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Reconstruction of the solution of inverse Sturm–Liouville problem

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Abstract

In this paper we are concerned with an inverse problem with Robin boundary conditions, which states that, when the potential on [0, 1/2] and the coefficient at the left end point are known a priori, a full spectrum uniquely determines its potential on the whole interval and the coefficient at the right end point. We shall give a new method for reconstructing the potential for this problem in terms of the Mittag-Leffler decomposition of entire functions associated with this problem. The new reconstructing method also deduces a necessary and sufficient condition for the existence issue.

Mathematics Subject Classification: Primary 15A29; secondary 34A55

Keywords: Eigenvalue; Mittag-Leffler expansion; Levin–Lyubarski interpolation theorem; Reconstruction

1 Introduction

Consider the Sturm–Liouville operator L given by

$$-u'' + q(x)u = \lambda^2 u, \quad x \in [0,1], \tag{1.1}$$

with the Robin boundary conditions

$$u'(0) - h_1 u(0) = 0, \tag{1.2}$$

$$u'(1) + h_2 u(1) = 0, \tag{1.3}$$

where $q \in L^2[0,1]$ is real-valued and h_1, h_2 are real constants.

In 1978, Hochstadt and Lieberman [6] proved the remarkable uniqueness theorem, which proved that, if the potential $q(x) \in L^1[0,1]$ in equation (1.1) together with real constants h_1, h_2 in (1.2)–(1.3) is known a priori on the half-interval (1/2, 1), then the spectrum $\sigma = \{\lambda_n^2\}_{n=0}^{+\infty}$ alone is sufficient for the unique specification of $q(x)$ on (0, 1/2). The Hochstadt–Lieberman problem was the first half-inverse problem. Various aspects of this so-called half-inverse spectral problem were investigated in [3, 7, 12, 13] and other works. In [12, 13], Martinyuk and Pivovarchik studied this type problems with Dirichlet and Robin boundary conditions respectively for a potential $q \in L^2[0,1]$. They proposed a method of recovering the potential on the whole interval and obtained the necessary and sufficient conditions of the Hochstadt–Lieberman problem solvability. Buterin [3] proved the uniqueness theorem for this half-inverse spectral problem for a second-order differential pencil with spectral parameter dependent boundary conditions by Weyl function. Using the transformation operator and the properties of Riesz basis, the necessary and sufficient conditions of the Hochstadt–Lieberman problem solvability have been obtained in [7] for singular potentials from the space $W_2^{-1}(0,1)$. Furthermore, a reconstruction algorithm was provided.

Our main goal in this paper is to provide a new method for reconstructing potentials on the half-interval $[1/2,1]$ and h_2 for the above inverse spectral problem. We also give a necessary and sufficient condition for the existence issue.

Let $u_+(x,\lambda)$ be the solution of equation (1.1) satisfying the initial conditions $u_+(1,\lambda)=1$ and $u_+'(1,\lambda)=-h_2$. In order to solve the half-inverse problem by finding the functions $u_+(1/2,\lambda)$ and $u_+'(1/2,\lambda)$, the method we use is to employ the Mittag-Leffler expansion for meromorphic functions, which has been used in [14] to uniquely reconstruct the potential for the interior transmission eigenvalue problem and the Sturm–Liouville problem with the potential function known on the subinterval $(0,a)$ $(a<1/2)$ [15], respectively. This can help us to use the Levin–Lyubarski interpolation formula to find the unknown $u_+(1/2,\lambda)$ and $u_+'(1/2,\lambda)$. Moreover, this decomposition also provides a well-suited situations for utilizing the Levin–Lyubarski interpolation formula to our problem. Let us mention that our reconstructing process also deduces the existence condition of solutions for the above half-inverse spectral problem. In fact, the necessary and sufficient conditions are similar to the conditions of Pivovarchik [12, 13]. The difference is the method by which the functions $u_+(1/2,\lambda)$ and $u_+'(1/2,\lambda)$ are constructed. Let us mention that our method can also be used to treat the case where potential q is known a priori on the interval $[0,a]$ with $a<1/2$ (see [8, 14] and the references therein).

Throughout this paper, we denote by $\mathcal{L}_a$ the class of entire functions of exponential type $\le a$ that belong to $L^2(-\infty,\infty)$ for real λ [10].

The paper is organized as follows. In Sect. 2 we give some preliminaries that will be needed subsequently. Section 3 presents our main results for inverse problems.

2 Preliminaries

In this section, we recall the spectral characteristics of the operator L and give some theorems we will use.

Let $u_-(x,\lambda)$ be the solution of equation (1.1) satisfying the initial conditions $u_-(0)=1$ and $u_-'(0)=h_1$. According to [11], one knows that

$$u_-(x,\lambda)=\cos\lambda x+K_1(x,x,h_1)\frac{\sin\lambda x}{\lambda}-\int_0^x\frac{\partial}{\partial t}K_1(x,t,h_1)\frac{\sin\lambda t}{\lambda}\,dt, \tag{2.1}$$

where

$$K_1(x,t,h_1)=h_1+\tilde{K}_1(x,t)-\tilde{K}_1(x,-t)+h_1\int_t^x\big(\tilde{K}_1(x,s)-\tilde{K}_1(x,-s)\big)\,ds,$$

and $\tilde{K}_1(x,t)$ is the solution of the integral equation

$$\tilde{K}_1(x,t)=\frac{1}{2}\int_0^{\frac{x+t}{2}}q(s)\,ds+\int_0^{\frac{x+t}{2}}d\alpha\int_0^{\frac{x-t}{2}}q(\alpha+\beta)\tilde{K}_1(\alpha+\beta,\alpha-\beta)\,d\beta. \tag{2.2}$$

The solution $\tilde{K}_1(x,t)$ possesses partial derivatives of first order with $\frac{\partial}{\partial t}\tilde{K}_1(x,t) \in L^2[0,1/2]$ and $\frac{\partial}{\partial x}\tilde{K}_1(x,t) \in L^2[0,1/2]$. Moreover,

$$\tilde{K}_1(x,x) = \frac{1}{2}\int_0^x q(t)\,dt, \quad \text{and} \quad \tilde{K}_1(x,0) = 0. \tag{2.3}$$

By using (2.1) we infer

$$\begin{aligned} u_-(1/2,\lambda) &= \cos(\lambda/2) + \frac{h_1 + K_1}{\lambda}\sin(\lambda/2) + \frac{\psi_{-,0}(\lambda)}{\lambda}, \\ u_-'(1/2,\lambda) &= -\lambda\sin(\lambda/2) + (h_1 + K_1)\cos(\lambda/2) + \psi_{-,1}(\lambda), \end{aligned} \tag{2.4}$$

where $K_1 = \tilde{K}_1(1/2,1/2)$ defined by (2.3) and $\psi_{-,j} \in \mathcal{L}_{1/2}$ for $j = 0,1$.

On the other hand, we denote by $u_+(x,\lambda)$ the solution of (1.1) satisfying the initial conditions $u_+(1,\lambda) = 1$ and $u_+'(1,\lambda) = -h_2$. One knows that $u_+(x,\lambda)$ has a similar representation as (2.1):

$$\begin{aligned} u_+(x,\lambda) = {} & \cos\lambda(1-x) + \frac{K_2(x,x,h_2)}{\lambda}\sin\lambda(1-x) \\ & - \int_x^1 \frac{\partial}{\partial t}K(x,t,h_2)\frac{\sin\lambda(1-t)}{\lambda}\,dt, \end{aligned} \tag{2.5}$$

where the function $K_2(x,t,h_2)$ satisfies the expression similar to (2.2). This gives the asymptotics of $u_+(x,\lambda)$ by

$$\begin{aligned} u_+(1/2,\lambda) &= \cos(\lambda/2) + \frac{K_2 + h_2}{\lambda}\sin(\lambda/2) + \frac{\psi_{+,0}(\lambda)}{\lambda}, \\ u_+'(1/2,\lambda) &= \lambda\sin(\lambda/2) - (K_2 + h_2)\cos(\lambda/2) + \psi_{+,1}(\lambda), \end{aligned} \tag{2.6}$$

where $K_2 = \frac{1}{2}\int_{1/2}^1 q(t)\,dt$ and $\psi_{+,j} \in \mathcal{L}_{1/2}$ for $j = 0,1$.

The eigenvalues $\{\lambda_n^2\}_{n=0}^{+\infty}$ of problem (1.1)–(1.3) coincide with the zeros of

$$\Delta(\lambda) = u_-'(1,\lambda) + h_2 u_-(1,\lambda), \tag{2.7}$$

which are called the characteristic function of (1.1)–(1.3). From (2.1), $\Delta(\lambda)$ has the following representation:

$$\Delta(\lambda) = -\lambda\sin\lambda + (h_1 + K_1 + h_2 + K_2)\cos\lambda + \hat{\psi}(\lambda), \tag{2.8}$$

where $\hat{\psi} \in \mathcal{L}_1$. It is known that the function $\Delta(\lambda)$ is entire in λ of type 1. The eigenvalues $\{\lambda_n^2\}_{n=0}^{+\infty}$ behave asymptotically as follows:

$$\lambda_n = n\pi + \frac{1}{n\pi}(h_1 + K_1 + h_2 + K_2) + \frac{\alpha_n}{n} \tag{2.9}$$

as $n \to \infty$, where $\{\alpha_n\}_{n=0}^{+\infty} \in l^2$, which implies that

$$h_2 + K_2 = \pi \lim_{n\to+\infty} n\bigl(\lambda_n - (n-1)\pi\bigr) - (h_1 + K_1). \tag{2.10}$$

Moreover, the specification of the spectrum $\{\lambda_n^2\}_{n=0}^{+\infty}$ uniquely determines the characteristic function $\Delta(\lambda)$ by the formula [4, Theorem 1.1.4]:

$$\Delta(\lambda) = \left(\lambda_0^2 - \lambda^2\right) \prod_{n=1}^{\infty} \frac{\lambda_n^2 - \lambda^2}{n^2\pi^2}. \tag{2.11}$$

We write the Mittag-Leffler theorem [1, Theorem 3.6.2] for the case of simple poles as follows.

Theorem 2.1 *Assume that $F(z)$ is a meromorphic function and has only simple poles $\{z_j\}_{j\in\mathbb{Z}}$ with z_j distinct and $|z_j| \to \infty$ as $j \to \infty$. Let c_j be the residues of poles z_j of $F(z)$. If*

$$\sum_{j\in\mathbb{Z}} \frac{|c_j|}{|z_j|} < \infty, \tag{2.12}$$

then there exists an entire function $f(z)$ such that

$$F(z) = f(z) + \sum_{j\in\mathbb{Z}} \frac{c_j}{z - z_j}, \tag{2.13}$$

where the series in the right-hand side of (2.13) converges uniformly on every bounded set of $\mathbb{C}$ not containing the points $\{z_j\}_{j\in\mathbb{Z}}$.

The following theorem [9, Theorem A] is corresponding to sine type functions, which plays an important role in our paper.

Theorem 2.2 (*Levin–Lyubarski interpolation formula*) *Let f be a sine type function with indicator diagram of width $2a$, and $\{z_k\}_{k\in\mathbb{Z}}$ be the set of its zeros. Then, for any sequence $\{c_k\}_{k\in\mathbb{Z}} \in l^p$ with $1 < p < \infty$, the interpolation series*

$$\phi(\lambda) = f(\lambda) \sum_{k\in\mathbb{Z}} \frac{c_k}{\dot{f}(z_k)(\lambda - z_k)} \tag{2.14}$$

converges uniformly on any compact subsets in $\mathbb{C}$ and also in the norm of $L^p(-\infty,\infty)$ on the real axis, which belongs to $\mathcal{L}_a$.

3 Inverse spectral problem

In this section, we show the way of recovering q on $[1/2, 1]$ and give conditions of the existence of the solution in an implicit form.

Denote by $\nu_-(x,\lambda)$ the solution of (1.1) satisfying the initial conditions $\nu_-(0,\lambda) = 0$ and $\nu_-'(0,\lambda) = 1$. We infer

$$\begin{aligned} \nu_-(1/2,\lambda) &= \frac{1}{\lambda}\sin(\lambda/2) + \frac{K_1 + h_1}{\lambda^2}\cos(\lambda/2) + \frac{\varphi_{-,0}(\lambda)}{\lambda^2}; \\ \nu_-'(1/2,\lambda) &= \cos(\lambda/2) - \frac{K_1 + h_1}{\lambda}\sin(\lambda/2) + \frac{\varphi_{-,1}(\lambda)}{\lambda}, \end{aligned} \tag{3.1}$$

where $\varphi_{+,j} \in \mathcal{L}_{1/2}$ for $j = 0, 1$.

We denote by $\{\mu_n\}_{n\in\mathbb{Z}}$ the zeros of $u_-(1/2,\lambda)$, then

$$\mu_n = (2n-1)\pi + \frac{K_1+h_1}{n\pi} + \frac{\kappa_n}{n}, \tag{3.2}$$

as $n\to\infty$, where $\{\kappa_n\}_{n\in\mathbb{Z}}\in l^2$. It is easy to say that $\frac{v_-(1/2,\lambda)\Delta(\lambda)}{u_-(1/2,\lambda)}$ is meromorphic and has only simple poles $\{\mu_n\}_{n\in\mathbb{Z}}$. Let e_n be the residues of $\frac{v_-(1/2,\lambda)\Delta(\lambda)}{u_-(1/2,\lambda)}$ at μ_n. One has

$$e_n = \frac{v_-(1/2,\mu_n)\Delta(\mu_n)}{\dot{u}_-(1/2,\mu_n)} \tag{3.3}$$

with $\dot{u}_- = \partial u_-/\partial\lambda$. By virtue of (2.4), (2.8), and (2.9), we have

$$e_n = \frac{K_2+h_2-K_1-h_1}{n\pi} + \frac{\zeta_n}{n}, \tag{3.4}$$

where $\{\zeta_n\}_{n\in\mathbb{Z}}\in l^2$, which together with (3.2) yields $\{e_n/\mu_n\}_{n\in\mathbb{Z}}\in l^1$.

By Mittag-Leffler expansion, there exists an entire function $a_0(\lambda)$ such that

$$\frac{v_-(1/2,\lambda)\Delta(\lambda)}{u_-(1/2,\lambda)} = a_0(\lambda) + \sum_{n\in\mathbb{Z}} \frac{e_n}{\lambda-\mu_n}. \tag{3.5}$$

Denote by

$$b_0(\lambda) = u_-(1/2,\lambda)\sum_{n\in\mathbb{Z}} \frac{e_n}{\lambda-\mu_n}. \tag{3.6}$$

Substituting (3.6) into (3.5), we arrive at

$$v_-(1/2,\lambda)\Delta(\lambda) = a_0(\lambda)u_-(1/2,\lambda) + b_0(\lambda). \tag{3.7}$$

It is easy to see that $a_0(\lambda)$ and $b_0(\lambda)$ are real-valued functions when $\lambda\in\mathbb{R}$.

Lemma 3.1 *Let $a_0(\lambda)$ and $b_0(\lambda)$ be defined by (3.5) and (3.6), respectively. If we assume that*

$$\begin{aligned} \varphi_0(\lambda) &= \lambda\big(a_0(\lambda)+1-\cos\lambda\big) + (h_1+K_1+h_2+K_2)\sin\lambda, \\ \varphi_1(\lambda) &= \lambda b_0(\lambda) - (h_1+K_1-h_2-K_2)\sin(\lambda/2), \end{aligned} \tag{3.8}$$

then $\varphi_0(\lambda)\in\mathcal{L}_1$ and $\varphi_1(\lambda)\in\mathcal{L}_{1/2}$.

Proof Recall that $\{\mu_n\}_{n\in\mathbb{Z}}$ are the zeros of $u_-(1/2,\lambda)$, then (3.7) yields

$$b_0(\mu_n) = v_-(1/2,\mu_n)\Delta(\mu_n). \tag{3.9}$$

By virtue of the second equation of (3.8) together with (3.9) and using the asymptotic formulae (3.1) and (2.8) we get

$$\begin{aligned}\varphi_1(\mu_n) &= \mu_n\left[v_-(1/2,\mu_n)\Delta(\mu_n) - \frac{h_1+K_1-h_2-K_2}{\mu_n}\sin(\mu_n/2)\right]\\ &= \mu_n\left[\frac{1}{2}\cos(3\mu_n/2) - \frac{1}{2}\cos(\mu_n/2) + \frac{h_2+K_2}{2\mu_n}\sin(3\mu_n/2)\right.\\ &\quad\left. - \frac{4h_1+4K_1-h_2-K_2}{2\mu_n}\sin(\mu_n/2)\right].\end{aligned} \tag{3.10}$$

Using (3.2) we get

$$\begin{aligned}&\cos(3\mu_n/2) = (-1)^{n-1}\frac{3(K_1+h_1)}{2n\pi} + \frac{\alpha_n}{n},\\ &\cos(\mu_n/2) = (-1)^{n}\frac{(K_1+h_1)}{2n\pi} + \frac{\xi_n}{n},\\ &\sin(3\mu_n/2) = (-1)^n + \vartheta_n,\\ &\sin(\mu_n/2) = (-1)^{n-1} + \delta_n,\end{aligned} \tag{3.11}$$

where $\{\alpha_n\}_{n\in\mathbb{Z},n\neq0}$, $\{\xi_n\}_{n\in\mathbb{Z},n\neq0}$, $\{\vartheta_n\}_{n\in\mathbb{Z},n\neq0}$, and $\{\delta_n\}_{n\in\mathbb{Z},n\neq0}$ all belong to l^2. Substituting (3.11) into (3.10), we obtain

$$\{\varphi_1(\mu_n)\}_{n\in\mathbb{Z}} \in l^2. \tag{3.12}$$

The function $u_-(1/2,\lambda)$ is of sine type, i.e., there exist positive numbers m, M, and p such that

$$me^{\frac{1}{2}|\mathrm{Im}\lambda|} \le |u_-(1/2,\lambda)| \le Me^{\frac{1}{2}|\mathrm{Im}\lambda|}$$

for $|\mathrm{Im}\lambda| > p$. Taking into account (3.12), we use the Levin–Lyubarski interpolation theorem (see Theorem 2.2 for details) and choose $\{\mu_n\}_{n\in\mathbb{Z}}$ as the nodes of interpolation for finding the function $\varphi_1(\lambda)$:

$$\varphi_1(\lambda) = u_-(1/2,\lambda)\sum_{n\in\mathbb{Z}}\frac{\varphi_1(\mu_n)}{\dot{u}_-(1/2,\mu_n)(\lambda-\mu_n)}. \tag{3.13}$$

Note that $u_-(1/2,\lambda)$ is a sine type function with the indicator diagram of width 1, thus $\varphi_1(\lambda) \in \mathcal{L}_{1/2}$ according to Theorem 2.2.

On the other hand, note that $\{\lambda_n\}_{n\in\mathbb{Z}}$ are the zeros of $\Delta(\lambda)$. In virtue of (3.7) we have

$$\varphi_0(\lambda_n) = \lambda_n\left[-\frac{b_0(\lambda_n)}{u_-(1/2,\lambda_n)} + 1 - \cos\lambda_n - \frac{h_1+K_1+h_2+K_2}{\lambda_n}\sin\lambda_n\right]. \tag{3.14}$$

It should be noted from (2.9) that

$$\begin{aligned} \sin(\lambda_n/2) &= \begin{cases} (-1)^k \frac{h_1+K_1+h_2+K_2}{n\pi} + \frac{\alpha_n}{n}, & \text{if } n = 2k+1, \\ (-1)^{k+1} + \frac{\alpha_n}{n}, & \text{if } n = 2k, \end{cases} \\ \cos(\lambda_n/2) &= \begin{cases} (-1)^k + \frac{\beta_n}{n}, & \text{if } n = 2k+1, \\ (-1)^{k+1} \frac{h_1+K_1+h_2+K_2}{n\pi} + \frac{\beta_n}{n}, & \text{if } n = 2k, \end{cases} \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} \sin\lambda_n &= (-1)^{n-1} \frac{h_1+K_1+h_2+K_2}{n\pi} + \vartheta_n, \\ \cos\lambda_n &= (-1)^{n-1} + \delta_n. \end{aligned} \tag{3.16}$$

Substituting (3.15)–(3.16) into (3.14), one obtains

$$\{\varphi_0(\lambda_n)\}_{n\in\mathbb{Z}} \in l^2. \tag{3.17}$$

Let $\Phi(\lambda) = \Delta(\lambda)/(\lambda_0 - \lambda)$. It is easy to see that $\Phi(\lambda)$ belongs to sine type functions. We choose $\{\lambda_n\}_{n\in\mathbb{Z}}$ as the nodes of interpolation for finding the function $\varphi_0(\lambda)$:

$$\varphi_0(\lambda) = \Phi(\lambda) \sum_{n\in\mathbb{Z}} \frac{\varphi_0(\lambda_n)}{\dot{\Phi}(\lambda_n)(\lambda - \lambda_n)}. \tag{3.18}$$

From (2.8), one knows that $\Phi(\lambda)$ is a sine type function with the indicator diagram of width 2, thus $\varphi_0(\lambda) \in \mathcal{L}_1$ by Theorem 2.2. Moreover, from (3.8) we have

$$\begin{aligned} a_0(\lambda) &= -1 + \cos\lambda + (h_1 + K_1 + h_2 + K_2)\frac{\sin\lambda}{\lambda} + \frac{\varphi_0(\lambda)}{\lambda}, \\ b_0(\lambda) &= (h_1 + K_1 - h_2 - K_2)\frac{\sin(\lambda/2)}{\lambda} + \frac{\varphi_1(\lambda)}{\lambda}. \end{aligned} \tag{3.19}$$

□

Lemma 3.2 *Let $a_0(\lambda)$ and $b_0(\lambda)$ be defined by (3.5) and (3.6), respectively. If we write*

$$b_1(\lambda) = v'_-(1/2,\lambda)\Delta(\lambda) - a_0(\lambda)u'_-(1/2,\lambda), \tag{3.20}$$

then

$$\begin{aligned} u_+(1/2,\lambda) &= u_-(1/2,\lambda) - b_0(\lambda) \\ u'_+(1/2,\lambda) &= u'_-(1/2,\lambda) - b_1(\lambda). \end{aligned} \tag{3.21}$$

Proof It should be noted that

$$v'_-(1/2,\lambda)u_-(1/2,\lambda) - v_-(1/2,\lambda)u'_-(1/2,\lambda) = 1. \tag{3.22}$$

From (3.20) and (3.7), by simple computation we have

$$b_1(\lambda)u_-(1/2,\lambda) - b_0(\lambda)u'_-(1/2,\lambda) = \Delta(\lambda). \tag{3.23}$$

Note that

$$\Delta(\lambda) = u_-'(1/2,\lambda)u_+(1/2,\lambda) - u_-(1/2,\lambda)u_+'(1/2,\lambda) \tag{3.24}$$

and $|b_0(\lambda)| < |u_-(1/2,\lambda)|$. (3.24) together with (3.23) yields that there exists $h(\lambda)$ satisfying

$$\frac{u_+(1/2,\lambda) + b_0(\lambda)}{u_-(1/2,\lambda)} = \frac{u_+'(1/2,\lambda) + b_1(\lambda)}{u_-'(1/2,\lambda)} = h(\lambda). \tag{3.25}$$

By virtue of (2.4) and (2.6), for $|\lambda - \mu_n| > 0$, we have

$$\lim_{\lambda\to\infty} \frac{u_+(1/2,\lambda) + b_0(\lambda)}{u_-(1/2,\lambda)} = 1,$$

thus $h(\lambda) = 1$. It follows that (3.21) remains true from (3.25). This completes the proof. □

By the above arguments, we have recovered the functions $b_0(\lambda)$, $a_0(\lambda)$ and then $b_1(\lambda)$ in terms of the given mixed spectral data consisting of q on $[0,1/2]$, h_1, and the set σ of eigenvalues of Sturm–Liouville problems. Thus we can reconstruct $u_+(1/2,\lambda)$ and $u_+'(1/2,\lambda)$ by (3.21), and hence q on $(1/2,1)$ via the Gelfand–Levitan–Marchenko method [11]. The method of reconstructing the potential $q(x)$ on the half-interval $[1/2,1]$ and constant h_2 can be summarized as follows.

Algorithm Let the input data set $\mathcal{D} = \{q(x) \in L^2[0,1/2], \sigma = \{\lambda_n^2\}_{n=0}^{+\infty}, h_1\}$ be given.

(1) Compute $h_2 + K_2$ in virtue of (2.10) and construct $\Delta(\lambda)$ in terms of (2.11).
(2) Compute the functions $u_-(1/2,\lambda)$, $u_-'(1/2,\lambda)$, $v_-(1/2,\lambda)$, and $v_-'(1/2,\lambda)$.
(3) Determine the sequences $\varphi_1(\mu_n)$ by (3.10), then construct the function $\varphi_1(\lambda)$ in virtue of (3.13).
(4) Construct $b_0(\lambda)$ in virtue of the second formula of (3.19) and compute the sequence $b_0(\lambda_n)$.
(5) Determine the sequence $\varphi_0(\lambda_n)$ by (3.14), then construct the function $\varphi_0(\lambda)$ in virtue of (3.18).
(6) Construct $a_0(\lambda)$ in terms of the first formula of (3.19).
(7) Construct the function $b_1(\lambda)$ by (3.20).
(8) Reconstruct $u_+(1/2,\lambda)$ and $u_+'(1/2,\lambda)$ by (3.21).
(9) Reconstruct the function q on $(1/2,1)$ from the zeros of $u_+(1/2,\lambda)$ and $u_+'(1/2,\lambda)$ via the Gelfand–Levitan–Marchenko method [11].
(10) Compute $h_2 = K_2 + h_2 - \int_{1/2}^1 q(x)\,dx$.

Let us mention that $(u_+/u_+')(\sqrt{\lambda})$ belongs to the Nevanlinna class, i.e., $(u+/u_+')(\sqrt{\lambda})$: $\mathbb{C}_+ \to \mathbb{C}_+$ is analytic with $\mathbb{C}_+$ being the open complex upper half-plane [10]. Our reconstructing process also deduces the following conclusion for the existence problem.

Theorem 3.3 *Assume that a real function $q_- \in L^2[0,1/2]$ is known together with the real constant h_1. Let a set of numbers $\{\lambda_n^2\}_{n=0}^{\infty}$ be given and satisfy the following asymptotics*:

$$\lambda_n = n\pi + \frac{A}{n\pi} + \frac{\alpha_n}{n}, \tag{3.26}$$

where $A \in \mathbb{R}$ and $\{\alpha_n\}_{n=0}^{\infty} \in l^2$. Let $u_-(x,\lambda)$ be the solution of (1.1) with the potential $q = q_-$ on $[0,1/2]$, which satisfies the initial conditions $u_-(0) = 1$, $u_-'(0) = h$, and let $u_+(\lambda)$ and $\hat{u}_+(\lambda)$ be given by

$$\begin{aligned} &u_+(\lambda) = u_-(1/2,\lambda) - b_0(\lambda), \\ &\hat{u}_+(\lambda) = u_-'(1/2,\lambda) - b_1(\lambda), \end{aligned} \tag{3.27}$$

with $b_0(\lambda)$ and $b_1(\lambda)$ being defined by (3.19) and (3.20), respectively.

Then there exists a unique real-valued function $q_+ \in L^2[1/2,1]$ and a real constant h_2 such that the spectrum σ of problem (1.1)–(1.3) with potential $q = q_-$ on $[0,1/2]$ and $q = q_+$ on $[1/2,1]$ coincides with the sequence $\{\lambda_n^2\}_{n=0}^{+\infty}$ if and only if the function $u_+/\hat{u}_+(\sqrt{\lambda})$ belongs to the Nevanlinna class.

Proof Suppose that there exists a real-valued function $q \in L^2(0,1)$ such that $\{\lambda_n^2\}_{n=0}^{+\infty}$ is the spectrum of the Sturm–Liouville operator defined by (1.1)–(1.3). Then, by the above discussion, $u_+(1/2,\lambda) = u_+(\lambda)$ and $u_+'(1/2,\lambda) = \hat{u}_+(\lambda)$. In this situation, it is known [5, 11] that $(u_+/\hat{u}_+)(\sqrt{\lambda})$ is the Weyl m-function [5] of Sturm–Liouville equation (1.1), which ensures that the function $(u_+/\hat{u}_+)(\sqrt{\lambda})$ belongs to the Nevanlinna class.

Since the spectrum $\sigma = \{\lambda_n^2\}_{n=0}^{+\infty}$ of the operator L is given, by (2.10) and (2.11) one obtains $K_2 + h_2$ and $\Delta(\lambda)$. If a real-valued function $q_- \in L^2(0,1/2)$ is known a priori, then both functions $u_-(1/2,\lambda)$ and $u_-'(1/2,\lambda)$ are also known. Thus by (3.13) and (3.19) we obtain $b_0(\lambda)$ and from Lemma 3.2 we obtain $b_1(\lambda)$. We therefore obtain $u_+(\lambda)$ and $\hat{u}_+(\lambda)$ from (3.27):

$$\begin{aligned} u_+(\lambda) &= u_-(1/2,\lambda) - b_0(\lambda) \\ &= \cos(\lambda/2) + \frac{h_2 + K_2}{\lambda}\sin(\lambda/2) + \frac{\psi_{+,0}(\lambda)}{\lambda} \end{aligned}$$

and

$$\begin{aligned} \hat{u}_+(\lambda) &= u_-'(1/2,\lambda) - b_1(\lambda) \\ &= \lambda\sin(\lambda/2) - (h_2 + K_2)\cos(\lambda/2) + \psi_{+,1}(\lambda). \end{aligned}$$

Here one knows that $\psi_{+,j}(\lambda) \in \mathcal{L}_{1/2}$ for $j = 0,1$ by computing from (3.20) and above formulae since $\psi_{-,j}(\lambda) \in \mathcal{L}_{1/2}$, $\varphi_0(\lambda) \in \mathcal{L}_1$, and $\varphi_1(\lambda) \in \mathcal{L}_{1/2}$. It is easy to see that their zeros, denoted by $\{\alpha_{n,D}\}_{n\in\mathbb{Z}}$ and $\{\alpha_{n,N}\}_{n\in\mathbb{Z}}$, satisfy the following conditions:

$$\begin{aligned} &\alpha_{n,D} = (2n+1)\pi + \frac{K_2 + h_2}{2n\pi} + \frac{\beta_n}{n}, \\ &\alpha_{n,N} = 2n\pi + \frac{K_2 + h_2}{2n\pi} + \frac{\hat{\beta}_n}{n}, \end{aligned} \tag{3.28}$$

where $\{\beta_n\}_{n\in\mathbb{Z}}$ and $\{\hat{\beta}_n\}_{n\in\mathbb{Z}}$ belong to l^2. Furthermore, if $(u_+/\hat{u}_+)(\sqrt{\lambda})$ belongs to the Nevanlinna class, then its zeros $\{\alpha_{n,D}^2\}_{n=0}^{+\infty}$ and poles $\{\alpha_{n,N}^2\}_{n=0}^{+\infty}$ are interlacing:

$$-\infty < \alpha_{0,N}^2 < \alpha_{0,D}^2 < \alpha_{1,N}^2 < \alpha_{1,D}^2 < \cdots.$$

Moreover, by (3.28) it is easy to check that the sequences $\{(\frac{\alpha_{n,D}}{2\pi})^2\}_{n=0}^{+\infty}$ and $\{(\frac{\alpha_{n,N}}{2\pi})^2\}_{n=0}^{+\infty}$ satisfy the conditions of Theorem 3.4.3 in [11]. By Borg's two-spectra theorem [2] there exists a unique real-valued function $q_+ \in L^2(1/2,1)$ such that $\{\alpha_{n,D}^2\}_{n=0}^{+\infty}$ and $\{\alpha_{n,N}^2\}_{n=0}^{+\infty}$ are exactly the Dirichlet–Dirichlet spectrum (under the boundary conditions $y(1/2) = 0 = y(1)$) and the Dirichlet–Neumann spectrum (under the boundary conditions $y(1/2) = 0 = y'(1)$) of two Sturm–Liouville operators defined on $(1/2,1)$ with potential q_+. On the other hand, it is easy to see that the known σ is the spectrum of Sturm–Liouville operators defined by (1.1)–(1.3) with potential $q = q_-$ on $(0,1/2)$ a.e. and $q = q_+$ on $(1/2,1)$. This completes the proof. □

Appendix

In this section, we supply the details of Marchenko's uniqueness theorem and Borg's two spectra theorem.

Let us introduce the Weyl–Titchmarsh m-function for the operator $L(q,h_1,h_2)$ as

$$m(x,\lambda) = \frac{u_+'(x,\lambda)}{u_+(x,\lambda)}. \tag{A.1}$$

Denote by $\tilde{m}(x,\lambda)$ by the Weyl–Titchmarsh m-function for the operator $L(\tilde{q},h_1,\tilde{h}_2)$.

Theorem A.1 (*Marchenko's uniqueness theorem*) *If* $m(a,\lambda) = \tilde{m}(a,\lambda)$, *then* $q(x) = \tilde{q}(x)$ *on* $[a,1]$.

Theorem A.2 (*Borg's two spectra theorem*) *Let* $h_2 \neq h_3$. *If the two spectra* $\sigma(q,h_1,h_2)$ *and* $\sigma(q,h_1,h_3)$ *are known a priori, then* $q(x)$ *on* $[0,1]$ *is uniquely determined.*

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