Azouz et al. *Boundary Value Problems* (2024) 2024:72
https://doi.org/10.1186/s13661-024-01869-9

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

On steady state of viscous compressible heat conducting full magnetohydrodynamic equations

Mohamed Azouz[1], Rachid Benabidallah[2] and François Ebobisse[3*]

*Correspondence:
francois.ebobissebille@uct.ac.za
[3]Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch 7700, South Africa
Full list of author information is available at the end of the article

Abstract

This paper is concerned with the study of equations of viscous compressible and heat-conducting full magnetohydrodynamic (MHD) steady flows in a horizontal layer under the gravitational force and a large temperature gradient across the layer. We assume as boundary conditions, periodic conditions in the horizontal directions, while in the vertical directions, slip-boundary is assumed for the velocity, vertical conditions for the magnetic field, and fixed temperature or fixed heat flux are prescribed for the temperature. The existence of stationary solution in a small neighborhood of a stationary profile close to hydrostatic state is obtained in Sobolev spaces as a fixed point of some nonlinear operator.

Mathematics Subject Classification: 76N10; 76W05; 35Q35; 35L60; 35L65

Keywords: Magnetohydrodynamics (MHD) equations; Compressible flows; Steady solutions; Existence

1 Introduction

Magnetohydrodynamics (MHD) studies the dynamics of electrically conducting fluids in the presence of a magnetic field or, more precisely, in their macroscopic interaction with a magnetic field. Magnetohydrodynamics is known to be of great importance in several directions. First of all, its applications cover a very wide range of areas in physics from liquid metals to cosmic plasmas. We can mention among others, in astrophysics (with the study of solar structure, the solar wind bathing the earth and other planets), in geophysics (with the planetary magnetism produced by currents deep in the planet), in high-speed aerodynamics, and in plasma physics. Magnetohydrodynamics is also important in connection with many engineering problems such as sustained plasma confinement for controlled thermonuclear fusion, liquid-metal cooling of nuclear reactors, magnetohydrodynamic power generation, electro-magnetic casting of metals, and plasma accelerators for ion thrusters for spacecraft propulsion. Due to their practical relevance, magnetohydrodynamic problems have long been the subject of intense cross-disciplinary research, but except for relatively simplified special cases, the rigorous mathematical analysis of such problems still presents many interesting aspects to be studied.

In magnetohydrodynamic flows, magnetic fields can induce currents in a moving conductive fluid, which create forces on the fluid, and also change the magnetic field itself, resulting in a complex interaction between the magnetic and fluid dynamic phenomena, and therefore, both hydrodynamic and electrodynamic effects have to be considered. The set of equations that describe compressible viscous magnetohydrodynamics are a combination of the compressible Navier–Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. In this paper, we consider the system of partial differential equations for the three-dimensional viscous compressible full magnetohydrodynamic stationary flows (see [23, 25, 26])

$$\varrho(v\cdot\nabla)v-\mu\Delta v-(\mu+\lambda)\nabla\nabla\cdot v=-R\nabla(\varrho T)-g\varrho e_3+(\nabla\times B)\times B, \tag{1.1}$$

$$\nabla\cdot(\varrho v)=0, \tag{1.2}$$

$$c_V\varrho(v\cdot\nabla)T-\kappa\Delta T+R\varrho T\nabla\cdot v=2\mu\left|D(v)\right|^2+\lambda(\nabla\cdot v)^2+\nu|\nabla\times B|^2 \tag{1.3}$$

$$\nu\nabla\times(\nabla\times B)-\nabla\times(v\times B)=0,\quad \nabla\cdot B=0, \tag{1.4}$$

where ϱ denotes the density, $v=(v_1,v_2,v_3)$ the velocity, T the absolute temperature, $B=(B_1,B_2,B_3)$ the magnetic field, and $D(v)$ is the deformation tensor defined by

$$\left[D(v)\right]_{i,j}=\frac{1}{2}(\partial_{x_i}v_j+\partial_{x_j}v_i).$$

The constants μ and λ are the first and second viscosity coefficients respectively and satisfy the physical restrictions $\mu>0$ and $2\mu+3\lambda\geq 0$. The positive constants c_V, κ, R, and ν are respectively the heat capacity, the ratio of the heat conductivity coefficient over the heat capacity, the universal constant of gases, and the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field, while g the gravity acceleration and $e_3=(0,0,1)$. In (1.1) and (1.3) we have assumed that the pressure p is given by the law of perfect gases

$$p=R\varrho T. \tag{1.5}$$

In magnetohydrodynamics, the displacement current can be neglected (see [23, 26]). As a consequence, equation (1.4) is called the induction equation, and the electric field can be written in terms of the magnetic field B and the velocity v,

$$E=\nu\nabla\times B-v\times B.$$

Although the electric field E does not appear in the MHD system (1.1)–(1.4), it is indeed induced according to the above identity by the moving conductive flow in the magnetic field.

We recall that, due to their physical importance, complexity, rich phenomena, and mathematical challenges, there have been many studies on magnetohydrodynamics by physicists and mathematicians (see, e.g., [12, 13, 15, 17, 18, 20, 21, 26, 44] and the references cited therein). About the steady MHD equations for incompressible or compressible flows, we refer the interested reader to the articles [1–3, 5, 6, 14, 22, 30, 47] for the existence of strong and weak solutions. In [19], Gerbeau et al. also considered several kinds of unsteady problems and did some numerical analysis (see also [40]). On unsteady MHD, there have been many studies by physicists and mathematicians in the recent years (see, e.g., [13, 16, 17, 27, 44]). In particular, the one-dimensional problem has been studied in many articles (see, e.g., [4, 9, 12, 29]. On the other hand, computational aspects of the system of MHD equations were also studied with considerable interest in developing accurate numerical methods for those systems (see, e.g., [31, 39, 41]).

Almost all the literature mentioned above is concerned with the Cauchy problem or the initial boundary value problem for compressible MHD equations, with the homogenous Dirichlet condition on the magnetic field. In contrast with the extensive research on unsteady MHD flow, we find that there are only few results concerning the steady flow. In [45], Yang et al. established the existence and uniqueness of a strong solution to the steady magnetohydrodynamic equations for the compressible barotropic fluids in a bounded smooth domain with a perfectly conducting boundary under the assumption that the external force field is small. In [5] the authors improved the work in [45] by considering large external forces and, instead of perfectly conducting boundary conditions on the magnetic field, they considered non-homogeneous Dirichlet boundary conditions that can possibly be large enough.

In this work, we consider the MHD system of equations (1.1)–(1.4) in an infinite horizontal layer, and we investigate the fundamental problem of the existence of a stationary solution in a small neighborhood of a stationary profile close to hydrostatic state. As for boundary conditions, in the horizontal directions, the choice of periodic boundary conditions on velocity, temperature, and magnetic field is made naturally. Such conditions lend themselves better to the study of infinite plane-parallel media than wall-like conditions, which are more characteristic of laboratory experiment simulations. For the vertical direction, slip-boundary conditions are considered on velocity, and the magnetic field has its lines assumed vertical, while fixed temperature or fixed heat flux are prescribed.

More precisely, we consider system (1.1)–(1.4) in the bounded domain

$$\Omega = \mathbb{T}^2 \times]0, h[,$$

where $\mathbb{T}^2$ is a two-dimensional torus with the following boundary conditions:

$$v|_{x_3=0} = 0, \qquad v_3|_{x_3=h} = 0, \qquad \partial_{x_3} v_i|_{x_3=h} = 0, \quad i = 1,2 \tag{1.6}$$

$$T|_{x_3=0} = T_0 + \varepsilon\big(x'\big), \qquad T|_{x_3=h} = T_0 - \frac{\gamma - 1}{\gamma R} gh, \tag{1.7}$$

$$B_1|_{x_3=0,h} = B_2|_{x_3=0,h} = 0, \qquad B_3|_{x_3=0,h} = B_0, \tag{1.8}$$

where $T_0 > 0$ and B_0 are given large constants, $\varepsilon(x')$ is a small perturbation of the given temperature T_0 on the lower plane of the horizontal layer, which will be assumed periodic, $x' = (x_1, x_2) \in \mathbb{T}^2$ so that $x = (x', x_3) \in \Omega$ and γ is the adiabatic exponent given by

$$\gamma = \frac{R}{C_V} + 1.$$

Finally, it is assumed that the total mass of the fluid is prescribed as

$$\int_\Omega \varrho \, dx = M_0. \tag{1.9}$$

From now on, we make the following assumptions:

$$T_0 \geq \bar{T}_0, \qquad M_0 \geq \bar{M}_0, \qquad \|\varepsilon\|_{H^3(\mathbb{T}^2)} \leq \varepsilon_0, \qquad \gamma > 1, \tag{1.10}$$

where the positive constants $\bar{T}_0$ and $\bar{M}_0$ are given large enough, while the constant $\varepsilon_0 > 0$ is sufficiently small.

Remark 1.1 If (see (1.7)) the perturbation $\varepsilon(x')$ is identically zero, then it is clear that the only stationary solution of the system of equations (1.1)–(1.4) with boundary conditions (1.6)–(1.8) and periodic conditions on the horizontal directions is given by the rest state $(0, B_{rs}, T_{rs}, \varrho_{rs})$, where

$$B_{rs} := B_0 e_3, \qquad T_{rs} := T_0 - \frac{\gamma - 1}{\gamma R} g x_3, \qquad \varrho_{rs}(x_3) =: \bar{C} T_{rs}^{\frac{1}{\gamma-1}}(x_3), \tag{1.11}$$

with the constant $\bar{C}$ determined by the condition $\int_\Omega \varrho_{rs}(x_3)\, dx = M_0$.

Notice that, if $\varepsilon(x')$ is a nonzero constant ε_0, then we can reduce ourselves to case (1.11) with T_0 replaced by $\bar{T}_0 = T_0 + \varepsilon_0$.

Remark 1.2 If the perturbation $\varepsilon(x') \not\equiv 0$, i.e., nonidentically zero, and if there exists a stationary solution (v, B, T, ϱ) to our problem, then necessarily v is not identically zero (see Remark 5.2 in Sect. 5). This is obtained from the main result (Theorem 2.1) in the next section. This simply means that the nonhomogeneous repartition of the temperature around T_0 in the lower plane $x_3 = 0$ can generate a stationary magnetoconvective motion close to the rest state $(0, B_{rs}, T_{rs}, \varrho_{rs})$ given by (1.11).

The system of equations (1.1)–(1.4) with boundary conditions (1.6)–(1.8) can be used as a simple model of a three-dimensional plane-parallel atmosphere made up of ideal gas (see [37, 38] and their exhaustive references on subjects related to this issue). More precisely, we consider in a horizontal layer a polytropic atmosphere with a large temperature gradient across the layer and in the presence of a vertical magnetic field. We will then focus on the existence of a magnetoconvective steady flow close to the equilibrium state. This coupling between natural convection (the fluids motions induced by a large temperature gradient) and magnetic field has aroused enormous interest following the observation of astrophysical and geophysical phenomena. The examples of application are numerous: we can cite among others the planetary magnetospheres, the sun, the stars, the solar and stellar winds, the interstellar clouds, the accretion disks, and the galaxies.

This work is organized as follows. In Sect. 2, we present some preliminaries of the proof of the main result. In Sect. 3, we introduce a linearized problem, which is crucial for the construction of the nonlinear operator S, for which we establish the existence of a fixed point through the Schauder fixed point theorem. The crucial part of this work is then to show that the operator S satisfies the assumptions of the Schauder fixed point theorem (see Sect. 5). This has required some tedious computations aimed at establishing suitable fine estimates (in Lemmas 4.1–4.10). These estimates rely on some techniques developed in the study of the equations of viscous gazes, in particular those in [8] and also in [35] (in the general case). We can also mention [33, 34, 42, 43].

2 Main result and some preliminaries to the proof

We look for a solution (v, B, T, ϱ) in a small neighborhood of the stationary profile $(0, \hat{B}, \hat{T}, \hat{\varrho})$ close to the equilibrium state $(0, B_{rs}, T_{rs}, \varrho_{rs})$ given in (1.11)). This stationary profile is defined by

$$\hat{B}(x', x_3) := B_{rs}, \qquad \hat{T}(x', x_3) := T_{rs}(x_3) + \left(1 - \frac{x_3}{h}\right)\varepsilon(x'), \tag{2.1}$$

$$\hat{\varrho}(x', x_3) := \hat{M}\frac{T_{rs}^{\frac{\gamma}{\gamma-1}}(x_3)}{\hat{T}(x', x_3)} \quad \text{with } \hat{M} := M_0\left[\int_\Omega \frac{T_{rs}^{\frac{\gamma}{\gamma-1}}(x_3)}{\hat{T}(x', x_3)}\,dx\right]^{-1}. \tag{2.2}$$

Notice that $\hat{T}$ satisfies the same boundary conditions (1.7) and $\hat{\varrho}$ satisfies (1.9) so that

$$\int_\Omega (\varrho - \hat{\varrho})\,dx = 0. \tag{2.3}$$

Given (2.1) and (2.2), easy computations show that

$$\partial_{x_i}(R\hat{T}\hat{\varrho}) = 0 \quad (i = 1, 2), \qquad -\partial_{x_3}(R\hat{T}\hat{\varrho}) - g\hat{\varrho} = \frac{g\hat{\varrho}}{T_{rs}}\left(1 - \frac{x_3}{h}\right)\varepsilon(x'), \tag{2.4}$$

$$C_V\hat{\varrho}\nabla\hat{T} - R\hat{T}\nabla\hat{\varrho} = (R + C_V)\hat{\varrho}\nabla\left(\left(1 - \frac{x_3}{h}\right)\varepsilon(x')\right) + \frac{g\hat{\varrho}}{T_{rs}}\left(1 - \frac{x_3}{h}\right)\varepsilon(x')e_3. \tag{2.5}$$

This being so, we then set

$$v = v, \qquad b = B - \hat{B}, \qquad \vartheta = T - \hat{T}, \qquad \sigma = \varrho - \hat{\varrho}, \tag{2.6}$$

and, since as we seek ϱ close to $\hat{\varrho}$, we can assume that

$$\|\varrho - \hat{\varrho}\|_{L^\infty} \le \frac{1}{2}\inf_{\bar{\Omega}}\hat{\varrho}. \tag{2.7}$$

By considering for a generic vector field M the identities

$$(\nabla \times M) \times M = (M \cdot \nabla)M - \frac{1}{2}\nabla|M|^2,$$
$$\nabla \times (v \times M) = v\nabla \cdot M - M\nabla \cdot v + (M \cdot \nabla)v - (v \cdot \nabla)M,$$
$$\nabla \times (\nabla \times M) = \nabla\nabla \cdot M - \Delta M,$$

and given (2.6), we then rewrite problem (1.1)–(1.9) with unknowns (T, v, B, ϱ) as a new problem with unknowns $u = (\vartheta, v, b, \sigma)$ as follows:

$$-\kappa\Delta\vartheta = G(u), \tag{2.8}$$

$$-\mu\Delta v - (\mu + \lambda)\nabla\nabla \cdot v = -R\nabla(\hat{T}\sigma + \hat{\varrho}\vartheta) - g\sigma e_3 + (\hat{B} \cdot \nabla)b - \hat{B} \cdot \nabla b + F(u), \tag{2.9}$$

$$-\nu\Delta b = -\hat{B}\nabla \cdot v + (\hat{B} \cdot \nabla)v + H(u), \quad \nabla \cdot b = 0, \tag{2.10}$$

$$\nabla \cdot (\sigma v) = -\nabla \cdot (\hat{\varrho}v), \tag{2.11}$$

with the boundary conditions

$$v|_{x_3=0} = 0, \qquad v_3|_{x_3=h} = 0, \qquad \partial_{x_3} v_i|_{x_3=h} = 0 \quad i = 1, 2, \tag{2.12}$$

$$\vartheta|_{x_3=0} = \vartheta|_{x_3=h} = 0, \tag{2.13}$$

$$b|_{x_3=0} = b|_{x_3=h} = 0, \tag{2.14}$$

and (see (2.6) and (2.3)) the mean value property

$$\int_\Omega \sigma \, dx = 0. \tag{2.15}$$

The functions $F(u)$, $G(u)$, and $H(u)$ are given by

$$F(u) := -(\hat{\varrho} + \sigma)(v \cdot \nabla)v - R\nabla(\sigma\vartheta) + g\frac{\hat{\varrho}}{T_{rs}}\left(1 - \frac{x_3}{h}\right)\varepsilon(x')e_3 + (\nabla \times b) \times b, \tag{2.16}$$

$$\begin{aligned} G(u) := {} & 2\mu D(v) : D(v) + \lambda(\nabla \cdot v)^2 + Rv \cdot (\vartheta\nabla\hat{\varrho} + \hat{T}\nabla\sigma + \vartheta\nabla\sigma) \\ & + C_V(\sigma\nabla\hat{T} + \hat{\varrho}\nabla\vartheta + \sigma\nabla\vartheta) \cdot v - (R + C_V)\hat{\varrho}v \cdot \nabla\left(\left(1 - \frac{x_3}{h}\right)\varepsilon(x')\right) \\ & - g\frac{\hat{\varrho}}{T_{rs}}\left(1 - \frac{x_3}{h}\right)\varepsilon(x')v_3 + \kappa\Delta\left(\left(1 - \frac{x_3}{h}\right)\varepsilon(x')\right) + \nu|\nabla \times b|^2, \end{aligned} \tag{2.17}$$

$$H(u) := v\nabla \cdot b - b\nabla \cdot v + (b \cdot \nabla)v - (v \cdot \nabla)b. \tag{2.18}$$

Thus, problem (1.1)–(1.8) is reduced to finding

$$u = (\vartheta, v, b, \sigma) = (T, v, B, \varrho) - (\hat{T}, 0, \hat{B}, \hat{\varrho})$$

solution of problem (2.8)–(2.15).

Notice that equation (2.10) is obtained from (1.4), which is written as

$$-\nu\Delta B - \nabla \times (v \times B) = 0, \quad B = b + \hat{B},$$

because $\nabla \times (\nabla \times B) = \nabla\nabla \cdot B - \Delta B$, $\nabla \cdot B = 0$, $\nabla \cdot \hat{B} = 0$, and so $\nabla \cdot b = 0$.

Our main result is the following theorem.

Theorem 2.1 *Under* (1.10), *the system of equations* (1.1)–(1.4) *with boundary conditions* (1.6)–(1.8) *has at least one solution*

$$(T, v, B, \varrho) \in H^2_\natural(\Omega) \times H^3_\natural(\Omega) \times H^2_\natural(\Omega) \times H^2_\natural(\Omega), \tag{2.19}$$

where $\natural$ *means periodicity in the horizontal directions.*

We look for a solution

$$(T, v, B, \varrho) = (\vartheta, v, b, \sigma) + (\hat{T}, 0, \hat{B}, \hat{\varrho}),$$

where $u = (\vartheta, v, b, \sigma)$ is the appropriate solution of problem (2.8)–(2.15) which will be obtained as a fixed point of some operator constructed from a suitable linearization of the system of equations (2.8)–(2.10).

3 Linearized equations and the nonlinear operator *S*

We set

$$\widehat{H}^2_\sharp(\Omega) := H^2(\Omega) \cap \widehat{H}^1_\sharp(\Omega), \qquad \widehat{H}^1_\sharp(\Omega) := \{b \in H^1_\sharp(\Omega) : \nabla \cdot b = 0\}, \tag{3.1}$$

$$V := \{u \in H^2_\sharp(\Omega) \times H^3_\sharp(\Omega) \times \widehat{H}^2_\sharp(\Omega) \times H^2_\sharp(\Omega) : u \text{ satisfies (2.12)–(2.15)}\}, \tag{3.2}$$

$$\|\cdot\|_V^2 := \|\cdot\|_{H^2}^2 + \|\cdot\|_{H^3}^2 + \|\cdot\|_{H^2}^2 + \|\cdot\|_{H^2}^2, \tag{3.3}$$

where $\|\cdot\|_{H^k}$ and $\|\cdot\|_{L^2}$ denote the usual norms of $H^k(\Omega)$ and $L^2(\Omega)$. Furthermore, we consider

$$V_0 := (V, \|\cdot\|_{V_0}) \quad \text{with } \|\cdot\|_{V_0}^2 := \|\cdot\|_{H^1}^2 + \|\cdot\|_{H^2}^2 + \|\cdot\|_{H^1}^2 + \|\cdot\|_{H^1}^2. \tag{3.4}$$

Let now $u' = (\vartheta', v', b', \sigma') \in V$ and $k > 0$ be given. We consider the system of equations

$$-\kappa \Delta \vartheta = G', \tag{3.5}$$

$$-\mu \Delta v - (\mu + \lambda)\nabla\nabla \cdot v = -R\nabla(\hat{\varrho}\vartheta + \hat{T}\sigma') - g\sigma' e_3 + (\hat{B}\cdot\nabla)b - \hat{B}\cdot\nabla b + F', \tag{3.6}$$

$$-\nu \Delta b = -\widehat{B}\nabla \cdot v + (\widehat{B}\cdot\nabla)v + H', \quad \nabla \cdot b = 0, \tag{3.7}$$

$$k(\sigma - \sigma') + \nabla \cdot (\sigma v) = -\nabla \cdot (\hat{\varrho} v), \tag{3.8}$$

with boundary conditions (2.12)–(2.14) and (2.15), where (see (2.16)–(2.18)),

$$G' = G(u'), \qquad F' = F(u'), \qquad H' = H(u'). \tag{3.9}$$

We obtain in the following lemma an existence result for system (3.5)–(3.8) with (2.12)–(2.14).

Lemma 3.1 *Let $u' = (\vartheta', v', b', \sigma') \in V$. If $k > 0$ is large enough, then system (3.5)–(3.8) with conditions (2.12)–(2.14) has a unique solution $u = (\vartheta, v, b, \sigma) \in V$ such that*

$$\|\vartheta\|_{H^2}^2 \le c_\Omega \|G'\|_{L^2}^2, \tag{3.10}$$

$$\|v\|_{H^3}^2 \le c_\Omega (\|\sigma'\|_{H^2}^2 + \|F'\|_{H^1}^2 + \|G'\|_{L^2}^2 + \|H'\|_{L^2}^2), \tag{3.11}$$

$$\|b\|_{H^2}^2 \le c_\Omega (\|\sigma'\|_{H^1}^2 + \|F'\|_{L^2}^2 + \|G'\|_{L^2}^2 + \|H'\|_{L^2}^2), \tag{3.12}$$

$$k\|\sigma\|_{H^2}^2 \le c_\Omega \|v\|_{H^3} \|\sigma\|_{H^2}^2 + k\|\sigma'\|_{H^2}^2 + c_\Omega \|v\|_{H^3}^2, \tag{3.13}$$

where c_Ω is a constant depending only on $\hat{T}$, $\hat{B}$, $\hat{\varrho}$, and Ω.

Proof Set

$$W = \{(\vartheta, v, b) \in H^1_\sharp(\Omega) \times H^1_\sharp(\Omega) \times \hat{H}^1_\sharp(\Omega) : \vartheta|_{x_3=0,h} = 0,\ v|_{x_3=0} = b|_{x_3=0,h} = 0\},$$

and let us start by showing that the system of equations (3.5)–(3.7) with boundary conditions (2.12)–(2.14) has a unique weak solution $(\vartheta, v, b) \in W$ satisfying

$$a_\alpha\big((\vartheta, v, b), (\chi, \varphi, \psi)\big) = L_\alpha(\chi, \varphi, \psi) \quad \forall(\chi, \varphi, \psi) \in W, \tag{3.14}$$

where

$$\begin{aligned}
&a_\alpha\big((\vartheta, v, b), (\chi, \varphi, \psi)\big)\\
&\quad := \alpha\kappa \int_\Omega \nabla\vartheta \cdot \nabla\chi \, dx + \int_\Omega \big(\mu\nabla v \cdot \nabla\varphi + (\mu + \lambda)\nabla \cdot v\nabla \cdot \varphi\big) \, dx\\
&\qquad + \nu \int_\Omega \nabla b \cdot \nabla\psi \, dx + R \int_\Omega \nabla(\hat{\varrho}\vartheta) \cdot \varphi \, dx + \int_\Omega \big[-(\hat{B} \cdot \nabla)b + \widehat{B} \cdot \nabla b)\big]\varphi \, dx\\
&\qquad + \int_\Omega \big(-(\hat{B} \cdot \nabla)v + \hat{B}\nabla \cdot v\big)\psi \, dx,\\
&L_\alpha(\chi, \varphi, \psi) := \int_\Omega \alpha G'\chi \, dx + \int_\Omega F'\varphi \, dx + \int_\Omega H'\psi \, dx - \int_\Omega \big(R\nabla\big(\hat{\varrho}^{\gamma-1}\sigma'\big) + g\sigma' e_3\big)\varphi \, dx,
\end{aligned}$$

and $\alpha > 0$ is a real number, which will be appropriately chosen thereafter.

Indeed, by endowing W with the norm $\|(\vartheta, v, b)\|_W = (\|\nabla\vartheta\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla b\|_{L^2}^2)^{\frac{1}{2}}$, which makes it a Hilbert space, we can easily see that the bilinear form a_α is continuous on W. Moreover, since

$$\begin{aligned}
&\int_\Omega \big[(\hat{B} \cdot \nabla)b - \nabla(b \cdot \hat{B})\big]v \, dx + \int_\Omega b \cdot \big(v\nabla \cdot \hat{B} + (\widehat{B} \cdot \nabla)v + \hat{B}\nabla \cdot v\big) \, dx = 0,\\
&R \int_\Omega \nabla(\hat{\varrho}\vartheta) \cdot v \, dx = -R \int_\Omega \hat{\varrho}\vartheta\nabla \cdot v \, dx \ge -\frac{\mu + \lambda}{2}\|\nabla \cdot v\|_{L^2}^2 - c_\Omega\|\nabla\vartheta\|_{L^2}^2,
\end{aligned}$$

we have

$$\begin{aligned}
a_\alpha\big((\vartheta, v, b), (\vartheta, v, b)\big) &= \alpha\kappa\|\nabla\vartheta\|_{L^2}^2 + \mu\|\nabla v\|_{L^2}^2 + (\mu + \lambda)\|\nabla \cdot v\|_{L^2}^2\\
&\quad + \nu\|\nabla b\|_{L^2}^2 + R \int_\Omega \nabla(\hat{\varrho}\vartheta) \cdot v \, dx\\
&\ge (\alpha\kappa - c_\Omega)\|\nabla\vartheta\|_{L^2}^2 + \mu\|\nabla v\|_{L^2}^2 + \frac{1}{2}(\mu + \lambda)\|\nabla \cdot v\|_{L^2}^2 + \nu\|\nabla b\|_{L^2}^2\\
&\ge \inf\left(\kappa, \mu, \frac{1}{2}(\mu + \lambda), \nu\right)\big\|(\vartheta, v, b)\big\|_W^2,
\end{aligned}$$

where we have chosen α such that $\alpha\kappa - c_\Omega \ge \kappa$.

So, the bilinear form a_α is coercive on W and, since the linear form L_α is obviously continuous on W, according to the Lax–Milgram theorem, problem (3.14) has a unique weak solution $(\vartheta, v, b) \in W$ such that

$$\|\nabla\vartheta\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 \le c_\Omega\big(\big\|\sigma'\big\|_{H^1}^2 + \big\|F'\big\|_{L^2}^2 + \big\|G'\big\|_{L^2}^2 + \big\|H'\big\|_{L^2}^2\big). \tag{3.15}$$

This being, from the classical theory of systems of linear elliptic pdes (see e.g., [24, 32]), according to (3.15), we have (3.10)–(3.12). As for the solution $\sigma \in H^2$ of (3.8) and its estimate (3.13), we refer, for instance, the interested reader to [3]. □

Following Lemma 3.1, we then define the nonlinear operator S as follows:

$$S : V \to V, \qquad u' = (\vartheta', v', b', \sigma'), \qquad S(u') = u, \tag{3.16}$$

where $u = (\vartheta, v, b, \sigma) \in V$ is the unique solution of system (3.5)–(3.8).

Lemma 3.2 *The nonlinear operator $S : V_0 \to V_0$ is continuous. More precisely, for any bounded subset D of V, there exists $c_D > 0$ such that*

$$\|S(u_1') - S(u_2')\|_{V_0} \le c_D \|u_1' - u_2'\|_{V_0} \quad \forall u_1', u_2' \in D. \tag{3.17}$$

Proof Let us first notice that the operator S maps bounded subsets into bounded sets. Indeed, taking into account (3.9), it is easy to see that, for every $u' = (\vartheta', v', b', \sigma') \in V$, we have

$$\|F'\|_{H^1} \le c_\Omega \big[\|v'\|_{H^2}^2 (1 + \|\sigma'\|_{H^2}) + \|\sigma'\|_{H^2}^2 + \|\vartheta'\|_{H^2}^2 + \|b'\|_{H^2}^2 + \|\varepsilon\|_{H^1(\mathbb{T}^2)}\big], \tag{3.18}$$

$$\begin{aligned} \|G'\|_{L^2} \le{}& c_\Omega \big(\|v'\|_{H^2}^2 + \|\vartheta'\|_{H^2}^2 + \|b'\|_{H^2}^2 + \|\sigma'\|_{H^2}^2\big) \\ &+ c_\Omega \|\vartheta'\|_{H^2} \|v'\|_{H^2} \|\sigma'\|_{H^2} + c_\Omega \big(1 + \|v'\|_{H^2}\big) \|\varepsilon\|_{H^2(\mathbb{T}^2)}, \end{aligned} \tag{3.19}$$

$$\|H'\|_{L^2} \le c_\Omega \big(\|v'\|_{H^2}^2 + \|b'\|_{H^2}^2\big). \tag{3.20}$$

Hence, by recalling the V-norm (see (3.2)), we obtain

$$\|F'\|_{H^1} \le c_\Omega \|u'\|_V^2 (1 + \|u'\|_V) + c_\Omega \|\varepsilon\|_{H^1(\mathbb{T}^2)}, \tag{3.21}$$

$$\|G'\|_{L^2} \le c_\Omega \|u'\|_V^2 (1 + \|u'\|_V) + c_\Omega (1 + \|u'\|_V) \|\varepsilon\|_{H^2(\mathbb{T}^2)}, \tag{3.22}$$

$$\|H'\|_{L^2} \le c_\Omega \|u'\|_V^2. \tag{3.23}$$

Let D be a bounded subset of V. If k is large enough, from (3.10)–(3.13) and the previous inequalities, we can easily show that

$$\|S(u')\|_V^2 = \|v\|_{H^3}^2 + \|\vartheta\|_{H^2}^2 + \|b\|_{H^2}^2 + \|\sigma\|_{H^2}^2 \le c_D \quad \forall u' \in D. \tag{3.24}$$

This means that $S(D)$ is a bounded set of V. Let now

$$u_i' = (\vartheta_i', v_i', b_i', \sigma_i') \in D, \qquad S(u_i') = u_i = (\vartheta_i, v_i, b_i, \sigma_i), \quad i = 1, 2, \tag{3.25}$$

$$u_1' - u_2' = (\vartheta', v', b', \sigma'), \qquad S(u_1') - S(u_2') = u_1 - u_2 = (\vartheta, v, b, \sigma). \tag{3.26}$$

By definition (3.16) of the operator S, we have

$$-\kappa \Delta \vartheta = G(u_1') - G(u_2'), \tag{3.27}$$

$$\begin{aligned} -\mu \Delta v - (\mu + \lambda) \nabla \nabla \cdot v ={}& -R \nabla (\hat{T} \sigma') - R \nabla (\hat{\varrho} \vartheta) - g \sigma' e_3 \\ &+ (\hat{B} \cdot \nabla) b - \hat{B} \cdot \nabla b + F(u_1') - F(u_2'), \end{aligned} \tag{3.28}$$

$$-\nu \Delta b = v \nabla \cdot \hat{B} - \hat{B} \nabla \cdot v + (\hat{B} \cdot \nabla) v + H(u_1') - H(u_2'), \tag{3.29}$$

$$k(\sigma - \sigma') + \nabla \cdot (\sigma v_1) = -\nabla \cdot \big((\hat{\varrho} + \sigma_2)v\big). \tag{3.30}$$

From (3.9) and (2.17)–(2.18), it is easy to see that

$$\begin{aligned}&\|G(u_1') - G(u_2')\|_{H^{-1}} + \|F(u_1') - F(u_2')\|_{L^2} + \|H(u_1') - H(u_2')\|_{H^{-1}} \\ &\quad\le c_\Omega\big(1 + \|u_1'\|_V^2 + \|u_2'\|_V^2\big)\big(\|v'\|_{H^2} + \|\vartheta'\|_{H^1}^2 + \|b'\|_{H^1} + \|\sigma'\|_{H^1}\big).\end{aligned}$$

Now, by applying the regularity results for systems of elliptic PDEs to (3.27)–(3.29), we obtain (see (3.26) and (3.4))

$$\begin{aligned}\|\vartheta\|_{H^1} + \|v\|_{H^2} + \|b\|_{H^1} &\le \|G(u_1') - G(u_2')\|_{H^{-1}} + \|F(u_1') - F(u_2')\|_{L^2} \\ &\quad + \|H(u_1') - H(u_2')\|_{H^{-1}} \\ &\le c_\Omega\big(1 + \|u_1'\|_V^2 + \|u_2'\|_V^2\big)\|u_1' - u_2'\|_{V_0}.\end{aligned} \tag{3.31}$$

Now, it remains to estimate in the H^1-norm the solution σ of equation (3.30). We first multiply (3.30) by σ, and we integrate over Ω. Applying integration by parts and using the inequality

$$\int_\Omega \nabla \cdot (\sigma v_1)\sigma = \frac{1}{2}\int_\Omega |\sigma|^2 \nabla \cdot v_1 \, dx \le c_\Omega \|v_1\|_{H^3}\|\sigma\|_{L^2}^2,$$

we easily obtain

$$k\|\sigma\|_{L^2}^2 \le c_\Omega\|v_1\|_{H^3}\|\sigma\|_{L^2}^2 + k\|\sigma'\|_{L^2}^2 + c_\Omega\big(1 + \|\sigma_2\|_{H^1}^2\big)\|v\|_{H^1}^2. \tag{3.32}$$

We next differentiate equation (3.30) by applying ∇ to both sides of this equation, then we take the scalar product with $\nabla\sigma$, and we integrate over Ω. Applying integration by parts and using the inequality

$$\begin{aligned}\int_\Omega \nabla\sigma \cdot \nabla\big(\nabla \cdot (\sigma v_1)\big)\, dx &= \frac{1}{2}\int_\Omega |\nabla\sigma|^2(\nabla \cdot v_1)\, dx + \int_\Omega (\nabla\sigma \cdot \nabla)v_1 \cdot \nabla\sigma \, dx \\ &\quad + \int_\Omega \sigma\big(\nabla(\nabla \cdot v_1)\big) \cdot \nabla\sigma \, dx \le c_\Omega\|v_1\|_{H^3}\|\sigma\|_{H^1}^2,\end{aligned}$$

we get

$$k\|\nabla\sigma\|_{L^2}^2 \le c_\Omega\|v_1\|_{H^3}\|\sigma\|_{H^1}^2 + k\|\nabla\sigma'\|_{L^2}^2 + c_\Omega\big(1 + \|\sigma_2\|_{H^2}^2\big)\|v\|_{H^2}^2. \tag{3.33}$$

Adding (3.32) and (3.33), we obtain, using (3.31) (see also (3.25) and (3.26)), that

$$\begin{aligned}k\|\sigma\|_{H^1}^2 &\le c_\Omega\|S(u_1')\|_V\|\sigma\|_{H^1}^2 + k\|\sigma'\|_{H^1}^2 + c_\Omega\big(1 + \|S(u_2')\|_V^2\big)\|v\|_{H^2}^2 \\ &\le c_\Omega\|S(u_1')\|_V\|\sigma\|_{H^1}^2 + c_\Omega\big(1 + \|S(u_2')\|_V^2\big)\big(1 + \|u_1'\|_V^2 + \|u_2'\|_V^2\big)\|u_1' - u_2'\|_{V_0}^2.\end{aligned}$$

Hence, if k is large enough so that (see (3.24)) $k - c_\Omega\|S(u_1')\|_V \ge k - c_\Omega c_D \ge 1$, we obtain

$$\|\sigma\|_{H^1} \le c_\Omega\big(1 + \|S(u_2')\|_V\big)\big(1 + \|u_1'\|_V + \|u_2'\|_V\big)\|u_1' - u_2'\|_{V_0}. \tag{3.34}$$

Now, adding up inequalities (3.34) and (3.31), we obtain

$$\begin{aligned} &\|v\|_{H^2} + \|\vartheta\|_{H^1} + \|b\|_{H^1} + \|\sigma\|_{H^1} \\ &\quad \le c_\Omega\big(1 + \|S(u_2')\|_V^2\big)\big(1 + \|u_1'\|_V^2 + \|u_2'\|_V^2\big)\|u_1' - u_2'\|_{V_0}, \end{aligned}$$

from which (see (3.26)) follows (3.17), and this completes the proof of Lemma 3.2. □

Remark 3.3 The V_0-continuous operator S (i.e., continuous in the norm of V_0) will be shown to satisfy the assumptions of the Schauder fixed point theorem. Following the work done in [7, 8], we will show that

$$S(B_V^a) \subseteq B_V^a, \tag{3.35}$$

where $B_V^a = \overline{B(0,a)}$ is the closed ball of V with radius a sufficiently small. Moreover, since the embedding $H^k(\Omega) \hookrightarrow H^{k-1}(\Omega)$ is compact, the closed ball B_V^a is (see (3.4)) V_0-compact, and hence, the operator S has a fixed point $u = (\vartheta, v, b, \sigma) \in B_V^a$. We will establish the crucial inclusion (3.35) in Sect. 5.

Lemma 3.4 *Let $a > 0$ be sufficiently small. If the operator S has a fixed point in B_V^a through* (3.35), *then the boundary value problem* (3.5)–(2.14) *has at least one solution u in V.*

Proof Indeed, if (3.35) is satisfied, then, according to Remark 3.3, the operator S has a fixed point $u \in B_V^a$. By the definition (see (3.16)) of the operator S, we get that $u \in V$ and satisfies the system of equations (2.8)–(2.11) with boundary conditions (2.12)–(2.14). □

Remark 3.5 If we set

$$T := \hat{T} + \vartheta, \qquad B := b + \hat{B}, \qquad \varrho := \hat{\varrho} + \sigma,$$

(see (2.6)), following Lemma 3.4, $(T, v, B, \varrho) \in H_\sharp^2(\Omega) \times H_\sharp^3(\Omega) \times H_\sharp^2(\Omega) \times H_\sharp^2(\Omega)$ and satisfies equations (1.1)–(1.4) with boundary conditions (1.6)–(1.8). Thus, our main result Theorem 2.1 is proved.

From Remark 3.3 and Lemma 3.4, we can then turn our focus to the crucial point (3.35) to be established. To this aim, we need adequate estimates of the solution $u = S(u')$ of the system of equations (3.5)–(3.8) with conditions (2.12)–(2.15) from which we will obtain (3.35) in Remark 3.3. These estimates will require a more elaborate treatment and hence will be discussed in the next technical section.

We will need some estimates of the nonlinear terms

$$F' = F(u'), \qquad G' = G(u')), \qquad H' = H(u'), \qquad v\sigma = v(u')\sigma(u'),$$

which appear in equations (3.5)–(3.8).

Indeed, let $u' = (\vartheta', v', b', \sigma') \in B_V^a$. If a is small enough then

$$\|F'\|_{H^1}^2 + \|G'\|_{L^2}^2 + \|H'\|_{L^2}^2 + \|v\|_{H^3}\|\sigma\|_{H^2}^2 \le c_\Omega a^3. \tag{3.36}$$

In fact, considering (3.13) and (3.11), we have

$$\|v\|_{H^3}^2 \le c_\Omega\big(\|\sigma'\|_{H^2}^2 + \|F'\|_{H^1}^2 + \|G'\|_{L^2}^2 + \|H'\|_{L^2}^2\big), \tag{3.37}$$

$$k\|\sigma\|_{H^2}^2 \le c_\Omega\|v\|_{H^3}\|\sigma\|_{H^2}^2 + k\|\sigma'\|_{H^2}^2 + c_\Omega\|v\|_{H^3}^2. \tag{3.38}$$

Since (see (3.21)–(3.23))

$$\|F'\|_{H^1} + \|G'\|_{L^2} + \|H'\|_{L^2} \le c_\Omega\|u'\|_V^2\big(1 + \|u'\|_V\big) + c_\Omega\|u'\|_V\|\varepsilon\|_{H^2}, \tag{3.39}$$

then for all $u' = (\vartheta', v', b', \sigma') \in B_V^a$, we have

$$\|\sigma'\|_{H^2} + \|F'\|_{H^1} + \|G'\|_{L^2} + \|H'\|_{L^2} \le c_\Omega\big[a + a^2(1+a) + a\|\varepsilon\|_{H^2(\mathbb{T}^2)}\big] \le c_a. \tag{3.40}$$

If k is large enough so that

$$k - c_\Omega\big(\|\sigma'\|_{H^2}^2 + \|F'\|_{H^1}^2 + \|G'\|_{L^2}^2 + \|H'\|_{L^2}^2\big) \ge k - c_a > 0,$$

then from (3.37) and (3.38) it follows that

$$\|\sigma\|_{H^2}^2 \le c_\Omega\big(\|\sigma'\|_{H^2}^2 + \|F'\|_{H^1}^2 + \|G'\|_{L^2}^2 + \|H'\|_{L^2}^2\big), \tag{3.41}$$

whence, given (3.37), we get

$$\|v\|_{H^3}\|\sigma\|_{H^2}^2 \le c_\Omega\big(\|\sigma'\|_{H^2} + \|F'\|_{H^1} + \|G'\|_{L^2} + \|H'\|_{L^2}\big)^3, \tag{3.42}$$

where (see (3.39))

$$\|F'\|_{H^1} + \|G'\|_{L^2} + \|H'\|_{L^2} \le c_\Omega\big(a^2(1+a) + a\|\varepsilon\|_{H^2(\mathbb{T}^2)}\big). \tag{3.43}$$

From (3.43), (3.42), and (3.40) it follows

$$\begin{aligned}
&\|F'\|_{H^1}^2 + \|G'\|_{L^2}^2 + \|H'\|_{L^2}^2 + \|v\|_{H^3}\|\sigma\|_{H^2}^2 \\
&\quad \le c_\Omega\big(a^4(1+a^2) + a^2\|\varepsilon\|_{H^2(\mathbb{T}^2)}^2\big) + c_\Omega\big[a + a^2(1+a) + a\|\varepsilon\|_{H^2(\mathbb{T}^2)}\big]^3.
\end{aligned}$$

By choosing (see (1.10)) $\varepsilon_0 \le a$, it is easy to see that for all $a \in]0,1[$, we have

$$c_\Omega\big(a^4(1+a^2) + a^2\|\varepsilon\|_{H^2(\mathbb{T}^2)}^2\big) + c_\Omega\big[a + a^2(1+a) + a\|\varepsilon\|_{H^2(\mathbb{T}^2)}\big]^3 \le c_\Omega a^3,$$

and hence we obtain (3.36). □

4 Estimates of the solutions $(v, \vartheta, b, \sigma)$

This section is devoted to the estimates of the solutions $u = (\vartheta, v, b, \sigma)$ of equations (3.5)–(3.8) with boundary conditions (2.12)–(2.14). While the estimate of ϑ is obtained in (3.10), the estimates of v, b, and σ will be obtained in the following Lemmas 4.1–4.10 and are based largely on the ideas developed in the works [7, 8]. We recall here that these lemmas will be proven under assumptions (1.10). Regarding the positive constant k, which appears in equation (3.8), it can be chosen arbitrarily large as in Lemma 3.1. We set

$$k := \frac{\bar{\kappa} k_1}{2} \quad \text{with } k_1 = \frac{R}{2\mu + \lambda} \hat{M} \left(T_0 - \frac{\gamma - 1}{\gamma R} gh \right)^{\frac{\gamma}{\gamma-1}}, \tag{4.1}$$

and $\bar{\kappa}$ being a large positive constant satisfying in particular

$$\bar{\kappa} \geq 16 \left(1 - \frac{\gamma - 1}{\gamma R T_0} gh \right)^{-\frac{\gamma}{\gamma-1}}. \tag{4.2}$$

In the following lemmas, we denote by C'_k ($k = 1, \dots 10$) the constants that depend on Ω and $\hat{B}$ but neither on T_0 nor on M_0 (provided that they are larger than some constant) and by $\tilde{C}_k$ ($k = 1, \dots 9$) the constants that depend on Ω, $\hat{B}$, T_0, and M_0. In addition, in the proof of each lemma, if it is not necessary to specify them, one will indicate by C_Ω the constants that depend neither on T_0 nor on M_0 and by $\hat{C}$ those that depend on T_0 and/or M_0.

Lemma 4.1 *Let $u' = (\vartheta', v', b', \sigma') \in V$ and $u = S(u') = (\vartheta, v, b, \sigma) \in V$ be the solution of system (3.5)–(3.7) whose existence is guaranteed by Lemma 3.1. Under hypothesis (1.10) and the assumption that the constant k given in $(4.1)_1$ is large, we have*

$$\begin{aligned} &\frac{\mu}{R} \|\nabla v\|_{L^2}^2 + \frac{\mu + \lambda}{R} \|\nabla \cdot v\|_{L^2}^2 + \frac{\nu}{R} \|\nabla b\|_{L^2}^2 + C'_1 T_0 \|\sigma'\|_{L^2}^2 + k \left\| \left(\hat{T} \hat{\varrho}^{-1}\right)^{\frac{1}{2}} \sigma \right\|_{L^2}^2 \\ &\quad \leq k \left\| \left(\hat{T} \hat{\varrho}^{-1}\right)^{\frac{1}{2}} \sigma' \right\|_{L^2}^2 + C'_1 \left(\|H'\|_{L^2}^2 + \|F'\|_{L^2}^2 + \|v\|_{H^3} \|\sigma\|_{L^2}^2 \right) + \tilde{C}_1 \|\vartheta\|_{H^1}^2. \end{aligned} \tag{4.3}$$

Proof Let us first observe that

$$\begin{aligned} &- \int_\Omega \left[v \nabla \left(\hat{T} \sigma' \right) + \frac{\hat{T}}{\hat{\varrho}} \sigma \nabla \cdot (\hat{\varrho} v) \right] dx \\ &\quad = - \int_\Omega \frac{\hat{T}}{\hat{\varrho}} \left(\sigma - \sigma' \right) \nabla \cdot (\hat{\varrho} v) \, dx - \int_\Omega \hat{T} (v \cdot \nabla \log \hat{\varrho}) \sigma' \, dx, \\ &\int_\Omega \left[(\hat{B} \cdot \nabla) b - \nabla (b \cdot \hat{B}) \right] v \, dx + \int_\Omega b \cdot \left((\hat{B} \cdot \nabla) v - \hat{B} \nabla \cdot v \right) dx = 0. \end{aligned}$$

We multiply equations (3.6)–(3.8) by $R^{-1} v$, b, and $\hat{T} \hat{\varrho}^{-1} \sigma$ respectively and integrate the resulting equations over Ω. By using the above identities, we find after integration by parts that

$$\begin{aligned} &\frac{\mu}{R} \|\nabla v\|_{L^2}^2 + \frac{\mu + \lambda}{R} \|\nabla \cdot v\|_{L^2}^2 + \frac{\nu}{R} \|\nabla b\|_{L^2}^2 + \frac{k}{2} \left\| \left(\hat{T} \hat{\varrho}^{-1}\right)^{\frac{1}{2}} \left(\sigma - \sigma' \right) \right\|_{L^2}^2 \\ &\quad + \frac{k}{2} \left\| \left(\hat{T} \hat{\varrho}^{-1}\right)^{\frac{1}{2}} \sigma \right\|_{L^2}^2 = \frac{k}{2} \left\| \left(\hat{T} \hat{\varrho}^{-1}\right)^{\frac{1}{2}} \sigma' \right\|_{L^2}^2 + \sum_{i=1}^{4} I_i, \end{aligned} \tag{4.4}$$

where

$$I_1 := - \int_\Omega \frac{\hat{T}}{\hat{\varrho}} \left(\sigma - \sigma' \right) \nabla \cdot (\hat{\varrho} v) \, dx, \qquad I_2; = \frac{1}{R} \int_\Omega \left(H' \cdot b + F' \cdot v + R \hat{\varrho} \vartheta \nabla \cdot v \right) dx,$$

$$I_3 := -\frac{g}{R}\int_\Omega (e_3\cdot v)\sigma'\,dx - \int_\Omega \bar{T}(v\cdot\nabla\log\hat{\varrho})\sigma'\,dx, \qquad I_4 := -\int_\Omega \frac{\hat{T}}{\hat{\varrho}}\sigma\nabla(v\sigma)\,dx.$$

Recalling the expressions of $\hat{T}$ and $\hat{\varrho}$ in (2.1) and using (1.10), one can easily see that $\|\nabla\log\hat{\varrho}\|_{L^\infty}$ is small enough so that

$$\|v\cdot\nabla\log\hat{\varrho}\|_{L^2}^2 \le \frac{\mu}{2\mu+\lambda}\|\nabla v\|_{L^2}^2. \tag{4.5}$$

Thus, by using (4.1) and (4.2), we obtain

$$\begin{aligned}
I_1 &\le \frac{k}{2}\big\|\big(\hat{T}\hat{\varrho}^{-1}\big)^{\frac12}\big(\sigma-\sigma'\big)\big\|_{L^2}^2 + \frac{2}{\bar{\kappa}k_1}\|\hat{T}\hat{\varrho}\|_{L^\infty}\big(\|\nabla\cdot v\|_{L^2}^2 + \|v\cdot\nabla\log\hat{\varrho})\|_{L^2}^2\big)\\
&\le \frac{k}{2}\big\|\big(\hat{T}\hat{\varrho}^{-1}\big)^{\frac12}\big(\sigma-\sigma'\big)\big\|_{L^2}^2 + \frac{\mu+\lambda}{16R}\|\nabla\cdot v\|_{L^2}^2 + \frac{\mu}{8R}\|\nabla v\|_{L^2}^2,\\
I_2 &\le \frac{\nu}{4R}\|\nabla b\|_{L^2}^2 + \frac{3}{16R}(\mu+\lambda)\|\nabla\cdot v\|_{L^2}^2 + \frac{\mu}{16R}\|\nabla v\|_{L^2}^2\\
&\quad + C_\Omega\big(\big\|F'\big\|_{L^2}^2 + \big\|H'\big\|_{L^2}^2\big) + \hat{C}\|\vartheta\|_{H^1}^2,\\
I_3 &\le \frac{\mu}{16R}\|\nabla v\|_{L^2}^2 + C_\Omega\big\|\sigma'\big\|_{L^2}^2.
\end{aligned}$$

As for the last term I_4, we have

$$I_4 = -\frac12\int_\Omega \frac{\hat{T}}{\hat{\varrho}}|\sigma|^2(\nabla\cdot v)\,dx + \frac12\int_\Omega |\sigma|^2 v\cdot\nabla\left(\frac{\hat{T}}{\hat{\varrho}}\right)dx \le C_\Omega\|v\|_{H^3}\|\sigma\|_{L^2}^2.$$

Combining these estimates with (4.4), we obtain

$$\begin{aligned}
&\frac{3\mu}{4R}\|\nabla v\|_{L^2}^2 + \frac{3}{4R}(\mu+\lambda)\|\nabla\cdot v\|_{L^2}^2 + \frac{3\nu}{4R}\|\nabla b\|_{L^2}^2 + \frac{k}{2}\big\|\big(\hat{T}\hat{\varrho}^{-1}\big)^{\frac12}\big(\sigma-\sigma'\big)\big\|_{L^2}^2\\
&\quad + \frac{k}{2}\big\|\big(\hat{T}\hat{\varrho}^{-1}\big)^{\frac12}\sigma\big\|_{L^2}^2 \le \frac{k}{2}\big\|\big(\hat{T}\hat{\varrho}^{-1}\big)^{\frac12}\sigma'\big\|_{L^2}^2 + C_\Omega\big\|\sigma'\big\|_{L^2}^2\\
&\quad + \hat{C}\|\vartheta\|_{H^1}^2 + C_\Omega\big(\big\|F'\big\|_{L^2}^2 + \big\|H'\big\|_{L^2}^2 + \|v\|_{H^3}\|\sigma\|_{L^2}^2\big).
\end{aligned} \tag{4.6}$$

Next, let us establish the L^2-estimate of σ', which appears in the right-hand side of (4.6).

To this aim, we introduce the following auxiliary problem:

$$\begin{cases} \nabla\cdot\varphi = \sigma' \text{ in } \Omega = \mathbb{T}^2\times]0,h[,\\ \varphi\big(x',0\big) = \varphi\big(x',h\big) = 0. \end{cases} \tag{4.7}$$

There exists (see [10]) at least one solution $\varphi\in H^1(\Omega)$ to problem (4.7) such that

$$\|\varphi\|_{H^1} \le c_\Omega\big\|\sigma'\big\|_{L^2}. \tag{4.8}$$

Now, we rewrite equation (3.6) as

$$-R\nabla\big(\hat{T}\sigma'\big) = -\mu\Delta v - (\mu+\lambda)\nabla\nabla\cdot v + R\nabla(\hat{\varrho}\vartheta) + g\sigma' e_3 - (\hat{B}\cdot\nabla)b + \hat{B}\cdot\nabla b - F',$$

multiply it by $R^{-1}\varphi$, and integrate over Ω. Integrating by parts, taking into account (4.7), we obtain that

$$\begin{aligned}\int_\Omega \hat{T}\left|\sigma'\right|^2 dx = \frac{g}{R}\int_\Omega \sigma' e_3\cdot\varphi\, dx\\ &+ \frac{1}{R}\int_\Omega\left[\mu\nabla v\cdot\nabla\varphi + (\mu+\lambda)(\nabla\cdot v)(\nabla\cdot\varphi) + R\nabla(\hat{\varrho}\vartheta)\cdot\varphi - F'\cdot\varphi\right]dx\\ &+ \frac{1}{R}\int_\Omega\left[-(\hat{B}\cdot\nabla)b + \hat{B}\cdot\nabla b\right]\cdot\varphi\, dx.\end{aligned} \tag{4.9}$$

Considering (4.8), we obtain that

$$T_0\left\|\sigma'\right\|_{L^2}^2 \le C_\Omega\left[\mu\|\nabla v\|_{L^2}^2 + (\mu+\lambda)\|\nabla\cdot v\|_{L^2}^2 + \nu\|\nabla\cdot b\|_{L^2}^2 + \left\|F'\right\|_{L^2}^2\right] + \hat{C}\|\vartheta\|_{H^1}^2.$$

If we multiply now the inequality above by $(4RC_\Omega)^{-1}$ and add it to (4.6), we obtain estimate (4.3), and this completes the proof the lemma. □

Lemma 4.2 *Under the same hypotheses of Lemma 4.1, we have*

$$\begin{aligned}&\sum_{i=1}^{2}\left[\frac{\mu}{R}\|\nabla\partial_{x_i}v\|_{L^2}^2 + \frac{\mu+\lambda}{R}\|\nabla\cdot\partial_{x_i}v\|_{L^2}^2 + \frac{\nu}{R}\|\nabla\cdot\partial_{x_i}b\|_{L^2}^2 + k\left\|\left(\hat{T}\hat{\varrho}^{-1}\right)^{\frac{1}{2}}\partial_{x_i}\sigma\right\|_{L^2}^2\right]\\ &\le \sum_{i=1}^{2}k\left\|\left(\hat{T}\hat{\varrho}^{-1}\right)^{\frac{1}{2}}\partial_{x_i}\sigma'\right\|_{L^2}^2 + C_2'\left\|\sigma'\right\|_{L^2}^2 + \tilde{C}_2\|\vartheta\|_{H^1}^2\\ &\quad + C_2'\left(\left\|F'\right\|_{L^2}^2 + \left\|H'\right\|_{L^2}^2 + \|v\|_{H^3}\|\sigma\|_{H^1}^2\right).\end{aligned} \tag{4.10}$$

Proof Let us notice that, since $\partial_{x_i}v$ and $\partial_{x_i}b$ $(i = 1,2)$ satisfy the same boundary conditions (2.12) and (2.14), we have

$$\begin{aligned}&-\int_\Omega\left[(\partial_{x_i}v)\nabla\cdot\left(\partial_{x_i}\left(\hat{T}\sigma'\right)\right) + \frac{\hat{T}}{\hat{\varrho}}\partial_{x_i}\nabla\cdot(\hat{\varrho}v)\partial_{x_i}\sigma\right]dx\\ &= -\int_\Omega\frac{\hat{T}}{\hat{\varrho}}\left(\partial_{x_i}\nabla\cdot(\hat{\varrho}v)\right)\left(\partial_{x_i}\sigma - \partial_{x_i}\sigma'\right)dx + \int_\Omega(\partial_{x_i}\hat{T})\sigma'\partial_{x_i}\nabla\cdot v\, dx\\ &\quad - \int_\Omega\hat{T}\left[(\partial_{x_i}\log\hat{\varrho})\nabla\cdot v + \frac{1}{\hat{\varrho}}\partial_{x_i}(v\cdot\nabla\hat{\varrho})\right]\partial_{x_i}\sigma'\, dx\\ &\int_\Omega\left[(\hat{B}\cdot\nabla)\partial_{x_i}b - \nabla(\partial_{x_i}b\cdot\hat{B})\right]\partial_{x_i}v\, dx + \int_\Omega\partial_{x_i}b\cdot\left((\hat{B}\cdot\nabla)\partial_{x_i}v - \hat{B}\nabla\cdot\partial_{x_i}v\right)dx = 0.\end{aligned}$$

Now, we differentiate equations (3.6)–(3.8) by applying on both sides ∂_{x_i} $(i = 1,2)$ and multiply them by $R^{-1}\partial_{x_i}v$, $R^{-1}\partial_{x_i}b$, and $\hat{T}\hat{\varrho}^{-1}\partial_{x_i}\sigma$ respectively. We integrate over Ω and apply the integration by parts to obtain

$$\begin{aligned}&\frac{\mu}{R}\|\nabla\partial_{x_i}v\|_{L^2}^2 + \frac{\mu+\lambda}{R}\|\nabla\cdot\partial_{x_i}v\|_{L^2}^2 + \frac{\nu}{R}\|\nabla\partial_{x_i}b\|_{L^2}^2 + \frac{k}{2}\left\|\left(\hat{T}\hat{\varrho}^{-1}\right)^{\frac{1}{2}}\partial_{x_i}\sigma\right\|_{L^2}^2\\ &\quad + \frac{k}{2}\left\|\left(\hat{T}\hat{\varrho}^{-1}\right)^{\frac{1}{2}}\left(\partial_{x_i}\sigma - \partial_{x_i}\sigma'\right)\right\|_{L^2}^2 = \frac{k}{2}\left\|\left(\hat{T}\hat{\varrho}^{-1}\right)^{\frac{1}{2}}\partial_{x_i}\sigma'\right\|_{L^2}^2 + \sum_{i=1}^{4}I_i,\end{aligned} \tag{4.11}$$

where

$$I_1 := -\int_\Omega \frac{\bar{T}}{\hat{\varrho}}\big(\partial_{x_i}\sigma - \partial_{x_i}\sigma'\big)\partial_{x_i}\nabla\cdot(\hat{\varrho}v)\,dx,$$

$$I_2 := \int_\Omega \big[(\partial_{x_i}\nabla\cdot v)\partial_{x_i}(\hat{\varrho}\vartheta) - R^{-1}\big(F'\cdot\partial_{x_i}\partial_{x_i}v - H'\cdot\partial_{x_i}\partial_{x_i}b\big)\big]\,dx,$$

$$I_3 := -\int_\Omega \sigma'\big[\partial_{x_i}\big(\hat{T}(\partial_{x_i}\log\hat{\varrho})\nabla\cdot v + \hat{\varrho}^{-1}\partial_{x_i}(v\cdot\nabla\hat{\varrho})\big) + (\partial_{x_i}\hat{T})\partial_{x_i}\nabla\cdot v\big]\,dx,$$

$$I_4 =: \frac{g}{R}\int_\Omega \sigma'\partial_{x_i}\partial_{x_i}v_3\,dx, \qquad I_5 := -\int_\Omega \frac{\hat{T}}{\hat{\varrho}}(\partial_{x_i}\sigma)\partial_{x_i}\nabla\cdot(\sigma v)\,dx.$$

Now, from the arguments similar to the ones used in the proof of Lemma 4.1 (particularly for the term I_1), we get

$$I_1 \le \frac{k}{2}\big\|\big(\hat{T}\hat{\varrho}^{-1}\big)^{\frac{1}{2}}(\partial_{x_i}\sigma - \partial_{x_i}\sigma'\big\|_{L^2}^2 + \frac{\mu+\lambda}{2R}\big\|\nabla\cdot(\partial_{x_i}v)\big\|_{L^2}^2 + \frac{\mu}{6R}\|\nabla\partial_{x_i}v\|_{L^2}^2,$$

$$I_2 \le \frac{\mu}{6R}\|\nabla\partial_{x_i}v\|_{L^2}^2 + \frac{\nu}{2R}\|\nabla\partial_{x_i}b\|_{L^2}^2 + C_\Omega\big(\big\|F'\big\|_{L^2}^2 + \big\|H'\big\|_{L^2}^2\big) + \hat{C}\|\vartheta\|_{H^1}^2,$$

$$I_3 + I_4 \le \frac{\mu}{6R}\|\nabla\partial_{x_i}v\|_{L^2}^2 + C_\Omega\big\|\sigma'\big\|_{L^2}^2.$$

As for the term I_5, we have

$$I_5 = -\frac{1}{2}\int_\Omega \frac{\hat{T}}{\hat{\varrho}}|\partial_{x_i}\sigma|^2(\nabla\cdot v)\,dx + \frac{1}{2}\int_\Omega |\partial_{x_i}\sigma|^2\big(v\cdot\nabla\big(\hat{T}\hat{\varrho}^{-1}\big)\big)\,dx$$
$$- \int_\Omega \bar{T}\hat{\varrho}^{-1}(\partial_{x_i}\sigma)\nabla\cdot(\sigma\partial_{x_i}v)\,dx \le C_\Omega\|v\|_{H^3}\|\sigma\|_{H^1}^2. \tag{4.12}$$

By adding these estimates to (4.11) and summing on $i = 1, 2$, we obtain (4.10). □

Lemma 4.3 *Under the same hypotheses of Lemma 4.1, we have*

$$\begin{aligned}(\bar{\kappa}+1)C_0T_0^2\|\partial_{x_3}\sigma\|_{L^2}^2 \le{}& (\bar{\kappa}-1)C_0T_0^2\big\|\partial_{x_3}\sigma'\big\|_{L^2}^2 \\ &+ C_3'\left(\sum_{i=1}^2\|\nabla\partial_{x_i}v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \big\|\nabla\sigma'\big\|_{L^2}^2 + \|\nabla b\|_{L^2}^2\right) \\ &+ \tilde{C}_3\big(\big\|F'\big\|_{L^2}^2 + \|\nabla\vartheta\|_{H^1}^2\big) + C_3'\|v\|_{H^3}\|\sigma\|_{H^1}^2,\end{aligned} \tag{4.13}$$

where $\bar{\kappa}$ is given by (4.2) and

$$C_0 := \frac{R}{2\mu+\lambda}\left(1 - \frac{\gamma-1}{\gamma RT_0}gh\right)^{\frac{\gamma}{\gamma-1}}. \tag{4.14}$$

Proof Using the identity

$$\Delta v_3 = \partial_{x_3}\nabla\cdot v + \partial_{x_1}(\partial_{x_1}v_3 - \partial_{x_3}v_1) + \partial_{x_2}(\partial_{x_2}v_3 - \partial_{x_3}v_2),$$

it follows from equation (3.7) that

$$\partial_{x_3}\nabla\cdot v = -\frac{\mu}{2\mu+\lambda}\big[\partial_{x_1}(\partial_{x_1}v_3 - \partial_{x_3}v_1) + \partial_{x_2}(\partial_{x_2}v_3 - \partial_{x_3}v_2)\big] \tag{4.15}$$

$$+ \frac{1}{2\mu+\lambda}\big[R\partial_{x_3}(\bar{\varrho}\vartheta) + R\hat{T}\partial_{x_3}\sigma' + (R\partial_{x_3}\hat{T}+g)\sigma' + \partial_{x_3}(b\cdot\hat{B}) - (\hat{B}\cdot\nabla)b_3 - F_3'\big].$$

Next we differentiate equation (3.8) by applying ∂_{x_3}, multiply the equation by $\partial_{x_3}\sigma$, and integrate over Ω. Integrating by parts, taking into account (4.15), we get

$$\int_\Omega k\left[\big(\partial_{x_3}\sigma - \partial_{x_3}\sigma'\big)\partial_{x_3}\sigma + \frac{R}{2\mu+\lambda}\hat{\varrho}\hat{T}(\partial_{x_3}\sigma)\big(\partial_{x_3}\sigma'\big)\right]dx = \sum_{i=1}^{6} I_i, \tag{4.16}$$

where

$$I_1 := \frac{-1}{2\mu+\lambda}\int_\Omega \hat{\varrho}\big(R\sigma'\partial_{x_3}\hat{T} + g\sigma'\big)\partial_{x_3}\sigma\,dx,$$

$$I_2 := \frac{1}{2\mu+\lambda}\int_\Omega \hat{\varrho}\big(F_3' - R\partial_{x_3}(\hat{\varrho}\vartheta)\big)\partial_{x_3}\sigma\,dx,$$

$$I_3 := \frac{\mu}{2\mu+\lambda}\int_\Omega \hat{\varrho}(\partial_{x_3}\sigma)\big(\partial_{x_1}(\partial_{x_1}v_3 - \partial_{x_3}v_1) + \partial_{x_2}(\partial_{x_2}v_3 - \partial_{x_3}v_2)\big)\,dx,$$

$$I_4 := -\int_\Omega \big(\partial_{x_3}(v\cdot\nabla\hat{\varrho}) + (\partial_{x_3}\hat{\varrho})\nabla\cdot v\big)\partial_{x_3}\sigma\,dx,$$

$$I_5 := \int_\Omega \hat{\varrho}\big((\hat{B}\cdot\nabla)b_3 - \hat{B}\cdot\partial_{x_3}b\big)\partial_{x_3}\sigma\,dx,$$

$$I_6 := -\int_\Omega (\partial_{x_3}\sigma)\nabla\cdot\big(\partial_{x_3}(v\sigma)\big)\,dx.$$

Using the following identity in $\mathbb{R}$

$$(X-Y)X + aXY = \frac{1+a}{2}X^2 + \frac{1-a}{2}(X-Y)^2 - \frac{1-a}{2}Y^2$$

and taking into account expressions (2.1) of $\hat{\varrho}$ and $\hat{T}$, one obtains for T_0 large enough that

$$\int_\Omega k\left[\big(\partial_{x_3}\sigma - \partial_{x_3}\sigma'\big)\partial_{x_3}\sigma + \frac{R}{2\mu+\lambda}\hat{\varrho}\hat{T}(\partial_{x_3}\sigma)\big(\partial_{x_3}\sigma'\big)\right]dx$$
$$\geq \frac{k+k_1}{2}\|\partial_{x_3}\sigma\|_{L^2}^2 - \frac{k-k_1}{2}\big\|\partial_{x_3}\sigma'\big\|_{L^2}^2 + \frac{k-k_1'}{2}\big\|\partial_{x_3}\sigma - \partial_{x_3}\sigma'\big\|_{L^2}^2,$$

where k_1 is the constant given in (4.1) and

$$k_1' := \frac{M_0 R}{2\mu+\lambda}T_0^{\frac{\gamma}{\gamma-1}}.$$

Moreover, from expressions (2.1) (see also (1.11)) of $\hat{\varrho}$ and $\hat{T}$, it follows that

$$I_1 \leq \frac{k-k_1'}{6}\big\|\partial_{x_3}\sigma - \partial_{x_3}\sigma'\big\|_{L^2}^2 + \frac{k_1}{12}\big\|\partial_{x_3}\sigma'\big\|_{L^2}^2 + C_\Omega M_0 T_0^{\frac{2-\gamma}{\gamma-1}}\big\|\nabla\sigma'\big\|_{L^2}^2,$$

$$I_2 \leq \hat{C}\big(\big\|F'\big\|_{L^2}^2 + \|\vartheta\|_{H^1}^2\big) + \frac{k_1}{8}\|\partial_{x_3}\sigma\|_{L^2}^2,$$

$$I_3 \leq \frac{k-k_1'}{6}\big\|\partial_{x_3}\sigma - \partial_{x_3}\sigma'\big\|_{L^2}^2 + \frac{k_1}{12}\big\|\partial_{x_3}\sigma'\big\|_{L^2}^2 + C_\Omega M_0 T_0^{\frac{2-\gamma}{\gamma-1}}\sum_{i=1}^{2}\|\nabla\partial_{x_i}v\|_{L^2}^2,$$

$$I_4 \le \frac{k-k_1'}{6}\left\|\partial_{x_3}\sigma - \partial_{x_3}\sigma'\right\|_{L^2}^2 + \frac{k_1}{12}\left\|\partial_{x_3}\sigma'\right\|_{L^2}^2 + C_\Omega M_0 T_0^{\frac{4-3\gamma}{\gamma-1}}\|v\|_{L^2}^2,$$
$$I_5 \le C_\Omega M_0 T_0^{\frac{\gamma-2}{\gamma-1}}\|\nabla b\|_{L^2}^2 + \frac{k_1}{8}\|\partial_{x_3}\sigma\|_{L^2}^2.$$

As for the last I_6 term, one has

$$\begin{aligned} I_6 &= -\frac{1}{2}\int_\Omega \frac{\hat{T}}{\hat{\varrho}}|\partial_{x_3}\sigma|^2(\nabla\cdot v)\,dx - \int_\Omega(\partial_{x_3}\sigma)(\nabla\sigma\cdot v)\,dx + \int_\Omega(\partial_{x_3}\sigma)\big(\nabla\cdot(\partial_{x_3}v)\big)\sigma\,dx \\ &\le C_\Omega\|v\|_{H^3}\|\sigma\|_{H^1}^2. \end{aligned}$$

Combining these estimates with (4.16), we obtain

$$\begin{aligned} \left(k+\frac{k_1}{2}\right)\|\partial_{x_3}\sigma\|_{L^2}^2 \le{}& \left(k-\frac{k_1}{2}\right)\left\|\partial_{x_3}\sigma'\right\|_{L^2}^2 + C_\Omega M_0 T_0^{\frac{2-\gamma}{\gamma-1}}\left(\left\|\nabla\sigma'\right\|_{L^2}^2 + \|\nabla b\|_{L^2}^2\right) \\ &+ M_0 C_\Omega T_0^{\frac{2-\gamma}{\gamma-1}}\sum_{i=1}^{2}\|\nabla\partial_{x_i}v\|_{L^2}^2 + C_\Omega M_0 T_0^{\frac{4-3\gamma}{\gamma-1}}\|v\|_{L^2}^2 \\ &+ \hat{C}\left(\left\|F'\right\|_{L^2}^2 + \|\vartheta\|_{H^1}^2\right) + C_\Omega\|v\|_{H^3}\|\sigma\|_{H^1}^2. \end{aligned}$$

By multiplying now both sides of this inequality by

$$\left(M_0 T_0^{\frac{2-\gamma}{\gamma-1}}\right)^{-1} = T_0^2\left(M_0 T_0^{\frac{\gamma}{\gamma-1}}\right)^{-1}$$

and taking into account (4.1), we obtain (4.13), which completes the proof of the lemma. □

Lemma 4.4 *Under the assumptions of Lemma 4.1, we have*

$$\begin{aligned} &\|v\|_{H^2}^2 + T_0^2\left\|\nabla\sigma'\right\|_{L^2}^2 \\ &\quad\le C_4'\sum_{i=1}^{2}\|\nabla\partial_{x_i}v\|_{L^2}^2 + C_4'\left(T_0^2\left\|\partial_{x_3}\sigma'\right\|_{L^2}^2 + \|\nabla b\|_{l^2}^2 + \|\nabla v\|_{L^2}^2 + \left\|F'\right\|_{L^2}^2\right). \end{aligned} \tag{4.17}$$

Proof We first rewrite equation (3.6) as the following Stokes problem in Ω:

$$\begin{aligned} &-\mu\Delta v + R\nabla\big(\hat{T}\sigma'\big) = -(\mu+\lambda)\nabla\nabla\cdot v - R\nabla(\hat{\varrho}\vartheta) - g\sigma' e_3 + (\hat{B}\cdot\nabla)b - \nabla(b\cdot\hat{B}) + F', \\ &\nabla\cdot v = \nabla\cdot v, \end{aligned}$$

with boundary conditions (2.12). From the classical estimates (see [11] and [29]) of the solutions of the Stokes problem, one has

$$\|v\|_{H^2}^2 + \left\|\nabla\big(\hat{T}\sigma'\big)\right\|_{L^2}^2 \le \hat{C}\|\vartheta\|_{H^1}^2 + C_\Omega\left(\|\nabla\cdot v\|_{H^1}^2 + \left\|F'\right\|_{L^2}^2 + \left\|\nabla\sigma'\right\|_{L^2}^2 + \|\nabla b\|_{L^2}^2\right). \tag{4.18}$$

Moreover, using (4.15), one can see easily that

$$\|\partial_{x_3}\nabla\cdot v\|_{L^2}^2$$

$$\leq C_\Omega \left\|\nabla\sigma'\right\|_{L^2}^2 + C_\Omega\left(\sum_{i=1}^{2}\|\nabla\partial_{x_i}v\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + T_0^2\left\|\partial_{x_3}\sigma'\right\|_{L^2}^2 + \left\|F'\right\|_{L^2}^2 + \|\vartheta\|_{H^1}^2\right),$$

and since σ (see (2.15)) has mean value equal to zero and T_0 is large enough, we get

$$\left\|\nabla\left(\bar{T}\sigma'\right)\right\|_{L^2}^2 \geq C_\Omega\left(T_0^2\left\|\nabla\sigma'\right\|_{L^2}^2 - \left\|\sigma'\right\|_{L^2}^2\right) \leq C'_\Omega T_0^2\left\|\nabla\sigma'\right\|_{L^2}^2.$$

Taking into account these inequalities, it results that estimate (4.17) follows from (4.18), and this completes the proof of the lemma. □

Lemma 4.5 *With the same assumptions as in Lemma 4.1, we have*

$$\|b\|_{H^2}^2 \leq C'_5\left(\|\nabla v\|_{L^2}^2 + \left\|H'\right\|_{L^2}^2\right). \tag{4.19}$$

Proof Since $\|b\|_{H^2} \leq C_\Omega\|\Delta b\|_{L^2}$ for all $b \in H_{\sharp}^2(\Omega)\cap H_0^1(\Omega)$, from equation (3.7) and boundary condition (2.14), we easily get (4.19). □

Lemma 4.6 *Under the same assumptions as in Lemma 4.1, we have*

$$\begin{aligned}
&\sum_{i,j=1}^{2}\left(\frac{\mu}{R}\|\nabla\partial_{x_i}\partial_{x_j}v\|_{L^2}^2 + \frac{\mu+\lambda}{R}\|\partial_{x_i}\partial_{x_j}\nabla\cdot v\|_{L^2}^2\right) + k\sum_{i,j=1}^{2}\left\|\left(\hat{T}\hat{\varrho}^{-1}\right)^{\frac{1}{2}}\partial_{x_i}\partial_{x_j}\sigma\right\|_{L^2}^2 \\
&\quad\leq k\sum_{i,j=1}^{2}\left\|\left(\hat{T}\hat{\varrho}^{-1}\right)^{\frac{1}{2}}\partial_{x_i}\partial_{x_j}\sigma'\right\|_{L^2}^2 + C'_6\left(\|v\|_{H^2}^2 + \left\|\nabla\sigma'\right\|_{L^2}^2 + \|b\|_{H^2}^2\right) + \tilde{C}_5\|\vartheta\|_{H^2}^2 \\
&\qquad + C'_6\left(\left\|F'\right\|_{H^1}^2 + \|v\|_{H^3}\|\sigma\|_{H^2}^2\right).
\end{aligned} \tag{4.20}$$

Proof We apply the differential operator $\partial_{x_i}\partial_{x_j}$ $(i,j = 1,2)$ to equations (3.6) and (3.8), and we multiply them by $R^{-1}\partial_{x_i}\partial_{x_j}v$ and $\hat{T}\hat{\varrho}^{-1}\partial_{x_i}\partial_{x_j}\sigma$ respectively, then we integrate the resulting equations on Ω. By integrating them by parts, using the fact that $\partial_{x_i}\partial_{x_j}v$ satisfies (2.12), we obtain

$$\begin{aligned}
&\frac{\mu}{R}\|\nabla\partial_{x_i}\partial_{x_j}v\|_{L^2}^2 + \frac{\mu+\lambda}{R}\|\partial_{x_i}\partial_{x_j}\nabla\cdot v\|_{L^2}^2 + \frac{k}{2}\left\|\left(\frac{\hat{T}}{\hat{\varrho}}\right)^{\frac{1}{2}}\partial_{x_i}\partial_{x_j}\sigma\right\|_{L^2}^2 \\
&\quad + \frac{k}{2}\left\|\left(\frac{\bar{T}}{\bar{\varrho}}\right)^{\frac{1}{2}}\left(\partial_{x_i}\partial_{x_j}\sigma - \partial_{x_i}\partial_{x_j}\sigma\right)\right\|_{L^2}^2 = \frac{k}{2}\left\|\left(\frac{\hat{T}}{\hat{\varrho}}\right)^{\frac{1}{2}}\partial_{x_i}\partial_{x_j}\sigma'\right\|_{L^2}^2 + \sum_{i=1}^{8}I_i,
\end{aligned} \tag{4.21}$$

where

$$I_1 := -\int_\Omega \frac{\hat{T}}{\hat{\varrho}}\left(\partial_{x_i}\partial_{x_j}\sigma - \partial_{x_i}\partial_{x_j}\sigma'\right)\partial_{x_j}\partial_{x_i}\nabla\cdot(\hat{\varrho}v)\,dx,$$

$$I_2 := -\int_\Omega \left[(\partial_{x_i}\hat{T})\partial_{x_j}\sigma' + (\partial_{x_j}\hat{T})\partial_{x_i}\sigma' + \sigma'\partial_{x_j}\partial_{x_i}\hat{T}\right]\partial_{x_j}\partial_{x_i}\nabla\cdot v\,dx,$$

$$I_3 := -\int_\Omega \partial_{x_i}\left[\hat{T}\hat{\varrho}^{-1}\left(\partial_{x_i}\partial_{x_j}(v\cdot\nabla\hat{\varrho}) + (\partial_{x_i}\partial_{x_j}\hat{\varrho})\nabla\cdot v\right)\right]\partial_{x_j}\sigma'\,dx,$$

$$I_4 := -\int_\Omega \partial_{x_i}\left[\hat{T}\hat{\varrho}^{-1}\left((\partial_{x_i}\hat{\varrho})\partial_{x_j}\nabla\cdot v + (\partial_{x_j}\hat{\varrho})\partial_{x_i}\nabla\cdot v\right)\right]\partial_{x_j}\sigma'\,dx,$$

$$I_5 := -\frac{g}{R}\int_\Omega \big(\partial_{x_i}\sigma'\big)\partial_{x_i}^2\partial_{x_j}v_3\,dx,$$

$$I_6 := \int_\Omega \big(-(\hat{B}\cdot\nabla)\partial_{x_j}\partial_{x_i}v + (\hat{B}\cdot\nabla\partial_{x_j}\partial_{x_i}v)\big)\cdot\partial_{x_i}\partial_{x_j}b\,dx,$$

$$I_7 := \int_\Omega \big[(\partial_{x_j}\partial_{x_i}\nabla\cdot v)\partial_{x_i}\partial_{x_j}(\hat{\varrho}\vartheta) - R^{-1}\big(\partial_{x_i}^2\partial_{x_j}v\big)\cdot\partial_{x_j}F'\big]\,dx,$$

$$I_8 := -\int_\Omega \frac{\hat{T}}{\hat{\varrho}}(\partial_{x_i}\partial_{x_j}\sigma)\partial_{x_i}\partial_{x_j}\nabla\cdot(v\sigma)\,dx.$$

By the same arguments as in Lemma 4.2, we can estimate the terms I_i ($i = 1,\dots,7$) so that

$$\begin{aligned} I_1 + \cdots + I_7 \le{}& \frac{k}{2}\left\|\left(\frac{\hat{T}}{\hat{\varrho}}\right)^{\frac{1}{2}}(\partial_{x_i}\partial_{x_j}\sigma - \partial_{x_i}\partial_{x_j}\sigma'\right\|_{L^2}^2 + \frac{\mu}{2R}\|\nabla\partial_{x_i}\partial_{x_j}v\|_{L^2}^2 \\ &+ \frac{\mu+\lambda}{2R}\|\partial_{x_i}\partial_{x_j}\nabla\cdot v\|_{L^2}^2 + C_\Omega\big(\|\nabla\sigma'\|_{L^2}^2 + \|b\|_{H^2}^2 + \|v\|_{H^2}^2 + \|F'\|_{H^1}^2\big) \\ &+ \hat{C}\|\vartheta\|_{H^2}^2. \end{aligned}$$

We estimate the term I_8 in a similar way as in (4.12), to get

$$I_8 \le C_\Omega\|v\|_{H^3}\|\sigma\|_{H^2}^2.$$

By adding the above estimates to (4.21), we obtain (4.20), and the proof of the lemma is completed. □

Lemma 4.7 *Under the assumptions of Lemma 4.1, we have*

$$\begin{aligned} (\bar{\kappa}+1)C_0T_0^2\sum_{i=1}^2\|\partial_{x_3}\partial_{x_i}\sigma\|_{L^2}^2 \le{}& (\bar{\kappa}-1)C_0T_0^2\sum_{i=1}^2\|\partial_{x_3}\partial_{x_i}\sigma'\|_{L^2}^2 \\ &+ C_7'\sum_{j=1}^2\|\nabla\partial_{x_i}\partial_{x_j}v\|_{L^2}^2 \\ &+ C_7'\big(T_0^{-2}\|v\|_{H^2}^2 + \|\nabla\sigma'\|_{L^2}^2 + \|b\|_{H^2}^2\big) \\ &+ \tilde{C}_6\big(\|F'\|_{H^1}^2 + \|\vartheta\|_{H^2}^2\big) + C_7'\|v\|_{H^3}\|\sigma\|_{H^2}^2, \end{aligned} \tag{4.22}$$

where $\bar{\kappa}$ and C_0 are given by (4.2) and (4.14) respectively.

Proof We differentiate equation (3.8) by applying the operator $\partial_{x_3}\partial_{x_i}$, ($i = 1,2$) on both sides of the equation and multiply the resulting equation by $\partial_{x_3}\partial_{x_i}\sigma$. We then obtain (4.22) by using the same arguments of Lemma 4.3. □

Lemma 4.8 *Under the assumptions of Lemma 4.1, one has*

$$\sum_{i=1}^2\big(\|\partial_{x_i}v\|_{H^2}^2 + T_0^2\|\nabla\partial_{x_i}\sigma'\|_{L^2}^2\big) \le C_8'\sum_{i,j=1}^2\|\nabla\partial_{x_i}\partial_{x_j}v\|_{L^2}^2 + \tilde{C}_7\|\vartheta\|_{H^2}^2 \tag{4.23}$$

$$+ C_8' \sum_{i=1}^{2} T_0^2 \|\partial_{x_3}\partial_{x_i}\sigma'\|_{L^2}^2$$
$$+ C_8' \big(\|v\|_{H^2}^2 + \|b\|_{H^2}^2 + \|F'\|_{H^1}^2 + \|\nabla\sigma'\|_{L^2}^2\big).$$

Proof We apply to equation (3.6) the differential operator ∂_{x_i} $(i = 1, 2)$, and we rewrite the obtained equation as the following Stokes problem:

$$\begin{aligned} &-\mu\Delta(\partial_{x_i}v) + R\nabla\big(\partial_{x_i}(\hat{T}\sigma')\big) \\ &\quad = -(\mu+\lambda)\partial_{x_i}\nabla\nabla\cdot v - R\nabla\partial_{x_i}(\hat{\varrho}\vartheta) - g\partial_{x_i}\sigma' e_3 + \partial_{x_i}\big((\hat{B}\cdot\nabla)b - \nabla(\hat{B}\cdot b)\big) + \partial_{x_i}F', \\ &\nabla\cdot\partial_{x_i}v = \nabla\cdot\partial_{x_i}v, \end{aligned}$$

with the same boundary conditions on $\partial_{x_i}v$ as (2.12). We obtain by the same arguments of the proof of Lemma 4.4 that

$$\begin{aligned} &\|\partial_{x_i}v\|_{H^2}^2 + \|\nabla\partial_{x_i}(\hat{T}\sigma')\|_{L^2}^2 \\ &\quad \le C_\Omega\big[\|\nabla\cdot\partial_{x_i}v\|_{H^1}^2 + \|F'\|_{H^1}^2 + \|\nabla\sigma'\|_{L^2}^2 + \|b\|_{H^2}^2\big] + \hat{C}\|\vartheta\|_{H^2}^2. \end{aligned} \tag{4.24}$$

In addition, let us notice that

$$\|\partial_{x_i}\nabla\cdot v\|_{H^1}^2 \le C_\Omega\left[\sum_{j=1}^{2}\|\partial_{x_i}\partial_{x_j}\nabla\cdot v\|_{L^2}^2 + \|\partial_{x_i}\partial_{x_3}\nabla\cdot v\|_{L^2}^2 + \|v\|_{H^2}^2\right] \tag{4.25}$$

and, in applying ∂_{x_i} $(i = 1, 2)$ to (4.15), it follows that

$$\begin{aligned} \|\partial_{x_i}\partial_{x_3}\nabla\cdot v\|_{L^2}^2 \le \hat{C}\|\vartheta\|_{H^2}^2 + C_\Omega\Bigg[&\|b\|_{H^2}^2 \\ &+ \sum_{j=1}^{2}\|\nabla\partial_{x_i}\partial_{x_j}v\|_{L^2}^2 + T_0^2\|\partial_{x_3}\partial_{x_i}\sigma'\|_{L^2}^2 + \|F'\|_{H^1}^2 + \|\nabla\sigma'\|_{L^2}^2\Bigg]. \end{aligned} \tag{4.26}$$

By substituting (4.25) and (4.26) into (4.24) and taking into account

$$\|\nabla\partial_{x_i}(\hat{T}\sigma')\|_{L^2}^2 \ge C_\Omega\big[T_0^2\|\nabla\partial_{x_i}\sigma'\|_{L^2}^2 - \|\sigma'\|_{H^1}^2\big] \ge C_\Omega\big[T_0^2\|\nabla\partial_{x_i}\sigma'\|_{L^2}^2 - \|\nabla\sigma'\|_{L^2}^2\big],$$

we obtain (4.23), which completes the proof of the lemma. □

Lemma 4.9 *Under the assumptions of Lemma 4.1, one has*

$$\begin{aligned} (\bar{\kappa}+1)C_0T_0^2\|\Delta\sigma\|_{L^2}^2 \le\ &(\bar{\kappa}-1)C_0T_0^2\|\Delta\sigma'\|_{L^2}^2 + C_9'T_0^{-2}\|v\|_{H^2}^2 \\ &+ \tilde{C}_8\big(\|F'\|_{H^1}^2 + \|\vartheta\|_{H^2}^2\big) \\ &+ C_9'\|v\|_{H^3}\|\sigma\|_{H^2}^2 + C_9'\big(\|\nabla\sigma'\|_{L^2}^2 + \|b\|_{H^2}^2\big), \end{aligned} \tag{4.27}$$

where $\bar{\kappa}$ and C_0 are given by (4.2) and (4.14) respectively.

Proof We apply the Laplacian operator to equation (3.8) and multiply the resulting equation by $\Delta\sigma$. By using the same arguments of Lemmas 4.2 and 4.3, we obtain (4.27). □

Lemma 4.10 *Under the same assumptions of Lemma 4.1, one has*

$$\begin{aligned} \|v\|_{H^3}^2 + T_0^2 \|\nabla\sigma'\|_{H^1}^2 &\leq C'_{10}\left(\sum_{i=1}^{2}\|\partial_{x_i}v\|_{H^2}^2 + \|v\|_{H^1}^2\right) \\ &\leq C'_{10}T_0^2\left(\|\Delta\sigma'\|_{L^2}^2 + \sum_{i=1}^{2}\|\nabla\partial_{x_i}\sigma'\|_{L^2}^2\right) \\ &\quad + C'_{10}\left(\|F'\|_{H^1}^2 + \|\nabla\sigma'\|_{L^2}^2 + \|b\|_{H^2}^2\right) + \tilde{C}_9\|\vartheta\|_{H^2}^2. \end{aligned} \tag{4.28}$$

Proof As in Lemma 4.4, according the well-known theory on the estimates of the Stokes problem, one can deduce from (4.18) with boundary conditions (2.12) that

$$\|v\|_{H^3}^2 + \|\nabla(\hat{T}\sigma')\|_{H^1}^2 \leq \hat{C}\|\vartheta\|_{H^2}^2 + C_\Omega\left(\|\nabla\cdot v\|_{H^2}^2 + \|\nabla\sigma'\|_{L^2}^2 + \|F'\|_{H^1}^2 + \|b\|_{H^2}^2\right). \tag{4.29}$$

Let us first notice that

$$\|\nabla\cdot v\|_{H^2}^2 \leq \|\partial_{x_3}\nabla\cdot v\|_{H^1}^2 + \sum_{i=1}^{2}\|\partial_{x_i}v\|_{H^2}^2 + \|v\|_{H^1}^2.$$

In addition, taking into account (4.15), we have

$$\begin{aligned} &\|\partial_{x_3}\nabla\cdot v\|_{H^1}^2 \\ &\quad\leq C_\Omega\|b\|_{H^2}^2 + C_\Omega\left(\sum_{i=1}^{2}\|\partial_{x_i}v\|_{H^2}^2 + T_0^2\|\nabla\partial_{x_3}\sigma'\|_{L^2}^2 + \|F'\|_{L^2}^2 + \|\nabla\sigma'\|_{L^2}^2\right) + \hat{C}\|\vartheta\|_{H^2}^2. \end{aligned}$$

Since

$$\begin{aligned} &\|\nabla(\hat{T}\sigma')\|_{H^1}^2 \geq C_\Omega\left(T_0^2\|\nabla\sigma'\|_{H^1}^2 - \|\sigma'\|_{H^1}^2\right) \geq C_\Omega\left(T_0^2\|\nabla\sigma'\|_{H^1}^2 - \|\nabla\sigma'\|_{L^2}^2\right) \\ &\|\nabla\partial_{x_3}\sigma'\|_{L^2}^2 \leq \|\Delta\sigma'\|_{L^2}^2 + \sum_{i=1}^{2}\|\nabla\partial_{x_i}\sigma'\|_{L^2}^2, \end{aligned}$$

by adding these inequalities to (4.29), we obtain (4.28). □

5 Fixed point of the operator *S*

Having proved Lemmas 4.1–4.10, we are now in a position to establish that the operator S has a fixed point. We recall that following Remark 3.3 and Lemma 3.4, it only remains to show that the crucial point (3.35) holds, which is the goal in the following lemma.

We recall that the nonlinear operator $S : V \to V$, $u' = (\vartheta', v', b', \sigma')$, $S(u') = u$, where $u = (\vartheta, v, b, \sigma) \in V$ is the unique solution of system (3.5)–(3.8).

Lemma 5.1 *There is a norm* $|\cdot|_V$ *equivalent (see (3.3)) to* $\|\cdot\|_V$ *such that*

$$S(B_V^a) \subseteq B_V^a, \quad B_V^a = \left\{u = (\vartheta, v, b, \sigma) \in V : |u|_V \leq a\right\}. \tag{5.1}$$

Proof Let $\lambda_1, \lambda_2, \dots, \lambda_9$ be positive numbers that will be suitably chosen thereafter. We set

$$\nu_1 := C_0(\bar{\kappa}+1)\lambda_9 T_0^2, \qquad \nu_2 := C_0(\bar{\kappa}+1)\lambda_7 T_0^2, \qquad \nu_3 := k\lambda_6,$$
$$\nu_4 := C_0(\bar{\kappa}+1)\lambda_3 T_0^2, \qquad \nu_5 := k\lambda_2, \qquad \nu_6 := k\lambda_1,$$

where C_0 is given in (4.14) and $\bar{\kappa}$ in (4.2).

For any $\varphi \in H^2$, we set

$$|\varphi|_{2,\Omega}^2 = \nu_1\|\Delta\varphi\|_{L^2}^2 + \nu_2\sum_{i=1}^{2}\|\partial_{x_i}\partial_{x_3}\varphi\|_{L^2}^2 + \nu_3\sum_{i,j=1}^{2}\left\|\left(\frac{\hat{T}}{\hat{\varrho}}\right)^{\frac{1}{2}}\partial_{x_i}\partial_{x_j}\varphi\right\|_{L^2}^2 + \nu_4\|\partial_{x_3}\varphi\|_{L^2}^2 + \nu_5\sum_{i=1}^{2}\left\|\left(\frac{\hat{T}}{\hat{\varrho}}\right)^{\frac{1}{2}}\partial_{x_i}\varphi\right\|_{L^2}^2 + \nu_6\left\|\left(\frac{\hat{T}}{\hat{\varrho}}\right)^{\frac{1}{2}}\varphi\right\|_{L^2}^2. \tag{5.2}$$

It is clear that $|\cdot|_{2,\Omega}$ is equivalent to the H^2-norm. Moreover, if we set

$$|u|_V^2 := \|v\|_{H^3}^2 + \|b\|_{H^2}^2 + \|\vartheta\|_{H^2}^2 + |\sigma|_{2,\Omega}^2, \quad u = (\vartheta, v, b, \sigma) \in V, \tag{5.3}$$

we obtain (see (3.3)) a norm equivalent to $\|\cdot\|_V$.

Let now

$$S(u') = u = (\vartheta, v, b, \sigma), \quad u' \in B_V^a. \tag{5.4}$$

If we multiply the estimates on the solution u established in Lemmas 4.1–4.10 by

$$\lambda_1, \lambda_2, \dots, \lambda_8, \lambda_9 \quad \text{and} \quad \lambda_{10} = 1$$

respectively, then adding them and taking into account (3.10), we obtain

$$\begin{aligned}
&\|v\|_{H^3}^2 + \|\vartheta\|_{H^2}^2 + \Lambda_4\|b\|_{H^2}^2 + |\sigma|_{2,\Omega}^2 + \Lambda_1\left[\sum_{i=1}^{2}\|\partial_{x_i}v\|_{H^2}^2 + T_0^2\sum_{i=1}^{2}\left\|\nabla\partial_{x_i}\sigma'\right\|_{L^2}^2\right] \\
&+ \Lambda_2\sum_{i,j=1}^{2}\|\nabla\partial_{x_i}\partial_{x_j}v\|_{L^2}^2 + \Lambda_3\|v\|_{H^2}^2 + \Lambda_5\sum_{i=1}^{2}\|\nabla\partial_{x_i}v\|_{L^2}^2 \\
&+ \Lambda_6\left(\|\nabla v\|_{L^2}^2 + \|\nabla b\|_{L^2}^2\right) + \Lambda_7\left\|\nabla\sigma'\right\|_{L^2}^2 + \Lambda_8\left\|\sigma'\right\|_{L^2}^2 \le N(\sigma') \\
&+ \tilde{C}\left(\left\|G'\right\|_{H^1}^2 + \left\|F'\right\|_{H^1}^2 + \left\|H'\right\|_{H^1}^2 + \|v\|_{H^3}\|\sigma\|_{H^2}^2\right)
\end{aligned} \tag{5.5}$$

with some positive constant $\tilde{C}$ while $N(\sigma')$ is given by

$$\begin{aligned}
N(\sigma') := &\left(C_{10}' + (\bar{\kappa}-1)C_0\lambda_9\right)T_0^2\left\|\Delta\sigma'\right\|_{L^2}^2 \\
&+ \left(C_8'\lambda_8 + C_0(\bar{\kappa}-1)\lambda_7\right)T_0^2\sum_{i=1}^{2}\left\|\partial_{x_i}\partial_{x_3}\sigma'\right\|_{L^2}^2 + k\lambda_6\sum_{i,j=1}^{2}\left\|\left(\frac{\hat{T}}{\hat{\varrho}}\right)^{\frac{1}{2}}\partial_{x_i}\partial_{x_j}\sigma'\right\|_{L^2}^2 \\
&+ \left(C_4'\lambda_4 + C_0(\bar{\kappa}-1)\lambda_3\right)T_0^2\left\|\partial_{x_3}\sigma'\right\|_{L^2}^2 + k\lambda_2\sum_{i=1}^{2}\left\|\left(\frac{\hat{T}}{\hat{\varrho}}\right)^{\frac{1}{2}}\partial_{x_i}\sigma'\right\|_{L^2}
\end{aligned} \tag{5.6}$$

$$+ k\lambda_1 \left\| \left(\frac{\hat{T}}{\hat{\varrho}}\right)^{\frac{1}{2}} \sigma' \right\|_{L^2}^2,$$

and the numbers Λ_i $(i = 1, 2, \dots, 8)$ are given by

$$\Lambda_1 := \lambda_8 - C'_{10}, \qquad \Lambda_2 := \frac{\mu}{R}\lambda_6 - C'_7\lambda_7 - C'_8\lambda_8,$$
$$\Lambda_3 := \lambda_4 - C'_6\lambda_6 - C'_7 T_0^{-2}\lambda_7 - C'_8\lambda_8 - C'_9 T_0^{-2}\lambda_9 - C'_{10},$$
$$\Lambda_4 := \lambda_5 - C'_6\lambda_6 - C'_7\lambda_7 - C'_8\lambda_8 - C'_9\lambda_9 - C'_{10},$$
$$\Lambda_5 := \frac{\mu}{R}\lambda_2 - C'_3\lambda_3 - C'_4\lambda_4, \qquad \Lambda_6 := \frac{\mu}{R}\lambda_1 - C'_3\lambda_3 - C'_5\lambda_5,$$
$$\Lambda_7 := C'_4\lambda_4 T_0^2 - C'_3\lambda_3 - C'_6\lambda_6 - C'_7\lambda_7 - C'_8\lambda_8 - C'_9\lambda_9 - C'_{10},$$
$$\Lambda_8 := C'_1 T_0\lambda_1 - C'_2\lambda_2.$$

If T_0 is large enough, we can easily see that it is possible to choose (see (1.10))

$$\lambda_i = \lambda_i(\bar{T}_0), \quad i = 9, 8, \dots, 1,$$

so that

$$\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_5 \geq 0, \quad \Lambda_4 \geq 1, \qquad N(\sigma') \leq (1 - \bar{t})|\sigma'|_2^2 \quad \text{for some } \bar{t} \in]0, 1[. \tag{5.7}$$

Indeed, if λ_j for $j = i + 1, \dots, 9$ and $i = 8, \dots, 1$ are given, as can be seen easily, considering constraints (5.7), we can choose λ_i large enough so that the inequalities containing only λ_i and λ_j are satisfied (and so we can proceed to the choices of λ_i, starting from λ_9 and then choosing successively λ_i for $i = 8, 6, \dots, 1$). The constants $\lambda_i = \lambda_i(\bar{T}_0)$ $(i = 9, \dots, 1)$ so determined imply in particular that $\Lambda_7 > 0$ and $\Lambda_8 > 0$ if T_0 is large enough, and we can choose λ_5 such that $\Lambda_4 \geq 1$.

Recalling now inequality (5.5) and using (5.7), we obtain

$$\begin{aligned} \|v\|_{H^3}^2 + \|\vartheta\|_{H^2}^2 + \|b\|_{H^2}^2 + |\sigma|_{2,\Omega}^2 &\leq (1 - \bar{t})|\sigma'|_{2,\Omega}^2 \\ &\quad + \tilde{C}\left(\|F'\|_{H^1}^2 + \|G'\|_{L^2}^2 + \|H'\|_{L^2}^2 + \|v\|_{H^3}\|\sigma\|_{H^2}^2\right), \end{aligned} \tag{5.8}$$

and given (3.36) (see also (5.3) and (5.4)), it follows from (5.8) that

$$|S(u')|_V^2 \leq (1 - \bar{t})|u'|_V^2 + c_\Omega a^3 \leq (1 - \bar{t})a^2 + c_\Omega a^3 \quad \forall u' \in B_V^a. \tag{5.9}$$

If a is small enough so that $a \leq \frac{\bar{t}}{2c_\Omega}$, it follows from (5.9) that

$$|S(u')|_V^2 \leq \left(1 - \frac{\bar{t}}{2}\right)a^2 \quad \forall u' \in B_V^a,$$

which means that $S(u') \in B_V^a$ for every $u' \in B_V^a$, and hence we obtain (5.1) and equivalently (3.35), thus the lemma is proven. □

Remark 5.2 As we have already seen in Remark 1.1, if the perturbation $\varepsilon(x')$ is identically zero or is a nonzero constant function, then $(0, B_{rs}, T_{rs}, \varrho_{rs})$ given by (1.11) is the unique stationary solution to the system of equations (1.1)–(1.4) with boundary conditions (1.6)–(1.8). However, it is important to establish whether this system can still have a stationary solution $(0, B_{st}, T_{st}, \varrho_{ts})$ when $\varepsilon(x')$ is not identically zero and is not identically a nonzero constant function.

Indeed, suppose that $\varepsilon(x') \not\equiv 0$ and is not identically a nonzero constant function, and suppose that there exists a stationary solution $(0, B_{st}, T_{st}, \varrho_{st})$ to equations (1.1)–(1.4) with boundary conditions (1.6)–(1.8). Hence, $(B_{st}, T_{st}, \varrho_{st})$ solves the system of equations

$$\nu\nabla\nabla\cdot B_{st} - \nu\Delta B_{st} = 0, \quad \nabla\cdot B_{st} = 0, \tag{5.10}$$

$$-\kappa\Delta T_{st} = \nu|\nabla\times B_{st}|^2, \tag{5.11}$$

$$R\nabla(T_{st}\varrho_{st}) + g\varrho e_3 = -(\nabla\times B_{st})\times B_{st}, \tag{5.12}$$

in $\Omega = \mathbb{T}^2\times]0,h[$ with the boundary condition

$$B_{st}\big(x',0\big) = B_{st}\big(x',h\big) = B_0 e_3, \tag{5.13}$$

$$T_{st}\big(x',0\big) = T_0 + \varepsilon\big(x'\big), \qquad T_{st}\big(x',h\big) = T_0 + \frac{\gamma-1}{\gamma R}gh. \tag{5.14}$$

Notice first that $B_{st} = B_0 e_3$ is the unique solution of (5.10) with (5.13) and, since $\nabla\times B_{st} = 0$, the boundary value problem (5.11) and (5.14) becomes

$$\Delta T_{st} = 0, \qquad T_{st}\big(x',0\big) = T_0 + \varepsilon\big(x'\big), \qquad T_{st}\big(x',h\big) = T_0 - \frac{\gamma-1}{\gamma R}gh. \tag{5.15}$$

It is obvious that the unique solution of (5.15) is given by

$$T_{st}\big(x',x_3\big) = T_{rs}(x_3) + \delta\big(x',x_3\big), \qquad T_{rs}(x_3) = T_0 - \frac{\gamma-1}{\gamma R}gx_3, \tag{5.16}$$

where δ is the unique solution of the boundary value problem

$$\Delta\delta = 0 \quad \text{in } \Omega = \mathbb{T}^2\times]0,h[, \qquad \delta\big(x',0\big) = \varepsilon\big(x'\big), \quad \delta\big(x',h\big) = 0. \tag{5.17}$$

Given (5.12) and $B_{st} = B_0 e_3$, by setting $P_{st} = RT_{st}\varrho_{st}$, we have

$$\partial_{x_i} P_{st} = 0 \quad (i = 1,2), \qquad \partial_{x_3}\log P_{st} = -\frac{g}{RT_{st}}. \tag{5.18}$$

Hence, $P_{st}(x',x_3) = p_{st}(x_3)$, and this contradicts $(5.18)_2$ unless (see (5.16)) $T_{st}(x',x_3)$ does not depend on $x' = (x_1,x_2)$. Thus, we get the following two cases for the only solution $\delta(x',x_3)$ of the boundary value problem (5.17): Either

(i) $\delta(x',x_3)$ is constant and therefore, given $(5.17)_3$, we get $\delta(x',x_3) \equiv 0$ for which $\varepsilon(x') = 0$ or

(ii) $\delta(x',x_3)$ is independent of $x' = (x_1,x_2)$, and this contradicts $(5.17)_2$ unless $\varepsilon(x') = \varepsilon_0$.

Hence, in both cases (i) and (ii), we get $\varepsilon(x')$ is zero or constant, which is a contradiction. Therefore, the system of equations (1.1)–(1.4) does not have a stationary solution $(0, B_{st}, T_{st}, \varrho_{st})$ when $\varepsilon(x')$ is not identically zero or a nonzero constant function.

6 Conclusion

To our knowledge, there are not results in the literature on the study of stationary full magnetohydrodynamic flows for viscous, compressible, and heat-conducting fluids. Our work aims to contribute to the mathematical modeling of magnetoconvection by analyzing a system of partial differential equations describing the motion of full magnetohydrodynamic equations. As a first contribution to this study, we considered a simple model of three-dimensional plane-parallel atmosphere made up of ideal gas in the presence of a magnetic field. More precisely, we consider a plane-parallel polytropic atmosphere between $x_3 = 0$ (bottom of the layer) and $x_3 = h$ (top of the layer) with a large temperature gradient across the layer, and in the presence of a vertical magnetic field. We then focus on the existence of a magnetoconvective steady flow close to the equilibrium state. As an ongoing research, we are currently studying the stability of such a stationary solution, which is inspired from the works [28, 36, 46].

Author contributions

M.A. typed the preliminary drat; R.B. conceived the research and typed the preliminary draft; F.E. checked the first draft, discussed it thoroughly with R.B and M.A. All authors were then involved in the review of all the iterates of the manuscript till the final version.

Funding

The research of Francois Ebobisse is funded by the University of Cape Town URC-Block Grant.

Data Availability

No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate

This was not required for the present study.

Competing interests

The authors declare no competing interests.

Author details

[1]Laboratoire de Mathématiques Pures et Appliquées (LMPA), Faculté des Sciences, Université M. Mammeri, Tizi-Ouzou 15000, Algeria. [2]Department of Mathematics, University M. Mammeri, Tizi-Ouzou 1500, Algeria. [3]Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch 7700, South Africa.

Received: 24 November 2023 Accepted: 1 May 2024 Published online: 05 June 2024

References

1. Alekseev, G.V., Brizitskii, R.V.: Solvability of the mixed boundary value problem for the stationary equations of magnetohydrodynamic equations of a viscous fluid. Dal'nevost. Mat. Zh. **3**(2), 285–301 (2002)
2. Alekseev, G.V., Brizitskii, R.V.: Solvability of the boundary value problem for stationary magnetohydrodynamic equations under mixed boundary conditions for the magnetic field. Appl. Math. Lett. **32**, 13–18 (2014)
3. Alekseev, G.V., Brizitskii, R.V., Pukhnachev, V.V.: Solvability of the inhomogeneous mixed boundary for stationary magnetohydrodynamic equations. Dokl. Phys. **59**, 467–471 (2014)
4. Beirao Da Veiga, H.: Boundary value problems for a class of first order partial differential equations in Sobolev spaces and application to the Euler flow. Rend. Semin. Mat. Univ. Padova **97**, 247–273 (1988)
5. Benabidallah, R., Ebobisse, F.: On the stationary solution of a viscous compressible MHD equations. J. Math. Anal. Appl. **519**, 126825 (2023)

6. Benabidallah, R., Ebobisse, F.: On the steady flows of viscous compressible MHD equations in an infinite horizontal layer. J. Math. Fluid Mech., Accepted
7. Benabidallah, R., Kessoum, H., Ebobisse, F.: On stationary convective motion of viscous compressible heat-conducting fluid. Appl. Anal. **101**(18), 6709–6735 (2022)
8. Benabidallah, R., Taleb, L., Fujita Yashima, H.: Existence d'une solution stationnaire d'un système d'équations d'un fluide visqueux compressible et calorifère modélisant l'atmosphère. Boll. Unione Mat. Ital., B **10**, 1101–1124 (2007)
9. Berirao Da Veiga, H.: An L_p-theory for the n-dimensional, stationary, compressible Navier-Stokes equations, and the incompressible limit for compressible fluids. The equilibrium solutions. Commun. Math. Phys. **109**, 229–248 (1987)
10. Bogovskij, M.E.: Solution de quelques problèmes d'analyse vectorielle convexe aux opérateurs div et grad (en russe). Tr. Semin. S.L. Soboleva **1**, 5–40 (1980)
11. Cattabriga, L.: Su un problema al contorno relativo al sistema di equazioni di Stokes. Rend. Semin. Mat. Univ. Padova **31**, 308–340 (1961)
12. Chen, G.Q., Wang, D.H.: Global solution on nonlinear magnetohydrodynamics with large initial data. J. Differ. Equ. **182**, 344–376 (2002)
13. Chen, G.Q., Wang, D.H.: Existence and continuous dependence of large solutions magnetohydrodynamic equations. Z. Angew. Math. Phys. **54**, 608–632 (2003)
14. Chunshan, Z., Kaitai, L.: On existence, uniqueness and L^r-exponential stability for stationary solution to the MHD equations in three-dimensional domain. Anzian J. **46**, 95–109 (2004)
15. Ducomet, B., Feireisl, E.: The equations of magnetohydrodynamics: on the interaction between matter and radiation in the evolution of gaseous stars. Commun. Math. Phys. **226**, 595–629 (2006)
16. Fan, J., Huang, S., Li, F.: Global strong solutions to the planar compressible magnetohydrodynamic equations with large initial data and vacuum. Preprint. arXiv:1206.3624
17. Fan, J., Jiang, S., Nakamura, G.: Vanishing shear viscosity limit in the magnetohydrodynamic equations. Commun. Math. Phys. **270**, 691–708 (2007)
18. Freistühler, H., Szmolyan, P.: Existence and bifurcation of viscous profiles for all intermediate magnetohydrodynamic shock waves. SIAM J. Math. Anal. **26**, 112–128 (1995)
19. Gerbeau, J.F., Le Bris, C., Lelièvre, T.: Mathematical Methods for the Magnetohydrodynamics of Liquid Metals. Numerical Mathematics and Scientific Computation, Oxford Science Publications. Oxford University Press, London (2006)
20. Hoff, D., Tsyganov, E.: Uniqueness and continuous dependence of weak solutions in compressible magnetohydrodynamics. Z. Angew. Math. Phys. **56**, 791–804 (2005)
21. Hu, X.P., Wang, D.H.: Global existence and large-time behavior of solutions to the three-dimensional equations of compressible magnetohydrodynamic flows. Arch. Ration. Mech. Anal. **197**, 203–238 (2010)
22. Jiang, S., Ju, Q., Li, F.: Incompressible limit of the compressible magnetohydrodynamic equations with periodic boundary conditions. Commun. Math. Phys. **297**, 371–400 (2010)
23. Kulikovskiy, A.G., Lyubimov, G.A.: Magnetohydrodynamics. Addison-Wesley, Reading (1965)
24. Ladyzhenskaya, O.A., Uraltseva, N.N.: Equations aux dérivées partielles de type elliptique. Dunod, Paris (1968)
25. Landau, L.L., Lifchitz, E.M.: Mécanique des fluides. Physique théorique, vol. VI. Mir, Moscow (1989)
26. Laudau, L.D., Lifshitz, E.M.: Electrodynamics of Continuous Media, 2nd edn. Pergamon, New York (1984)
27. Li, F., Yu, H.: Optimal rate of classical solutions to the compressible magnetohydrodynamic equations. Proc. R. Soc. Edinb. **141A**, 109–126 (2011)
28. Li, Y., Yang, X.: Stability of stationary for the compressible viscous magnetohydrodynamic equations with large force in bounded domain. J. Differ. Equ. **262**, 3169–3193 (2017)
29. Mazya, V.G., Plamenevski, B.A., Stupyalis, L.I.: The three-dimensional problem of steady-state motion of a fluid with a free surface. Am. Math. Soc. Transl., Sov. Reg. Conf. **123**, 171–268 (1984)
30. Meir, A.J.: The equation of stationary, incompressible magnetohydrodynamic with mixed boundary conditions. Comput. Math. Appl. **25**, 13–29 (1993)
31. Murawski, K.: Numerical solutions of magnetohydrodynamic equations. Bull. Pol. Acad. Sci., Tech. Sci. **59**(2), 219–226 (2011)
32. Nečas, J.: Méthodes directes en théorie des équations elliptiques. Masson, Paris (1967)
33. Novovtný, A.: About the steady transport equation I: L^p-approach in domains with smooth boundaries. Comment. Math. Univ. Carol. **37**(1), 43–89 (1996)
34. Novovtný, A., Padula, M.: Existence et unicité de la solution stationnaire des équations d'un fluide compressible visqueux et calorifère en présence d'une grande force extérieure potentielle et d'une petite non-potentiel. Sib. Mat. Zh. **34**, 120–146 (1993)
35. Novovtný, A., Straškraba, I.: Introduction to the Mathematical Theory of Compressible Flow. Oxford univ. Press, Oxford (2004)
36. Padula, M., Pokorny, M.: Stability and decay to zero of the L^2-norm of perturbations to a viscous compressible heat conductive fluid motion exterior to a ball. J. Math. Fluid Mech. **3**, 342–357 (2001)
37. Rincon, F.: Dynamique des grandes échelles de la convection dans la photosphère solaire. Thèse de Doctorat - Univ. Toulouse III - Paul Sabatier (2004)
38. Robert, B.: MHD Waves in the Solar Atmosphere. Cambridge University Press, Cambridge (2019)
39. Rossmanith, J.A.: An unstaggered, high-resolution constrained transport method for magnetohydrodynamic flows. SIAM J. Sci. Comput. **28**, 1766–1797 (2006)
40. Sermange, M., Temam, R.: Some mathematical questions related to the MHD equations. Commun. Pure Appl. Math. **36**, 635–664 (1983)
41. Tóth, G.: The $\nabla \cdot B = 0$ constraint in shock-capturing magnetohydrodynamics codes. J. Comp. Physiol. **161**, 605–652 (2000)
42. Valli, A.: Periodic and stationary solutions for compressible Navier-Stokes equations via a stability method. Ann. Sc. Norm. Super. Pisa, Cl. Sci. **10**, 607–647 (1983)
43. Valli, A.: On the existence of stationary solutions to compressible Navier-Stokes equations. Ann. Inst. Henri Poincaré **4**, 99–113 (1987)

44. Wang, D.: Large solutions to the initial boundary value problem for planar magnetohydrodynamics. SIAM J. Appl. Math. **63**, 1424–1441 (2003)
45. Yang, Y.F., Gu, X., Dou, C.: Existence and uniqueness of steady solution to the magnetohydrodynamic equation of compressible flows. J. Math. Anal. Appl. **409**, 846–850 (2014)
46. Zhao, C., Li, K.: On existence, uniqueness and L^r exponential stability for stationary solution of the MHD equation in the three-dimensional domains decay. ANZIAM J. **46**, 95–109 (2004)
47. Zhou, C.H.: Existence of weak solution to the three-dimensional steady compressible magnetohydrodynamic equations. Acta Math. Sin. Engl. Ser. **28**, 2431–2442 (2012)

Publisher's Note