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Ψ-Bielecki-type norm inequalities for a generalized Sturm–Liouville–Langevin differential equation involving Ψ-Caputo fractional derivative

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Abstract

The present research work investigates some new results for a fractional generalized Sturm–Liouville–Langevin (FGSLL) equation involving the Ψ-Caputo fractional derivative with a modified argument. We prove the uniqueness of the solution using the Banach contraction principle endowed with a norm of the Ψ-Bielecki-type. Meanwhile, the fixed-point theorems of the Leray–Schauder and Krasnoselskii type associated with the Ψ-Bielecki-type norm are used to derive the existence properties by removing some strong conditions. We use the generalized Gronwall-type inequality to discuss Ulam–Hyers (UH), generalized Ulam–Hyers (GUH), Ulam–Hyers–Rassias (UHR), and generalized Ulam–Hyers–Rassias (GUHR) stability of these solutions. Lastly, three examples are provided to show the effectiveness of our main results for different cases of (FGSLL)-problem such as Caputo-type Sturm–Liouville, Caputo-type Langevin, Caputo–Erdélyi–Kober-type Langevin problems.

Mathematics Subject Classification: Primary 26A33; 34A08; 34A12; secondary 34B15

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1 Introduction

During the last century, fractional differential equations (FDEs) have fascinated the interest of many researchers due to their various applications in many fields of science, like physics, chemistry, biology, economics, engineering, signal processing, electromagnetics, etc. (see [1–3]). In many references, the basic notions and tools of fractional calculus can be observed; see, e.g., [4–6]. Recently, Almeida [7] defined a new fractional derivative called Ψ-Caputo fractional derivative and he published several scientific research works [8, 9]. Afterwards, several mathematicians concentrated their research on the generalized fractional operators; we cite them as examples [10–15]. In this direction, researchers have focused their interests on the investigation of existence, uniqueness, and Ulam stability of FDEs using a number of definitions related to fractional derivatives as can be found in some works like [16–21] and references therein, as well as in [22, 23].

In 1908, Paul Langevin formulated a new equation, named the Langevin equation, to define the evolution of some physical phenomena in fluctuating environments, like Brownian motion [24]. After this, other extensions of the Langevin equation have been studied in the works of many researchers [25–31]. Nowadays, the existence, uniqueness, and stability of solutions for Langevin nonlinear BVPs have been established by many researchers using different kinds of fractional derivatives by applying Banach, Krasnoselskii, Shaefer, and Leray–Schauder classical fixed point theorems. For more information on this topic, the reader is advised to refer to [32–39]. The results on the existence and Ulam–Hyers stability of solutions of Langevin fractional equation have been discussed in [29]. Motivated by the works cited above, several other types of stability will be discussed in this article for an advanced combined differential equation. More precisely, consider the fractional generalized Sturm–Liouville–Langevin (FGSLL) problem:

$$\begin{cases} {}^{C}\mathcal{D}_{0^+}^{\alpha_1,\Psi}(\eta(\mathfrak{z})\,{}^{C}\mathcal{D}_{0^+}^{\alpha_2,\Psi}u(\mathfrak{z}) + \chi(\mathfrak{z})u(\mathfrak{z})) = f(\mathfrak{z},u(\mathfrak{z})), \quad \mathfrak{z}\in I=[0,R],\\ u(0)=0, \qquad \eta(R)\,{}^{C}\mathcal{D}_{0^+}^{\sigma_1,\Psi}u(R) + \chi(R)\,{}^{C}\mathcal{D}_{0^+}^{\sigma_2,\Psi}u(R) = 0. \end{cases} \tag{1}$$

Here, $\eta\in C(I,\mathbb{R}^\star)$, $\chi\in C(I,\mathbb{R})$, $f: I\times\mathbb{R}\to\mathbb{R}$ is continuous, ${}^{C}\mathcal{D}_{0^+}^{\gamma,\Psi}$ is the Ψ-Caputo fractional derivative depending on an increasing function Ψ of order $\gamma\in\{\alpha_1,\alpha_2,\sigma_1,\sigma_2\}$, $0<\alpha_1,\alpha_2\le 1$, and $0<\sigma_1,\sigma_2<\alpha_2$.

To show the novelty and generality of our BVP, we note that:

1. If $\chi(\mathfrak{z})=0$, for each $\mathfrak{z}\in I$, the (FGSLL)-problem (1) reduces to the standard form of the fractional Sturm–Liouville (FSL) problem for a nonlinear FDE, which is as follows:

$$\begin{cases} {}^{C}\mathcal{D}_{0^+}^{\alpha_1,\Psi}(\eta(\mathfrak{z})\,{}^{C}\mathcal{D}_{0^+}^{\alpha_2,\Psi}u(\mathfrak{z})) = f(\mathfrak{z},u(\mathfrak{z})), \quad \mathfrak{z}\in I,\\ u(0)=0, \qquad \eta(R)\,{}^{C}\mathcal{D}_{0^+}^{\sigma_1,\Psi}u(R) = 0, \end{cases} \tag{2}$$

   and the considered (FSL)-problem (2) contains some problems involving different fractional derivative operators, for various choices of the function Ψ. Among these are interesting extensions:
   - If $\Psi(x)=x$, then the (FSL)-problem (2) reduces to the Caputo-type Sturm–Liouville (CSL) problem.
   - If $\Psi(x)=x^\nu$, then the (FSL)-problem (2) becomes the Caputo–Erdélyi–Kober-type Sturm–Liouville (CEKSL) problem.
   - If $\Psi(x)=\ln(x)$, then the (FSL)-problem (2) represents the Caputo–Hadamard-type Sturm–Liouville (CHSL) problem.
2. By choosing $\eta(\mathfrak{z})\equiv 1$, $\chi(\mathfrak{z})\equiv\lambda$ $(\lambda\in\mathbb{R})$, for $\mathfrak{z}\in I$, the (FGSLL)-problem (1) reduces to the standard form of the fractional Langevin (FL) problem for a nonlinear FDE, which is as follows:

$$\begin{cases} {}^{C}\mathcal{D}_{0^+}^{\alpha_1,\Psi}({}^{C}\mathcal{D}_{0^+}^{\alpha_2,\Psi}u(\mathfrak{z}) + \lambda u(\mathfrak{z})) = f(\mathfrak{z},u(\mathfrak{z})), \quad \mathfrak{z}\in I,\\ u(0)=0, \qquad {}^{C}\mathcal{D}_{0^+}^{\sigma_1,\Psi}u(R) + \lambda\,{}^{C}\mathcal{D}_{0^+}^{\sigma_2,\Psi}u(R) = 0, \end{cases} \tag{3}$$

and the considered (FL)-equation (3) contains some problems involving many classical fractional derivative operators, for various choices of a function Ψ. Among these are interesting extensions:

- If $\Psi(x) = x$, then the (FL)-problem (3) reduces to the Caputo-type Langevin (CL) problem.
- If $\Psi(x) = x^{\nu}$, then the (FL)-problem (3) represents the Caputo–Erdélyi–Kober-type Langevin (CEKL) problem.
- If $\Psi(x) = \ln(x)$, then the (FL)-problem (3) becomes the Caputo–Hadamard-type Langevin (CHL) problem.

Now, to organize the paper in a standard form for the readers, we arrange it as follows. In Sect. 2, we propose some definitions and lemmas that will be used to establish our theorems. In Sect. 3, we investigate the existence and uniqueness of the solution for the main (FGSLL)-problem (1) under some Ψ-Bielecki-type norm inequalities, and Sect. 4 presents the study of some stability results for the solutions of the (FGSLL)-problem (1), such as Ulam–Hyers, Ulam–Hyers–Rassias, and their generalizations, with the help of the generalized Gronwall inequality. Our main tools in this study are three fixed point theorems: the Banach contraction principle, Leray–Schauder, and Krasnoselskii theorems under some norm inequalities of the Ψ-Bielecki type. After that we give, in Sect. 5, three examples to illustrate our theoretical results. Finally, we complete the paper by a conclusion with some perspectives.

2 Essential concepts and basic tools

Some concepts are recalled in this section, and also some lemmas are proved.

Definition 2.1 ([7]) Let $\mu > 0$, $n \in \mathbb{N}$, $I = [a,b]$ with $-\infty \leq a < b \leq \infty$, $\varphi : I \to \mathbb{R}$ be an integrable function, and $\Psi \in \mathcal{C}^1(I,\mathbb{R})$ increasing with $\Psi'(\mathfrak{z}) \neq 0$ for any $\mathfrak{z} \in I$. The Ψ-Riemann–Liouville (R–L) fractional integral of order μ for φ that depends on Ψ is given as

$$\mathcal{I}_{a^+}^{\mu;\Psi}\varphi(\mathfrak{z}) = \frac{1}{\Gamma(\mu)}\int_a^{\mathfrak{z}} \Psi'(s)\big(\Psi(\mathfrak{z}) - \Psi(s)\big)^{\mu-1}\varphi(s)\,ds. \tag{4}$$

Definition 2.2 ([7]) Consider an interval $I \subset \mathbb{R}$ and let $\mu \in (n-1,n)$. Let also $\varphi : I \to \mathbb{R}$ be an integrable function and Ψ be as given in Definition 2.1. Then, the Ψ-R–L fractional derivative of the order μ of the function φ with respect to Ψ is given as

$$\begin{aligned}\mathcal{D}_{a^+}^{\mu;\Psi}\varphi(\mathfrak{z}) &= \left(\frac{1}{\Psi'(\mathfrak{z})}\frac{d}{d\mathfrak{z}}\right)^n \mathcal{I}_{a^+}^{n-\mu;\Psi}\varphi(\mathfrak{z}) \\ &= \frac{1}{\Gamma(n-\mu)}\left(\frac{1}{\Psi'(\mathfrak{z})}\frac{d}{d\mathfrak{z}}\right)^n \int_a^{\mathfrak{z}} \Psi'(s)\big(\Psi(\mathfrak{z}) - \Psi(s)\big)^{n-\mu-1}\varphi(s)\,ds,\end{aligned} \tag{5}$$

where $n = [\mu] + 1$ and $[\mu]$ indicates the integer part of μ.

Definition 2.3 ([7]) Let $\mu > 0$, $n \in \mathbb{N}$, $I = [a,b]$ with $-\infty \leq a < b \leq \infty$, $\Psi, \varphi \in C^n(I,\mathbb{R})$ be functions so that Ψ is increasing and $\Psi'(\mathfrak{z}) \neq 0$ for any $\mathfrak{z} \in I$. The left-sided Ψ-Caputo fractional derivative of order μ for φ is defined by

$${}^c\mathcal{D}_{a^+}^{\mu;\Psi}\varphi(\mathfrak{z}) = \mathcal{I}_{a^+}^{n-\mu;\Psi}\left(\frac{1}{\Psi'(\mathfrak{z})}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{z}}\right)^n \varphi(\mathfrak{z}),$$

where $n = [\mu] + 1$ for $\mu \notin \mathbb{N}$ and $n = \mu$ for $\mu \in \mathbb{N}$.

To simplify the notation, we put $\varphi_{\Psi}^{[n]}(\mathfrak{z}) = (\frac{1}{\Psi'(\mathfrak{z})}\frac{\mathrm{d}}{\mathrm{d}\mathfrak{z}})^n\varphi(\mathfrak{z})$. Then, from the definition we can write

$$ {}^{c}\mathcal{D}_{a^+}^{\mu;\Psi}\varphi(\mathfrak{z}) = \begin{cases} \int_a^{\mathfrak{z}} \frac{\Psi'(s)(\Psi(\mathfrak{z})-\Psi(s))^{n-\mu-1}}{\Gamma(n-\mu)}\varphi_{\Psi}^{[n]}(s)\,\mathrm{d}s & \text{if } \mu \notin \mathbb{N}, \\ \varphi_{\Psi}^{[n]}(\mathfrak{z}) & \text{if } \mu \in \mathbb{N}. \end{cases} $$

Lemma 2.4 ([7]) *Let $\mu > 0$ and $\varphi : [a,b] \to \mathbb{R}$. The properties given below hold*:

- *If $\varphi \in \mathcal{C}([a,b])$, then ${}^{C}\mathcal{D}_{a^+}^{\mu,\Psi}\mathcal{I}_{a^+}^{\mu,\Psi}\varphi(\mathfrak{z}) = \varphi(\mathfrak{z})$.*
- *If $\varphi \in \mathcal{C}^{n-1}([a,b])$, then*

$$ \mathcal{I}_{a^+}^{\mu,\Psi}\,{}^{C}\mathcal{D}_{a^+}^{\alpha,\Psi}\varphi(\mathfrak{z}) = \varphi(\mathfrak{z}) - \sum_{k=0}^{n-1} c_k\bigl(\Psi(\mathfrak{z}) - \Psi(a)\bigr)^k, $$

where $c_k = \frac{\varphi_{\Psi}^{[k]}(a)}{k!}$, $\varphi_{\Psi}^{[k]}(a) = [\frac{1}{\Psi'(\mathfrak{z})}\frac{d}{d\mathfrak{z}}]^k\varphi(a)$, $n-1 < \mu \le n$.
In particular, for $\mu \in (0,1)$, we have $\mathcal{I}_{a+}^{\mu,\Psi}\,{}^{C}\mathcal{D}_{a+}^{\mu,\Psi}\varphi(\mathfrak{z}) = \varphi(\mathfrak{z}) - \varphi(a)$.

Now, we define the norms $\|\cdot\|_C : C([a,b]) \to \mathbb{R}$ and $\|\cdot\|_{C_{\Psi}^{[n]}} : C^n([a,b]) \to \mathbb{R}$ by

$$ \|\varphi\|_C := \max_{\mathfrak{z}\in[a,b]}\bigl|\varphi(\mathfrak{z})\bigr| \quad \text{and} \quad \|\varphi\|_{C_{\Psi}^{[n]}} := \sum_{j=0}^{n}\bigl\|\varphi_{\Psi}^{[j]}\bigr\|_C. $$

Lemma 2.5 ([11]) *If $\varphi : C([a,b]) \to \mathbb{R}$, then $I_{a+}^{\mu,\Psi}\varphi(\mathfrak{z})$ is well-defined for every $\mathfrak{z} \in [a,b]$. In addition, we have*

(i) $\mathcal{I}_{a+}^{\mu,\Psi}\varphi(a) = 0$;
(ii) $\|\mathcal{I}_{a+}^{\mu,\Psi}\varphi\|_C \le \frac{(\Psi(\mathfrak{z})-\Psi(a))^{\mu}}{\Gamma(\mu+1)}\|\varphi\|_C$.

Proof From (4), we derive the following inequality:

$$ \bigl|\mathcal{I}_{a+}^{\mu,\Psi}\varphi(\mathfrak{z})\bigr| \le \frac{(\Psi(\mathfrak{z}) - \Psi(a))^{\mu}}{\Gamma(\mu+1)}\|\varphi\|_C, $$

which gives immediately $\mathcal{I}_{a+}^{\mu,\Psi}\varphi(a) = 0$ and $\|\mathcal{I}_{a+}^{\mu,\Psi}\varphi\|_C \le \frac{(\Psi(\mathfrak{z})-\Psi(a))^{\mu}}{\Gamma(\mu+1)}\|\varphi\|_C$. □

Lemma 2.6 ([7]) *The Ψ-Caputo derivatives of the fractional order are bounded and, for any $\mu > 0$, we have*

$$ \|{}^{C}\mathcal{D}_{a+}^{\mu,\Psi}\varphi\|_C \le \frac{(\Psi(b) - \Psi(a))^{n-\mu}}{\Gamma(n+1-\mu)}\|\varphi\|_{C_{\Psi}^{[n]}}. $$

Remark 2.7 From equality (5), we can easily obtain

$$ \bigl|{}^{c}\mathcal{D}_{a+}^{\mu,\Psi}\varphi(\mathfrak{z})\bigr| \le \frac{(\Psi(\mathfrak{z}) - \Psi(a))^{n-\mu}}{\Gamma(n+1-\mu)}\|\varphi\|_{C_{\Psi}^{[n]}}, $$

which allows us to conclude that ${}^{C}\mathcal{D}_{a+}^{\mu,\Psi}\varphi(a) = 0$.

Lemma 2.8 *Let $\mu, \theta > 0$. We have*

$$ \mathcal{I}_{0^+}^{\mu;\Psi}e^{\theta(\Psi(\mathfrak{z})-\Psi(0))} \le \frac{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}{\theta^{\mu}}, \quad 0 \le \mathfrak{z} \le R, \tag{6} $$

and

$$\int_0^{\mathfrak{z}_1} \frac{\Psi'(s)(\Psi(\mathfrak{z}_2)-\Psi(s))^{\mu-1}}{\Gamma(\mu)} e^{\theta(\Psi(s)-\Psi(0))}\, ds \le \frac{e^{\theta(\Psi(\mathfrak{z}_2)-\Psi(0))}}{\theta^\mu}, \quad 0 \le \mathfrak{z}_1 < \mathfrak{z}_2 \le R. \tag{7}$$

Proof By applying the Ψ-R–L fractional operator $\mathcal{I}_{0^+}^{\mu_1,\Psi}$ to the function $\mathfrak{z} \mapsto e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}$ together with the replacement of variables $\mathrm{y} = \Psi(\mathfrak{z}) - \Psi(s)$ and $z = \theta\mathrm{y}$, we have

$$\begin{aligned}
\mathcal{I}_{0^+}^{\mu;\Psi} e^{\theta(\Psi(\mathfrak{z})-\Psi(0))} &= \frac{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}{\Gamma(\mu)} \int_0^{\Psi(\mathfrak{z})-\Psi(0)} \mathrm{y}^{\mu-1} e^{-\theta\mathrm{y}}\, \mathrm{dy} \\
&= \frac{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}{\Gamma(\mu)\theta^\mu} \int_0^{\theta(\Psi(\mathfrak{z})-\Psi(0))} z^{\mu-1} e^{-z}\, dz \\
&\le \frac{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}{\Gamma(\mu)\theta^\mu} \int_0^{\infty} \mathrm{z}^{\mu-1} e^{-z}\, \mathrm{dz} = \frac{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}{\theta^\mu}.
\end{aligned}$$

For the proof of the inequality (7), we again use the same replacement of variables $\mathrm{y} = \Psi(\mathfrak{z}_2) - \Psi(s)$ and $z = \theta\mathrm{y}$, and we obtain

$$\begin{aligned}
&\int_0^{\mathfrak{z}_1} \frac{\Psi'(s)(\Psi(\mathfrak{z}_2)-\Psi(s))^{\mu-1}}{\Gamma(\mu)} e^{\theta(\Psi(s)-\Psi(0))}\, ds \\
&\quad = -\frac{e^{\theta(\Psi(\mathfrak{z}_2)-\Psi(0))}}{\Gamma(\mu)} \int_{\Psi(\mathfrak{z}_2)-\Psi(0)}^{\Psi(\mathfrak{z}_2)-\Psi(\mathfrak{z}_1)} \mathrm{y}^{\mu-1} e^{-\theta\mathrm{y}}\, \mathrm{dy} \\
&\quad = \frac{e^{\theta(\Psi(\mathfrak{z}_2)-\Psi(0))}}{\Gamma(\mu)\theta^\mu} \int_{\theta(\Psi(\mathfrak{z}_2)-\Psi(\mathfrak{z}_1))}^{\theta(\Psi(\mathfrak{z}_2)-\Psi(0))} z^{\mu-1} e^{-z}\, dz \\
&\quad \le \frac{e^{\theta(\Psi(\mathfrak{z}_2)-\Psi(0))}}{\Gamma(\mu)\theta^\mu} \int_0^{\infty} z^{\mu-1} e^{-z}\, dz = \frac{e^{\theta(\Psi(\mathfrak{z}_2)-\Psi(0))}}{\theta^\mu}.
\end{aligned}$$

The proof is now complete. □

Lemma 2.9 *Let* $0 < \alpha_1, \alpha_2 \le 1$, $\alpha_3 > 0$, *and* $0 < \sigma_1$, $\sigma_2 < \alpha_2$. *Suppose that* $h \in \mathcal{C}(I,\mathbb{R})$, $\eta \in \mathcal{C}(I,\mathbb{R}^\star)$, *and* $\chi \in \mathcal{C}(I,\mathbb{R})$. *Then, u is a solution of*

$$ {}^C\mathcal{D}_{0^+}^{\alpha_1,\Psi}\left(\eta(\mathfrak{z})\, {}^C\mathcal{D}_{0^+}^{\alpha_2,\Psi} u(\mathfrak{z}) + \chi(\mathfrak{z})u(\mathfrak{z})\right) = h(\mathfrak{z}), \tag{8}$$

$$u(0) = 0, \tag{9}$$

$$\eta(R)\, {}^C\mathcal{D}_{0^+}^{\sigma_1,\Psi} u(R) + \chi(R)\, {}^C\mathcal{D}_{0^+}^{\sigma_2,\Psi} u(R) = 0 \tag{10}$$

if and only if it fulfills the integral equation given below:

$$\begin{aligned}
u(\mathfrak{z}) &= \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{1}{\eta(\mathfrak{z})}\mathcal{I}_{0^+}^{\alpha_1,\Psi} h(\mathfrak{z})\right) - \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\chi(\mathfrak{z})u(\mathfrak{z})}{\eta(\mathfrak{z})}\right) \\
&\quad + \frac{\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{\eta(\mathfrak{z})})}{\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)}) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})} \\
&\quad \times \Bigg[\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{\chi(R)u(R)}{\eta(R)}\right) - \eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi} h(R)\right) \\
&\quad - \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi} h(R)\right) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{\chi(R)u(R)}{\eta(R)}\right)\Bigg],
\end{aligned} \tag{11}$$

where

$$\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{1}{\eta(R)}\right)+\chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{1}{\eta(R)}\right)\neq 0.$$

Proof By applying the Ψ-R–L fractional operators $\mathcal{I}_{0^+}^{\alpha_1,\Psi}$ and $\mathcal{I}_{0^+}^{\alpha_2,\Psi}$ on both sides of equation (8) and utilizing Lemma 2.4, we obtain two real numbers c_0 and c_1 such that

$$u(\mathfrak{z})=\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{1}{\eta(\mathfrak{z})}\mathcal{I}_{0^+}^{\alpha_1,\Psi}h(\mathfrak{z})\right)-\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\chi(\mathfrak{z})}{\eta(\mathfrak{z})}\mathcal{I}_{0^+}^{\alpha_3,\Psi}u(\mathfrak{z})\right)+c_0\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{1}{\eta(\mathfrak{z})}\right)+c_1, \tag{12}$$

where c_0 and c_1 belong to $\mathbb{R}$.

From the boundary condition (9), together with Lemma 2.5, it follows that $c_1=0$, and by using the second boundary condition (10), as well as taking into account the assumption

$$\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{1}{\eta(R)}\right)+\chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{1}{\eta(R)}\right)\neq 0,$$

after some computations we obtain

$$\begin{aligned} c_0 &= \frac{1}{\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)})+\chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})} \\ &\quad\times\left[\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{\chi(R)u(R)}{\eta(R)}\right)-\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi}h(R)\right)\right. \\ &\quad\left.-\chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi}h(R)\right)+\chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{\chi(R)u(R)}{\eta(R)}\right)\right]. \end{aligned}$$

Replacing c_0 with its value in (12), we get

$$\begin{aligned} u(\mathfrak{z}) &= \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{1}{\eta(\mathfrak{z})}\mathcal{I}_{0^+}^{\alpha_1,\Psi}h(\mathfrak{z})\right)-\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\chi(\mathfrak{z})u(\mathfrak{z})}{\eta(\mathfrak{z})}\right) \\ &\quad+\frac{\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{\eta(\mathfrak{z})})}{\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)})+\chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})} \\ &\quad\times\left[\chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{\chi(R)u(R)}{\eta(R)}\right)+\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{\chi(R)u(R)}{\eta(R)}\right)\right. \\ &\quad\left.-\chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi}h(R)\right)-\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi}h(R)\right)\right]. \end{aligned} \tag{13}$$

For the reverse case, taking the Ψ-Caputo operator ${}^C\mathcal{D}_{0^+}^{\alpha_2,\Psi}$ on both sides of equation (13) and applying again the operator ${}^C\mathcal{D}_{0^+}^{\alpha_1,\Psi}$ after multiplying the obtained equation by η, and finally by exploiting Lemma 2.4, we find

$${}^C\mathcal{D}_{0^+}^{\alpha_1,\Psi}\left(\eta(\mathfrak{z}){}^C\mathcal{D}_{0^+}^{\alpha_2,\Psi}u(\mathfrak{z})+\chi(\mathfrak{z})u(\mathfrak{z})\right)=h(\mathfrak{z}).$$

To examine the boundary conditions, it is trivial to verify them using (13).

As a result, u is a solution to the problem (1), and the proof of Lemma 2.9 is now finished. □

Now, we pay attention to the space $\mathfrak{C} = \mathcal{C}(I,\mathbb{R})$ equipped with the well-known Ψ-Bielecki-type norm $\|u\|_{\theta,\alpha}$ proposed by previous works (see [40]) defined by

$$\|u\|_{\theta,\alpha} = \sup_{\mathfrak{z}\in I} \frac{|u(\mathfrak{z})|}{\mathbb{E}_\alpha[\theta(\Psi(\mathfrak{z}) - \Psi(0))^\alpha]}, \quad \theta,\alpha > 0,$$

where $\mathbb{E}_\alpha$ indicates the Mittag-Leffler function of one-parameter that is given as

$$\mathbb{E}_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \quad \alpha > 0.$$

If we take $\alpha \to 1$ in the above norm $\|u\|_{\theta,\alpha}$, we obtain

$$\|u\|_\theta := \sup_{\mathfrak{z}\in I} \frac{|u(\mathfrak{z})|}{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}, \quad \theta > 0,$$

and $(\mathfrak{C}, \|u\|_\theta)$ is a Banach space. We now focus on the key findings of our study.

3 Main results

For a good and straightforward continuation of our work, we propose the hypotheses as given below:

(H1) $f : [0,R] \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H2) For some positive real constant L_f, we have

$$\left|f(\mathfrak{z},u_1) - f(\mathfrak{z},u_2)\right| \le L_f|u_1 - u_2|, \quad \text{for each } u_1,u_2 \in \mathbb{R}, \mathfrak{z} \in [0,R].$$

(H3) $|f(\mathfrak{z},u)| \le \mathcal{K}_f(\mathfrak{z})$, $\forall(\mathfrak{z},u) \in [0,R] \times \mathbb{R}$, with $\mathcal{K}_f \in \mathcal{C}([0,R],\mathbb{R}_+)$.

(H4) A function $\mathfrak{g} \in \mathcal{C}([0,R],\mathbb{R}_+)$ and a real constant $d > 0$ exist such that

$$\left|f(\mathfrak{z},u)\right| \le \mathfrak{g}(\mathfrak{z}) + d|u|, \quad \forall(\mathfrak{z},u) \in [0,R] \times \mathbb{R}.$$

(H5) A positive real constant M exists such that

$$\frac{M(1 - d\Lambda_\theta - \nabla_\theta)}{\Lambda_\theta\|\mathfrak{g}\|_\theta} > 1.$$

Furthermore, to analyze the stability of UHR and GUHR, we adopt the assumption as given below:

(H6) A nondecreasing function $\Upsilon \in \mathcal{C}([0,R],\mathbb{R}_+)$ and a real constant $\gamma_{\Upsilon,\alpha_1+\alpha_2} > 0$ exist such that for any $\mathfrak{z} \in [0,R]$, we have

$$\mathcal{I}_{0^+}^{\alpha_1+\alpha_2,\Psi}\Upsilon(\mathfrak{z}) \le \gamma_{\Upsilon,\alpha_1+\alpha_2}\Upsilon(\mathfrak{z}). \tag{14}$$

In light of Lemma 2.9, we can define the following operator:

$$\mathcal{N} : \mathfrak{C} \to \mathfrak{C},$$

$$\begin{aligned}\mathcal{N}u(\mathfrak{z}) &= \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{1}{\eta(\mathfrak{z})}\mathcal{I}_{0^+}^{\alpha_1,\Psi}f\big(\mathfrak{z},u(\mathfrak{z})\big)\right) - \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\chi(\mathfrak{z})u(\mathfrak{z})}{\eta(\mathfrak{z})}\right)\\
&\quad+ \frac{\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{\eta})(\mathfrak{z})}{\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)}) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})}\\
&\quad\times\left[\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{\chi(R)}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_3,\Psi}u(R)\right) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{\chi(R)u(R)}{\eta(R)}\right)\right.\\
&\quad- \eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi}f\big(R,u(R)\big)\right)\\
&\quad\left.- \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi}f\big(R,u(R)\big)\right)\right],\end{aligned} \tag{15}$$

where

$$\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{1}{\eta(R)}\right) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{1}{\eta(R)}\right) \neq 0.$$

Now, we express the operator $\mathcal{N}$ as a sum of two operators $\mathcal{N}_1$ and $\mathcal{N}_2$ as follows:

$$\begin{aligned}\mathcal{N}_1u(\mathfrak{z}) &= \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{1}{\eta(\mathfrak{z})}\mathcal{I}_{0^+}^{\alpha_1,\Psi}f\big(\mathfrak{z},u(\mathfrak{z})\big)\right)\\
&\quad- \frac{\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{\eta(\mathfrak{z})})}{\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)}) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})}\\
&\quad\times\left[\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi}f\big(R,u(R)\big)\right)\right.\\
&\quad\left.+ \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi}f\big(R,u(R)\big)\right)\right],\\
\mathcal{N}_2u(\mathfrak{z}) &= -\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\chi(\mathfrak{z})u(\mathfrak{z})}{\eta(\mathfrak{z})}\right) + \frac{\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{\eta(\mathfrak{z})})}{\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)}) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})}\\
&\quad\times\left[\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{\chi(R)u(R)}{\eta(R)}\right) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{\chi(R)}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_3,\Psi}u(R)\right)\right].\end{aligned}$$

To facilitate the reading of the work, we utilize the following notations:

$$\begin{aligned}&M_f := \sup_{\mathfrak{z}\in I}\big|f(\mathfrak{z},0)\big| < \infty, \qquad M_{f,\theta} := \sup_{\mathfrak{z}\in I}\frac{|f(\mathfrak{z},0)|}{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}} < \infty,\\
&\underline{\eta} := \inf_{\mathfrak{z}\in I}\big|\eta(\mathfrak{z})\big|, \qquad \overline{\chi} := \sup_{\mathfrak{z}\in I}\big|\chi(\mathfrak{z})\big|,\end{aligned}$$

and, for more convenience, we put

$$\begin{aligned}\Lambda &= \frac{(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2+1)} + \frac{\frac{(\Psi(R)-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}|}\\
&\quad\times\left[\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_2+1)} + \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_1+1)}\right],\end{aligned} \tag{16}$$

$$\begin{aligned}\nabla = {}& \frac{\overline{\chi}(\Psi(R)-\Psi(0))^{\alpha_2}}{\underline{\eta}\Gamma(\alpha_2+1)} + \frac{\frac{(\Psi(R)-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}|} \\ & \times \left[\frac{|\chi(R)|\overline{\chi}(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_2-\sigma_2+1)} + \frac{|\eta(R)|\overline{\chi}(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_2-\sigma_1+1)}\right],\end{aligned} \tag{17}$$

$$\begin{aligned}\Lambda_\theta = {}& \frac{1}{\underline{\eta}\theta^{\alpha_1+\alpha_2}} + \frac{\frac{(\Psi(R)-\Psi(0)))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}|} \\ & \times \left[\frac{|\chi(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_2}} + \frac{|\eta(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_1}}\right],\end{aligned} \tag{18}$$

$$\begin{aligned}\nabla_\theta := {}& \frac{\overline{\chi}}{\underline{\eta}\theta^{\alpha_2}} \\ & + \frac{\frac{(\Psi(R)-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}|}\left[\frac{|\chi(R)|\overline{\chi}}{\underline{\eta}\theta^{\alpha_2-\sigma_2}} + \frac{|\eta(R)|\overline{\chi}}{\underline{\eta}\theta^{\alpha_2-\sigma_1}}\right],\end{aligned} \tag{19}$$

and

$$\mathcal{J} = \Lambda L_f + \nabla. \tag{20}$$

3.1 Uniqueness of solution by using Banach contraction principle

To prove the results, we first provide the Banach contraction principle as a reminder.

Lemma 3.1 ([40]) *Let (U,d) be a complete metric space, and $\mathbb{T}: U \to U$ a contraction. Then there is a unique fixed point of $\mathbb{T}$ in U.*

Theorem 3.2 *Suppose that* (H1) *and* (H2) *are satisfied. Then the* (*FGSLL*)*-problem* (1) *has a unique solution if $\mathcal{J} < 1$, where $\mathcal{J}$ is defined by* (20).

Proof First, we choose r_1 such that

$$r_1 \geq \frac{\Lambda M_f}{1-\mathcal{J}}.$$

Briefly, our aim is to show that $\mathcal{N}\mathcal{B}_{r_1} \subseteq \mathcal{B}_{r_1}$, where

$$\mathcal{B}_{r_1}(u) = \left\{u \in \mathfrak{C} : \|u\| \leq r_1\right\}$$

is a nonempty, closed, and convex subset of the Banach space $\mathfrak{C}$.

For each $\mathfrak{z} \in [0,R]$ and $u \in \mathcal{B}_{r_1}$, we get

$$\left|f(\mathfrak{z},u)\right| \leq \left|f(\mathfrak{z},u) - f(\mathfrak{z},0)\right| + \left|f(\mathfrak{z},0)\right| \leq L_f|u| + \left|f(\mathfrak{z},0)\right|,$$

which implies that

$$\sup_{\mathfrak{z}\in[0,R]} \left|f(\mathfrak{z},u)\right| \leq L_f\|u\| + M_f.$$

Let $u \in \mathcal{B}_{r_1}$, then

$$\begin{aligned}
&\left|\mathcal{N}u(\mathfrak{z})\right| \\
&\le \left|\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{1}{\eta(\mathfrak{z})}\mathcal{I}_{0^+}^{\alpha_1,\Psi}f\big(\mathfrak{z},u(\mathfrak{z})\big)\right)\right| + \left|\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\chi(\mathfrak{z})u(\mathfrak{z})}{\eta(\mathfrak{z})}\right)\right| \\
&\quad + \frac{|\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{\eta(\mathfrak{z})})|}{|\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)}) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})|} \\
&\quad \times \left[\left|\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{\chi(R)u(R)}{\eta(R)}\right)\right| + \left|\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi}f\big(R,u(R)\big)\right)\right|\right. \\
&\quad \left. + \left|\chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi}f\big(R,u(R)\big)\right)\right| + \left|\chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{\chi(R)u(R)}{\eta(R)}\right)\right|\right].
\end{aligned}$$

By using the property $||\kappa| - |\ell|| \le |\kappa + \ell|$ and taking into consideration

$$\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)} \neq \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)},$$

we get

$$\begin{aligned}
\left|\mathcal{N}u(\mathfrak{z})\right| &\le \frac{(L_f\|u\| + M_f)(\Psi(\mathfrak{z})-\Psi(0))^{\alpha_1+\alpha_2}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2+1)} + \frac{\overline{\chi}\|u\|(\Psi(\mathfrak{z})-\Psi(0))^{\alpha_2}}{\underline{\eta}\Gamma(\alpha_2+1)} \\
&\quad + \frac{\frac{(\Psi(\mathfrak{z})-\Psi(0))^{\alpha_2}}{\underline{\eta}\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_2-\sigma_1+1)}|} \\
&\quad \times \left[\frac{|\chi(R)|\overline{\chi}\|u\|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_2-\sigma_2+1)} + \frac{|\eta(R)|\overline{\chi}\|u\|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_2-\sigma_1+1)}\right. \\
&\quad + \frac{|\chi(R)|(L_f\|u\| + M_f)(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_2+1)} \\
&\quad \left. + \frac{|\eta(R)|(L_f\|u\| + M_f)(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_1+1)}\right],
\end{aligned}$$

which gives

$$\begin{aligned}
\left|\mathcal{N}u(\mathfrak{z})\right| &\le \left(\frac{\frac{(\Psi(R)-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}|}\right. \\
&\quad \times \left[\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_2+1)} + \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_1+1)}\right] \\
&\quad \left. + \frac{(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2+1)}\right)\big(L_f\|u\| + M_f\big) \\
&\quad + \|u\|\left(\frac{\frac{(\Psi(R)-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}|}\right. \\
&\quad \times \left[\frac{|\chi(R)|\overline{\chi}(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\underline{\eta}\alpha_2-\sigma_2+1)} + \frac{|\eta(R)|\overline{\chi}(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(+\alpha_2-\sigma_1+1)}\right]
\end{aligned}$$

$$+ \frac{\overline{\chi}(\Psi(R) - \Psi(0))^{\alpha_2}}{\underline{\eta}\Gamma(\alpha_2 + 1)}\Bigg);$$

that is,

$$\begin{aligned}
\left|\mathcal{N}u(\mathfrak{z})\right| &\le \Lambda\left(L_f\|u\| + M_f\right) + \|u\|\nabla \\
&\le \Lambda L_f r_1 + \Lambda M_f + r_1\nabla \\
&\le r_1,
\end{aligned}$$

which implies that $\|\mathcal{N}u\| \le r_1$. Thus, $\mathcal{N}$ maps $\mathcal{B}_{r_1}$ into itself.

The last step is to show that $\mathcal{N}$ is a contraction mapping. Letting $u_1, u_2 \in \mathcal{B}_{r_1}$ and $\mathfrak{z} \in [0, R]$, we have

$$\begin{aligned}
&\left|\mathcal{N}u_1(\mathfrak{z}) - \mathcal{N}u_2(\mathfrak{z})\right| \\
&\le \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{1}{|\eta(\mathfrak{z})|}\mathcal{I}_{0^+}^{\alpha_1,\Psi}\left|f\left(\mathfrak{z}, u_1(\mathfrak{z})\right) - f\left(\mathfrak{z}, u_2(\mathfrak{z})\right)\right|\right) + \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{|\chi(\mathfrak{z})|u_1(\mathfrak{z}) - u_2(\mathfrak{z})||}{|\eta(\mathfrak{z})|}\right) \\
&\quad + \frac{\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{|\eta(\mathfrak{z})|})}{|\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)}) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})|} \\
&\quad \times \Bigg[\left|\eta(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{|\chi(R)||u_1(R) - u_2(R)|}{|\eta(R)|}\right) \\
&\quad + \left|\eta(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{1}{|\eta(R)|}\mathcal{I}_{0^+}^{\alpha_1,\Psi}\left|f\left(R, u_1(R)\right) - f\left(R, u_2(R)\right)\right|\right)(R) \\
&\quad + \left|\chi(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{|\chi(R)||u_1(R) - u_2(R)|}{|\eta(R)|}\right) \\
&\quad + \left|\chi(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{1}{|\eta(R)|}\mathcal{I}_{0^+}^{\alpha_1,\Psi}\left|f\left(R, u_1(R)\right) - f\left(R, u_2(R)\right)\right|\right)\Bigg] \\
&\le \frac{L_f\|u_1 - u_2\|(\Psi(\mathfrak{z}) - \Psi(0))^{\alpha_1+\alpha_2}}{\underline{\eta}\Gamma(\alpha_1 + \alpha_2 + 1)} + \frac{\overline{\chi}\|u_1 - u_2\|(\Psi(\mathfrak{z}) - \Psi(0))^{\alpha_2}}{\underline{\eta}\Gamma(\alpha_2 + 1)} \\
&\quad + \frac{\frac{(\Psi(\mathfrak{z}) - \Psi(0))^{\alpha_2}}{\underline{\eta}\Gamma(\alpha_2+1)}}{||\eta(R)|\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\underline{\eta}})(R) - |\chi(R)|\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\underline{\eta}})(R)|} \\
&\quad \times \Bigg[\frac{|\chi(R)|\overline{\chi}\|u_1 - u_2\|(\Psi(R) - \Psi(0))^{\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_2 - \sigma_2 + 1)} \\
&\quad + \frac{|\eta(R)|\overline{\chi}\|u_1 - u_2\|(\Psi(R) - \Psi(0))^{\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_2 - \sigma_1 + 1)} \\
&\quad + \frac{L_f|\chi(R)|\|u_1 - u_2\|(\Psi(R) - \Psi(0))^{\alpha_1+\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_1 + \alpha_2 - \sigma_2 + 1)} \\
&\quad + \frac{L_f|\eta(R)|\|u_1 - u_2\|(\Psi(R) - \Psi(0))^{\alpha_1+\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_1 + \alpha_2 - \sigma_1 + 1)}\Bigg].
\end{aligned}$$

Thus,

$$\|\mathcal{N}u_1 - \mathcal{N}u_2\|$$

$$\leq \Bigg(\frac{L_f(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2+1)}+\frac{\overline{\chi}(\Psi(R)-\Psi(0))^{\alpha_3+\alpha_2}}{\underline{\eta}\Gamma(\alpha_3+\alpha_2+1)}$$
$$+\frac{\frac{(\Psi(R)-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)}-\frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}|}$$
$$\times\Bigg[\frac{|\chi(R)|\overline{\chi}(\Psi(R)-\Psi(0))^{\alpha_3+\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_3+\alpha_2-\sigma_2+1)}+\frac{|\eta(R)|\overline{\chi}(\Psi(R)-\Psi(0))^{\alpha_3+\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_3+\alpha_2-\sigma_1+1)}$$
$$+\frac{L_f|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_2+1)}$$
$$+\frac{L_f|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_1+1)}\Bigg]\Bigg)\|u_1-u_2\|,$$

consequently, we get

$$\|\mathcal{N}u_1-\mathcal{N}u_2\|\leq\mathcal{J}\|u_1-u_2\|.$$

Since $\mathcal{J}<1$, hence $\mathcal{N}$ is a contraction mapping. Consequently, by the Banach contraction principle 3.1, we conclude that $\mathcal{N}$ has a unique fixed point in $\mathcal{B}_{r_1}$. Hence, the (FGSLL)-problem (1) has a unique solution on $[0,R]$. □

Now, we would like to prove Theorem 3.2 using the Ψ-Bielecki-type norm inequalities. Here, the strong condition $\mathcal{J}<1$ is removed.

Theorem 3.3 *Let* (H1) *and* (H2) *be satisfied. Then the* (*FGSLL*)*-problem* (1) *has a unique solution on* $[0,R]$.

Proof Let us choose

$$r_2\geq\frac{\Lambda_\theta M_{f,\theta}}{1-(L_f\Lambda_\theta+\nabla_\theta)},$$

where Λ_θ, ∇_θ, and $M_{f,\theta}$ are three constants defined previously.

Claim 1: One has $\mathcal{N}\mathcal{B}_{r_2,\theta}\subseteq\mathcal{B}_{r_2,\theta}$, where $\mathcal{B}_{r_2,\theta}(u)=\{u\in\mathfrak{C},\|u\|_\theta\leq r_2\}$ is a nonempty, closed, and convex subset of the Banach space $\mathfrak{C}$.

For each $\mathfrak{z}\in[0,R]$ and $u\in\mathcal{B}_{r_2,\theta}$, we have

$$\frac{|f(\mathfrak{z},u)|}{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}\leq\frac{|f(\mathfrak{z},u)-f(\mathfrak{z},0)|}{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}+\frac{|f(\mathfrak{z},0)|}{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}\leq\frac{L_f|u|}{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}+\frac{|f(\mathfrak{z},0)|}{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}},$$

which implies that

$$\sup_{\mathfrak{z}\in[0,R]}\frac{|f(\mathfrak{z},u)|}{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}\leq L_f\|u\|_\theta+M_{f,\theta}.$$

Let $u \in \mathcal{B}_{r_2,\theta}$, then

$$\begin{aligned}
\left|\mathcal{N}u(\mathfrak{z})\right| \le{}& \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{1}{|\eta(\mathfrak{z})|}\mathcal{I}_{0^+}^{\alpha_1,\Psi}\frac{|f(\mathfrak{z},u(\mathfrak{z}))|e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}\right)\\
&+\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{|\chi(\mathfrak{z})|}{|\eta(\mathfrak{z})|}\mathcal{I}_{0^+}^{\alpha_3,\Psi}\frac{|u(\mathfrak{z})|e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}\right)\\
&+\frac{\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{|\eta(\mathfrak{z})|})}{|\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{|\eta(R)|})+\chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{|\eta(R)|})|}\\
&\times\Bigg[\left|\eta(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{|\chi|}{|\eta|}\frac{|u(R)|e^{\theta(\Psi(R)-\Psi(0))}}{e^{\theta(\Psi(R)-\Psi(0))}}\right)\\
&+\left|\eta(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{1}{|\eta|}\mathcal{I}_{0^+}^{\alpha_1,\Psi}\frac{|f(R,u(R))|e^{\theta(\Psi(R)-\Psi(0))}}{e^{\theta(\Psi(R)-\Psi(0))}}\right)\\
&+\left|\chi(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{1}{|\eta|}\mathcal{I}_{0^+}^{\alpha_1,\Psi}\frac{|f(R,u(R))|e^{\theta(\Psi(R)-\Psi(0))}}{e^{\theta(\Psi(R)-\Psi(0))}}\right)\\
&+\left|\chi(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{\chi}{|\eta|}\frac{|u(R)|e^{\theta(\Psi(R)-\Psi(0))}}{e^{\theta(\Psi(R)-\Psi(0))}}\right)\Bigg].
\end{aligned}$$

Using the estimate $||\kappa|-|\ell||\le|\kappa+\ell|$ and taking into account

$$\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)}\ne\frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)},$$

we obtain

$$\begin{aligned}
\left|\mathcal{N}u(\mathfrak{z})\right| \le{}& \frac{(L_f\|u\|_\theta+M_{f,\theta})}{\underline{\eta}}\mathcal{I}_{0^+}^{\alpha_1+\alpha_2,\Psi}\left(e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}\right)\\
&+\frac{\overline{\chi}\|u\|_\theta}{\underline{\eta}}\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}\right)\\
&+\frac{\frac{(\Psi(\mathfrak{z})-\Psi(0))^{\alpha_2}}{\underline{\eta}\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_2-\sigma_2+1)}-\frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_2-\sigma_1+1)}|}\\
&\times\Bigg[\frac{|\eta(R)|\overline{\chi}\|u\|_\theta}{\underline{\eta}}\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(e^{\theta(\Psi(R)-\Psi(0))}\right)\\
&+\frac{(L_f\|u\|_\theta+M_{f,\theta})|\eta(R)|}{\underline{\eta}}\mathcal{I}_{0^+}^{\alpha_1+\alpha_2-\sigma_1,\Psi}\left(e^{\theta(\Psi(R)-\Psi(0))}\right)\\
&+\frac{(L_f\|u\|_\theta+M_{f,\theta})|\chi(R)|}{\underline{\eta}}\mathcal{I}_{0^+}^{\alpha_1+\alpha_2-\sigma_2,\Psi}\left(e^{\theta(\Psi(R)-\Psi(0))}\right)\\
&+\frac{\overline{\chi}\|u\|_\theta}{\underline{\eta}}\left|\chi(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(e^{\theta(\Psi(R)-\Psi(0))}\right)\Bigg].
\end{aligned}$$

By exploiting (6), we get

$$\left|\mathcal{N}u(\mathfrak{z})\right|\le\Bigg(\frac{L_f\|u\|_\theta+M_{f,\theta}}{\underline{\eta}\theta^{\alpha_1+\alpha_2}}+\frac{\overline{\chi}\|u\|_\theta}{\underline{\eta}\theta^{\alpha_2}}$$

$$+\frac{\frac{(\Psi(\mathfrak{z})-\Psi(0))^{\alpha_2}}{\underline{\eta}\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_2-\sigma_2+1)}-\frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_2-\sigma_1+1)}|}$$
$$\times\left[\frac{|\chi(R)|\overline{\chi}\|u\|_\theta}{\underline{\eta}\theta^{\alpha_2-\sigma_2}}+\frac{|\eta(R)|\overline{\chi}\|u\|_\theta}{\underline{\eta}\theta^{\alpha_2-\sigma_1}}+\frac{|\chi(R)|(L_f\|u\|_\theta+M_{f,\theta})}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_2}}\right.$$
$$\left.+\frac{|\eta(R)|(L_f\|u\|_\theta+M_{f,\theta})}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_1}}\right]\Bigg)e^{\theta(\Psi(\mathfrak{z})-\Psi(0))},$$

which yields

$$\|\mathcal{N}u\|_\theta\le(\Lambda_\theta L_f+\nabla_\theta)r_2+\Lambda_\theta M_{f,\theta}\le r_2. \tag{21}$$

This means that $\mathcal{N}$ maps $\mathcal{B}_{r_2,\theta}$ into itself.

Claim 2: Operator $\mathcal{N}$ is a contraction mapping.

Let $u_1,u_2\in\mathcal{B}_{r_2,\theta}$ and $\mathfrak{z}\in[0,R]$, we have

$$\begin{aligned}
&\left|\mathcal{N}u_1(\mathfrak{z})-\mathcal{N}u_2(\mathfrak{z})\right|\\
&\le\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{1}{|\eta(\mathfrak{z})|}\mathcal{I}_{0^+}^{\alpha_1,\Psi}\left(\frac{|f(\mathfrak{z},u_1(\mathfrak{z}))-f(\mathfrak{z},u_2(\mathfrak{z}))|e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}\right)\right)\\
&\quad+\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{|\chi(\mathfrak{z})|}{|\eta(\mathfrak{z})|}\mathcal{I}_{0^+}^{\alpha_3,\Psi}\left(\frac{|u_1(\mathfrak{z})-u_2(\mathfrak{z})|e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}\right)\right)\\
&\quad+\frac{\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{|\eta(\mathfrak{z})|})}{|\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)})+\chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})|}\\
&\quad\times\left[\left|\eta(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{|\chi(R)|}{|\eta(R)|}\frac{|u_1(R)-u_2(R)|e^{\theta(\Psi(R)-\Psi(0))}}{e^{\theta(\Psi(R)-\Psi(0))}}\right)\right.\\
&\quad+\left|\eta(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{1}{|\eta(R)|}\mathcal{I}_{0^+}^{\alpha_1,\Psi}\left(\frac{|f(R,u_1(R))-f(R,u_2(R))|e^{\theta(\Psi(R)-\Psi(0))}}{e^{\theta(\Psi(R)-\Psi(0))}}\right)\right)\\
&\quad+\left|\chi(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{|\chi(R)|}{|\eta(R)|}\frac{|u_1(R)-u_2(R)|e^{\theta(\Psi(R)-\Psi(0))}}{e^{\theta(\Psi(R)-\Psi(0))}}\right)\\
&\quad\left.+\left|\chi(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{1}{|\eta(R)|}\mathcal{I}_{0^+}^{\alpha_1,\Psi}\left(\frac{|f(R,u_1(R))-f(R,u_2(R))|e^{\theta(\Psi(R)-\Psi(0))}}{e^{\theta(\Psi(R)-\Psi(0))}}\right)\right)\right].
\end{aligned}$$

Simple computations give us

$$\begin{aligned}
&\left|\mathcal{N}u_1(\mathfrak{z})-\mathcal{N}u_2(\mathfrak{z})\right|\\
&\le\frac{L_f\|u_1-u_2\|_\theta}{\underline{\eta}}\mathcal{I}_{0^+}^{\alpha_1+\alpha_2,\Psi}\left(e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}\right)+\frac{\overline{\chi}\|u_1-u_2\|_\theta}{\underline{\eta}}\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}\right)\\
&\quad+\frac{\|u_1-u_2\|_\theta\frac{(\Psi(\mathfrak{z})-\Psi(0))^{\alpha_2}}{\underline{\eta}\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_2-\sigma_2+1)}-\frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_2-\sigma_1+1)}|}\\
&\quad\times\left[\frac{|\eta(R)|\overline{\chi}}{\underline{\eta}}\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(e^{\theta(\Psi(R)-\Psi(0))}\right)+\frac{L_f|\eta(R)|}{\underline{\eta}}\mathcal{I}_{0^+}^{\alpha_1+\alpha_2-\sigma_1,\Psi}\left(e^{\theta(\Psi(R)-\Psi(0))}\right)\right.\\
&\quad\left.+\frac{\overline{\chi}|\chi(R)|}{\underline{\eta}}\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(e^{\theta(\Psi(R)-\Psi(0))}\right)+\frac{L_f|\chi(R)|}{\underline{\eta}}\mathcal{I}_{0^+}^{\alpha_1+\alpha_2-\sigma_2,\Psi}\left(e^{\theta(\Psi(R)-\Psi(0))}\right)\right].
\end{aligned}$$

By using (6), we get

$$
\begin{aligned}
&\left|\mathcal{N}u_1(\mathfrak{z})-\mathcal{N}u_2(\mathfrak{z})\right| \\
&\leq \left(\frac{L_f}{\underline{\eta}\theta^{\alpha_1+\alpha_2}}+\frac{\overline{\chi}}{\underline{\eta}\theta^{\alpha_2}}+\frac{\frac{(\Psi(\mathfrak{z})-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{\left|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)}-\frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}\right|}\right. \\
&\quad\left.\times\left[\frac{|\eta(R)|\overline{\chi}}{\underline{\eta}\theta^{\alpha_2-\sigma_1}}+\frac{L_f|\eta(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_1}}+\frac{\overline{\chi}|\chi(R)|}{\underline{\eta}\theta^{\alpha_2-\sigma_2}}+\frac{L_f|\chi(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_2}}\right]\right)e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}\|u_1-u_2\|_\theta.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\|\mathcal{N}u_1-\mathcal{N}u_2\|_\theta \\
&\leq \left(\frac{L_f}{\underline{\eta}\theta^{\alpha_1+\alpha_2}}+\frac{\overline{\chi}}{\underline{\eta}\theta^{\alpha_2}}+\frac{\frac{(\Psi(R)-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{\left|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)}-\frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}\right|}\right. \\
&\quad\left.\times\left[\frac{|\eta(R)|\overline{\chi}}{\underline{\eta}\theta^{\alpha_2-\sigma_1}}+\frac{L_f|\eta(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_1}}+\frac{\overline{\chi}|\chi(R)|}{\underline{\eta}\theta^{\alpha_2-\sigma_2}}+\frac{L_f|\chi(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_2}}\right]\right)\|u_1-u_2\|_\theta.
\end{aligned}
$$

Hence, we obtain

$$
\|\mathcal{N}u_1-\mathcal{N}u_2\|_\theta \leq (L_f\Lambda_\theta+\nabla_\theta)\|u_1-u_2\|_\theta.
$$

By choosing $\theta > 0$ large enough such that

$$
\begin{aligned}
&\left(\frac{L_f}{\underline{\eta}\theta^{\alpha_1+\alpha_2}}+\frac{\overline{\chi}}{\underline{\eta}\theta^{\alpha_2+\alpha_3}}+\frac{\frac{(\Psi(R)-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{\left|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)}-\frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}\right|}\right. \\
&\quad\left.\times\left[\frac{|\eta(R)|\overline{\chi}}{\underline{\eta}\theta^{\alpha_2-\sigma_1}}+\frac{L_f|\eta(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_1}}+\frac{\overline{\chi}|\chi(R)|}{\underline{\eta}\theta^{\alpha_2-\sigma_2}}+\frac{L_f|\chi(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_2}}\right]\right)<1,
\end{aligned}
$$

we conclude that the mapping $\mathcal{N}$ is a contraction relative to the Ψ-Bielecki norm. Exploiting the Banach fixed point Theorem 3.1, it follows that $\mathcal{N}$ has a unique fixed point which is a unique solution to the (FGSLL)-problem (1). □

Corollary 3.4 *Let* (H1) *and* (H2) *be satisfied. Then,*

- *If* $\chi(\mathfrak{z}) = 0$ *for* $\mathfrak{z} \in I$, *then we have* $\overline{\chi} = 0$ *and one solution is guaranteed for the (FSL)-problem* (2) *on* I.
- *If* $\eta(\mathfrak{z}) = 1$ *and* $\chi(\mathfrak{z}) = \lambda$ $(\lambda \in \mathbb{R})$ *for* $\mathfrak{z} \in I$, *then we have* $\underline{\eta} = 1$ *and* $\overline{\chi} = |\lambda|$, *and so the (FL)-problem* (3) *has a unique solution on* I.

3.2 Application of Krasnoselskii's fixed point theorem for existence results

First, we recall Arzelà–Ascoli and Krasnoselskii theorems and then give our main results.

Lemma 3.5 ([40]) *A family of functions in* $\mathcal{C}([a_1,a_2])$ *is relatively compact if it is both equicontinuous and uniformly bounded on* $[a_1,a_2]$.

Lemma 3.6 ([40]) *Consider a nonempty subset* M *of a Banach space* U *that is bounded, closed, and convex. Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be operators so that*:

1. $\mathcal{P}x + \mathcal{Q}y \in M$ *whenever* $x, y \in M$,
2. $\mathcal{Q}$ *is a contraction*,
3. $\mathcal{P}$ *is compact and continuous*,

Then there exists $\varpi \in M$ *so that* $\varpi = \mathcal{P}\varpi + \mathcal{Q}\varpi$.

Now, we present the following existence theorem which is proved using the above lemmas.

Theorem 3.7 *Suppose that* (H1) *and* (H3) *hold. The* (*FGSLL*)*-problem* (1) *has at least one solution defined on* $[0,R]$ *under the following condition*:

$$\nabla < 1. \tag{22}$$

Proof We fix $r_3 \geq \frac{\Lambda\|\mathcal{K}_f\|}{1-\nabla}$ with $\|\mathcal{K}_f\| = \sup_{\mathfrak{z}\in[0,R]} |\mathcal{K}_f(\mathfrak{z})|$, and consider the closed ball $\mathcal{B}_{r_3}(u) = \{u \in \mathfrak{C}, \|u\| \leq r_3\}$ which is a convex and nonempty subset of the Banach space $\mathfrak{C}$. For each $\mathfrak{z} \in [0,R]$ and any $x \in \mathcal{B}_{r_3}$, we have

$$\left|\mathcal{N}u(\mathfrak{z})\right| \leq \left|\mathcal{N}_1 u(\mathfrak{z})\right| + \left|\mathcal{N}_2 u(\mathfrak{z})\right|$$

which implies that

$$\|\mathcal{N}u\| \leq \|\mathcal{N}_1 u\| + \|\mathcal{N}_2 u\|. \tag{23}$$

Claim 1: For $u, v \in \mathcal{B}_{r_3}$ we show that $\mathcal{N}_1 u + \mathcal{N}_2 v \in \mathcal{B}_{r_3}$.

Let $u \in \mathcal{B}_{r_3}$, then

$$\begin{aligned}\left|\mathcal{N}_1 u(\mathfrak{z})\right| \leq{}& \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{1}{|\eta(\mathfrak{z})|}\mathcal{I}_{0^+}^{\alpha_1,\Psi}\left|f\left(\mathfrak{z},u(\mathfrak{z})\right)\right|\right)\\ &+\frac{\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{|\eta(\mathfrak{z})|})}{|\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)}) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})|}\\ &\times\left[\left|\eta(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{1}{|\eta(R)|}\mathcal{I}_{0^+}^{\alpha_1,\Psi}\left|f\left(R,u(R)\right)\right|\right)\right.\\ &\left.+\left|\chi(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{1}{|\eta(R)|}\mathcal{I}_{0^+}^{\alpha_1,\Psi}\left|f\left(R,u(R)\right)\right|\right)\right].\end{aligned}$$

By using $||\kappa| - |\ell|| \leq |\kappa + \ell|$, where

$$\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)} \neq \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)},$$

we get

$$\begin{aligned}\left|\mathcal{N}_1 u(\mathfrak{z})\right| \leq{}& \left(\frac{(\Psi(\mathfrak{z})-\Psi(0))^{\alpha_1+\alpha_2}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2+1)} + \frac{\frac{(\Psi(\mathfrak{z})-\Psi(0))^{\alpha_2}}{\underline{\eta}\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_2-\sigma_1+1)}|}\right.\\ &\left.\times\left[\frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_1+1)} + \frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_2+1)}\right]\right)\|\mathcal{K}_f\|,\end{aligned}$$

which means that

$$\|\mathcal{N}_1 u\| \leq \Bigg(\frac{(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2+1)} + \frac{\frac{(\Psi(R)-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}|} \times \left[\frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_1+1)} + \frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_2+1)} \right] \Bigg) \|\mathcal{K}_f\|$$

and yields

$$\|\mathcal{N}_1 u\| \leq \Lambda \|\mathcal{K}_f\|. \tag{24}$$

Similarly, if $v \in \mathcal{B}_{r_3}$, then

$$\begin{aligned} \left|\mathcal{N}_2 v(\mathfrak{z})\right| &\leq \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{|\chi(\mathfrak{z})||v(\mathfrak{z})|}{|\eta(\mathfrak{z})|}\right) + \frac{\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{|\eta(\mathfrak{z})|})}{|\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)}) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})|} \\ &\quad \times \left[|\eta(R)| \mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{|\chi(R)v(R)|}{|\eta(R)|}\right) + |\chi(R)| \mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{|\chi(R)||v(R)|}{|\eta(R)|}\right)\right] \\ &\leq \frac{\overline{\chi}\|v\|(\Psi(\mathfrak{z})-\Psi(0))^{\alpha_2}}{\underline{\eta}\Gamma(\alpha_2+1)} + \frac{\frac{(\Psi(\mathfrak{z})-\Psi(0))^{\alpha_2}}{\underline{\eta}\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_2-\sigma_1+1)}|} \\ &\quad \times \left[\frac{|\chi(R)|\overline{\chi}\|v\|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_2-\sigma_2+1)} + \frac{|\eta(R)|\overline{\chi}\|v\|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_2-\sigma_1+1)} \right]. \end{aligned}$$

This implies that

$$\|\mathcal{N}_2 v\| \leq \Bigg(\frac{\overline{\chi}(\Psi(R)-\Psi(0))^{\alpha_2}}{\underline{\eta}\Gamma(\alpha_2+1)} + \frac{\frac{(\Psi(R)-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}|} \times \left[\frac{|\chi(R)|\overline{\chi}(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_2-\sigma_2+1)} + \frac{|\eta(R)|\overline{\chi}(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_2-\sigma_1+1)} \right] \Bigg) \|v\|,$$

yielding

$$\|\mathcal{N}_2 v\| \leq \nabla \|v\|. \tag{25}$$

Inserting (24) and (25) into (23), we get

$$\|\mathcal{N}_1 u + \mathcal{N}_2 v\| \leq \Lambda_\theta \|\mathcal{K}_f\| + \nabla r_3 \leq r_3, \tag{26}$$

which implies that $\mathcal{N}_1 u + \mathcal{N}_2 v \in \mathcal{B}_{r_3}$ for all $u, v \in \mathcal{B}_{r_3}$. Thus assumption 1 of Lemma 3.6 is verified.

Claim 2: We show that $\mathcal{N}_2$ is contraction.

For each $u_1, u_2 \in \mathcal{B}_{r_3}$ and $\mathfrak{z} \in [0, R]$, we have

$$\begin{aligned}
&\left|\mathcal{N}_2 u_1(\mathfrak{z}) - \mathcal{N}_2 u_2(\mathfrak{z})\right| \\
&\leq \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{|\chi(\mathfrak{z})||u_1(\mathfrak{z}) - u_2(\mathfrak{z})|}{|\eta(\mathfrak{z})|}\right) + \frac{\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{|\eta(\mathfrak{z})|})}{|\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)}) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})|} \\
&\quad \times \left[\left|\eta(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{|\chi(R)||u_1(R) - u_2(R)|}{|\eta(R)|}\right)\right. \\
&\quad \left. + \left|\chi(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{|\chi(R)||u_1(R) - u_2(R)|}{|\eta(R)|}\right)\right] \\
&\leq \frac{\overline{\chi}\|u_1 - u_2\|(\Psi(\mathfrak{z}) - \Psi(0))^{\alpha_2}}{\underline{\eta}\Gamma(\alpha_2 + 1)} + \frac{\frac{(\Psi(\mathfrak{z})-\Psi(0))^{\alpha_2}}{\underline{\eta}\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_2-\sigma_1+1)}|} \\
&\quad \times \left[\frac{|\chi(R)|\overline{\chi}\|u_1 - u_2\|(\Psi(R) - \Psi(0))^{\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_2 - \sigma_2 + 1)}\right. \\
&\quad \left. + \frac{|\eta(R)|\overline{\chi}\|u_1 - u_2\|(\Psi(R) - \Psi(0))^{\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_2 - \sigma_1 + 1)}\right] \\
&\leq \left(\frac{\overline{\chi}(\Psi(\mathfrak{z}) - \Psi(0))^{\alpha_2}}{\underline{\eta}\Gamma(\alpha_2 + 1)} + \frac{\frac{(\Psi(\mathfrak{z})-\Psi(0))^{\alpha_2}}{\underline{\eta}\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}|}\right. \\
&\quad \left. \times \left[\frac{|\chi(R)|\overline{\chi}(\Psi(R) - \Psi(0))^{\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_2 - \sigma_2 + 1)} + \frac{|\eta(R)|\overline{\chi}(\Psi(R) - \Psi(0))^{\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_2 - \sigma_1 + 1)}\right]\right)\|u_1 - u_2\|,
\end{aligned}$$

which yields

$$\|\mathcal{N}_2 u_1 - \mathcal{N}_2 u_2\| \leq \nabla\|u_1 - u_2\|.$$

Hence, by (22), $\mathcal{N}_2$ is a contraction.

Claim 3: Assumption 3 in Lemma 3.6 holds.

Take a sequence $\{u_n\}_{n\in\mathbb{N}}$ with $u_n \to u \in \mathfrak{C}$ as $n \to \infty$. For $\mathfrak{z} \in [0, R]$, we get

$$\begin{aligned}
&\left|\mathcal{N}_1 u_n(\mathfrak{z}) - \mathcal{N}_1 u(\mathfrak{z})\right| \\
&\leq \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\mathcal{I}_{0^+}^{\alpha_1,\Psi}|f(\mathfrak{z}, u_n(\mathfrak{z})) - f(\mathfrak{z}, u(\mathfrak{z}))|}{|\eta(\mathfrak{z})|}\right) \\
&\quad + \frac{\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{|\eta(\mathfrak{z})|})}{|\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)}) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})|} \\
&\quad \times \left[\left|\eta(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{\mathcal{I}_{0^+}^{\alpha_1,\Psi}|f(R, u_n(R)) - f(R, u(R))|}{|\eta(R)|}\right)\right. \\
&\quad \left. + \left|\chi(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{\mathcal{I}_{0^+}^{\alpha_1,\Psi}|f(R, u_n(R)) - f(R, u(R))|}{|\eta(R)|}\right)\right] \\
&\leq \left(\frac{(\Psi(\mathfrak{z}) - \Psi(0))^{\alpha_1+\alpha_2}}{\underline{\eta}\Gamma(\alpha_1 + \alpha_2 + 1)} + \frac{\frac{(\Psi(\mathfrak{z})-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}|}\right.
\end{aligned}$$

$$\times\left[\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_2+1)}+\frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_1+1)}\right]\Bigg)\|f_n-f\|,$$

with $\|f_n - f\| = \sup_{\mathfrak{z}\in[0,R]} |f(\mathfrak{z}, u_n(\mathfrak{z})) - f(\mathfrak{z}, u(\mathfrak{z}))|$. Thus

$$\begin{aligned}&\|\mathcal{N}_1u_n-\mathcal{N}_1u\|\\&\quad\le\Bigg(\frac{(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2+1)}+\frac{\frac{(\Psi(R)-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)}-\frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}|}\\&\qquad\times\left[\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_2+1)}+\frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_1+1)}\right]\Bigg)\|f_n-f\|,\end{aligned}$$

where

$$\begin{aligned}&\Bigg(\frac{(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2+1)}\\&\quad+\frac{\frac{(\Psi(R)-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)}-\frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}|}\\&\quad\times\left[\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_2+1)}+\frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_1+1)}\right]\Bigg)<\infty.\end{aligned}$$

The Lebesgue's dominated convergence theorem and continuity of f lead to the conclusion that $\|\mathcal{N}_1u_n - \mathcal{N}_1u\| \to 0$ as $n \to \infty$. Therefore, $\mathcal{N}_1$ is continuous. Furthermore, $\mathcal{N}_1$ is uniformly bounded on $\mathcal{B}_{r_3}$ as $\|\mathcal{N}_1u\| \le \Lambda\|\mathcal{K}_f\|$ due to (24). Also, $\mathcal{N}_1$ is equicontinuous. Indeed, letting $u \in \mathcal{B}_{r_3}$, for $\mathfrak{z}_1, \mathfrak{z}_2 \in [0, R]$, $\mathfrak{z}_1 < \mathfrak{z}_2$, we have

$$\begin{aligned}&\left|\mathcal{N}_1u(\mathfrak{z}_2)-\mathcal{N}_1u(\mathfrak{z}_1)\right|\\&\quad\le\left|\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\mathcal{I}_{0^+}^{\alpha_1,\Psi}f(\mathfrak{z}_2,u(\mathfrak{z}_2))}{\eta(\mathfrak{z}_2)}\right)-\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\mathcal{I}_{0^+}^{\alpha_1,\Psi}f(\mathfrak{z}_1,u(\mathfrak{z}_1))}{\eta(\mathfrak{z}_1)}\right)\right|\\&\qquad+\frac{|\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{\eta(\mathfrak{z}_2)})-\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{\eta(\mathfrak{z}_1)})|}{|\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)})+\chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})|}\\&\qquad\times\left|\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{\mathcal{I}_{0^+}^{\alpha_1,\Psi}f(R,u(R))}{\eta(R)}\right)+\chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{\mathcal{I}_{0^+}^{\alpha_1,\Psi}f(R,u(R))}{\eta(R)}\right)\right|,\end{aligned}$$

i.e.,

$$\begin{aligned}&\left|\mathcal{N}_1u(\mathfrak{z}_2)-\mathcal{N}_1u(\mathfrak{z}_1)\right|\\&\quad\le\frac{1}{\underline{\eta}\Gamma(\alpha_2)\Gamma(\alpha_1)}\\&\qquad\times\left[\int_0^{\mathfrak{z}_1}\Psi'(s)\left|\big(\Psi(\mathfrak{z}_2)-\Psi(s)\big)^{\alpha_2-1}-\big(\Psi(\mathfrak{z}_1)-\Psi(s)\big)^{\alpha_2-1}\right|\right.\\&\qquad\times\int_0^s\Psi'(x)\big(\Psi(s)-\Psi(x)\big)^{\alpha_1-1}\left|f\big(x,u(x)\big)\right|dx\,ds\\&\qquad\left.+\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\Psi'(s)\big(\Psi(\mathfrak{z}_2)-\Psi(s)\big)^{\alpha_2-1}\int_0^s\Psi'(x)\big(\Psi(s)-\Psi(x)\big)^{\alpha_1-1}\left|f\big(x,u(x)\big)\right|dx\,ds\right]\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|\eta(R)|\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{|\eta(R)|})-|\chi(R)|\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})}\\
&\times\Bigg[\frac{1}{\underline{\eta}\Gamma(\alpha_2)}\bigg(\int_0^{\mathfrak{z}_1}\Psi'(s)\big|\big(\Psi(\mathfrak{z}_2)-\Psi(s)\big)^{\alpha_2-1}-\big(\Psi(\mathfrak{z}_1)-\Psi(s)\big)^{\alpha_2-1}\big|\,ds\\
&+\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\Psi'(s)\big(\Psi(\mathfrak{z}_2)-\Psi(s)\big)^{\alpha_2-1}\,ds\bigg)\Bigg]\\
&\times\Bigg[\big|\eta(R)\big|\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\bigg(\frac{\mathcal{I}_{0^+}^{\alpha_1,\Psi}|f(R,u(R))|}{|\eta(R)|}\bigg)\\
&+\big|\chi(R)\big|\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\bigg(\frac{\mathcal{I}_{0^+}^{\alpha_1,\Psi}|f(R,u(R))|}{|\eta(R)|}\bigg)\Bigg]\\
\le{}&\frac{\|\mathcal{K}_f\|}{\underline{\eta}\Gamma(\alpha_1+1)\Gamma(\alpha_2)}\bigg[\int_0^{\mathfrak{z}_1}\Psi'(s)\Big[\big(\Psi(\mathfrak{z}_2)-\Psi(s)\big)^{\alpha_2-1}\big(\Psi(s)-\Psi(0)\big)^{\alpha_1}\,ds\\
&-\int_0^{\mathfrak{z}_1}\Psi'(s)\big(\Psi(\mathfrak{z}_1)-\Psi(s)\big)^{\alpha_2-1}\big(\Psi(s)-\Psi(0)\big)^{\alpha_1}\,ds\\
&+\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\Psi'(s)\big(\Psi(\mathfrak{z}_2)-\Psi(s)\big)^{\alpha_2-1}\big(\Psi(s)-\Psi(0)\big)^{\alpha_1}\,ds\Big]\\
&+\frac{1}{\big|\frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_2-\sigma_1+1)}-\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_2-\sigma_2+1)}\big|}\\
&\times\Bigg[\frac{1}{\underline{\eta}\Gamma(\alpha_2+1)}\Big(\big(\Psi(\mathfrak{z}_2)-\Psi(s)\big)^{\alpha_2}\big|_0^{\mathfrak{z}_1}-\big(\Psi(\mathfrak{z}_1)-\Psi(s)\big)^{\alpha_2}\big|_0^{\mathfrak{z}_1}\\
&+\big(\Psi(\mathfrak{z}_2)-\Psi(s)\big)^{\alpha_2}\big|_{\mathfrak{z}_1}^{\mathfrak{z}_2}\Big)\Bigg]\\
&\times\bigg[\frac{|\eta(R)|\|\mathcal{K}_f\|(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_1+1)}+\frac{|\chi(R)|\|\mathcal{K}_f\|(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_2+1)}\bigg].
\end{aligned}
$$

Finally, we get

$$
\begin{aligned}
&\big|\mathcal{N}_1u(\mathfrak{z}_2)-\mathcal{N}_1u(\mathfrak{z}_1)\big|\\
&\le\Bigg[\frac{(\Psi(R)-\Psi(0))^{\alpha_1}((\Psi(\mathfrak{z}_1)-\Psi(0))^{\alpha_2}-(\Psi(\mathfrak{z}_2)-\Psi(0))^{\alpha_2})}{\underline{\eta}\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\\
&+\frac{(\Psi(\mathfrak{z}_1)-\Psi(0))^{\alpha_2}-(\Psi(\mathfrak{z}_2)-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)\big|\frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}-\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)}\big|}\\
&\times\bigg(\frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_1+1)}+\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_2+1)}\bigg)\Bigg]\|\mathcal{K}_f\|.
\end{aligned}
\tag{27}
$$

The right-hand side of (27) is clearly independent of u and $|\mathcal{N}_1u(\mathfrak{z}_2)-\mathcal{N}_1u(\mathfrak{z}_1)|\to 0$ as $\mathfrak{z}_2\to\mathfrak{z}_1$. Hence, this implies that $\mathcal{N}_1\mathcal{B}_{r_3}$ is equicontinuous and $\mathcal{N}_1$ maps bounded subsets into relatively compact subsets, which implies that $\mathcal{N}_1\mathcal{B}_{r_3}$ is relatively compact.

Therefore, using Lemma 3.5, we determine that $\mathcal{N}_1$ is compact in $\mathcal{B}_{r_3}$. Then, in view of Lemma 3.6, this guarantees at least one solution for the problem (1) in $[0,R]$. □

Before stating and proving the results via Krasnoselskii and Leray–Schauder fixed point theorems under the Ψ-Bielecki's norm, we provide an auxiliary lemma which is related to the proof of the equicontinuity property.

Lemma 3.8 *For a given* $\eta \in C(I, \mathbb{R}^\star)$, *let* (H1) *and* (H3) *hold. For all* $\theta > 0$ *and with* $0 < \alpha_i \leq 1$, $i \in \{1,2\}$, *we have*

$$\begin{aligned} &\left| \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\mathcal{I}_{0^+}^{\alpha_1,\Psi} f(\mathfrak{z}_2, u(\mathfrak{z}_2))}{\eta(\mathfrak{z}_2)}\right) - \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\mathcal{I}_{0^+}^{\alpha_1,\Psi} f(\mathfrak{z}_1, u(\mathfrak{z}_1))}{\eta(\mathfrak{z}_1)}\right)\right| \\ &\quad \leq \frac{1}{\underline{\eta}}\left(\frac{e^{\theta(\Psi(\mathfrak{z}_2)-\Psi(0))}}{\theta^{\alpha_1+\alpha_2}} - \frac{e^{\theta(\Psi(\mathfrak{z}_1)-\Psi(0))}}{\theta^{\alpha_1+\alpha_2}}\right. \\ &\qquad \left. + \frac{1}{\theta^{\alpha_1}\Gamma(\alpha_2)}\int_{\mathfrak{z}_1}^{\mathfrak{z}_2} \Psi'(s)\big(\Psi(\mathfrak{z}_2)-\Psi(s)\big)^{\alpha_2-1} e^{\theta(\Psi(s)-\Psi(0))}\,ds\right)\|\mathcal{K}_f\|_\theta, \end{aligned} \tag{28}$$

and

$$\begin{aligned} &\left| \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\chi(\mathfrak{z}_1)u(\mathfrak{z}_1)}{\eta(\mathfrak{z}_1)}\right) - \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\chi(\mathfrak{z}_2)u(\mathfrak{z}_2)}{\eta(\mathfrak{z}_2)}\right)\right| \\ &\quad \leq \frac{\overline{\chi}}{\underline{\eta}}\left(\frac{e^{\theta(\Psi(\mathfrak{z}_1)-\Psi(0))}}{\theta^{\alpha_1+\alpha_2}} - \frac{e^{\theta(\Psi(\mathfrak{z}_2)-\Psi(0))}}{\theta^{\alpha_1+\alpha_2}}\right. \\ &\qquad \left. - \frac{1}{\theta^{\alpha_1}\Gamma(\alpha_2)}\int_{\mathfrak{z}_1}^{\mathfrak{z}_2} \Psi'(s)\big(\Psi(\mathfrak{z}_2)-\Psi(s)\big)^{\alpha_2-1} e^{\theta(\Psi(s)-\Psi(0))}\,ds\right)\|u\|_\theta. \end{aligned} \tag{29}$$

Proof Let $\mathfrak{z}_1, \mathfrak{z}_2 \in [0, R]$ where $\mathfrak{z}_1 < \mathfrak{z}_2$, we have

$$\begin{aligned} &\left| \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\mathcal{I}_{0^+}^{\alpha_1,\Psi} f(\mathfrak{z}_2, u(\mathfrak{z}_2))}{\eta(\mathfrak{z}_2)}\right) - \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\mathcal{I}_{0^+}^{\alpha_1,\Psi} f(\mathfrak{z}_1, u(\mathfrak{z}_1))}{\eta(\mathfrak{z}_1)}\right)\right| \\ &\quad \leq \left| \int_0^{\mathfrak{z}_1}\left[\frac{\Psi'(s)(\Psi(\mathfrak{z}_2)-\Psi(s))^{\alpha_2-1}}{\eta(s)\Gamma(\alpha_2)} - \frac{\Psi'(s)(\Psi(\mathfrak{z}_1)-\Psi(s))^{\alpha_2-1}}{\eta(s)\Gamma(\alpha_2)}\right]\right. \\ &\qquad \times \int_0^s \frac{\Psi'(x)(\Psi(s)-\Psi(x))^{\alpha_1-1}}{\Gamma(\alpha_1)} f\big(x, u(x)\big)\,dx\,ds \\ &\qquad \left. + \int_{\mathfrak{z}_1}^{\mathfrak{z}_2} \frac{\Psi'(s)(\Psi(\mathfrak{z}_2)-\Psi(s))^{\alpha_2-1}}{|\eta|(s)\Gamma(\alpha_2)} \int_0^s \frac{\Psi'(x)(\Psi(s)-\Psi(x))^{\alpha_1-1}}{\Gamma(\alpha_1)} f\big(x, u(x)\big)\,dx\,ds\right| \\ &\quad \leq \int_0^{\mathfrak{z}_1}\left|\frac{\Psi'(s)(\Psi(\mathfrak{z}_2)-\Psi(s))^{\alpha_2-1}}{\eta(s)\Gamma(\alpha_2)} - \frac{\Psi'(s)(\Psi(\mathfrak{z}_1)-\Psi(s))^{\alpha_2-1}}{\eta(s)\Gamma(\alpha_2)}\right| \\ &\qquad \times \int_0^s \frac{\Psi'(x)(\Psi(s)-\Psi(x))^{\alpha_1-1}|f(x,u(x))| e^{\theta(\Psi(x)-\Psi(0))}}{\Gamma(\alpha_1)e^{\theta(\Psi(x)-\Psi(0))}}\,dx\,ds \\ &\qquad + \int_{\mathfrak{z}_1}^{\mathfrak{z}_2} \frac{\Psi'(s)(\Psi(\mathfrak{z}_2)-\Psi(s))^{\alpha_2-1}}{|\eta|(s)\Gamma(\alpha_2)} \\ &\qquad \times \int_0^s \frac{\Psi'(x)(\Psi(s)-\Psi(x))^{\alpha_1-1}|f(x,u(x))| e^{\theta(\Psi(x)-\Psi(0))}}{\Gamma(\alpha_1)e^{\theta(\Psi(x)-\Psi(0))}}\,dx\,ds. \end{aligned}$$

By using (7), we get

$$\left| \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\mathcal{I}_{0^+}^{\alpha_1,\Psi} f(\mathfrak{z}_2, u(\mathfrak{z}_2))}{\eta(\mathfrak{z}_2)}\right) - \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\mathcal{I}_{0^+}^{\alpha_1,\Psi} f(\mathfrak{z}_1, u(\mathfrak{z}_1))}{\eta(\mathfrak{z}_1)}\right)\right|$$

$$\leq \left(\int_0^{\mathfrak{z}_1}\left|\frac{\Psi'(s)(\Psi(\mathfrak{z}_2)-\Psi(s))^{\alpha_2-1}}{\eta(s)\Gamma(\alpha_2)}-\frac{\Psi'(s)(\Psi(\mathfrak{z}_1)-\Psi(s))^{\alpha_2-1}}{\eta(s)\Gamma(\alpha_2)}\right|\frac{e^{\theta(\Psi(s)-\Psi(0))}}{\theta^{\alpha_1}}\,ds + \int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\frac{\Psi'(s)(\Psi(\mathfrak{z}_2)-\Psi(s))^{\alpha_2-1}e^{\theta(\Psi(s)-\Psi(0))}}{\underline{\eta}\Gamma(\alpha_2)\theta^{\alpha_1}}\,ds\right)\|\mathcal{K}_f\|_\theta,$$

thus, we have

$$\begin{aligned}
&\left|\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\mathcal{I}_{0^+}^{\alpha_1,\Psi}f(\mathfrak{z}_2,u(\mathfrak{z}_2))}{\eta(\mathfrak{z}_2)}\right)-\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\mathcal{I}_{0^+}^{\alpha_1,\Psi}f(\mathfrak{z}_1,u(\mathfrak{z}_1))}{\eta(\mathfrak{z}_1)}\right)\right| \\
&\leq \frac{1}{\underline{\eta}\Gamma(\alpha_2)\theta^{\alpha_1}}\left(\int_0^{\mathfrak{z}_1}\Psi'(s)\big(\Psi(\mathfrak{z}_2)-\Psi(s)\big)^{\alpha_2-1}e^{\theta(\Psi(s)-\Psi(0))}\,ds\right. \\
&\quad -\int_0^{\mathfrak{z}_1}\Psi'(s)\big(\Psi(\mathfrak{z}_1)-\Psi(s)\big)^{\alpha_2-1}e^{\theta(\Psi(s)-\Psi(0))}\,ds \\
&\quad \left.+\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\Psi'(s)\big(\Psi(\mathfrak{z}_2)-\Psi(s)\big)^{\alpha_2-1}e^{\theta(\Psi(s)-\Psi(0))}\,ds\right)\|\mathcal{K}_f\|_\theta \\
&\leq \frac{1}{\underline{\eta}\theta^{\alpha_1}}\left(\frac{e^{\theta(\Psi(\mathfrak{z}_2)-\Psi(0))}}{\theta^{\alpha_2}}-\frac{e^{\theta(\Psi(\mathfrak{z}_1)-\Psi(0))}}{\theta^{\alpha_2}}\right. \\
&\quad \left.+\frac{1}{\Gamma(\alpha_2)}\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\Psi'(s)\big(\Psi(\mathfrak{z}_2)-\Psi(s)\big)^{\alpha_2-1}e^{\theta(\Psi(s)-\Psi(0))}\,ds\right)\|\mathcal{K}_f\|_\theta.
\end{aligned}$$

Similarly, for $\mathfrak{z}_1,\mathfrak{z}_2\in[0,R]$ where $\mathfrak{z}_1<\mathfrak{z}_2$, we get

$$\begin{aligned}
&\left|\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\chi(\mathfrak{z}_1)u(\mathfrak{z}_1)}{\eta(\mathfrak{z}_1)}\right)-\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\chi(\mathfrak{z}_2)u(\mathfrak{z}_2)}{\eta(\mathfrak{z}_2)}\right)\right| \\
&\leq \int_0^{\mathfrak{z}_1}\left|\frac{\Psi'(s)(\Psi(\mathfrak{z}_1)-\Psi(s))^{\alpha_2-1}}{\Gamma(\alpha_2)}-\frac{\Psi'(s)(\Psi(\mathfrak{z}_2)-\Psi(s))^{\alpha_2-1}}{\Gamma(\alpha_2)}\right| \\
&\quad \times\frac{|\chi(s)||u(s)|e^{\theta(\Psi(s)-\Psi(0))}}{|\eta(s)|e^{\theta(\Psi(s)-\Psi(0))}|}\,ds \\
&\quad +\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\frac{\Psi'(s)(\Psi(\mathfrak{z}_2)-\Psi(s))^{\alpha_2-1}}{\Gamma(\alpha_2)}\frac{|\chi(s)||u(s)|e^{\theta(\Psi(s)-\Psi(0))}}{|\eta(s)|e^{\theta(\Psi(s)-\Psi(0))}|}\,ds \\
&\leq \frac{\overline{\chi}\|u\|_\theta}{\underline{\eta}\Gamma(\alpha_2)}\left[\int_0^{\mathfrak{z}_1}\Psi'(s)\big(\Psi(\mathfrak{z}_1)-\Psi(s)\big)^{\alpha_2-1}e^{\theta(\Psi(s)-\Psi(0))}\,ds\right. \\
&\quad -\int_0^{\mathfrak{z}_1}\Psi'(s)\big(\Psi(\mathfrak{z}_2)-\Psi(s)\big)^{\alpha_2-1}e^{\theta(\Psi(s)-\Psi(0))}\,ds \\
&\quad \left.+\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\Psi'(s)\big(\Psi(\mathfrak{z}_2)-\Psi(s)\big)^{\alpha_2-1}e^{\theta(\Psi(s)-\Psi(0))}\,ds\right].
\end{aligned}$$

By using (6) and (7), we obtain

$$\begin{aligned}
&\left|\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\chi(\mathfrak{z}_1)u(\mathfrak{z}_1)}{\eta(\mathfrak{z}_1)}\right)-\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\chi(\mathfrak{z}_2)u(\mathfrak{z}_2)}{\eta(\mathfrak{z}_2)}\right)\right| \\
&\leq \frac{\overline{\chi}}{\underline{\eta}}\left(\frac{e^{\theta(\Psi(\mathfrak{z}_1)-\Psi(0))}}{\theta^{\alpha_2}}-\frac{e^{\theta(\Psi(\mathfrak{z}_2)-\Psi(0))}}{\theta^{\alpha_2}}\right. \\
&\quad \left.+\frac{1}{\Gamma(\alpha_2)}\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\Psi'(s)\big(\Psi(\mathfrak{z}_2)-\Psi(s)\big)^{\alpha_2-1}e^{\theta(\Psi(s)-\Psi(0))}\,ds\right)\|u\|_\theta.
\end{aligned}$$

The proof is complete. □

Now, we discuss existence results by using the Krasnoselskii fixed point theorem and some inequalities of the Ψ-Bielecki's norm-type.

Theorem 3.9 *Let* (H1) *and* (H3) *hold. Then, at least one solution exists for the* (*FGSLL*)*-problem* (1) *on* $[0,R]$.

Proof We fix $r_4 \geq \frac{\Lambda_\theta \|\mathcal{K}_f\|_\theta}{1-\nabla_\theta}$, where Λ_θ and ∇_θ are constants defined by (18) and (19) and focus on the nonempty closed ball $\mathcal{B}_{r_4,\theta}(u) = \{u \in \mathfrak{C}, \|\mathrm{u}\|_\theta \leq r_4\}$ which is convex in the Banach space $\mathfrak{C}$.

For each $\mathfrak{z} \in [0,R]$ and $x \in \mathcal{B}_{r_4,\theta}$,

$$\frac{|\mathcal{N}u(\mathfrak{z})|}{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}} \leq \frac{|\mathcal{N}_1 u(\mathfrak{z})|}{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}} + \frac{|\mathcal{N}_2 u(\mathfrak{z})|}{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}},$$

which implies that

$$\|\mathcal{N}u\|_\theta \leq \|\mathcal{N}_1 u\|_\theta + \|\mathcal{N}_2 u\|_\theta. \tag{30}$$

Claim 1: For $u, v \in \mathcal{B}_{r_4,\theta}$, one has $\mathcal{N}_1 u + \mathcal{N}_2 v \in \mathcal{B}_{r_4,\theta}$.

To show this, let $u \in \mathcal{B}_{r_4,\theta}$. Then

$$\begin{aligned}\left|\mathcal{N}_1 u(\mathfrak{z})\right| \leq{}& \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{1}{|\eta(\mathfrak{z})|}\mathcal{I}_{0^+}^{\alpha_1,\Psi}\left(\frac{|f(\mathfrak{z},u(\mathfrak{z}))|e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}\right)\right) \\ &+ \frac{\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{|\eta(\mathfrak{z})|})}{|\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)}) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})|} \\ &\times \left[\left|\eta(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{1}{|\eta(R)|}\mathcal{I}_{0^+}^{\alpha_1,\Psi}\left(\frac{|f(R,u(R))|e^{\theta(\Psi(R)-\Psi(0))}}{e^{\theta(\Psi(R)-\Psi(0))}}\right)\right)\right. \\ &\left.+ \left|\chi(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{1}{|\eta(R)|}\mathcal{I}_{0^+}^{\alpha_1,\Psi}\left(\frac{|f(R,u(R))|e^{\theta(\Psi(R)-\Psi(0))}}{e^{\theta(\Psi(R)-\Psi(0))}}\right)\right)\right].\end{aligned}$$

By using $||a| - |b|| \leq |a+b|$ and taking into account

$$\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)} \neq \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)},$$

we find

$$\begin{aligned}\left|\mathcal{N}_1 u(\mathfrak{z})\right| \leq{}& \left(\frac{\mathcal{I}_{0^+}^{\alpha_1+\alpha_2,\Psi}(e^{\theta(\Psi(\mathfrak{z})-\Psi(0))})}{\underline{\eta}}\right. \\ &+ \frac{\frac{1}{\underline{\eta}}(\mathcal{I}_{0^+}^{\alpha_2,\Psi}1)(\mathfrak{z})}{\left|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_2-\sigma_1+1)}\right|} \\ &\times \left[\frac{|\eta(R)|}{\underline{\eta}}\mathcal{I}_{0^+}^{\alpha_1+\alpha_2-\sigma_1,\Psi}\left(e^{\theta(\Psi(R)-\Psi(0))}\right)\right. \\ &\left.\left.+ \frac{|\chi(R)|}{\underline{\eta}}\mathcal{I}_{0^+}^{\alpha_1+\alpha_2-\sigma_2,\Psi}\left(e^{\theta(\Psi(R)-\Psi(0))}\right)\right]\right)\|\mathcal{K}_f\|_\theta\end{aligned}$$

$$\leq \left(\frac{1}{\underline{\eta}\theta^{\alpha_1+\alpha_2}} + \frac{\frac{(\Psi(\mathfrak{z})-\Psi(0))^{\alpha_2}}{\underline{\eta}\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_2-\sigma_1+1)}|} \times \left[\frac{|\eta(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_1}} + \frac{|\chi(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_2}}\right]\right) e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}\|\mathcal{K}_f\|_\theta.$$

Consequently,

$$\|\mathcal{N}_1 u\| \leq \left(\frac{1}{\underline{\eta}\theta^{\alpha_1+\alpha_2}} + \frac{\frac{(\Psi(R)-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}|} \times \left[\frac{|\eta(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_1}} + \frac{|\chi(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_2}}\right]\right)\|\mathcal{K}_f\|_\theta,$$

which means that

$$\|\mathcal{N}_1 u\| \leq \Lambda_\theta \|\mathcal{K}_f\|_\theta. \tag{31}$$

Similarly, if $\nu \in \mathcal{B}_{r_4,\theta}$, then

$$\begin{aligned}
\left|\mathcal{N}_2\nu(\mathfrak{z})\right| &\leq \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{|\chi(\mathfrak{z})||\nu(\mathfrak{z})|}{|\eta(\mathfrak{z})|e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}} e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}\right) \\
&\quad + \frac{\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{|\eta(\mathfrak{z})|})}{|\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)}) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})|} \\
&\quad \times \left[\left|\eta(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{|\chi(R)||\nu(R)|}{|\eta(R)|e^{\theta(\Psi(R)-\Psi(0))}} e^{\theta(\Psi(R)-\Psi(0))}\right)\right. \\
&\quad \left. + \left|\chi(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{|\chi(R)||\nu(R)|}{|\eta(R)|e^{\theta(\Psi(R)-\Psi(0))}} e^{\theta(\Psi(R)-\Psi(0))}\right)\right] \\
&\leq \left(\frac{\overline{\chi}}{\underline{\eta}\theta^{\alpha_2}} + \frac{\frac{(\Psi(\mathfrak{z})-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}|}\right. \\
&\quad \left. \times \left[\frac{|\chi(R)|\overline{\chi}}{\underline{\eta}\theta^{\alpha_2-\sigma_2}} + \frac{|\eta(R)|\overline{\chi}}{\underline{\eta}\theta^{\alpha_2-\sigma_1}}\right]\right)\|\nu\|_\theta e^{\theta(\Psi(\mathfrak{z})-\Psi(0))},
\end{aligned}$$

implying the following inequality:

$$\|\mathcal{N}_2\nu\|_\theta \leq \left(\frac{\overline{\chi}}{\underline{\eta}\theta^{\alpha_2}} + \frac{\frac{(\Psi(R)-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}|} \times \left[\frac{|\chi(R)|\overline{\chi}}{\underline{\eta}\theta^{\alpha_2-\sigma_2}} + \frac{|\eta(R)|\overline{\chi}}{\underline{\eta}\theta^{\alpha_2-\sigma_1}}\right]\right)\|\nu\|_\theta.$$

This yields

$$\|\mathcal{N}_2\nu\|_\theta \leq \nabla_\theta \|\nu\|_\theta. \tag{32}$$

Inserting (31) and (32) into (30) gives

$$\|\mathcal{N}_1 u + \mathcal{N}_2\nu\|_\theta \leq \Lambda_\theta\|\mathcal{K}_f\|_\theta + \nabla_\theta r_4 \leq r_4,$$

which implies that $\mathcal{N}_1 u + \mathcal{N}_2 v \in \mathcal{B}_{r_4,\theta}$ for all $u, v \in \mathcal{B}_{r_4,\theta}$, and so assumption 1 of Lemma 3.6 is satisfied.

Claim 2: We show that $\mathcal{N}_2$ is a contraction.

For each $u_1, u_2 \in \mathcal{B}_{r_4,\theta}$, $\mathfrak{z} \in [0, R]$, we estimate

$$\begin{aligned}
\left|\mathcal{N}_2 u_1(\mathfrak{z}) - \mathcal{N}_2 u_2(\mathfrak{z})\right| &\leq \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{|\chi(\mathfrak{z})||u_1(\mathfrak{z}) - u_2(\mathfrak{z})|}{|\eta(\mathfrak{z})|e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}} e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}\right) \\
&\quad + \frac{\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{|\eta(\mathfrak{z})|})}{|\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)}) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})|} \\
&\quad \times \left[\left|\eta(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{|\chi(R)||u_1(R) - u_2(R)|}{|\eta(R)|e^{\theta(\Psi(R)-\Psi(0))}} e^{\theta(\Psi(R)-\Psi(0))}\right)\right. \\
&\quad \left. + \left|\chi(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{|\chi(R)||u_1(R) - u_2(R)|}{|\eta(R)|e^{\theta(\Psi(R)-\Psi(0))}} e^{\theta(\Psi(R)-\Psi(0))}\right)\right] \\
&\leq \left(\frac{\overline{\chi}}{\underline{\eta}\theta^{\alpha_2}} + \frac{\frac{(\Psi(\mathfrak{z})-\Psi(0))^{\alpha_2}}{\underline{\eta}\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_2-\sigma_1+1)}|}\right. \\
&\quad \left. \times \left[\frac{|\chi(R)|\overline{\chi}}{\underline{\eta}\theta^{\alpha_2-\sigma_2}} + \frac{|\eta(R)|\overline{\chi}}{\underline{\eta}\theta^{\alpha_2-\sigma_1}}\right]\right) e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}\|u_1 - u_2\|_\theta \\
&\leq \left(\frac{\overline{\chi}}{\underline{\eta}\theta^{\alpha_2}} + \frac{\frac{(\Psi(R)-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}|}\right. \\
&\quad \left. \times \left[\frac{|\chi(R)|\overline{\chi}}{\underline{\eta}\theta^{\alpha_2-\sigma_2}} + \frac{|\eta(R)|\overline{\chi}}{\underline{\eta}\theta^{\alpha_2-\sigma_1}}\right]\right)\|u_1 - u_2\|_\theta.
\end{aligned}$$

Then, this gives

$$\|\mathcal{N}_2 u_1 - \mathcal{N}_2 u_2\|_\theta \leq \nabla_\theta \|u_1 - u_2\|_\theta.$$

By choosing $\theta > 0$ large enough so that

$$\begin{aligned}
&\left(\frac{\overline{\chi}}{\underline{\eta}\theta^{\alpha_2}} + \frac{\frac{(\Psi(R)-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}|}\right. \\
&\quad \left. \times \left[\frac{|\chi(R)|\overline{\chi}}{\underline{\eta}\theta^{\alpha_2-\sigma_2}} + \frac{|\eta(R)|\overline{\chi}}{\underline{\eta}\theta^{\alpha_2-\sigma_1}}\right]\right) = \nabla_\theta < 1,
\end{aligned}$$

it follows that $\mathcal{N}_2$ is a contraction.

Claim 3: Next, we will verify that condition 3 of Lemma 3.6 holds.

Consider a sequence u_n so that $u_n \to u \in \mathfrak{C}$ as $n \to \infty$. For $\mathfrak{z} \in [0, R]$, we get the following inequality:

$$\begin{aligned}
&\left|\mathcal{N}_1 u_n(\mathfrak{z}) - \mathcal{N}_1 u(\mathfrak{z})\right| \\
&\quad \leq \frac{1}{\underline{\eta}}\mathcal{I}_{0^+}^{\alpha_1+\alpha_2,\Psi}\left(\frac{|f(\mathfrak{z}, u_n(\mathfrak{z})) - f(\mathfrak{z}, u(\mathfrak{z}))|}{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}} e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}\right) \\
&\qquad + \frac{\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{|\eta(\mathfrak{z})|})}{|\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)}) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})|}
\end{aligned}$$

$$\times \left[\frac{|\eta(R)|}{\underline{\eta}} \mathcal{I}_{0^+}^{\alpha_1+\alpha_2-\sigma_1,\Psi} \left(\frac{|f(R,u_n(R)) - f(R,u(R))|}{e^{\theta(\Psi(R)-\Psi(0))}} e^{\theta(\Psi(R)-\Psi(0))} \right) + \frac{|\chi(R)|}{\underline{\eta}} \mathcal{I}_{0^+}^{\alpha_1+\alpha_2-\sigma_2,\Psi} \left(\frac{|f(R,u_n(R)) - f(R,u(R))|}{e^{\theta(\Psi(R)-\Psi(0))}} e^{\theta(\Psi(R)-\Psi(0))} \right) \right].$$

Thus

$$\left|\mathcal{N}_1 u_n(\mathfrak{z}) - \mathcal{N}_1 u(\mathfrak{z})\right| \leq \left(\frac{1}{\underline{\eta}\theta^{\alpha_1+\alpha_2}} + \frac{\frac{(\Psi(\mathfrak{z})-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{\left|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}\right|} \times \left[\frac{|\chi(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_2}} + \frac{|\eta(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_1}} \right] \right) e^{\theta(\Psi(\mathfrak{z})-\Psi(0))} \|f_n - f\|_\theta,$$

and so

$$\|\mathcal{N}_1 u_n - \mathcal{N}_1 u\|_\theta \leq \left(\frac{1}{\underline{\eta}\theta^{\alpha_1+\alpha_2}} + \frac{\frac{(\Psi(R)-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{\left|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}\right|} \times \left[\frac{|\chi(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_2}} + \frac{|\eta(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_1}} \right] \right) \|f_n - f\|_\theta,$$

with

$$\|f_n - f\|_\theta = \sup_{t\in[0,R]} \frac{|f(\mathfrak{z},u_n(\mathfrak{z})) - f(\mathfrak{z},u(\mathfrak{z}))|}{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}$$

and

$$\left(\frac{1}{\underline{\eta}\theta^{\alpha_1+\alpha_2}} + \frac{\frac{(\Psi(R)-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{\left|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}\right|} \times \left[\frac{|\chi(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_2}} + \frac{|\eta(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_1}} \right] \right) < \infty.$$

The Lebesgue's dominated convergence theorem, along with the continuity of f, leads to the conclusion that $\|\mathcal{N}_1 u_n - \mathcal{N}_1 u\|_\theta \to 0$ as $\mathfrak{z} \to \infty$. Therefore, $\mathcal{N}_1$ is continuous. Besides, $\mathcal{N}_1$ is uniformly bounded on $\mathcal{B}_{r_4,\theta}$ as $\|\mathcal{N}_1 v\|_\theta \leq \Lambda \|\mathcal{K}_f\|_\theta$, due to (31).

Also, $\mathcal{N}_1$ is equicontinuous. Indeed, let $u \in \mathcal{B}_{r_4,\theta}$. Then for $\mathfrak{z}_1, \mathfrak{z}_2 \in [0,R]$, $\mathfrak{z}_1 < \mathfrak{z}_2$, we have

$$\begin{aligned}
&\left|\mathcal{N}_1 u(\mathfrak{z}_2) - \mathcal{N}_1 u(\mathfrak{z}_1)\right| \\
&\quad\leq \left| \mathcal{I}_{0^+}^{\alpha_2,\Psi} \left(\frac{1}{\eta(\mathfrak{z}_2)} \mathcal{I}_{0^+}^{\alpha_1,\Psi} f(\mathfrak{z}_2, u(\mathfrak{z}_2)) \right) - \mathcal{I}_{0^+}^{\alpha_2,\Psi} \left(\frac{1}{\eta(\mathfrak{z}_1)} \mathcal{I}_{0^+}^{\alpha_1,\Psi} f(\mathfrak{z}_1, u(\mathfrak{z}_1)) \right) \right| \\
&\qquad + \frac{|\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{\eta(\mathfrak{z}_1)}) - \mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{\eta(\mathfrak{z}_2)})|}{|\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)}) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})|}
\end{aligned}$$

$$\times \left[\left|\eta(R)\right| \mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi} \left(\frac{1}{|\eta(R)|} \mathcal{I}_{0^+}^{\alpha_1,\Psi} \left|f\left(R,u(R)\right)\right| \right) + \left|\chi(R)\right| \mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi} \left(\frac{1}{|\eta(R)|} \mathcal{I}_{0^+}^{\alpha_1,\Psi} \left|f\left(R,u(R)\right)\right| \right) \right].$$

By using (28), we get

$$\begin{aligned} \left|\mathcal{N}_1 u(\mathfrak{z}_2) - \mathcal{N}_1 u(\mathfrak{z}_1)\right| \le \frac{1}{\underline{\eta}} \Bigg(& \frac{e^{\theta(\Psi(\mathfrak{z}_2)-\Psi(0))}}{\theta^{\alpha_1+\alpha_2}} - \frac{e^{\theta(\Psi(\mathfrak{z}_1)-\Psi(0))}}{\theta^{\alpha_1+\alpha_2}} \\ & + \frac{1}{\Gamma(\alpha_2)\theta^{\alpha_1}} \int_{\mathfrak{z}_1}^{\mathfrak{z}_2} \Psi'(s)\left(\Psi(\mathfrak{z}_2)-\Psi(s)\right)^{\alpha_2-1} e^{\theta(\Psi(s)-\Psi(0))}\, ds \\ & + \frac{\frac{(\Psi(\mathfrak{z}_1)-\Psi(s))^{\alpha_2}-(\Psi(\mathfrak{z}_2)-\Psi(s))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{\left|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}\right|} \\ & \times \left[\frac{|\eta(R)|}{\theta^{\alpha_1+\alpha_2-\sigma_1}} + \frac{|\chi(R)|}{\theta^{\alpha_1+\alpha_2-\sigma_2}} \right] \Bigg) e^{\theta(\Psi(R)-\Psi(0))} \|\mathcal{K}_f\|_\theta. \end{aligned} \tag{33}$$

The independence of the right-hand side of (33) with respect to u is apparent and $|\mathcal{N}_1 u(\mathfrak{z}_2) - \mathcal{N}_1 u(\mathfrak{z}_1)| \to 0$ as $\mathfrak{z}_2 \to \mathfrak{z}_1$. Hence, $\mathcal{N}_1\mathcal{B}_{r_4,\theta}$ is equicontinuous and $\mathcal{N}_1$ maps bounded sets to relatively compact sets, so that $\mathcal{N}_1\mathcal{B}_{r_4,\theta}$ is relatively compact. Using the Arzelà–Ascoli theorem, we can conclude that $\mathcal{N}_1$ is compact in $\mathcal{B}_{r_4,\theta}$.

Then because Lemma 3.6 is verified, this shows that the (FGSLL)-problem (1) has at least one solution defined on $[0,R]$. □

Remark 3.10 The advantage of proving Theorem 3.7 by using the Ψ-Bielecki-type norm is that the strong condition $\nabla_\theta < 1$ is removed.

Corollary 3.11 *Let* (H1) *and* (H3) *hold. Then*

- *If* $\chi(\mathfrak{z}) = 0$ *for all* $\mathfrak{z} \in I$, *then we get* $\overline{\chi} = 0$ *and find that the (FSL)-problem* (2) *has at least one solution defined on* I.
- *If* $\eta(\mathfrak{z}) = 1$ *and* $\chi(\mathfrak{z}) = \lambda$ $(\lambda \in \mathbb{R})$ *for* $t \in I$, *then we have* $\underline{\eta} = 1$ *and* $\overline{\chi} = |\lambda|$. *We also find that the (FL)-problem* (3) *has at least one solution defined on* I.

3.3 Existence results via Leray–Schauder fixed point theorem

First, we recall Leray–Schauder nonlinear alternative theorem and then give our main results.

Lemma 3.12 ([40]) *Assume that* U *is a Banach space,* $\mathcal{C}$ *is a convex and closed subset of* U, $\mathcal{M}$ *is an open subset of* $\mathcal{C}$, *and* 0 *belongs to* $\mathcal{M}$. *Let* $\mathbb{T} : \overline{\mathcal{M}} \to \mathcal{C}$ *be a map that is continuous and compact, i.e.,* $\mathbb{T}(\overline{\mathcal{M}})$ *is a relatively compact subset of* $\mathcal{C}$. *Then either*

- $\mathbb{T}$ *has a fixed point in* $\overline{\mathcal{M}}$, *or*
- *There exists a point* $x \in \partial\mathcal{M}$, *where* $\partial\mathcal{M}$ *denotes the boundary of* $\mathcal{M}$ *in* $\mathcal{C}$, *and then there is a scalar* $\lambda \in (0,1)$ *such that* $\lambda\mathbb{T}(x) = x$.

Theorem 3.13 *Let* (H1) *and* (H3)–(H5) *hold. Then at least one solution exists for the (FGSLL)-problem* (1) *on* $[0,R]$.

Proof Pay attention to the operator $\mathcal{N} : \mathfrak{C} \to \mathfrak{C}$ given by (15).

Claim 1: Operator $\mathcal{N}$ maps bounded sets to bounded sets in $\mathfrak{C}$.

For $r_5 > 0$, assume that $\mathcal{B}_{r_5,\theta}(u) = \{u \in \mathfrak{C}, \|u\|_\theta \leq r_5\}$ is a bounded set in $\mathfrak{C}$. Let $u \in \mathcal{B}_{r_5,\theta}$, then

$$\begin{aligned}
\left|\mathcal{N}u(\mathfrak{z})\right| \leq{}& \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{1}{|\eta(\mathfrak{z})|}\mathcal{I}_{0^+}^{\alpha_1,\Psi}\left(\frac{|f(\mathfrak{z},u(\mathfrak{z}))|}{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}\right)\right) \\
&+ \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{|\chi(\mathfrak{z})||u(\mathfrak{z})|}{|\eta(\mathfrak{z})|e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}\right) \\
&+ \frac{\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{|\eta(\mathfrak{z})|})}{|\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)}) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})|} \\
&\times \Bigg[\left|\eta(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{1}{|\eta(R)|}\mathcal{I}_{0^+}^{\alpha_1,\Psi}\left(\frac{|f(R,u(R))|}{e^{\theta(\Psi(R)-\Psi(0))}}e^{\theta(\Psi(R)-\Psi(0))}\right)\right) \\
&+ \left|\chi(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{1}{|\eta(R)|}\mathcal{I}_{0^+}^{\alpha_1,\Psi}\left(\frac{|f(R,u(R))|}{e^{\theta(\Psi(R)-\Psi(0))}}e^{\theta(\Psi(R)-\Psi(0))}\right)\right) \\
&+ \left|\eta(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{|\chi(R)|u(R)||}{|\eta(R)|e^{\theta(\Psi(R)-\Psi(0))}}e^{\theta(\Psi(R)-\Psi(0))}\right) \\
&+ \left|\chi(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{|\chi(R)|u(R)||}{|\eta(R)|e^{\theta(\Psi(R)-\Psi(0))}}e^{\theta(\Psi(R)-\Psi(0))}\right)\Bigg].
\end{aligned}$$

By exploiting the well-known inequality $||\kappa| - |\ell|| \leq |\kappa + \ell|$ and taking into account

$$\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)} \neq \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)},$$

we get

$$\begin{aligned}
\left|\mathcal{N}u(\mathfrak{z})\right| \leq{}& \left(\frac{e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}{\underline{\eta}\theta^{\alpha_1+\alpha_2}} + \frac{\frac{(\Psi(\mathfrak{z})-\Psi(0))^{\alpha_2}}{\underline{\eta}\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_2-\sigma_1+1)}|}\right. \\
&\left.\times\left[\frac{|\eta(R)|e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_1}} + \frac{|\chi(R)|e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_2}}\right]\right)\|\mathfrak{g}\|_\theta \\
&+ d\|u\|_\theta \\
&+ \left(\frac{\overline{\chi}e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}{\underline{\eta}\theta^{\alpha_2}} + \frac{\frac{(\Psi(\mathfrak{z})-\Psi(0))^{\alpha_2}}{\underline{\eta}\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_2-\sigma_1+1)}|}\right. \\
&\left.\times\left[\frac{|\chi(R)|\overline{\chi}e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}{\underline{\eta}\theta^{\alpha_2-\sigma_2}} + \frac{|\eta(R)|\overline{\chi}e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}}{\underline{\eta}\theta^{\alpha_2-\sigma_1}}\right]\right)\|u\|_\theta.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|\mathcal{N}u\|_\theta \leq{}& \left(\frac{1}{\underline{\eta}\theta^{\alpha_1+\alpha_2}} + \frac{\frac{(\Psi(R)-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}|}\right. \\
&\left.\times\left[\frac{|\eta(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_1}} + \frac{|\chi(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_2}}\right]\right)\|\mathfrak{g}\|_\theta + dr_5
\end{aligned}$$

$$+\left(\frac{\overline{\chi}}{\underline{\eta}\theta^{\alpha_2}}+\frac{\frac{(\Psi(R)-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)}-\frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}|}\right.$$
$$\left.\times\left[\frac{|\chi(R)|\overline{\chi}}{\underline{\eta}\theta^{\alpha_2-\sigma_2}}+\frac{|\eta(R)|\overline{\chi}}{\underline{\eta}\theta^{\alpha_2-\sigma_1}}\right]\right)r_5,$$

which yields

$$\|\mathcal{N}u\|_\theta\leq\Lambda_\theta\|\mathfrak{g}\|_\theta+dr_5+\nabla_\theta r_5=l.$$

Claim 2: Operator $\mathcal{N}$ maps bounded sets to equicontinuous sets in $\mathfrak{C}$.

Assuming that the points $\mathfrak{z}_1,\mathfrak{z}_2\in[0,R]$ are arbitrary with $\mathfrak{z}_1<\mathfrak{z}_2$ and $u\in\mathcal{B}_{r_5,\theta}$, where $\mathcal{B}_{r_5,\theta}$ is a bounded set in $\mathfrak{C}$, we get

$$\begin{aligned}
&\left|\mathcal{N}u(\mathfrak{z}_2)-\mathcal{N}u(\mathfrak{z}_1)\right|\\
&\leq\left|\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{1}{\eta(\mathfrak{z}_2)}\mathcal{I}_{0^+}^{\alpha_1,\Psi}f(\mathfrak{z}_2,u(\mathfrak{z}_2))\right)-\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{1}{\eta(\mathfrak{z}_1)}\mathcal{I}_{0^+}^{\alpha_1,\Psi}f(\mathfrak{z}_1,u(\mathfrak{z}_1))\right)\right|\\
&\quad+\left|\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\chi(\mathfrak{z}_1)u(\mathfrak{z}_1)}{\eta(\mathfrak{z}_1)}\right)-\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\chi(\mathfrak{z}_2)u(\mathfrak{z}_2)}{\eta(\mathfrak{z}_2)}\right)\right|\\
&\quad+\frac{|\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{\eta(\mathfrak{z}_1)})-\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{\eta(\mathfrak{z}_2)})|}{|\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)})+\chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})|}\\
&\quad\times\left[\left|\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{\chi(R)u(R)}{\eta(R)}\right)-\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi}f(R,u(R))\right)\right.\right.\\
&\quad\left.\left.-\chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi}f(R,u(R))\right)+\chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{\chi(R)u(R)}{\eta(R)}\right)\right|\right].
\end{aligned}$$

By using (28) and (29), we get

$$\begin{aligned}
&\left|\mathcal{N}u(\mathfrak{z}_2)-\mathcal{N}u(\mathfrak{z}_1)\right|\\
&\leq\frac{1}{\underline{\eta}}\left(\frac{e^{\theta(\Psi(\mathfrak{z}_2)-\Psi(0))}}{\theta^{\alpha_1+\alpha_2}}-\frac{e^{\theta(\Psi(\mathfrak{z}_1)-\Psi(0))}}{\theta^{\alpha_1+\alpha_2}}\right.\\
&\quad\left.+\frac{1}{\theta^{\alpha_1}\Gamma(\alpha_2)}\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\Psi'(s)\big(\Psi(\mathfrak{z}_2)-\Psi(s)\big)^{\alpha_2-1}e^{\theta(\Psi(s)-\Psi(0))}\,ds\right)\|\mathcal{K}_f\|_\theta\\
&\quad+\frac{\overline{\chi}}{\underline{\eta}}\left(\frac{e^{\theta(\Psi(\mathfrak{z}_1)-\Psi(0))}}{\theta^{\alpha_1+\alpha_2}}-\frac{e^{\theta(\Psi(\mathfrak{z}_2)-\Psi(0))}}{\theta^{\alpha_1+\alpha_2}}\right.\\
&\quad\left.-\frac{1}{\theta^{\alpha_1}\Gamma(\alpha_2)}\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\Psi'(s)\big(\Psi(\mathfrak{z}_2)-\Psi(s)\big)^{\alpha_2-1}e^{\theta(\Psi(s)-\Psi(0))}\,ds\right)\|u\|_\theta\\
&\quad+\frac{\frac{(\Psi(\mathfrak{z}_1)-\Psi(s))^{\alpha_2}-(\Psi(\mathfrak{z}_2)-\Psi(s))^{\alpha_2}}{\Gamma(\alpha_2+1)}e^{\theta(\Psi(R)-\Psi(0))}}{|\frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)}-\frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)}|}\\
&\quad\times\left[\frac{|\eta(R)|\|\mathcal{K}_f\|_\theta}{\theta^{\alpha_1+\alpha_2-\sigma_1}}+\frac{|\chi(R)|\|\mathcal{K}_f\|_\theta}{\theta^{\alpha_1+\alpha_2-\sigma_2}}+\frac{\overline{\chi}|\eta(R)|\|u\|_\theta}{\theta^{\alpha_1+\alpha_2-\sigma_1}}+\frac{\overline{\chi}|\chi(R)|\|u\|_\theta}{\theta^{\alpha_1+\alpha_2-\sigma_2}}\right].
\end{aligned}$$

Observe that, as $\mathfrak{z}_1\to\mathfrak{z}_2$, the right-hand side goes to zero uniformly. This means that it does not depend on u. Furthermore, by Lemma 3.5, the operator $\mathcal{N}:\mathfrak{C}\to\mathfrak{C}$ is completely continuous.

Eventually, we prove that the set of all solutions of the equation $\lambda\mathcal{N}(u) = u$ is bounded for $\lambda \in (0,1)$.

Following similar computations as in the first claim, we have

$$\left|u(\mathfrak{z})\right| = \left|\lambda\mathcal{N}u(\mathfrak{z})\right| \leq \left[\Lambda_\theta\left(\|\mathfrak{g}\|_\theta + d\|u\|_\theta\right) + \nabla_\theta\|u\|_\theta\right]e^{\theta(\Psi(\mathfrak{z})-\Psi(0))}.$$

Taking the norm for $t \in [0,R]$, we have the following:

$$\|u\|_\theta \leq \Lambda_\theta\|\mathfrak{g}\|_\theta + (d\Lambda_\theta + \nabla_\theta)\|u\|_\theta,$$

which leads to

$$\frac{\|u\|_\theta(1 - d\Lambda_\theta - \nabla_\theta)}{\Lambda_\theta\|\mathfrak{g}\|_\theta} \leq 1.$$

In accordance with (H4), then there exists $M > 0$ such that $\|u\|_\theta \neq M$. Define a set

$$\mathcal{M}_\theta = \left\{u \in \mathfrak{C} : \|\mathrm{u}\|_\theta < M\right\},$$

and consider the fact that $\mathcal{N} : \overline{\mathcal{M}_\theta} \to \mathcal{C}$ is continuous and completely continuous. The choice of $\mathcal{M}_\theta$ gives that there is no $x \in \partial\mathcal{M}_\theta$ such that $\lambda\mathcal{N}(u) = u$ for some $\lambda \in (0,1)$. As a result, we conclude by Lemma 3.12 that $\mathcal{N}$ has a fixed point $u \in \overline{\mathcal{M}_\theta}$ that corresponds to a solution of the (FGSLL)-problem (1). □

Corollary 3.14 *Let* (H1),(H4), *and* (H5) *hold.*

- *If* $\chi(\mathfrak{z}) = 0$ *for* $t \in I$, *then we get* $\overline{\chi} = 0$ *and obtain that at least one solution for the* (*FSL*)*-problem* (2) *is guaranteed on* I.
- *If* $\eta(\mathfrak{z}) = 1$ *and* $\chi(\mathfrak{z}) = \lambda$ *for* $t \in I$ *and* $\lambda \in \mathbb{R}$, *then we have* $\underline{\eta} = 1$ *and* $\overline{\chi} = |\lambda|$. *We also conclude that at least one solution for the* (*FL*)*-problem* (3) *is guaranteed on* I.

4 Stability analysis

This section analyzes the stability property. In other words, in the present section, we will discuss UH, GUH, UHR, and GUHR stability of the given (FGSLL)-problem (1).

4.1 Ulam stability

Let $\varepsilon > 0$, $L_f > 0$, and let $\phi : [0,R] \to \mathbb{R}_+$ be continuous. We will examine the set of inequalities as below ($\mathfrak{z} \in [0,R]$):

$$\left|{}^C\mathcal{D}_{0^+}^{\alpha_1,\Psi}\left(\eta(\mathfrak{z}){}^C\mathcal{D}_{0^+}^{\alpha_2,\Psi}\breve{u}(\mathfrak{z}) + \chi(\mathfrak{z})\breve{u}(\mathfrak{z})\right) - f\left(\mathfrak{z},\breve{u}(\mathfrak{z})\right)\right| \leq \varepsilon, \tag{34}$$

$$\left|{}^C\mathcal{D}_{0^+}^{\alpha_1,\Psi}\left(\eta(\mathfrak{z}){}^C\mathcal{D}_{0^+}^{\alpha_2,\Psi}\breve{u}(\mathfrak{z}) + \chi(\mathfrak{z})\breve{u}(\mathfrak{z})\right) - f\left(\mathfrak{z},\breve{u}(\mathfrak{z})\right)\right| \leq \phi(\mathfrak{z}), \tag{35}$$

$$\left|{}^C\mathcal{D}_{0^+}^{\alpha_1,\Psi}\left(\eta(\mathfrak{z}){}^C\mathcal{D}_{0^+}^{\alpha_2,\Psi}\breve{u}(\mathfrak{z}) + \chi(\mathfrak{z})\breve{u}(\mathfrak{z})\right) - f\left(\mathfrak{z},\breve{u}(\mathfrak{z})\right)\right| \leq \varepsilon\phi(\mathfrak{z}). \tag{36}$$

Definition 4.1 ([34]) The (FGSLL)-problem (1) is UH stable if there exists $\mathcal{C}_f > 0$ so that for any $\varepsilon > 0$ and each solution $\breve{u} \in \mathcal{C}([0,R],\mathbb{R})$ of the inequality (34), there exists $u \in \mathcal{C}([0,R],\mathbb{R})$ as a solution of the (FGSLL)-problem (1) with

$$\left|\breve{u}(\mathfrak{z}) - u(\mathfrak{z})\right| \leq \mathcal{C}_f\varepsilon, \quad \mathfrak{z} \in [0,R].$$

Definition 4.2 ([34]) The (FGSLL)-problem (1) has GUH stability if there exists a positive constant $\mathcal{C}_f$ so that for any $\varepsilon > 0$ and for any solution $\breve{u} \in \mathcal{C}([0,R],\mathbb{R})$ of the inequality (34), there exists $u \in \mathcal{C}([0,R],\mathbb{R})$ as a solution of the (FGSLL)-problem (1) with

$$\left|\breve{u}(\mathfrak{z}) - u(\mathfrak{z})\right| \leq \Upsilon(\varepsilon), \quad \mathfrak{z} \in [0,R].$$

Definition 4.3 ([34]) The (FGSLL)-problem (1) is UHR stable asymptotically if and only if there exists $C > 0$ so that for each $\varepsilon > 0$ and for each solution $\breve{u} \in \mathcal{C}([0,R],\mathbb{R})$ of the inequality (36), there exists $u \in \mathcal{C}([0,R],\mathbb{R})$ as a solution of (FGSLL)-problem (1) with

$$\left|\breve{u}(\mathfrak{z}) - u(\mathfrak{z})\right| \leq \varepsilon \mathcal{C}_{f,\Upsilon}\Upsilon(\mathfrak{z}), \quad \mathfrak{z} \in [0,R].$$

Definition 4.4 ([34]) The (FGSLL)-problem (1) is GUHR stable with respect to Υ if there exists a real number $\mathcal{C}_{f,\Upsilon} > 0$ so that for any solution $\breve{u} \in \mathcal{C}([0,R],\mathbb{R})$ of the inequality (35), there exists $u \in \mathcal{C}([0,R],\mathbb{R})$ as a solution of the (FGSLL)-problem (1) with

$$\left|\breve{u}(\mathfrak{z}) - u(\mathfrak{z})\right| \leq \mathcal{C}_{f,\Upsilon}\Upsilon(\mathfrak{z}), \quad \mathfrak{z} \in [0,R].$$

Remark 4.5 (1) Definition 4.2 is implied by Definition 4.1,
(2) Definition 4.4 is implied by Definition 4.3,
(3) Definition 4.1 is implied by Definition 4.3 for $\Upsilon(\cdot) = 1$.

Remark 4.6 A continuous function $\breve{u} \in \mathcal{C}([0,R],\mathbb{R})$ is a solution of the inequality (34) iff there exists $g \in \mathcal{C}([0,R],\mathbb{R})$, a continuous function depending on $\breve{u}$ such that

(1) $\left|g(\mathfrak{z})\right| \leq \varepsilon, \quad \mathfrak{z} \in [0,R]$,

(2) ${}^{C}\mathcal{D}_{0^+}^{\alpha_1,\Psi}\left(\eta(\mathfrak{z})^{C}\mathcal{D}_{0^+}^{\alpha_2,\Psi}\breve{u}(\mathfrak{z}) + \chi(\mathfrak{z})\breve{u}(\mathfrak{z})\right) = f\left(\mathfrak{z},\breve{u}(\mathfrak{z})\right) + g(\mathfrak{z}), \quad \mathfrak{z} \in [0,R]$,

hold.

Remark 4.7 The essential condition for a function $\breve{u} \in \mathcal{C}([0,R],\mathbb{R})$ to satisfy inequality (36) is the existence of a function $w \in \mathcal{C}([0,R],\mathbb{R})$ that depends on the solution $\breve{u}$ and satisfies the following conditions:

(1) $\left|w(\mathfrak{z})\right| \leq \varepsilon\Upsilon(\mathfrak{z}), \quad \mathfrak{z} \in [0,R]$,

(2) ${}^{C}\mathcal{D}_{0^+}^{\alpha_1,\Psi}\left(\eta(\mathfrak{z})^{C}\mathcal{D}_{0^+}^{\alpha_2,\Psi}\breve{u}(\mathfrak{z}) + \chi(\mathfrak{z})\breve{u}(\mathfrak{z})\right) = f\left(\mathfrak{z},\breve{u}(\mathfrak{z})\right) + w(\mathfrak{z}), \quad \mathfrak{z} \in [0,R]$.

The following lemma, a generalized version of Gronwall inequality, plays a crucial role in establishing our main stability results.

Lemma 4.8 ([41]) *Suppose that u, v are two functions in $L^1([0,R])$ and g in $\mathcal{C}([0,R])$. Let $\Psi \in \mathcal{C}^1[0,R]$ be an increasing function so that $\Psi'(\mathfrak{z}) \neq 0$, $\forall \mathfrak{z} \in [0,R]$. Suppose, in addition, that*
(1) *u and v are nonnegative*;
(2) *g is nonnegative and nondecreasing.*

If

$$u(\mathfrak{z}) \leq v(\mathfrak{z}) + g(\mathfrak{z}) \int_0^{\mathfrak{z}} \Psi'(\tau)\big(\Psi(\mathfrak{z}) - \Psi(\tau)\big)^{\alpha-1} u(\tau)\,d\tau,$$

then

$$u(\mathfrak{z}) \leq v(\mathfrak{z}) + \int_0^{\mathfrak{z}} \sum_{k=1}^{\infty} \frac{[g(\mathfrak{z})\Gamma(\alpha)]^k}{\Gamma(\alpha k)} \Psi'(\tau)\big[\Psi(\mathfrak{z}) - \Psi(\tau)\big]^{\alpha k-1} v(\tau)\,d\tau, \quad \forall \mathfrak{z} \in [0,R].$$

Furthermore, if v is nondecreasing, then

$$u(\mathfrak{z}) \leq v(\mathfrak{z}) \mathbb{E}_\alpha\big(g(\mathfrak{z})\Gamma(\alpha)\big[\Psi(\mathfrak{z}) - \Psi(\tau)\big]^{\alpha}\big), \quad \forall \mathfrak{z} \in [0,R].$$

Proof See [42]. □

Remark 4.9 ([41]) Let $\alpha > 0$, $I = [0,R]$, and $\Psi \in \mathcal{C}^1(I,\mathbb{R})$ be increasing with $\Psi'(\mathfrak{z}) \neq 0$ for all $\mathfrak{z} \in I$. Assume that v is a nonnegative function with the local integrability on $[0,R]$ and let u be nonnegative and locally integrable on $[0,R]$ with

$$u(\mathfrak{z}) \leq v(\mathfrak{z}) + R \int_0^{\mathfrak{z}} \Psi'(\tau)\big[\Psi(\mathfrak{z}) - \Psi(\tau)\big]^{\alpha-1} u(\tau)\,d\tau, \quad \forall \mathfrak{z} \in [0,R].$$

Then

$$u(\mathfrak{z}) \leq v(\mathfrak{z}) + \int_0^{\mathfrak{z}} \sum_{k=1}^{\infty} \frac{[R\Gamma(\alpha)]^k}{\Gamma(\alpha k)} \Psi'(\tau)\big[\Psi(\mathfrak{z}) - \Psi(\tau)\big]^{\alpha k-1} v(\tau)\,d\tau, \quad \forall \mathfrak{z} \in [0,R].$$

Lemma 4.10 *Let $\breve{u} \in \mathcal{C}([0,R],\mathbb{R})$ is a solution of the inequality* (34) *and $\alpha_i \in (0,1]$, $i \in \{1,2\}$. Then $\breve{u} \in \mathcal{C}([0,R],\mathbb{R})$ satisfies*

$$\big|\breve{u}(\mathfrak{z}) - \mathcal{Z}(\mathfrak{z}) - \mathcal{I}_{a^+}^{\alpha_1+\alpha_2;\Psi} f\big(\mathfrak{z}, \breve{u}(\mathfrak{z})\big)\big| \leq \Lambda\epsilon, \tag{37}$$

where

$$\begin{aligned}
\mathcal{Z}(\mathfrak{z}) = {} & -\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\chi(\mathfrak{z})\breve{u}(\mathfrak{z})}{\eta(\mathfrak{z})}\right) + \frac{\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{\eta(\mathfrak{z})})}{\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)}) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})} \\
& \times \left[\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{\chi(R)\breve{u}(R)}{\eta(R)}\right) - \eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi} f\big(R,\breve{u}(R)\big)\right)\right. \\
& \left. - \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi} f\big(R,\breve{u}(R)\big)\right) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{\chi(R)\breve{u}(R)}{\eta(R)}\right)\right],
\end{aligned} \tag{38}$$

with Λ given by (16).

Proof Let $\breve{u}$ be a solution of (34). By Lemma 2.9 and Remark 4.6(2), we get

$$\begin{cases} {}^C\mathcal{D}_{0^+}^{\alpha_1,\Psi}(\eta(\mathfrak{z})^C\mathcal{D}_{0^+}^{\alpha_2,\Psi}\breve{u}(\mathfrak{z}) + \chi(\mathfrak{z})\breve{u}(\mathfrak{z})) = f(\mathfrak{z},\breve{u}(\mathfrak{z})) + g(\mathfrak{z}), & \mathfrak{z} \in (0,R), \\ \breve{u}(0) = 0, \quad \eta(R)^C\mathcal{D}_{0^+}^{\sigma_1,\Psi}\breve{u}(R) + \chi(R)^C\mathcal{D}_{0^+}^{\sigma_2,\Psi}\breve{u}(R) = 0, \end{cases} \tag{39}$$

and then the solution of problem (39) can be given as

$$\begin{aligned}
\breve{u}(\mathfrak{z}) &= \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{1}{\eta}\mathcal{I}_{0^+}^{\alpha_1,\Psi}f\big(\mathfrak{z},\breve{u}(\mathfrak{z})\big)\right) + \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{1}{\eta(\mathfrak{z})}\mathcal{I}_{0^+}^{\alpha_1,\Psi}g(\mathfrak{z})\right) - \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\chi(\mathfrak{z})\breve{u}(\mathfrak{z})}{\eta(\mathfrak{z})}\right)\\
&\quad - \frac{\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{\eta(\mathfrak{z})})}{\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)}) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})}\\
&\quad \times \Bigg[\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi}f\big(R,\breve{u}(R)\big)\right)\\
&\quad + \eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi}g(R)\right) - \eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{\chi(R)\breve{u}(R)}{\eta(R)}\right)\\
&\quad + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi}f\big(R,\breve{u}(R)\big)\right)\\
&\quad + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi}g(R)\right) - \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{\chi(R)\breve{u}(R)}{\eta(R)}\right)\Bigg].
\end{aligned}$$

Due to Remark 4.6(1), we can write

$$\begin{aligned}
\left|\breve{u}(\mathfrak{z}) - \mathcal{Z}(\mathfrak{z}) - \mathcal{I}_{a^+}^{\alpha_1+\alpha_2;\Psi}f\big(\mathfrak{z},\breve{u}(\mathfrak{z})\big)\right| &\le \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{1}{|\eta(\mathfrak{z})|}\mathcal{I}_{0^+}^{\alpha_1,\Psi}\left|g(\mathfrak{z})\right|\right)\\
&\quad + \frac{\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{|\eta(\mathfrak{z})|})}{|\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)}) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})|}\\
&\quad \times \Bigg[\left|\eta(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{1}{|\eta(R)|}\mathcal{I}_{0^+}^{\alpha_1,\Psi}\left|g(R)\right|\right)\\
&\quad + \left|\chi(R)\right|\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{1}{|\eta(R)|}\mathcal{I}_{0^+}^{\alpha_1,\Psi}\left|g(R)\right|\right)\Bigg].
\end{aligned}$$

By using Remark 4.6(1), we acquire

$$\begin{aligned}
&\left|\breve{u}(\mathfrak{z}) - \mathcal{Z}(\mathfrak{z}) - \mathcal{I}_{a^+}^{\alpha_1+\alpha_2;\Psi}f\big(\mathfrak{z},\breve{u}(\mathfrak{z})\big)\right|\\
&\quad\le \Bigg(\frac{(\Psi(\mathfrak{z})-\Psi(0))^{\alpha_1+\alpha_2}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2+1)} + \frac{\frac{(\Psi(\mathfrak{z})-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{||\eta(R)|\frac{(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)} - |\chi(R)|\frac{(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)}|}\\
&\qquad \times \left[\frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_1+1)} + \frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_2}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_2+1)}\right]\Bigg)\varepsilon\\
&\quad\le \Bigg(\frac{(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2+1)} + \frac{\frac{(\Psi(R)-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{||\eta(R)|\frac{(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)} - |\chi(R)|\frac{(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)}|}\\
&\qquad \times \left[\frac{|\eta(R)|(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_1}}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_1+1)} + \frac{|\chi(R)|(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2-\sigma_2}}{\underline{\eta}.\Gamma(\alpha_1+\alpha_2-\sigma_2+1)}\right]\Bigg)\varepsilon.
\end{aligned}$$

The proof of (37) is finished. □

Theorem 4.11 *Let* (H1) *and* (H2) *hold. The* (*FGSLL*)*-problem* (1) *is* UH *stable in* $\mathcal{C}([0,R],\mathbb{R})$.

Proof Let $\breve{u} \in \mathcal{C}([0,R],\mathbb{R})$ be a solution of (34), and $u \in \mathcal{C}([0,R],\mathbb{R})$ be a unique solution of (1). By using Lemma 4.10, it gives

$$u = \mathcal{X}(\mathfrak{z}) + \mathcal{I}_{a^+}^{\alpha_1+\alpha_2;\Psi} f\big(t,\breve{u}(\mathfrak{z})\big),$$

where

$$\begin{aligned}\mathcal{X}(\mathfrak{z}) = &-\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\chi(\mathfrak{z})u(\mathfrak{z})}{\eta(\mathfrak{z})}\right)\\ &+\frac{\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{\eta(\mathfrak{z})})}{\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)})+\chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})}\\ &\times\left[\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{\chi(R)u(R)}{\eta(R)}\right)-\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi}f\big(R,u(R)\big)\right)\right.\\ &\left.-\chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi}f\big(R,u(R)\big)\right)+\chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{\chi(R)u(R)}{\eta(R)}\right)\right].\end{aligned} \tag{40}$$

Clearly, if $u(0) = \breve{u}(0)$ and

$$\eta(R)^C\mathcal{D}_{0^+}^{\sigma_1,\Psi}u(R)+\chi(R)^C\mathcal{D}_{0^+}^{\sigma_2,\Psi}u(R)=\eta(R)^C\mathcal{D}_{0^+}^{\sigma_1,\Psi}\breve{u}(R)+\chi(R)^C\mathcal{D}_{0^+}^{\sigma_2,\Psi}\breve{u}(R),$$

then we obtain that $\mathcal{X}(\mathfrak{z}) = \mathcal{Z}(\mathfrak{z})$.

By the help of Lemma 4.10 and the known inequality $|u+v| \le |u|+|v|$ for any $\mathfrak{z} \in [0,R]$, we get

$$\begin{aligned}\big|\breve{u}(\mathfrak{z})-u(\mathfrak{z})\big| &= \big|\breve{u}(\mathfrak{z})-\mathcal{X}(\mathfrak{z})-\mathcal{I}_{a^+}^{\alpha_1+\alpha_2;\Psi}f\big(\mathfrak{z},\breve{u}(\mathfrak{z})\big)\big|\\ &\le \big|\breve{u}(\mathfrak{z})-\mathcal{Z}(\mathfrak{z})-\mathcal{I}_{a^+}^{\alpha_1+\alpha_2;\Psi}f\big(\mathfrak{z},u(\mathfrak{z})\big)\big|\\ &\quad+\mathcal{I}_{a^+}^{\alpha_1+\alpha_2;\Psi}\big|f\big(\mathfrak{z},u(\mathfrak{z})\big)-f\big(\mathfrak{z},\breve{u}(\mathfrak{z})\big)\big|+\big|\mathcal{Z}(\mathfrak{z})-\mathcal{X}(\mathfrak{z})\big|\\ &\le \Lambda\epsilon+\frac{L_f}{\Gamma(\alpha_1+\alpha_2)}\int_0^{\mathfrak{z}}\Psi'(s)\big(\Psi(\mathfrak{z})-\Psi(s)\big)^{\alpha_1+\alpha_2-1}\big|\breve{u}(s)-u(s)\big|\,ds\\ &\le \Lambda\epsilon\left(1+\int_0^{\mathfrak{z}}\sum_{k=1}^{\infty}\frac{L_f^k}{\Gamma(k(\alpha_1+\alpha_2)+1)}\Psi'(s)\big(\Psi(\mathfrak{z})-\Psi(s)\big)^{k(\alpha_1+\alpha_2)-1}\,ds\right)\\ &\le \Lambda\epsilon\sum_{k=0}^{\infty}\frac{L_f^k(\Psi(\mathfrak{z})-\Psi(0))^{k(\alpha_1+\alpha_2)}}{\Gamma(k(\alpha_1+\alpha_2)+1)}\\ &\le \Lambda\epsilon\sum_{k=0}^{\infty}\frac{L_f^k(\Psi(R)-\Psi(0))^{k(\alpha_1+\alpha_2)}}{\Gamma(k(\alpha_1+\alpha_2)+1)}\\ &= \Lambda\epsilon\mathbb{E}_{\alpha_1+\alpha_2}\big(L_f\big(\Psi(R)-\Psi(0)\big)^{\alpha_1+\alpha_2}\big).\end{aligned} \tag{41}$$

For simplicity, we take $\mathcal{C}_f := \Lambda\mathbb{E}_{\alpha_1+\alpha_2}(L_f(\Psi(R)-\Psi(0))^{\alpha_1+\alpha_2})$. Then (41) becomes

$$\big|\breve{u}(\mathfrak{z})-u(\mathfrak{z})\big| \le \mathcal{C}_f\epsilon.$$

Thus, the (FGSLL)-problem (1) is UH stable. □

Corollary 4.12 *Let* (H1) *and* (H2) *hold.*

- *If* $\chi(\mathfrak{z}) = 0$ *for all* $\mathfrak{z} \in I$, *then we have* $\overline{\chi} = 0$ *and the* (*FSL*)*-problem* (2) *is* UH *stable in* $\mathcal{C}([0,R],\mathbb{R})$.
- *If* $\eta(\mathfrak{z}) = 1$ *and* $\chi(\mathfrak{z}) = \lambda$ $(\lambda \in \mathbb{R})$ *for* $\mathfrak{z} \in I$, *then we have* $\underline{\eta} = 1$ *and* $\overline{\chi} = |\lambda|$. *We also find that the* (*FL*)*-problem* (3) *is* UH *stable in* $\mathcal{C}([0,R],\mathbb{R})$.

Now, if $\Upsilon(\varepsilon) = \varepsilon\mathcal{C}_f$ with $\Upsilon(0) = 0$, we have a corollary as follows.

Corollary 4.13 *Let* (H1) *and* (H2) *hold. Then the* (*FGSLL*)*-problem* (1) *is* GUH *stable in* $\mathcal{C}([0,R],\mathbb{R})$.

- *If* $\chi(\mathfrak{z}) = 0$ *for all* $\mathfrak{z} \in I$, *then* $\overline{\chi} = 0$ *and the* (*FSL*)*-problem* (2) *is* GUH *stable in* $\mathcal{C}([0,R],\mathbb{R})$.
- *If* $\eta(\mathfrak{z}) = 1$ *and* $\chi(\mathfrak{z}) = \lambda$ $(\lambda \in \mathbb{R})$ *for* $\mathfrak{z} \in I$, *then we have* $\underline{\eta} = 1$ *and* $\overline{\chi} = |\lambda|$. *We also have that the* (*FL*)*-problem* (3) *is* GUH *stable in* $\mathcal{C}([0,R],\mathbb{R})$.

In the sequel, we focus on the UHR and generalized UHR stability.

Lemma 4.14 *Let* $\alpha_i \in (0,1]$, $i \in \{1,2\}$, *and suppose* $\breve{u} \in \mathcal{C}([0,R],\mathbb{R})$ *is a solution of* (34). *Then* $\breve{u} \in \mathcal{C}([0,R],\mathbb{R})$ *satisfies*

$$\left|\breve{u}(\mathfrak{z}) - \mathcal{Z}(\mathfrak{z}) - \mathcal{I}_{a^+}^{\alpha_1+\alpha_2;\Psi} f\big(\mathfrak{z},\breve{u}(\mathfrak{z})\big)\right| \le \Psi\varepsilon\gamma_\Upsilon\Upsilon(\mathfrak{z}),$$

where

$$\Psi := \frac{1}{\underline{\eta}} + \frac{\frac{(\Psi(R)-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{\left||\eta(R)|\frac{(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)} - |\chi(R)|\frac{(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)}\right|}\left[\frac{|\eta(R)|}{\underline{\eta}} + \frac{|\chi(R)|}{\underline{\eta}}\right], \tag{42}$$

and $\mathcal{Z}$ *is given by* (38).

Proof Assuming that $\breve{u}$ is a solution of (36), we can utilize Lemma 2.9 and Remark 4.7(2) to obtain

$$\begin{cases} {}^c\mathcal{D}_{a+}^{\alpha,\Psi}({}^c\mathcal{D}_{a+}^{\alpha,\Psi}\breve{u}(\mathfrak{z}) + \mu\breve{u}(\mathfrak{z})) = f(\mathfrak{z},\breve{u}(\mathfrak{z})) + w(\mathfrak{z}), & \mathfrak{z} \in (0,R), \\ \breve{u}(0) = 0, \qquad \eta(R){}^C\mathcal{D}_{0^+}^{\sigma_1,\Psi}\breve{u}(R) + \chi(R){}^C\mathcal{D}_{0^+}^{\sigma_2,\Psi}\breve{u}(R) = 0, \end{cases} \tag{43}$$

and then the solution of problem (43) may be given as

$$\begin{aligned} \breve{u}(\mathfrak{z}) &= \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{1}{\eta(\mathfrak{z})}\mathcal{I}_{0^+}^{\alpha_1,\Psi} f\big(\mathfrak{z},\breve{u}(\mathfrak{z})\big)\right) + \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{1}{\eta(\mathfrak{z})}\mathcal{I}_{0^+}^{\alpha_1,\Psi} w(\mathfrak{z})\right) - \mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{\chi(\mathfrak{z})\breve{u}(\mathfrak{z})}{\eta(\mathfrak{z})}\right) \\ &\quad + \frac{\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{\eta(\mathfrak{z})})}{\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)}) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})} \\ &\quad \times \Bigg[-\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi} f\big(R,\breve{u}(R)\big)\right) \\ &\quad - \eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi} w(R)\right) + \eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{\chi(R)\breve{u}(R)}{\eta(R)}\right) \end{aligned}$$

$$- \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi}f\big(R,\breve{u}(R)\big)\right)$$
$$- \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi}w(R)\right) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{\chi(R)\breve{u}(R)}{\eta(R)}\right)\Bigg].$$

Thanks to Remark 4.7(2) and assumption (H6), we have

$$\begin{aligned}
&\left|\breve{u}(\mathfrak{z}) - \mathcal{Z}(\mathfrak{z}) - \mathcal{I}_{a^+}^{\alpha_1+\alpha_2;\Psi}f\big(\mathfrak{z},\breve{u}(\mathfrak{z})\big)\right| \\
&= \Bigg|\mathcal{I}_{0^+}^{\alpha_2,\Psi}\left(\frac{1}{\eta(\mathfrak{z})}\mathcal{I}_{0^+}^{\alpha_1,\Psi}w(\mathfrak{z})\right) + \frac{\mathcal{I}_{0^+}^{\alpha_2,\Psi}(\frac{1}{\eta(\mathfrak{z})})}{\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}(\frac{1}{\eta(R)}) + \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}(\frac{1}{\eta(R)})} \\
&\quad \times \left[-\eta(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi}w(R)\right) - \chi(R)\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}\left(\frac{1}{\eta(R)}\mathcal{I}_{0^+}^{\alpha_1,\Psi}w(R)\right)\right]\Bigg|.
\end{aligned}$$

By using Remark 4.7(1), we get

$$\begin{aligned}
&\left|\breve{u}(\mathfrak{z}) - \mathcal{Z}(\mathfrak{z}) - \mathcal{I}_{a^+}^{\alpha_1+\alpha_2;\Psi}f\big(\mathfrak{z},\breve{u}(\mathfrak{z})\big)\right| \\
&\leq \frac{1}{\underline{\eta}}\mathcal{I}_{0^+}^{\alpha_1+\alpha_2,\Psi}\big(\varepsilon\Upsilon(\mathfrak{z})\big) + \frac{\frac{1}{\underline{\eta}}(\mathcal{I}_{0^+}^{\alpha_2,\Psi}1)(\mathfrak{z})}{|\frac{|\eta(R)|}{\underline{\eta}}(\mathcal{I}_{0^+}^{\alpha_2-\sigma_1,\Psi}1)(R) - \frac{|\chi(R)|}{\underline{\eta}}(\mathcal{I}_{0^+}^{\alpha_2-\sigma_2,\Psi}1)(R)|} \\
&\quad \times \left[\frac{|\eta(R)|}{\underline{\eta}}\mathcal{I}_{0^+}^{\alpha_1+\alpha_2-\sigma_1,\Psi}\big(\varepsilon\Upsilon(R)\big) + \frac{|\chi(R)|}{\underline{\eta}}\mathcal{I}_{0^+}^{\alpha_1+\alpha_2-\sigma_2,\Psi}\big(\varepsilon\Upsilon(R)\big)\right].
\end{aligned}$$

In view of inequality (14), it follows that

$$\begin{aligned}
&\left|\breve{u}(\mathfrak{z}) - \mathcal{Z}(\mathfrak{z}) - \mathcal{I}_{a^+}^{\alpha_1+\alpha_2;\Psi}f\big(\mathfrak{z},\breve{u}(\mathfrak{z})\big)\right| \\
&\leq \Bigg(\frac{\gamma_{\Upsilon,\alpha_1+\alpha_2}}{\underline{\eta}} + \frac{\frac{(\Psi(\mathfrak{z})-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{||\eta(R)|\frac{(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)} - |\chi(R)|\frac{(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)}|} \\
&\quad \times \left[\frac{\gamma_{\Upsilon,\alpha_1+\alpha_2-\sigma_1}|\eta(R)|}{\underline{\eta}} + \frac{\gamma_{\Upsilon,\alpha_1+\alpha_2-\sigma_2}|\chi(R)|}{\underline{\eta}}\right]\Bigg)\varepsilon\Upsilon(\mathfrak{z}).
\end{aligned}$$

Finally, we conclude that

$$\begin{aligned}
&\left|\breve{u}(\mathfrak{z}) - \mathcal{Z}(\mathfrak{z}) - \mathcal{I}_{a^+}^{\alpha_1+\alpha_2;\Psi}f\big(\mathfrak{z},\breve{u}(\mathfrak{z})\big)\right| \\
&\leq \Bigg(\frac{1}{\underline{\eta}} + \frac{\frac{(\Psi(R)-\Psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}}{||\eta(R)|\frac{(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_1}}{\Gamma(\alpha_2-\sigma_1+1)} - |\chi(R)|\frac{(\Psi(R)-\Psi(0))^{\alpha_2-\sigma_2}}{\Gamma(\alpha_2-\sigma_2+1)}|} \\
&\quad \left[\frac{|\eta(R)|}{\underline{\eta}} + \frac{|\chi(R)|}{\underline{\eta}}\right]\Bigg)\varepsilon\gamma_{\Upsilon}\Upsilon(\mathfrak{z}).
\end{aligned}$$

The proof of (4.10) is now complete. □

Theorem 4.15 *Let* (H1), (H2), *and* (H6) *hold. Then the* (*FGSLL*)*-problem* (1) *is* UHR *stable in* $\mathcal{C}([0,R],\mathbb{R})$.

Proof Let $\breve{u} \in \mathcal{C}([0,R],\mathbb{R})$ be a solution of (36) and u be a unique solution for the (FGSLL)-problem (1). By applying Lemma 4.14, it yields that

$$u = \mathcal{X}(\mathfrak{z}) + \mathcal{I}_{a^+}^{\alpha_1+\alpha_2;\Psi} f(\mathfrak{z}, \breve{u}(\mathfrak{z})),$$

where $\mathcal{X}$ is given by (40). Similarly, if $u(0) = \breve{u}(0)$ and

$$\eta(R)^C\mathcal{D}_{0^+}^{\sigma_1,\Psi} u(R) + \chi(R)^C\mathcal{D}_{0^+}^{\sigma_2,\Psi} u(R) = \eta(R)^C\mathcal{D}_{0^+}^{\sigma_1,\Psi} \breve{u}(R) + \chi(R)^C\mathcal{D}_{0^+}^{\sigma_2,\Psi} \breve{u}(R),$$

then $\mathcal{X}(\mathfrak{z}) = \mathcal{Z}(\mathfrak{z})$.

Applying Lemma 4.14, the triangle inequality, and inequality (14), for any $t \in [0,R]$, we then may write

$$\begin{aligned}
&\left|\breve{u}(\mathfrak{z}) - u(\mathfrak{z})\right| \\
&\quad = \left|\breve{u}(\mathfrak{z}) - \mathcal{X}(\mathfrak{z}) - \mathcal{I}_{a^+}^{\alpha_1+\alpha_2;\Psi} f(\mathfrak{z}, \breve{u}(\mathfrak{z}))\right| \\
&\quad \le \left|\breve{u}(\mathfrak{z}) - \mathcal{Z}(\mathfrak{z}) - \mathcal{I}_{a^+}^{\alpha_1+\alpha_2;\Psi} f(\mathfrak{z}, u(\mathfrak{z}))\right| \\
&\qquad + \mathcal{I}_{a^+}^{\alpha_1+\alpha_2;\Psi} \left|f(\mathfrak{z}, \breve{u}(\mathfrak{z})) - f(\mathfrak{z}, u(\mathfrak{z}))\right| + \left|\mathcal{Z}(\mathfrak{z}) - \mathcal{X}(\mathfrak{z})\right| \\
&\quad \le \Psi \varepsilon \gamma_\Upsilon \Upsilon(\mathfrak{z}) + \frac{L_f}{\Gamma(\alpha_1+\alpha_2)} \int_0^{\mathfrak{z}} \Psi'(s)\big(\Psi(\mathfrak{z}) - \Psi(s)\big)^{\alpha_1+\alpha_2-1} \left|\breve{u}(s) - u(s)\right| ds \\
&\quad \le \Psi \varepsilon \gamma_\Upsilon \left[\Upsilon(\mathfrak{z}) + \int_0^{\mathfrak{z}} \sum_{k=1}^{\infty} \frac{L_f^k}{\Gamma(k(\alpha_1+\alpha_2))} \Psi'(s)\big(\Psi(\mathfrak{z}) - \Psi(s)\big)^{k(\alpha_1+\alpha_2)-1} \Upsilon(s)\, ds\right].
\end{aligned}$$

Since Υ is nondecreasing (see condition (H6)), for all $s \in [0,\mathfrak{z}]$, we obtain $\Upsilon(s) \le \Upsilon(\mathfrak{z})$ and can write

$$\begin{aligned}
\left|\breve{u}(\mathfrak{z}) - u(\mathfrak{z})\right| &\le \Psi \varepsilon \gamma_\Upsilon \Upsilon(\mathfrak{z}) \left[1 + \sum_{k=1}^{\infty} \frac{L_f^k}{\Gamma(k(\alpha_1+\alpha_2))} \frac{(\Psi(\mathfrak{z}) - \Psi(0))^{k(\alpha_1+\alpha_2)}}{k(\alpha_1+\alpha_2)}\right] \\
&\le \Psi \varepsilon \gamma_\Upsilon \Upsilon(\mathfrak{z}) \sum_{k=0}^{\infty} \frac{L_f^k (\Psi(R) - \Psi(0))^{k(\alpha_1+\alpha_2)}}{\Gamma(k(\alpha_1+\alpha_2)+1)} \\
&= \varepsilon \gamma_\Upsilon \Upsilon(\mathfrak{z}) \Psi \mathbb{E}_{\alpha_1+\alpha_2}\left(L_f\big(\Psi(R) - \Psi(0)\big)^{\alpha_1+\alpha_2}\right),
\end{aligned}$$

where Ψ is provided by (42). Thus,

$$\left|\breve{u}(\mathfrak{z}) - u(\mathfrak{z})\right| \le \mathcal{C}_{f,\Upsilon} \Upsilon(\mathfrak{z}) \varepsilon,$$

with

$$\mathcal{C}_{f,\Upsilon} := \gamma_\Upsilon \Psi \mathbb{E}_{\alpha_1+\alpha_2}\left(L_f\big(\Psi(R) - \Psi(0)\big)^{\alpha_1+\alpha_2}\right).$$

Then, the (FGSLL)-problem (1) is UHR stable. □

Corollary 4.16 *Let the assumptions* (H1), (H2), *and* (H6) *hold.*

- *If* $\chi(\mathfrak{z}) = 0$ *for all* $\mathfrak{z} \in I$, *then* $\overline{\chi} = 0$ *and the* (*FSL*)*-problem* (2) *is* UHR *stable in* $\mathcal{C}([0,R],\mathbb{R})$.

- *If* $\eta(\mathfrak{z}) = 1$ *and* $\chi(\mathfrak{z}) = \lambda$ $(\lambda \in \mathbb{R})$ *for* $\mathfrak{z} \in I$, *then we have* $\underline{\eta} = 1$ *and* $\overline{\chi} = |\lambda|$. *Furthermore, the* (*FL*)*-problem* (3) *is* UHR *stable in* $\mathcal{C}([0,R],\mathbb{R})$.

Now, we take $\varepsilon = 1$ in $|\breve{u}(\mathfrak{z}) - u(\mathfrak{z})| \leq \mathcal{C}_{f,\Upsilon}\Upsilon(\mathfrak{z})\varepsilon$ with $\Upsilon(0) = 0$. Then we have the following.

Corollary 4.17 *Suppose that* (H1), (H2), *and* (H6) *are fulfilled. Then the* (*FGSLL*)*-problem* (1) *is* GUHR *stable in* $\mathcal{C}([0,R],\mathbb{R})$.

- *If* $\chi(\mathfrak{z}) = 0$ *for all* $\mathfrak{z} \in I$, *then we have* $\overline{\chi} = 0$ *and the* (*FSL*)*-problem* (2) *is* GUHR *stable in* $\mathcal{C}([0,R],\mathbb{R})$.
- *If* $\eta(\mathfrak{z}) = 1$ *and* $\chi(\mathfrak{z}) = \lambda$ $(\lambda \in \mathbb{R})$ *for* $\mathfrak{z} \in I$, *then we have* $\underline{\eta} = 1$ *and* $\overline{\chi} = |\lambda|$. *In addition, the* (*FL*)*-problem* (3) *is* GUHR *stable in* $\mathcal{C}([0,R],\mathbb{R})$.

5 Illustrative examples

Here, three test examples are used to show the effectiveness of the proposed techniques.

Example 5.1 Two cases are formulated that require less restrictive conditions for a unique solution. Then we analyze the stability results based on the (FGSLL)-problem (1).

• *First case.* We fix $\alpha_1 = \frac{1}{3}, \alpha_2 = \frac{5}{6}, \sigma_1 = \frac{2}{3}, \sigma_2 = \frac{1}{2}, \Psi(\mathfrak{z}) = \mathfrak{z}$ for $\mathfrak{z} \in [0,1]$, $\eta(\mathfrak{z}) = e^{10^2} + 10^{-3}\mathfrak{z}$, and $\chi(\mathfrak{z}) = 0$ for $\mathfrak{z} \in [0,1]$. We have $\overline{\chi} = \chi(1) = 0$, $\eta(1) = e^{10^2} + 10^{-3}$ and $\underline{\eta} = e^{10^2}$. In this case, the (FGSLL)-problem (1) is reduced to the (CSL)-problem (Caputo-type Sturm–Liouville)

$$\begin{cases} {}^{C}D_{0^+}^{\frac{1}{3}}((e^{10^2} + 10^{-3}\mathfrak{z}){}^{C}D_{0^+}^{\frac{5}{6}}\mathrm{u}(\mathfrak{z})) = \frac{|u(\mathfrak{z})|e^{\mathfrak{z}}}{(1+|u(\mathfrak{z})|)(9+\mathfrak{z})^2}, & \mathfrak{z} \in [0,1], \\ \mathrm{u}(0) = 0, \qquad (e^{10^2} + 10^{-3}){}^{C}D_{0^+}^{\frac{2}{3}}\mathrm{u}(1) = 0. \end{cases} \tag{44}$$

The conditions (H1) and (H2) are satisfied so that

$$\begin{aligned} \left|f(\mathfrak{z},u_2) - f(\mathfrak{z},u_1)\right| &\leq \left|\frac{e^{\mathfrak{z}}}{(9+\mathfrak{z})^2}\frac{|u_1(\mathfrak{z})|}{1+|u_1(\mathfrak{z})|} - \frac{e^{\mathfrak{z}}}{(9+\mathfrak{z})^2}\frac{|u_2(\mathfrak{z})|}{1+|u_2(\mathfrak{z})|}\right| \\ &\leq \left|\frac{e^{\mathfrak{z}}}{(9+\mathfrak{z})^2}\right||u_1 - u_2| \\ &\leq \frac{e}{10^2}|u_1 - u_2|, \quad \text{for all } u_1, u_2 \in \mathbb{R}, \mathfrak{z} \in [0,1]. \end{aligned}$$

Then, we have $L_f = \frac{e}{10^2}$. Hence,

$$\begin{aligned} \mathcal{J} &= \frac{L_f}{\underline{\eta}\Gamma(\alpha_1 + \alpha_2 + 1)} + \frac{\frac{1}{\Gamma(\alpha_2+1)}}{|\frac{|\eta(R)|}{\Gamma(\alpha_2-\sigma_1+1)}|}\left[\frac{L_f|\eta(R)|}{\underline{\eta}\Gamma(\alpha_1 + \alpha_2 - \sigma_1 + 1)}\right] \\ &= \frac{e}{10^2 e^{10^2}\Gamma(\frac{13}{6})} + \frac{1}{\Gamma(\frac{11}{6})|\frac{e^{10^2}+10^{-3}}{\Gamma(\frac{7}{6})}|}\left[\frac{(e^{10^2} + 10^{-3})e}{10^2 e^{10^2}\Gamma(\frac{3}{2})}\right] < 1, \end{aligned}$$

where $\mathcal{J}$ is given by (20). Now, all the assumptions of Theorem 3.2 are satisfied. Thus the (CSL)-problem (44) has a unique solution on $[0,1]$.

Similarly, by choosing $\theta > 0$ large enough such that

$$L_f\Lambda_\theta + \nabla_\theta = L_f\left(\frac{1}{\underline{\eta}\theta^{\alpha_1+\alpha_2}} + \frac{\frac{1}{\Gamma(\alpha_2+1)}}{|\frac{|\eta(R)|}{\Gamma(\alpha_2-\sigma_1+1)}|}\left[\frac{|\eta(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_1}}\right]\right)$$

$$= \frac{e}{10^2 e^{10^2}\theta^{\frac{7}{6}}} + \frac{1}{\Gamma(\frac{11}{6})|\frac{e^{10^2}+10^{-3}}{\Gamma(\frac{7}{6})}|}\left[\frac{(e^{10^2}+10^{-3})e}{10^2 e^{10^2}\theta^{\frac{1}{2}}}\right] < 1,$$

where Λ_θ and ∇_θ are the constants given by (18) and (19), the conditions of Theorem 3.3 are fulfilled. Thus the (CSL)-problem (44) has a unique solution on $[0,1]$. Moreover, we have

$$\begin{aligned}
\mathcal{C}_f &= \frac{1}{\underline{\eta}\Gamma(\alpha_1+\alpha_2+1)} + \frac{\frac{1}{\Gamma(\alpha_2+1)}}{|\frac{|\eta(R)|}{\Gamma(\alpha_2-\sigma_1+1)}|}\left[\frac{|\eta(R)|}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_1+1)}\right]\mathbb{E}_{\alpha_1+\alpha_2}(L_f) \\
&= \frac{1}{e^{10^2}\Gamma(\frac{13}{6})} + \frac{1}{\Gamma(\frac{11}{6})|\frac{e^{10^2}+10^{-3}}{\Gamma(\frac{7}{6})}|}\left[\frac{e^{10^2}+10^{-3}}{e^{10^2}\Gamma(\frac{3}{2})}\right]\mathbb{E}_{\frac{7}{6}}\left(10^{-2}e\right) > 0.
\end{aligned}$$

Hence, from Theorem 4.11, the (CSL)-problem (44) is UH and GUH stable on $[0,1]$.

By taking $\Upsilon(\mathfrak{z}) = (\Psi(\mathfrak{z}) - \Psi(0))^{\frac{1}{6}} = \mathfrak{z}^{\frac{1}{6}}$, it follows that

$$\begin{aligned}
&\mathcal{I}_{0^+}^{\alpha_1+\alpha_2,\Psi}\Upsilon(\mathfrak{z}) = \frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{7}{3})}\mathfrak{z}^{8}\Upsilon(\mathfrak{z}) \le \gamma_{\Upsilon,\alpha_1+\alpha_2}\Upsilon(\mathfrak{z}), \quad \text{where } \gamma_{\Upsilon,\alpha_1+\alpha_2} = \frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{7}{3})}, \\
&\mathcal{I}_{0^+}^{\alpha_1+\alpha_2-\sigma_1,\Psi}\Upsilon(\mathfrak{z}) = \frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{5}{3})}\mathfrak{z}^{4}\Upsilon(\mathfrak{z}) \le \gamma_{\Upsilon,\alpha_1+\alpha_2-\sigma_1}\Upsilon(\mathfrak{z}), \quad \text{where } \gamma_{\Upsilon,\alpha_1+\alpha_2-\sigma_1} = \frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{5}{3})}, \\
&\mathcal{I}_{0^+}^{\alpha_1+\alpha_2-\sigma_2,\Psi}\Upsilon(\mathfrak{z}) = \frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{11}{6})}\mathfrak{z}^{5}\Upsilon(\mathfrak{z}) \le \gamma_{\Upsilon,\alpha_1+\alpha_2-\sigma_2}\Upsilon(\mathfrak{z}), \quad \text{where } \gamma_{\Upsilon,\alpha_1+\alpha_2-\sigma_2} = \frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{11}{6})}.
\end{aligned}$$

The inequality (14) is satisfied with

$$\gamma_\Upsilon = \max\{\gamma_{\Upsilon,\alpha_1+\alpha_2-\sigma_2}, \gamma_{\Upsilon,\alpha_1+\alpha_2-\sigma_1}, \gamma_{\Upsilon,\alpha_1+\alpha_2}\} = \frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{5}{3})} > 0,$$

where

$$\Psi = \frac{1}{\underline{\eta}} + \frac{\frac{1}{\Gamma(\alpha_2+1)}}{|\frac{|\eta(R)|}{\Gamma(\alpha_2-\sigma_1+1)}|}\left[\frac{|\eta(R)|}{\underline{\eta}}\right] = e^{-10^2} + \frac{1}{\Gamma(\frac{11}{6})|\frac{e^{10^2}+10^{-3}}{\Gamma(\frac{7}{6})}|}\left[\frac{e^{10^2}+10^{-3}}{e^{10^2}}\right].$$

Then

$$\begin{aligned}
\mathcal{C}_{f,\Upsilon} &= \gamma_\Upsilon \Psi \mathbb{E}_{\alpha_1+\alpha_2}(L_f) \\
&= \frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{5}{3})}\left(e^{-10^2} + \frac{1}{\Gamma(\frac{11}{6})|\frac{e^{10^2}+10^{-3}}{\Gamma(\frac{7}{6})}|}\left[\frac{e^{10^2}+10^{-3}}{e^{10^2}}\right]\right)\mathbb{E}_{\frac{7}{6}}\left(10^{-2}e\right) > 0.
\end{aligned}$$

Therefore, in view of Theorem 4.15, the (CSL)-problem (44) is UHR and GUHR stable on $[0,1]$.

• *Second case.* We fix $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{5}{6}$, $\sigma_1 = \frac{2}{3}$, $\sigma_2 = \frac{1}{2}$, $\Psi(\mathfrak{z}) = \mathfrak{z}$, $\eta(\mathfrak{z}) = 1$ for $\mathfrak{z} \in [0,1]$, and $\chi(\mathfrak{z}) = 10^{-3}$ for $\mathfrak{z} \in [0,1]$. We get $\eta(1) = \underline{\eta} = 1$ and $\chi(1) = \overline{\chi} = 10^{-3}$. In this case, the (FGSLL)-

problem (1) is reduced to (CL)-problem (Caputo-type Langevin)

$$\begin{cases} {}^{C}D_{0^+}^{\frac{1}{3}}\left({}^{C}D_{0^+}^{\frac{5}{6}}\mathrm{u}(\mathfrak{z}) + 10^{-3}\mathrm{u}(\mathfrak{z})\right) = \frac{|u(\mathfrak{z})|e^{\mathfrak{z}}}{(1+|u(\mathfrak{z})|)(9+\mathfrak{z})^2}, & \mathfrak{z} \in [0,1], \\ \mathrm{u}(0) = 0, \qquad {}^{C}D_{0^+}^{\frac{2}{3}}\mathrm{u}(1) + 10^{-3}\,{}^{C}D_{0^+}^{\frac{1}{2}}\mathrm{u}(1) = 0. \end{cases} \tag{45}$$

The conditions (H1) and (H2) are satisfied with $L_f = \frac{e}{10^2}$. Hence,

$$\begin{aligned} \mathcal{J} &= \frac{L_f}{\underline{\eta}\Gamma(\alpha_1+\alpha_2+1)} + \frac{\overline{\chi}}{\underline{\eta}\Gamma(\alpha_2+1)} + \frac{1}{\Gamma(\alpha_2+1)\left|\frac{|\chi(R)|}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|}{\Gamma(\alpha_2-\sigma_1+1)}\right|} \\ &\quad \times \left[\frac{|\chi(R)|\overline{\chi}}{\underline{\eta}\Gamma(\alpha_2-\sigma_2+1)} + \frac{|\eta(R)|\overline{\chi}}{\underline{\eta}\Gamma(\alpha_2-\sigma_1+1)} + \frac{L_f|\chi(R)|}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_2+1)} \right. \\ &\quad \left. + \frac{L_f|\eta(R)|}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_1+1)}\right] \\ &= \frac{e}{10^2\Gamma(\frac{13}{6})} + \frac{1}{10^3\Gamma(\frac{11}{6})} \\ &\quad + \frac{1}{\Gamma(\frac{11}{6})\left|\frac{1}{10^3\Gamma(\frac{4}{3})} - \frac{1}{\Gamma(\frac{7}{6})}\right|}\left[\frac{1}{10^6\Gamma(\frac{4}{3})} + \frac{1}{10^3\Gamma(\frac{7}{6})} + \frac{e}{10^5\Gamma(\frac{5}{3})} + \frac{e}{10^2\Gamma(\frac{3}{2})}\right] \\ &< 1. \end{aligned}$$

All the assumptions of Theorem 3.2 hold. Hence, the (CL)-problem (45) has a unique solution on $[0,1]$. Similarly, by choosing $\theta > 0$ large enough such that

$$\begin{aligned} L_f\Lambda_\theta + \nabla_\theta &= L_f\left(\frac{1}{\underline{\eta}\theta^{\alpha_1+\alpha_2}} + \frac{\frac{1}{\Gamma(\alpha_2+1)}}{\left|\frac{|\chi(R)|}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|}{\Gamma(\alpha_2-\sigma_1+1)}\right|}\left[\frac{|\chi(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_2}} + \frac{|\eta(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_1}}\right]\right) \\ &\quad + \frac{\overline{\chi}}{\underline{\eta}\theta^{\alpha_2}} + \frac{\frac{1}{\Gamma(\alpha_2+1)}}{\left|\frac{|\chi(R)|}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|}{\Gamma(\alpha_2-\sigma_1+1)}\right|}\left[\frac{|\chi(R)|\overline{\chi}}{\underline{\eta}\theta^{\alpha_2-\sigma_2}} + \frac{|\eta(R)|\overline{\chi}}{\underline{\eta}\theta^{\alpha_2-\sigma_1}}\right] \\ &= \frac{e}{10^2\theta^{\frac{7}{6}}} + \frac{1}{10^3\theta^{\frac{5}{6}}} \\ &\quad + \frac{1}{\Gamma(\frac{11}{6})\left|\frac{1}{10^3\Gamma(\frac{4}{3})} - \frac{1}{\Gamma(\frac{7}{6})}\right|}\left[\frac{1}{10^3\theta^{\frac{1}{6}}} + \frac{e}{10^2\theta^{\frac{1}{2}}} + \frac{1}{10^6\theta^{\frac{1}{3}}} + \frac{e}{10^5\theta^{\frac{2}{3}}}\right] < 1, \end{aligned}$$

all the conditions of Theorem 3.3 are fulfilled. Then the (CL)-problem (45) admits one solution uniquely on $[0,1]$. In addition, we have

$$\begin{aligned} \mathcal{C}_f &= \frac{1}{\underline{\eta}\Gamma(\alpha_1+\alpha_2+1)} + \frac{\frac{1}{\Gamma(\alpha_2+1)}}{\left|\frac{|\chi(R)|}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|}{\Gamma(\alpha_2-\sigma_1+1)}\right|} \\ &\quad \times \left[\frac{|\chi(R)|}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_2+1)} + \frac{|\eta(R)|}{\underline{\eta}\Gamma(\alpha_1+\alpha_2-\sigma_1+1)}\right]\mathbb{E}_{\alpha_1+\alpha_2}(L_f) \\ &= \frac{1}{\Gamma(\frac{13}{6})} + \frac{1}{\Gamma(\frac{11}{6})\left|\frac{1}{10^3\Gamma(\frac{4}{3})} - \frac{1}{\Gamma(\frac{7}{6})}\right|}\left[\frac{1}{10^3\Gamma(\frac{5}{3})} + \frac{1}{\Gamma(\frac{3}{2})}\right]\mathbb{E}_{\frac{7}{6}}\left(\frac{e}{10^2}\right) > 0. \end{aligned}$$

From Theorem 4.11, it follows that the (CL)-problem (45) is UH and GUH stable on $[0,1]$. Taking $\Upsilon(\mathfrak{z}) = (\Psi(\mathfrak{z}) - \Psi(0))^{\frac{1}{6}} = \mathfrak{z}^{\frac{1}{6}}$, we obtain

$$\begin{aligned}
&\mathcal{I}_{0^+}^{\alpha_1+\alpha_2,\Psi}\Upsilon(\mathfrak{z}) = \frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{7}{3})}\mathfrak{z}^2\Upsilon(\mathfrak{z}) \leq \frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{7}{3})}\Upsilon(\mathfrak{z}) = \gamma_{\Upsilon,\alpha_1+\alpha_2}\Upsilon(\mathfrak{z}), \gamma_{\Upsilon,\alpha_1+\alpha_2} = \frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{7}{3})},\\
&\mathcal{I}_{0^+}^{\alpha_1+\alpha_2-\sigma_1,\Psi}\Upsilon(\mathfrak{z}) = \frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{5}{3})}\mathfrak{z}^4\Upsilon(\mathfrak{z}) \leq \frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{5}{3})}\Upsilon(\mathfrak{z}) = \gamma_{\Upsilon,\alpha_1+\alpha_2-\sigma_1}\Upsilon(\mathfrak{z}), \gamma_{\Upsilon,\alpha_1+\alpha_2-\sigma_1} = \frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{5}{3})},\\
&\mathcal{I}_{0^+}^{\alpha_1+\alpha_2-\sigma_2,\Psi}\Upsilon(\mathfrak{z}) = \frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{11}{6})}\mathfrak{z}^5\Upsilon(\mathfrak{z}) \leq \frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{11}{6})}\Upsilon(\mathfrak{z}) = \gamma_{\Upsilon,\alpha_1+\alpha_2-\sigma_2}\Upsilon(\mathfrak{z}), \gamma_{\Upsilon,\alpha_1+\alpha_2-\sigma_2} = \frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{5}{3})}.
\end{aligned}$$

The inequality (14) is satisfied with

$$\gamma_\Upsilon = \max\{\gamma_{\Upsilon,\alpha_1+\alpha_2-\sigma_2}, \gamma_{\Upsilon,\alpha_1+\alpha_2-\sigma_1}, \gamma_{\Upsilon,\alpha_1+\alpha_2}\} = \frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{7}{3})} > 0,$$

where

$$\begin{aligned}
\Psi &= \frac{1}{\underline{\eta}} + \frac{\frac{1}{\Gamma(\alpha_2+1)}}{|\frac{|\eta(R)|}{\Gamma(\alpha_2-\sigma_1+1)} - \frac{|\chi(R)|}{\Gamma(\alpha_2-\sigma_2+1)}|}\left[\frac{|\eta(R)|}{\underline{\eta}} + \frac{|\chi(R)|}{\underline{\eta}}\right]\\
&= 1 + \frac{1001}{10^3\Gamma(\frac{11}{6})|\frac{1}{10^3\Gamma(\frac{4}{3})} - \frac{1}{\Gamma(\frac{7}{6})}|}.
\end{aligned}$$

Then

$$\mathcal{C}_{f,\Upsilon} = \frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{7}{3})}\left(1 + \frac{1001}{10^3\Gamma(\frac{11}{6})|\frac{1}{10^3\Gamma(\frac{4}{3})} - \frac{1}{\Gamma(\frac{7}{6})}|}\right)\mathbb{E}_{\frac{7}{6}}\left(\frac{e}{10^2}\right) > 0.$$

Therefore, from Theorem 4.15, the (CL)-problem (45) is UHR and GUHR stable on $[0,1]$.

Example 5.2 We start with the (FGSLL)-problem (1) and choose $\alpha_1 = \frac{4}{5}$, $\alpha_2 = \frac{\sqrt{5}}{7}$, $\sigma_1 = \frac{2}{7}$, $\sigma_2 = \frac{1}{4}$, $\Psi(x) = x^3$. For $\mathfrak{z} \in [0,1]$, $\eta(\mathfrak{z}) = 1$ and $\chi(\mathfrak{z}) = 10^{-4}$ for $\mathfrak{z} \in [0,1]$, we have $\underline{\eta} = \eta(1) = 1$ and $\chi(1) = \overline{\chi} = 10^{-4}$. In this case, the (FGSLL)-problem (1) is reduced to (CEKL)-problem (Caputo–Erdélyi–Kober-type Langevin)

$$\begin{cases} {}^C\mathcal{D}_{0^+}^{\frac{4}{5}}({}^C\mathcal{D}_{0^+}^{\frac{\sqrt{5}}{7}}\mathrm{u}(\mathfrak{z}) + 10^{-4}\mathrm{u}(\mathfrak{z})) = \frac{|u(\mathfrak{z})|e^{\mathfrak{z}}}{(9+\mathfrak{z})^2(1+|u(\mathfrak{z})|)}, & \mathfrak{z} \in [0,1],\\ \mathrm{u}(0) = 0, \qquad {}^C\mathcal{D}_{0^+}^{\frac{2}{7}}\mathrm{u}(1) + 10^{-4}\,{}^C\mathcal{D}_{0^+}^{\frac{1}{4}}\mathrm{u}(1) = 0. \end{cases} \tag{46}$$

The conditions (H1) and (H3) are satisfied with

$$\left|\frac{|u(\mathfrak{z})|e^{\mathfrak{z}}}{(9+\mathfrak{z})^2(1+|u(\mathfrak{z})|)}\right| \leq \frac{e^{\mathfrak{z}}}{(9+\mathfrak{z})^2} = \mathcal{K}_f(\mathfrak{z}).$$

Hence,

$$\begin{aligned}\nabla &= \frac{\overline{\chi}}{\underline{\eta}\Gamma(\alpha_2+1)} + \frac{\frac{1}{\Gamma(\alpha_2+1)}}{\left|\frac{|\chi(R)|}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|}{\Gamma(\alpha_2-\sigma_1+1)}\right|}\left[\frac{|\chi(R)|\overline{\chi}}{\underline{\eta}\Gamma(\alpha_2-\sigma_2+1)} + \frac{|\eta(R)|\overline{\chi}}{\underline{\eta}\Gamma(\alpha_2-\sigma_1+1)}\right] \\ &= \frac{1}{10^4\Gamma(\frac{7+\sqrt{5}}{7})} \\ &\quad + \frac{1}{\Gamma(\frac{7+\sqrt{5}}{7})\left|\frac{1}{10^4\Gamma(\frac{21+4\sqrt{5}}{28})} - \frac{1}{10^4\Gamma(\frac{5+\sqrt{5}}{7})}\right|}\left[\frac{1}{10^8\Gamma(\frac{21+4\sqrt{5}}{28})} + \frac{1}{10^4\Gamma(\frac{5+\sqrt{5}}{7})}\right] < 1.\end{aligned} \tag{47}$$

The assumptions of Theorem 3.7 are met. Hence, the (CEKL)-problem (46) has at least one solution defined on $[0,1]$. Similarly, by choosing $\theta > 0$ large enough such that

$$\begin{aligned}\nabla_\theta &= \frac{\overline{\chi}}{\underline{\eta}\theta^{\alpha_2}} + \frac{\frac{1}{\Gamma(\alpha_2+1)}}{\left|\frac{|\chi(R)|}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|}{\Gamma(\alpha_2-\sigma_1+1)}\right|}\left[\frac{|\chi(R)|\overline{\chi}}{\underline{\eta}\theta^{\alpha_2-\sigma_2}} + \frac{|\eta(R)|\overline{\chi}}{\underline{\eta}\theta^{\alpha_2-\sigma_1}}\right] \\ &= \frac{1}{10^4\theta^{\frac{\sqrt{5}}{7}}} + \frac{1}{\Gamma(\frac{7+\sqrt{5}}{7})\left|\frac{1}{10^4\Gamma(\frac{21+4\sqrt{5}}{28})} - \frac{1}{10^4\Gamma(\frac{5+\sqrt{5}}{7})}\right|}\left[\frac{1}{10^8\theta^{\frac{4\sqrt{5}-7}{7}}} + \frac{1}{\underline{\eta}\theta^{\frac{\sqrt{5}-2}{7}}}\right] < 1,\end{aligned}$$

and by utilizing Theorem 3.9, we conclude the (CEKL)-problem (46) has at least one solution defined on $[0,1]$.

Example 5.3 Based on the (FGSLL) problem (1), we take $\alpha_1 = \frac{3}{4}$, $\alpha_2 = \frac{\sqrt{5}}{7}$, $\sigma_1 = \frac{2}{7}$, $\sigma_2 = \frac{1}{4}$, $\Psi(x) = x^3$, $\eta(\mathfrak{z}) = 1$ for $\mathfrak{z} \in [0,1]$, and $\chi(\mathfrak{z}) = 10^{-2}$ for $\mathfrak{z} \in [0,1]$. We have $\underline{\eta} = \eta(1) = 1$ and $\chi(1) = \overline{\chi} = 10^{-2}$. In this case, the (FGSLL)-problem (1) is reduced to (CEKL)-problem (Caputo–Erdélyi–Kober-type Langevin)

$$\begin{cases} {}^C\mathcal{D}_{0^+}^{\frac{3}{4}}({}^C\mathcal{D}_{0^+}^{\frac{\sqrt{5}}{7}}\mathrm{u}(\mathfrak{z}) + 10^{-2}\mathrm{u}(\mathfrak{z})) = \frac{3+|u(\mathfrak{z})|}{9e^{\mathfrak{z}^2}\sqrt{1+\mathfrak{z}^4}(5+|u(\mathfrak{z})|)}, & \mathfrak{z} \in [0,1], \\ \mathrm{u}(0) = 0, \qquad {}^C\mathcal{D}_{0^+}^{\frac{2}{3}}\mathrm{u}(1) + 10^{-2}\,{}^C\mathcal{D}_{0^+}^{\frac{1}{2}}\mathrm{u}(1) = 0. \end{cases} \tag{48}$$

The conditions (H1), (H3), (H4), and (H5) are satisfied with

$$\left|\frac{3+|u(\mathfrak{z})|}{9e^{\mathfrak{z}^2}\sqrt{1+\mathfrak{z}^4}(5+|u(\mathfrak{z})|)}\right| \le \frac{1}{9e^{\mathfrak{z}^2}\sqrt{1+\mathfrak{z}^4}} = \mathcal{K}_f(\mathfrak{z}),$$

and

$$\begin{aligned}\left|\frac{3+|u(\mathfrak{z})|}{9e^{\mathfrak{z}^2}\sqrt{1+\mathfrak{z}^4}(5+|u(\mathfrak{z})|)}\right| &\le \frac{3}{9e^{\mathfrak{z}^2}\sqrt{1+\mathfrak{z}^4}(5+|u(\mathfrak{z})|)} + \frac{|u(\mathfrak{z})|}{9e^{\mathfrak{z}^2}\sqrt{1+\mathfrak{z}^4}(5+|u(\mathfrak{z})|)} \\ &\le \mathfrak{g}(\mathfrak{z}) + d|u|,\end{aligned}$$

such that $\mathfrak{g}(\mathfrak{z}) = \frac{1}{3e\mathfrak{z}^2\sqrt{1+\mathfrak{z}^4}}$, $d = 1$, and $\|\mathfrak{g}\|_\theta = \frac{1}{3}$, where

$$\begin{aligned}\Lambda_\theta &= \frac{1}{\underline{\eta}\theta^{\alpha_1+\alpha_2}} + \frac{\frac{1}{\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|}{\Gamma(\alpha_2-\sigma_1+1)}|}\left[\frac{|\chi(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_2}} + \frac{|\eta(R)|}{\underline{\eta}\theta^{\alpha_1+\alpha_2-\sigma_1}}\right] \\ &= \frac{1}{\theta^{\frac{21+4\sqrt{5}}{28}}} + \frac{1}{\Gamma(\frac{7+\sqrt{5}}{7})|\frac{1}{10^2\Gamma(\frac{21+4\sqrt{5}}{28})} - \frac{1}{\Gamma(\frac{5+\sqrt{5}}{7})}|}\left[\frac{1}{10^2\theta^{\frac{7+2\sqrt{5}}{14}}} + \frac{1}{\theta^{\frac{13+4\sqrt{5}}{28}}}\right], \\ \nabla_\theta &= \frac{\overline{\chi}}{\underline{\eta}\theta^{\alpha_2}} + \frac{\frac{1}{\Gamma(\alpha_2+1)}}{|\frac{|\chi(R)|}{\Gamma(\alpha_2-\sigma_2+1)} - \frac{|\eta(R)|}{\Gamma(\alpha_2-\sigma_1+1)}|}\left[\frac{|\chi(R)|\overline{\chi}}{\underline{\eta}\theta^{\alpha_2-\sigma_2}} + \frac{|\eta(R)|\overline{\chi}}{\underline{\eta}\theta^{\alpha_2-\sigma_1}}\right] \\ &= \frac{1}{10^2\theta^{\frac{\sqrt{5}}{7}}} + \frac{1}{\Gamma(\frac{7+\sqrt{5}}{7})|\frac{1}{10^2\Gamma(\frac{21+4\sqrt{5}}{28})} - \frac{1}{\Gamma(\frac{5+\sqrt{5}}{7})}|}\left[\frac{1}{10^4\theta^{\frac{4\sqrt{5}-7}{28}}} + \frac{1}{10^2\theta^{\frac{\sqrt{5}-2}{7}}}\right],\end{aligned}$$

and

$$\frac{M(1-d\Lambda_\theta-\nabla_\theta)}{\Lambda_\theta\|\mathfrak{g}\|_\theta} > 1.$$

Hence, from Theorem 3.13, we conclude that the (CEKL)-problem (48) has at least one solution defined on $[0,1]$.

6 Conclusion

We conclude this paper with some useful findings. First, we studied the existence and uniqueness of solutions for a new class generalizing the differential equations of Sturm–Liouville–Langevin (1) including two fractional derivative operators in the Ψ-Caputo sense. When $\chi(\mathfrak{z}) = 0$ for $\mathfrak{z} \in I$, we obtained the (FSL)-differential equation (2) (Sturm–Liouville problem), and if $\eta(\mathfrak{z}) = 1$ and $\chi(\mathfrak{z}) = \lambda$ ($\lambda \in \mathbb{R}$) for $\mathfrak{z} \in I$, we obtained the (FL)-differential equation (3) (Langevin problem). The acquired results have been established via Banach's contraction, Krasnoselskii and Leray–Schauder fixed point theorems using some norm inequalities of the Ψ-Bielecki-type. Moreover, we proved different kinds of stability in the sense of Ulam, such as Ulam–Hyers, Ulam–Hyers–Rassias, generalized Ulam–Hyers and generalized Ulam–Hyers–Rassias. Also, to prove our results, we applied the generalized Gronwall integral inequality.

The second main idea of the current research was to use the Ψ-Bielecki-type norm to reduce the constraints of the (FGSLL)-problem (1) to prove the results of existence and uniqueness. The advantage of this norm (Bielecki's norm) can be found by comparing the conditions of Theorems 3.2 and 3.3, and by removing the strong condition $\mathcal{J} < 1$ that appeared in proving Theorem 3.2 using the classical supremum norm, while Theorem 3.3 does not require this condition. It is also done by comparing the conditions of Theorems 3.7 and 3.9. In a future work, researchers may consider using the Ψ-Hilfer or other fractional derivative operators, such as the fractal-fractional derivative, to establish the existence, uniqueness, and stability of solutions to the (FGSLL)-problem (1).

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Author contributions

H.S. and B.T. and S.E. and I.A. dealt with the conceptualization, supervision, methodology, investigation, and writing-original draft preparation. H.S. and B.T. and S.E. and I.A. and S.R. made the formal analysis, writing-review, editing. All authors read and approved the final manuscript.

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