2}\int_0^1 t\left[f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)-f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right]dt$$

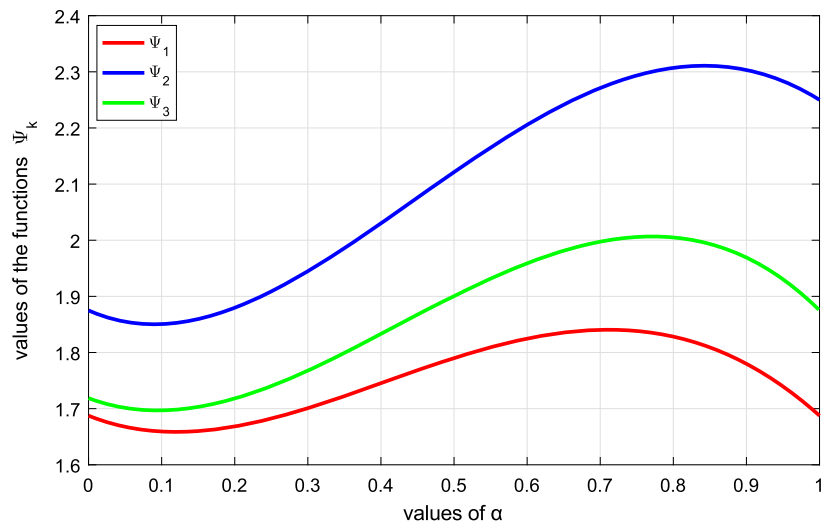

Figure 1 The graph of three parts of inequality (3) in Example 1, which is computed and drawn by MATLAB program, depending on $\alpha \in (0,1)$

$$
\begin{aligned}
&+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-4}\int_0^1 t^{2-2\alpha}\left[f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)-f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right]dt \\
&=\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{(\frac{a+b}{2})^+}D^{\alpha}_{b}f(b)+{}^{PC}_{(\frac{a+b}{2})^-}D^{\alpha}_{a}f(a)\right] \\
&\quad-\left(\alpha^2(b-a)^{\alpha}2^{-\alpha}f\left(\frac{a+b}{2}\right)+(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}f'\left(\frac{a+b}{2}\right)\right).
\end{aligned}
$$

Proof Integrating by parts, we get

$$
\int_0^1 tf'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt=\frac{2}{b-a}f\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_0^1 f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt
$$

and

$$
\begin{aligned}
&\int_0^1 t^{2-2\alpha}f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt \\
&\quad=\frac{2}{b-a}f'\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_0^1 t^{1-2\alpha}f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt.
\end{aligned}
$$

Using a variable change, multiplying the results by $\alpha^2(b-a)^{\alpha+1}2^{-\alpha-1}$ and $(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}$, and summing side by side, we arrive at the following result:

$$
\begin{aligned}
&\alpha^2(b-a)^{\alpha+1}2^{-\alpha-1}\int_0^1 tf'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt \\
&\quad+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}\int_0^1 t^{2-2\alpha}f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt \\
&=\alpha^2(b-a)^{\alpha}2^{-\alpha}f\left(\frac{a+b}{2}\right)+(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}f'\left(\frac{a+b}{2}\right) \\
&\quad-\frac{2^{1-\alpha}}{(b-a)^{1-\alpha}}\int_a^{\frac{a+b}{2}}\left[\alpha^2(\tau-a)^{\alpha}f(\tau)+(1-\alpha)^2(\tau-a)^{1-\alpha}f'(\tau)\right](\tau-a)^{-\alpha}\,d\tau.
\end{aligned}
\tag{4}
$$

Following similar steps, we obtain

$$\begin{aligned}
&\alpha^2(b-a)^{\alpha+1}2^{-\alpha-1}\int_0^1 tf'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt \\
&\quad+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}\int_0^1 t^{2-2\alpha}f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt \\
&=-\alpha^2(b-a)^{\alpha}2^{-\alpha}f\left(\frac{a+b}{2}\right)-(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}f'\left(\frac{a+b}{2}\right) \\
&\quad+\frac{2^{1-\alpha}}{(b-a)^{1-\alpha}}\int_{\frac{a+b}{2}}^{b}\left[\alpha^2(b-\tau)^{\alpha}f(\tau)+(1-\alpha)^2(b-\tau)^{1-\alpha}f'(\tau)\right](b-\tau)^{-\alpha}\,d\tau.
\end{aligned} \tag{5}$$

Subtracting (4) from (5), we have

$$\begin{aligned}
&\alpha^2(b-a)^{\alpha+1}2^{-\alpha-2}\int_0^1 t\left[f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)-f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right]dt \\
&\quad+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-4}\int_0^1 t^{2-2\alpha}\left[f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)-f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right]dt \\
&=\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{(\frac{a+b}{2})^+}D^{\alpha}_{b}f(b)+{}^{PC}_{(\frac{a+b}{2})^-}D^{\alpha}_{a}f(a)\right] \\
&\quad-\left(\alpha^2(b-a)^{\alpha}2^{-\alpha}f\left(\frac{a+b}{2}\right)+(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}f'\left(\frac{a+b}{2}\right)\right),
\end{aligned}$$

which completes the proof. □

Theorem 4 *Let $f: I\subset\mathbb{R}^+\to\mathbb{R}$ be a twice differentiable function on I^o, where $a,b\in I^o$ satisfy $a<b$, and let $f,f',f''\in L_1[a,b]$. If $|f'|^q$ and $|f''|^q$ are convex on $[a,b]$ for $q\geq 1$, then we have the following inequality*:

$$\begin{aligned}
&\left|\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{(\frac{a+b}{2})^+}D^{\alpha}_{b}f(b)+{}^{PC}_{(\frac{a+b}{2})^-}D^{\alpha}_{a}f(a)\right]\right. \\
&\quad\left.-\left(\alpha^2(b-a)^{\alpha}2^{-\alpha}f\left(\frac{a+b}{2}\right)+(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}f'\left(\frac{a+b}{2}\right)\right)\right| \\
&\leq\alpha^2(b-a)^{\alpha+1}2^{-\alpha-3}\left[\left(\frac{1}{3}\left|f'(a)\right|^q+\frac{2}{3}\left|f'(b)\right|^q\right)^{\frac{1}{q}}+\left(\frac{2}{3}\left|f'(a)\right|^q+\frac{1}{3}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right] \\
&\quad+(1-\alpha)(b-a)^{2-\alpha}2^{\alpha-4}\frac{1}{(3-2\alpha)}\frac{1}{(8-2\alpha)^{\frac{1}{q}}}\left[\left((3-2\alpha)\left|f''(a)\right|^q+5\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(5\left|f''(a)\right|^q+(3-2\alpha)\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{6}$$

Proof Firstly, let $q=1$. By the convexity of $|f'|$ and $|f''|$, from Lemma 1 it follows that

$$\begin{aligned}
&\left|\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{(\frac{a+b}{2})^+}D^{\alpha}_{b}f(b)+{}^{PC}_{(\frac{a+b}{2})^-}D^{\alpha}_{a}f(a)\right]\right. \\
&\quad\left.-\left(\alpha^2(b-a)^{\alpha}2^{-\alpha}f\left(\frac{a+b}{2}\right)+(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}f'\left(\frac{a+b}{2}\right)\right)\right| \\
&\leq\alpha^2(b-a)^{\alpha+1}2^{-\alpha-2}\int_0^1 t\left[\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|+\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|\right]dt
\end{aligned}$$

$$
\begin{aligned}
&+ (1-\alpha)(b-a)^{2-\alpha}2^{\alpha-4}\int_0^1 t^{2-2\alpha}\left[\left|f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|+\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|\right]dt \\
&\le \alpha^2(b-a)^{\alpha+1}2^{-\alpha-2}\int_0^1 \frac{t^2}{2}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)+\frac{2t-t^2}{2}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)dt \\
&\quad + (1-\alpha)(b-a)^{2-\alpha}2^{\alpha-4}\int_0^1 \frac{t^{3-2\alpha}}{2}\left(\left|f''(a)\right|+\left|f''(b)\right|\right) \\
&\quad + \frac{2t^{2-2\alpha}-t^{3-2\alpha}}{2}\left(\left|f''(a)\right|+\left|f''(b)\right|\right)dt \\
&= \alpha^2(b-a)^{\alpha+1}2^{-\alpha-2}\left(\frac{|f'(a)|+|f'(b)|}{2}\right) \\
&\quad + (1-\alpha)(b-a)^{2-\alpha}2^{\alpha-3}\frac{1}{(3-2\alpha)}\left(\frac{|f''(a)|+|f''(b)|}{2}\right).
\end{aligned}
$$

Secondly, consider $q > 1$. In view of Lemma 1, using the power mean inequality and the convexity of $|f'|^q$ and $|f''|^q$, we obtain

$$
\begin{aligned}
&\left|\frac{\Gamma(1-\alpha)}{2^\alpha(b-a)^{-\alpha+1}}\left[{}^{PC}_{(\frac{a+b}{2})^+}D^\alpha_b f(b)+{}^{PC}_{(\frac{a+b}{2})^-}D^\alpha_a f(a)\right]\right. \\
&\quad \left. -\left(\alpha^2(b-a)^\alpha 2^{-\alpha}f\left(\frac{a+b}{2}\right)+(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}f'\left(\frac{a+b}{2}\right)\right)\right| \\
&\le \alpha^2(b-a)^{\alpha+1}2^{-\alpha-2}\left[\left(\int_0^1 t\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\int_0^1 t\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&\quad + (1-\alpha)(b-a)^{2-\alpha}2^{\alpha-4}\left[\left(\int_0^1 t^{2-2\alpha}\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t^{2-2\alpha}\left|f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\int_0^1 t^{2-2\alpha}\,dt\right)^{\frac{1}{p}}\left(\int_0^1 t^{2-2\alpha}\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&\le \alpha^2(b-a)^{\alpha+1}2^{-\alpha-2}\frac{1}{2^{\frac{1}{p}}}\left[\left(\int_0^1 \frac{t^2}{2}\left|f'(a)\right|^q+\frac{2t-t^2}{2}\left|f'(b)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\int_0^1 \frac{2t-t^2}{2}\left|f'(a)\right|^q+\frac{t^2}{2}\left|f'(b)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&\quad + (1-\alpha)(b-a)^{2-\alpha}2^{\alpha-4}\frac{1}{(3-2\alpha)^{\frac{1}{p}}}\left[\left(\int_0^1 \frac{t^{3-2\alpha}}{2}\left|f''(a)\right|^q\right.\right. \\
&\quad \left.+\frac{2t^{2-2\alpha}-t^{3-2\alpha}}{2}\left|f''(b)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad \left.+\left(\int_0^1 \frac{2t^{2-2\alpha}-t^{3-2\alpha}}{2}\left|f''(a)\right|^q+\frac{t^{3-2\alpha}}{2}\left|f''(b)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&= \alpha^2(b-a)^{\alpha+1}2^{-\alpha-2}\frac{1}{2^{\frac{1}{p}}}\left[\left(\frac{1}{6}\left|f'(a)\right|^q+\frac{1}{3}\left|f'(b)\right|^q\right)^{\frac{1}{q}}+\left(\frac{1}{3}\left|f'(a)\right|^q+\frac{1}{6}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

$$+ (1-\alpha)(b-a)^{2-\alpha}2^{\alpha-4}\frac{1}{(3-2\alpha)^{\frac{1}{p}}}\left[\left(\frac{|f''(a)|^q}{8-2\alpha}+\frac{5|f''(b)|^q}{(8-2\alpha)(3-2\alpha)}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{5|f''(a)|^q}{(8-2\alpha)(3-2\alpha)}+\frac{|f''(b)|^q}{(8-2\alpha)}\right)^{\frac{1}{q}}\right].$$

This ends the proof. □

Now we illustrate our theorem by an example.

Example 2 Let the function $f:[1,3]\to\mathbb{R}$ be defined by $f(x)=x^3+x^2$. Then the right-hand side of inequality (6) is

$$\frac{\alpha^2}{4}\left[\left(\frac{1}{3}5^q+\frac{2}{3}33^q\right)^{\frac{1}{q}}+\left(\frac{2}{3}5^q+\frac{1}{3}33^q\right)^{\frac{1}{q}}\right]$$
$$+\frac{(1-\alpha)}{4}\frac{1}{3-2\alpha}\frac{1}{(8-2\alpha)^{\frac{1}{q}}}\left[\left((3-2\alpha)8^q+5.20^q\right)^{\frac{1}{q}}+\left(5.8^q+(3-2\alpha)20^q\right)^{\frac{1}{q}}\right]:=\Omega_1.$$

On the other hand, we obtain

$$\left|\frac{\Gamma(1-\alpha)}{2^\alpha(b-a)^{-\alpha+1}}\left[{}^{PC}_{(\frac{a+b}{2})^+}D^\alpha_b f(b)+{}^{PC}_{(\frac{a+b}{2})^-}D^\alpha_a f(a)\right]\right.$$
$$\left.-\left(\alpha^2(b-a)^\alpha 2^{-\alpha}f\left(\frac{a+b}{2}\right)+(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}f'\left(\frac{a+b}{2}\right)\right)\right|$$
$$=\alpha^2\frac{7}{3}+\frac{(1-\alpha)^2}{2}\left(\frac{22}{3-2\alpha}-\frac{11}{2-2\alpha}\right)-8(1-\alpha):=\Omega_2.$$

As we can see in Fig. 2, the left-hand side of inequality (6) is always below the right-hand side of this inequality for all values of $\alpha\in(0,1)$ and $q\geq 1$.

Remark 2 Taking the limit as $\alpha\to 1$ and putting $q=1$ in Theorem 4, it follows that

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{(b-a)}{8}\left(|f'(a)|+|f'(b)|\right),$$

as it was proved by Kırmacı [22].

Corollary 1 *In the limiting case $\alpha=0$ in Theorem 4, we obtain*

$$\left|\frac{1}{b-a}\left(\int_{\frac{a+b}{2}}^b f(x)\,dx-\int_a^{\frac{a+b}{2}}f(x)\,dx\right)-\left(\frac{b-a}{4}\right)f'\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)^2}{48}\left[\left(\frac{3}{8}|f''(a)|^q+\frac{5}{8}|f''(b)|^q\right)^{1/q}+\left(\frac{5}{8}|f''(a)|^q+\frac{3}{8}|f''(b)|^q\right)^{1/q}\right].$$

Moreover, taking $\alpha=\frac{1}{2}$, we have

$$\left|\frac{1}{b-a}\left\{\int_a^b f(x)\,dx+f(b)-f(a)\right\}-\left[f\left(\frac{a+b}{2}\right)+\frac{1}{2}f'\left(\frac{a+b}{2}\right)\right]\right|$$

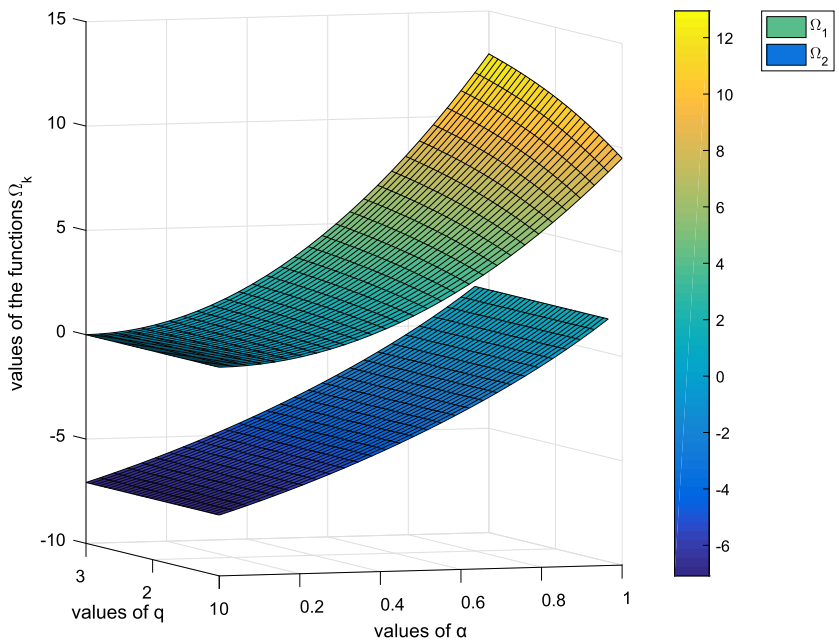

Figure 2 The graph of both sides of inequality (6) according to Example 2, which is computed and drawn by MATLAB program, depending on $\alpha \in (0,1)$ and $q \in [1,3]$

$$\leq \frac{b-a}{8}\left\{\left[\frac{1}{3}\left|f'(a)\right|^q + \frac{2}{3}\left|f'(b)\right|\right]^{1/q} + \left[\frac{2}{3}\left|f'(a)\right|^q + \frac{1}{3}\left|f'(b)\right|^q\right]^{1/q} + \left[\frac{2}{7}\left|f''(a)\right|^q + \frac{5}{7}\left|f'(b)\right|^q\right]^{1/q} + \left[\frac{5}{7}\left|f'(a)\right|^q + \frac{2}{7}\left|f'(b)\right|^q\right]^{1/q}\right\}.$$

Theorem 5 *Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a twice differentiable function on I^o, where $a, b \in I^o$ satisfy $a < b$, and let $f, f', f'' \in L_1[a,b]$. If $|f'|^q$ and $|f''|^q$ are convex on $[a,b]$ for $q > 1$, then we have the following inequality*:

$$\begin{aligned}
&\left|\frac{\Gamma(1-\alpha)}{2^\alpha (b-a)^{-\alpha+1}}\left[{}^{PC}_{(\frac{a+b}{2})^+}D^\alpha_b f(b) + {}^{PC}_{(\frac{a+b}{2})^-}D^\alpha_a f(a)\right]\right. \\
&\quad \left. - \left(\alpha^2 (b-a)^\alpha 2^{-\alpha} f\left(\frac{a+b}{2}\right) + (1-\alpha)(b-a)^{1-\alpha} 2^{\alpha-2} f'\left(\frac{a+b}{2}\right)\right)\right| \\
&\leq \alpha^2 (b-a)^{\alpha+1} 2^{-\alpha-2}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left(\left|f'(a)\right| + \left|f'(b)\right|\right) \\
&\quad + (1-\alpha)(b-a)^{2-\alpha} 2^{\alpha-4}\left(\frac{4}{(2-2\alpha)p+1}\right)^{\frac{1}{p}}\left(\left|f''(a)\right| + \left|f''(b)\right|\right),
\end{aligned} \tag{7}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Employing the Hölder inequality and the convexity of $|f'|^q$ and $|f''|^q$, by Lemma 1 we have

$$\left|\frac{\Gamma(1-\alpha)}{2^\alpha (b-a)^{-\alpha+1}}\left[{}^{PC}_{(\frac{a+b}{2})^+}D^\alpha_b f(b) + {}^{PC}_{(\frac{a+b}{2})^-}D^\alpha_a f(a)\right]\right.$$

$$
\begin{aligned}
&\quad - \left(\alpha^2(b-a)^\alpha 2^{-\alpha} f\left(\frac{a+b}{2}\right) + (1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2} f'\left(\frac{a+b}{2}\right)\right)\Bigg| \\
&\le \alpha^2(b-a)^{\alpha+1}2^{-\alpha-2}\left[\left(\int_0^1 t^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\int_0^1 t^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&\quad + (1-\alpha)(b-a)^{2-\alpha}2^{\alpha-4}\left[\left(\int_0^1 t^{2p-2\alpha p}\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\int_0^1 t^{2p-2\alpha p}\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&\le \alpha^2(b-a)^{\alpha+1}2^{-\alpha-2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\int_0^1 \frac{t}{2}\left|f'(a)\right|^q + \frac{2-t}{2}\left|f'(b)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\int_0^1 \frac{2-t}{2}\left|f'(a)\right|^q + \frac{t}{2}\left|f'(b)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&\quad + (1-\alpha)(b-a)^{2-\alpha}2^{\alpha-4}\left(\frac{1}{2p-2\alpha p+1}\right)^{\frac{1}{p}} \\
&\quad \times \left[\left(\int_0^1 \frac{t}{2}\left|f''(a)\right|^q + \frac{2-t}{2}\left|f''(b)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\int_0^1 \frac{2-t}{2}\left|f''(a)\right|^q + \frac{t}{2}\left|f''(b)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&= \alpha^2(b-a)^{\alpha+1}2^{-\alpha-2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{1}{4}\left|f'(a)\right|^q\right.\right. \\
&\quad \left.\left. + \frac{3}{4}\left|f'(b)\right|^q\right)^{\frac{1}{q}} + \left(\frac{3}{4}\left|f'(a)\right|^q + \frac{1}{4}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right] \\
&\quad + (1-\alpha)(b-a)^{2-\alpha}2^{\alpha-4}\left(\frac{1}{2p-2\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{1}{4}\left|f''(a)\right|^q + \frac{3}{4}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\frac{3}{4}\left|f''(a)\right|^q + \frac{1}{4}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Also, by the inequality

$$
\sum_{k=1}^{n}(a_k+b_k)^s \le \sum_{k=1}^{n} a_k^s + \sum_{k=1}^{n} b_k^s
$$

for $0 \le s < 1$ and $a_k, b_k \ge 0$, $k \in \{1,2,\dots,n\}$, taking $a_1 = |f'(a)|^q$, $b_1 = 3|f'(b)|^q$, $a_2 = 3|f'(a)|^q$, $b_2 = |f'(b)|^q$, $m_1 = |f''(a)|^q$, $n_1 = 3|f''(b)|^q$, $m_2 = 3|f''(a)|^q$, $n_2 = |f''(b)|^q$, we derive

$$
\begin{aligned}
&\left|\frac{\Gamma(1-\alpha)}{2^\alpha(b-a)^{-\alpha+1}}\left[{}^{PC}_{(\frac{a+b}{2})^+}D^\alpha_b f(b) + {}^{PC}_{(\frac{a+b}{2})^-}D^\alpha_a f(a)\right]\right. \\
&\quad \left. - \left(\alpha^2(b-a)^\alpha 2^{-\alpha} f\left(\frac{a+b}{2}\right) + (1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2} f'\left(\frac{a+b}{2}\right)\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \alpha^2(b-a)^{\alpha+1}2^{-\alpha-4}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left(3^{\frac{1}{q}}+1\right)\left(\left|f'(a)\right|+\left|f'(b)\right|\right) \\
&\quad + (1-\alpha)(b-a)^{2-\alpha}2^{\alpha-4}\frac{1}{4}\left(\frac{4}{(2-2\alpha)p+1}\right)^{\frac{1}{p}}\left(3^{\frac{1}{q}}+1\right)\left(\left|f''(a)\right|+\left|f''(b)\right|\right) \\
&\leq \alpha^2(b-a)^{\alpha+1}2^{-\alpha-2}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left(\left|f'(a)\right|+\left|f'(b)\right|\right) \\
&\quad + (1-\alpha)(b-a)^{2-\alpha}2^{\alpha-4}\left(\frac{4}{(2-2\alpha)p+1}\right)^{\frac{1}{p}}\left(\left|f''(a)\right|+\left|f''(b)\right|\right),
\end{aligned}
$$

which completes the proof. □

To illustrate the inequality in Theorem 5, derived using the Hölder inequality, we provide the following.

Example 3 Consider the function f defined in Example 2. We can evaluate the expression on the right-hand side of inequality (7) as follows:

$$
19\alpha^2\left(\frac{4}{p+1}\right)^{\frac{1}{p}}+7(1-\alpha)\left(\frac{4}{(2-2\alpha)p+1}\right)^{\frac{1}{p}} := \Phi_1.
$$

On the other hand, we know that

$$
\begin{aligned}
&\left|\frac{\Gamma(1-\alpha)}{2^{\alpha}(b-a)^{-\alpha+1}}\left[{}^{PC}_{(\frac{a+b}{2})^+}D^{\alpha}_{b}f(b)+{}^{PC}_{(\frac{a+b}{2})^-}D^{\alpha}_{a}f(a)\right]\right. \\
&\qquad \left.-\left(\alpha^2(b-a)^{\alpha}2^{-\alpha}f\left(\frac{a+b}{2}\right)+(1-\alpha)(b-a)^{1-\alpha}2^{\alpha-2}f'\left(\frac{a+b}{2}\right)\right)\right| \\
&= \alpha^2\frac{7}{3}+\frac{(1-\alpha)^2}{2}\left(\frac{22}{3-2\alpha}-\frac{11}{2-2\alpha}\right)-8(1-\alpha) := \Phi_2.
\end{aligned}
$$

Thus from Fig. 3 we can observe that for all values of $\alpha \in (0,1)$ and $p > 1$, the left-hand side of inequality (7) is consistently lower than the right-hand side.

Remark 3 In the limiting case $\alpha = 1$ in Theorem 5, we get

$$
\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4}\left(\frac{4}{p+1}\right)^{1/p}\left(\left|f'(a)\right|+\left|f'(b)\right|\right),
$$

as it was proved by Kırmacı [22].

Corollary 2 *Taking the limit as $\alpha \to 0$ in Theorem 5, we have*

$$
\begin{aligned}
&\left|\frac{1}{b-a}\left(\int_{\frac{a+b}{2}}^{b} f(x)\,dx - \int_a^{\frac{a+b}{2}} f(x)\,dx\right)-\left(\frac{b-a}{4}\right)f'\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{(b-a)^2}{16}\left(\frac{4}{2p+1}\right)^{1/p}\left(\left|f''(a)\right|+\left|f''(b)\right|\right).
\end{aligned}
$$

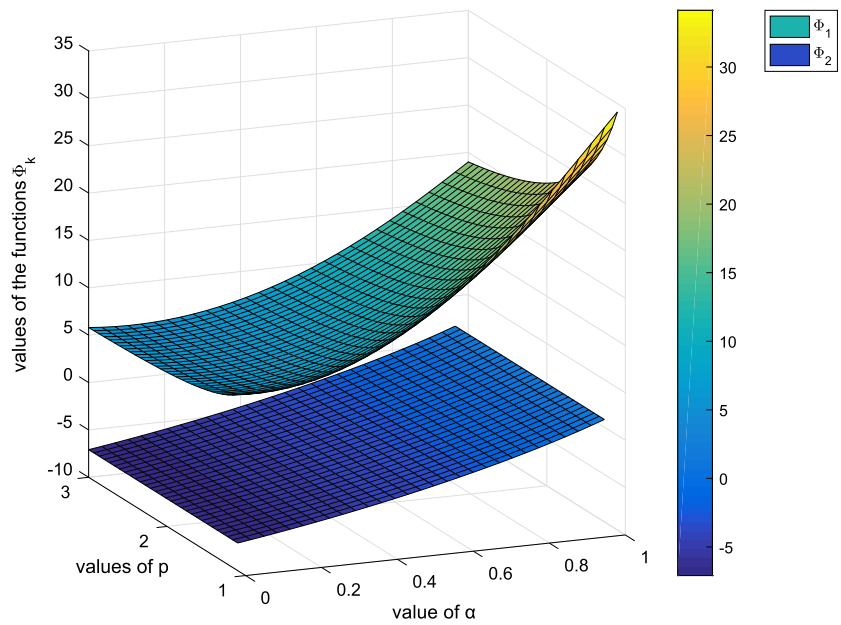

Figure 3 The graph of both sides of inequality (7) according to Example 3, which is computed and drawn by MATLAB program, depending on $\alpha \in (0,1)$ and $p \in (1,3]$

Also, taking $\alpha = \frac{1}{2}$, *we obtain*

$$\left|\frac{1}{b-a}\left(\int_a^b f(x)\,dx + f(b) - f(a)\right) - \left(f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\right)\right|$$
$$\leq \frac{b-a}{4}\left(\frac{4}{p+1}\right)^{1/p}\left(\left|f'(a)\right| + \left|f'(b)\right| + \left|f''(a)\right| + \left|f''(b)\right|\right).$$

3 Conclusions

This work uses a proportional Caputo-hybrid operator to establish novel Hermite–Hadamard-type integral inequalities for twice-differentiable convex mappings. To do this, we begin by proving a new integral identity of the Hermite–Hadamard type associated with the proportional Caputo-hybrid operator. Later, using convexity, the Hölder inequality, and the power mean inequality, we give several Hermite–Hadamard-type inequalities. Compared to classical calculus, our findings are more helpful because they present the particular situation of previously established boundaries as $\alpha \to 1$. So we believe that our approach and findings will encourage the readers to learn more about this topic. Similar inequalities for different fractional integrals can be investigated in future studies, and by utilizing different types of convexity new Hermite–Hadamard-type inequalities can be obtained.

Author contributions
T.T. and İ.D. wrote the main manuscript text and İ.D. prepared Figs. 1-3. All authors reviewed the manuscript.

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Data availability
Data sharing is not applicable to this paper as no data sets were generated or analyzed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

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