Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

On qualitative analysis of a fractional hybrid Langevin differential equation with novel boundary conditions

Gohar Ali[1], Rahman Ullah Khan[2], Kamran[1*], Ahmad Aloqaily[3] and Nabil Mlaiki[3*]

*Correspondence:
kamran.maths@icp.edu.pk;
nmlaiki@psu.edu.sa
[1]Department of Mathematics, Islamia College Peshawar, Peshawar, Khyber Pakhtoon Khwa, Pakistan
[3]Department of Mathematics and Sciences, Prince Sultan University, Riyadh, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

A hybrid system interacts with the discrete and continuous dynamics of a physical dynamical system. The notion of a hybrid system gives embedded control systems a great advantage. The Langevin differential equation can accurately depict many physical phenomena and help researchers effectively represent anomalous diffusion. This paper considers a fractional hybrid Langevin differential equation, including the ψ-Caputo fractional operator. Furthermore, some novel boundaries selected are considered to be a problem. We used the Schauder and Banach fixed-point theorems to prove the existence and uniqueness of solutions to the considered problem. Additionally, the Ulam-Hyer stability is evaluated. Finally, we present a representative example to verify the theoretical outcomes of our findings.

Keywords: Langevin hybrid differential equation; ψ-Caputo fractional operator; Fixed point theorems; Stability

1 Introduction

When implemented in a variety of numerical algorithms, fractional differential equations demonstrated their effectiveness and versatility in modeling and analyzing in many scientific fields, including engineering, material science, chemistry, bloodstream-based models, and also in image processing, for example, in electronics [33], physics [7], engineering [32], biology [8], and chemistry [20]. Kilbs et al. [23] examined recent advances in the field of fractional differential and fractional integro-differential equations, covering various operators of fractional calculus with significant potential utility. Podlubny described methods of solving differential equations of arbitrary real order using integrals and derivatives of arbitrary real order and applied these methods in various fields [26]. In recent years, numerous studies have consistently shown the beneficial effects of fractional differential equations. The definition of fractional-order operators within the framework of fractional calculus played an important role in achieving these results and advances, demonstrating their accuracy in describing phenomena and modeling processes occurring in the Universe. Fractional differential equations and their various branches, such as the Hybrid equation, Langevin equation, and Sturm-Liouville equation, have gained a strong reputation due to the huge number of articles and books written on the topic throughout the

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world. These equations have applications in many fields, including engineering and science.

In the last decade, researchers have introduced many operators. Almeida [10] has recently introduced the ψ-Caputo fractional derivative, which serves as a broad generalization covering various other formulations of fractional derivatives, including Caputo and Caputo-Hadamard types. They review recent advances and findings concerning initial and boundary value problems incorporating the ψ-Caputo fractional derivative. Subsequently, they explore the existing theory's obstacles to solving fractional systems in various abstract fractional boundary value problems. In these studies, the essential and definitive approach combined the well-known ideas of fractional calculus theory with fixed-point theory. This approach has proven useful in producing valuable existence results, such as Baleanu et al. [14] presented a model of childhood disease epidemics that uses a new fractional derivative method proposed by Caputo and Fabrizio. Khan et al. [21] proposed a mathematical model of tuberculosis using fractal-fractional-order principles, aiming to investigate existence, conduct numerical simulations, and analyze its stability. Baleanu et al. [15] developed an innovative approach to modeling boundary value problems on the glucose graph. In 2020, Baleanu et al. [17] unveiled a new fractional model of the human liver, incorporating the Caputo-Fabrizio derivative with an exponential kernel. Tuan et al. [31] presented a mathematical model of the transmission of COVID-19 using the Caputo fractional-order derivative. Khan et al. [22] showed that waterborne diseases result from the transmission of pathogenic bacteria through water, which affects human health. In 2020, using a mathematical model, Thabet et al. [30] investigated the existence, stability, and numerical findings of a novel coronavirus disease (COVID-19). To verify the existence and uniqueness of solutions to the mathematical model concerning the transmission dynamics of COVID-19, we refer interested readers to [29]. Ahmad et al. [4] examined the existence of solutions for a nonlocal hybrid boundary value problem involving Caputo fractional integro-differential equations. In 2019, Abdeljawad et al. [1] explored solutions to the nonlinear integral equation and fractional differential equation using the technique of a fixed point and a numerical experiment. Alsaedi et al. [12] explored the solvability of coupled nonlinear fractional differential equations of varying orders, accompanied by nonlocal coupled boundary conditions over a general domain. Many authors have investigated the Hyers-Ulam stability for fractional differential equations. Numerous authors have explored the Hyers-Ulam stability in fractional differential equations. They have explored numerous Hyers-Ulam stability issues concerning various types of fractional differential equations, such as Langevin systems, employing a variety of methodologies. Adjabi et al. [2] investigate a variant of Langevin differential equations incorporating ordinary and Hadamard fractional derivatives coupled with three-point local boundary conditions. Almalahi et al. [9] explore qualitative aspects of a nonlinear Langevin integro-fractional differential equation through their investigation. Ahmad et al. [5] explore the existence and Hyers-Ulam stability of solutions to a nonlinear neutral stochastic fractional differential system. The fixed-point theory has found numerous practical applications over recent decades. Its utility extends to optimization theory, game theory, conflict resolution, and mathematical modeling in quality management, presenting fascinating and valuable insights across these domains.

Developing novel fixed-point theorems and related approaches has facilitated investigating and researching boundary value problem models, including hybrid fractional boundary value problems. The mathematical analysis of fractional dynamical systems has recently expanded to include hybrid fractional differential equations. Many researchers have studied the hybrid fractional differential equations using various approaches. Fredj et al. [19] examined the existence, uniqueness, and Hyers-Ulam stability of hybrid sequential fractional differential equations featuring multiple fractional derivatives of Caputo type with varying orders. Samei et al. [27] discuss the existence of solutions to a class of hybrid Caputo-Hadamard fractional differential inclusions with Dirichlet boundary conditions. We investigate the existence of solutions for a fractional hybrid integro-differential equation with mixed hybrid integral boundary value conditions, as seen in [16]. Matar [24] delves into the qualitative characteristics of a set of hybrid nonlinear fractional differential equations. In 2021, Boutiara et al. [18] undertook a study to explore the existence of solutions for a novel category of hybrid Hilfer fractional differential equations, considering hybrid boundary conditions. Matar explores qualitative properties within a category of hybrid nonlinear fractional differential equations [24]. Examining the existence of solutions for fractional neutral hybrid differential equations with finite delay [25], Ali et al. [6] formulated certain conditions that are sufficient to ensure both the existence and uniqueness of solutions for the interconnected set of fractional hybrid differential equations.

Motivated by previous work, we propose nonlinear hybrid fractional Langevin equations in this work. This study explores the existence, uniqueness, and stability (according to the Ulam-Hyers notion) of solutions to the following problem:

$$\begin{cases} {}^{C}D_{c^+}^{\sigma,\psi}[{}^{C}D_{c^+}^{\varsigma,\psi}[\frac{u(\zeta)}{g(\zeta,u(\zeta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\zeta))}] - \mu u(\zeta)] = f(\zeta,u(\zeta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\zeta)), \\ \quad \zeta \in \mathbf{J} = [c,d], \\ u(c) = 0, \qquad {}^{C}D_{c^+}^{\varsigma,\psi}\frac{u(\zeta)}{g(\zeta,u(\zeta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\zeta))}|_{\zeta=a} = 0, \qquad u(d) = \xi u(\delta), \quad \delta \in (c,d), \end{cases} \tag{1}$$

where the ψ-Caputo fractional derivatives of order $\sigma \in (0,1]$ and $\varsigma \in (1,2]$ are denoted by the expressions ${}^{C}D_{c^+}^{\sigma,\psi}$ and ${}^{C}D_{c^+}^{\varsigma,\psi}$, respectively. The given functions are continuous: $f : \mathbf{J} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{J} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R} \setminus \{0\}$.

2 Preliminaries

To achieve our main objectives, we first explore some supplementary concepts necessary for the existence of this work.

The set of continuous real-valued functions from $\mathbf{J}$ to $\mathbf{R}$ can be represented using the notation $\mathbf{C} = C(\mathbf{J},\mathbf{R})$. Consequently, it is evident that $\mathbf{C}$ is a Banach space with the supremum norm defined as follows:

$$\|u\| = \sup_{\zeta\in\mathbf{J}}|u(\zeta)|.$$

The considered ψ-fractional integrals and derivatives are given. For further information, please refer to [24].

Definition 2.1 [10] For every $\zeta \in \mathbf{J}$, $\psi'(\zeta) \neq 0$. Let $\varsigma > 0$, and let $\psi : \mathbf{J} \to \mathbf{R}$ be an increasing function. An integrable function u on $\mathbf{J}$ has a left-sided ψ-Riemann-Liouville integral, which is defined as follows with respect to ψ:

$$I_{c^+}^{\varsigma,\psi} u(\zeta) = \frac{1}{\Gamma(\varsigma)} \int_a^\zeta \psi'(s)\big(\psi(\zeta) - \psi(s)\big)^{\varsigma-1} u(s)\, ds \tag{2}$$

for all $\zeta \in \mathbf{J}$.

When $\psi(\zeta) = \zeta$ and $\psi(\zeta) = \ln(\zeta)$, respectively, one can derive the Hadamard fractional integral and the Riemann-Liouville integral from Equation (2).

Definition 2.2 [10] Assuming $n = [\varsigma] + 1$ and $n \in \mathbb{N}$, the left-sided ψ-Caputo fractional derivative of $u \in C^n(\mathbf{J}, \mathbf{R})$ with respect to any $\zeta \in \mathbf{J}$ is defined, where ψ is a strictly increasing function

$${}^C D_{c^+}^{\varsigma,\psi} u(\zeta) = I_{c^+}^{n-\varsigma,\psi} \left(\frac{1}{\psi'(\zeta)} \frac{d}{dz} \right)^n u(\zeta). \tag{3}$$

Lemma 2.3 [10] *Given $\varsigma, \sigma > 0$ and $u \in L^1(J, \mathbf{R})$, we obtain*

$$I_{c^+}^{\varsigma,\psi} I_{c^+}^{\sigma,\psi} u(\zeta) = I_{c^+}^{\varsigma+\sigma;\psi} u(\zeta), \quad \zeta \in \mathbf{J}.$$

Lemma 2.4 [10] *Let $\varsigma > 0$.*

(1) *If $u \in C(\mathbf{J}, \mathbf{R})$, then*

$${}^C D_{c^+}^{\varsigma,\psi} I_{c^+}^{\varsigma,\psi} u(\zeta) = u(\zeta) \quad \zeta \in \mathbf{J}.$$

(2) *If $u \in C^n(\mathbf{J}, \mathbf{R})$ and $\varsigma \in (n-1, n)$, then*

$$I_{c^+}^{\varsigma,\psi} {}^C D_{c^+}^{\varsigma,\psi} u(\zeta) = u(\zeta) - \sum_{k=0}^{n-1} \frac{(\frac{1}{\psi'} \frac{d}{dz})^k u(c)}{k} \big[\psi(\zeta) - \psi(c)\big]^k, \quad \zeta \in \mathbf{J}$$

for some constants c_k, $k = 0, 1, 2, \ldots, n-1$.

Lemma 2.5 [11] *Consider $\zeta > a$, where $\varsigma \geq 0$ and $\sigma > 0$. Then,*

$$(a_1): \quad I_{a+}^{\varsigma,\psi} \big(\psi(\zeta) - \psi(c)\big)^{\sigma-1}(\zeta) = \frac{\Gamma(\sigma)}{\Gamma(\varsigma+\sigma)} \big(\psi(\zeta) - \psi(c)\big)^{\varsigma+\sigma-1}$$

$$(a_2): \quad {}^C D_{c^+}^{\varsigma,\psi} \big(\psi(\zeta) - \psi(c)\big)^{\sigma-1}(\zeta) = \frac{\Gamma(\varsigma)}{\Gamma(\varsigma+\sigma)} \big(\psi(\zeta) - \psi(c)\big)^{\sigma+\varsigma-1}$$

$$(a_3): \quad {}^C D_{c^+}^{\varsigma,\psi} \big(\psi(\zeta) - \psi(c)\big)^k(\zeta) = 0, \quad \textit{for } k = 0, \ldots, n-1; n \in \mathbb{N}.$$

Remark 2.6 From Lemmas (2.4) and (2.5), it is clear that under given general boundary conditions, we obtain

$$I_{a^+}^{\varsigma,\psi} {}^C D_{c^+}^{\varsigma,\psi} u(\zeta) = u(\zeta) + \sum_{k=0}^{n-1} c_k \big(\psi(\zeta) - \psi(c)\big)^k, \quad \zeta \in \mathbf{J}.$$

Below, we provide some contextual information about fixed point theory.

Definition 2.7 [3] A mapping $\mathbb{T}:\mathbf{C}\to\mathbf{C}$ is a contraction mapping or contraction, if there exists constant $L_{\mathbb{T}}$ with $L_{\mathbb{T}}<1$ such that

$$\left\|\mathbb{T}(u)-\mathbb{T}(v)\right\|\le L_{\mathbb{T}}\|u-v\| \tag{4}$$

for every $u,v\in\mathbf{C}$.

Theorem 1 (*Banach fixed point*) [3] *A contraction mapping $\mathbb{T}$ from the set C to itself possesses precisely one fixed point.*

Theorem 2 (*Schauder's fixed point*) [28] *In a Banach space* $\mathbf{C}$, *consider a non-empty, closed, convex subset $\mathcal{K}$. If $\mathbb{T}:\mathcal{K}\to\mathcal{K}$ is a compact operator, then there exists a fixed point of $\mathbb{T}$ within $\mathcal{K}$.*

3 Main results

This section addresses problem (1) using an arbitrary function ψ. We require the following lemma to examine whether solutions to (1) exist.

Lemma 3.1 *Let* $\frac{g(d)}{g(\delta)}\frac{\psi(d)-\psi(c)}{\psi(\delta)-\psi(c)}\neq\xi$, *then the solution of the problem*

$$\begin{cases} D_{c^+}^{\sigma,\psi}[{}^C D_{c^+}^{\varsigma,\psi}[\frac{u(\zeta)}{g(\zeta)}]-\mu u(\zeta)]=f(\zeta), & \zeta\in\mathbf{J}=[c,d],\\ u(c)=0, \quad {}^C D_{c^+}^{\varsigma,\psi}[\frac{u(\zeta)}{g(\zeta)}]_{\zeta=c}=0, & u(d)=\xi u(\delta), \quad \delta\in(c,d), \end{cases} \tag{5}$$

is given by

$$u(\zeta)=g(\zeta)\left[h(\zeta)-\frac{[g(d)h(d)-\xi g(\delta)h(\delta)](\psi(\zeta)-\psi(c))}{g(d)(\psi(d)-\psi(c))-\xi g(\delta)(\psi(\delta)-\psi(c))}\right], \tag{6}$$

where

$$h(\zeta)=I_{c^+}^{\sigma+\varsigma,\psi}f(\zeta)+\mu I_{c^+}^{\varsigma,\psi}u(\zeta).$$

In particular, if $\xi=\frac{g(d)}{g(\delta)}$, *then*

$$u(\zeta)=g(\zeta)\left[h(\zeta)-\frac{\psi(\zeta)-\psi(c)}{\psi(d)-\psi(\delta)}\left[h(d)-h(\delta)\right]\right].$$

Proof Using Lemma (2.4) and applying the σth ψ-Riemann-Liouville fractional integral to both sides of equation (5), we get:

$${}^C D_{c^+}^{\varsigma,\psi}\left(\frac{u(\zeta)}{g(\zeta)}\right)=I_{c^+}^{\sigma,\psi}f(\zeta)+\mu u(\zeta)+c_0.$$

We definitively establish that $c_0=0$ by employing the initial and boundary conditions. The following integral form is obtained by further applying the ςth ψ-Riemann-Liouville fractional integral and using Lemma (2.4):

$$\frac{u(\zeta)}{g(\zeta)}=h(\zeta)+c_1\big(\psi(\zeta)-\psi(c)\big)+c_2. \tag{7}$$

From the first boundary condition, it follows that $c_2 = 0$. Consequently, upon examining the last boundary condition, we deduce:

$$c_1 = \frac{\xi g(\delta)h(\delta) - g(d)h(d)}{g(d)(\psi(d) - \psi(c)) - \xi g(\delta)(\psi(\delta) - \psi(c))}.$$

Equation (6) can be obtained by replacing these constants in Equation (7).

Conversely, the function in (6) satisfies Equation (5) and the associated boundary conditions. □

This mild solution to equation (1) is clearly defined in Lemma (3.1).

Definition 3.2 A mild solution to equation (1) is defined as a function $u \in \mathbf{C}$ if it satisfies the following equation

$$\begin{aligned} u(\zeta) = {} & g\big(\zeta, u(\zeta), {}^{C}D_{c^+}^{\varsigma,\psi} u(\zeta)\big) \\ & \times \Bigg[hu(\zeta) \\ & - \frac{(\psi(\zeta) - \psi(c))[g(d, u(d), {}^{C}D_{c^+}^{\varsigma,\psi} u(d))hu(d) - \xi g(\delta, u(\delta), {}^{C}D_{c^+}^{\varsigma,\psi} u(\delta))hu(\delta)]}{g(d, u(d), {}^{C}D_{c^+}^{\varsigma,\psi} u(d))(\psi(d) - \psi(c)) - \xi g(\delta, u(\delta), {}^{C}D_{c^+}^{\varsigma,\psi} u(\delta))(\psi(\delta) - \psi(c))} \Bigg], \end{aligned} \tag{8}$$

where

$$h(\zeta) = I_{c^+}^{\sigma+\varsigma,\psi} f\big(\zeta, u(\zeta), {}^{C}D_{c^+}^{\varsigma,\psi} u(\zeta)\big) + \mu I_{c^+}^{\varsigma,\psi} u(\zeta), \quad \zeta \in \mathbf{J}. \tag{9}$$

We establish the following assumptions:

($\mathbb{Z}_1$): The function g is continuous, where g is defined over the set $\mathbf{J} \times \mathbf{R} \times \mathbf{R}$ and maps to the nonzero real numbers. Additionally, there exists a positive function φ with a supremum value represented by $\|\varphi\|$

$$\big|g(\zeta, u_1, u_2) - g(\zeta, v_1, v_2)\big| \le \varphi(\zeta)\big(|u_1 - v_1| + |u_1 - v_2|\big)$$

for all $(\zeta, u_1, v_1), (\zeta, u_2, v_2) \in \mathbf{J} \times \mathbf{R} \times \mathbf{R}$.

Furthermore, there exists a constant $\theta > 0$ such that

$$\big|g(d, u_1, u_2)\big(\psi(d) - \psi(c)\big) - \xi g(\delta, u_1, u_2)\big(\psi(\delta) - \psi(c)\big)\big| \ge \theta > 0$$

for all $u_1, u_2 \in R$.

($\mathbb{Z}_2$): The continuous function f maps to the real numbers and is defined over the set $\mathbf{J} \times \mathbf{R} \times \mathbf{R}$. Furthermore, a nondecreasing function Υ belongs to the set of continuous functions from the interval $[0, \infty)$ to the open interval $(0, \infty)$, and a function p belongs to the set of continuous functions from **J** to the positive real numbers.

$$\left|f(\zeta,u_1,u_2)\right| \le p(\zeta)\Upsilon\big(|u_1|+|u_2|\big) \tag{10}$$

for all $(\zeta,u_1,u_2)\in \mathbf{J}\times\mathbf{R}\times\mathbf{R}$.

($\mathbb{Z}_3$): There exists $r>0$ such that

$$\frac{g_0 F_r}{1-\|\varphi\|F_r}\le r$$

and

$$\|\varphi\|F_r<1, \tag{11}$$

where $g_0=\sup_{\zeta\in\mathbf{J}}|g(\zeta,0,0)|$, and

$$\begin{aligned}F_r={}&\frac{|g|\|p\|(\psi(d)-\psi(c))^{\sigma+\varsigma}}{\theta\Gamma(\sigma+\varsigma+1)}\big[\big(\theta+2g_0\big(\psi(d)-\psi(c)\theta\big)\big)\big]\Upsilon(r)\\&+\frac{2(\psi(d)-\psi(c))^{\sigma+\varsigma+1}\|p\|\|\varphi\|}{\theta\Gamma(\sigma+\varsigma+1)}r\Upsilon(r)+\frac{|\mu|(\psi(d)-\psi(c))^{\varsigma+1}\|\varphi\|}{\theta\Gamma(\varsigma+1)}r^2\\&+\frac{|\mu|(\psi(d)-\psi(c))^{\varsigma}}{\theta\Gamma(\varsigma+1)}(\theta+2g_0\big(\psi(d)-\psi(c)\big)r.\end{aligned} \tag{12}$$

The next result relies on Schauder's fixed-point theorem.

Theorem 3 *Problem* (1) *can have at least one mild solution if the conditions* ($\mathbb{Z}_1$)–($\mathbb{Z}_3$) *are met.*

Proof Let us define $\mathbf{H}=\{u\in\mathbf{C}:\|u\|\le r\}$. Certainly, **H** is a bounded, closed, convex subset of the Banach space **C**. We define the operator $\mathbb{T}:\mathbf{C}\to\mathbf{C}$ as follows, in compliance with Definition (3.2)

$$\begin{aligned}\mathbb{T}u(\zeta)={}&g\big(\zeta,u(\zeta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\zeta)\big)\\&\times\Bigg[hu(\zeta)\\&-\frac{(\psi(\zeta)-\psi(c))[g(d,u(d),{}^{C}D_{c^+}^{\varsigma,\psi}u(d))hu(d)-\xi g(\delta,u(\delta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\delta))hu(\delta)}{g(d,u(d),{}^{C}D_{c^+}^{\varsigma,\psi}u(\zeta))(\psi(d)-\psi(c))-\xi g(\delta,u(\delta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\delta))(\psi(\delta)-\psi(c))}\Bigg].\end{aligned}$$

Step 1: Take any $u\in\mathbf{H}$ and $\zeta\in\mathbf{J}$, and prove that $\mathbb{T}$ transfers bounded sets to bounded sets in **C**. Next, we have:

$$\begin{aligned}\big|hu(\zeta)\big|&\le I_{c^+}^{\sigma+\varsigma,\psi}\big|f\big(\zeta,u(\zeta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\zeta)\big)\big|+|\mu|I_{c^+}^{\varsigma,\psi}\big|u(\zeta)\big|\\&\le\frac{(\psi(\zeta)-\psi(c))^(\sigma+\varsigma)\|p\|\Upsilon(r)}{\Gamma(\sigma+\varsigma+1)}+\frac{|\mu|(\psi(\zeta)-\psi(c))^{\varsigma}r}{\Gamma(\varsigma+1)}.\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\left|\mathbb{T}(u)(\zeta)\right| \\
&\quad \le \left|g\big(\zeta,u(\zeta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\zeta)\big)\right|\left|hu(\zeta)\right| \\
&\qquad + \frac{(\psi(\zeta)-\psi(c))[|g(d,u(d)^{C}D_{c^+}^{\varsigma,\psi}u(d))||hu(d)|+|\xi||g(\delta,u(\delta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\delta))||hu(\delta)|]}{|g(d,u(d),{}^{C}D_{c^+}^{\varsigma,\psi}u(d))(\psi(d)-\psi(c))-\xi g(\delta,u(\delta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\delta))(\psi(\delta)-\psi(c))|} \\
&\quad \le \frac{|g|(\psi(\zeta)-\psi(c))^{\sigma+\varsigma}\|p\|\Upsilon(r)}{\Gamma(\sigma+\varsigma+1)} + \frac{|\mu|(\psi(\zeta)-\psi(c))^{\varsigma}r}{\Gamma(\varsigma+1)} \\
&\qquad + \frac{(\psi(\zeta)-\psi(c))|g(b,u(d),{}^{C}D_{c^+}^{\varsigma,\psi}u(d))|}{\theta} \\
&\qquad \times \left(\frac{(\psi(d)-\psi(c))^{\sigma+\varsigma}\|p\|\Upsilon(r)}{\Gamma(\sigma+\varsigma+1)} + \frac{|\mu|(\psi(d)-\psi(c))^{\varsigma}r}{\Gamma(\varsigma+1)}\right) \\
&\qquad + \frac{|\nu|(\psi(\zeta)-\psi(c))|g(\delta,u(\delta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\delta))|}{\theta} \\
&\qquad \times \left(\frac{(\psi(\delta)-\psi(c))^{\sigma+\varsigma}\|p\|\Upsilon(r)}{\Gamma(\sigma+\varsigma+1)} + \frac{|\mu|(\psi(\delta)-\psi(c))^{\varsigma}r}{\Gamma(\varsigma+1)}\right) \\
&\quad \le \frac{|g|\|p\|(\psi(d)-\psi(c))^{\sigma+\varsigma}}{\theta\Gamma(\sigma+\varsigma+1)} + \big[\theta+2g_0\big(\psi(d)-\psi(c)\big)\big]\Upsilon(r) \\
&\qquad + \frac{2(\psi(d)-\psi(c))^{\sigma+\varsigma+1}\|p\|\|\varphi\|}{\Gamma(\sigma+\varsigma+1)}r\Upsilon(r) + \frac{\mu|(\psi(d)-\psi(c))^{\varsigma+1}\|\varphi\|}{(\Gamma(\varsigma+1)}r^2 \\
&\qquad + \frac{\mu|(\psi(d)-\psi(c))^{\varsigma}}{\theta\Gamma(\varsigma+1)} + \big[\theta+2g_0\big(\psi(d)-\psi(c)\big)\big]r.
\end{aligned} \tag{13}
$$

Hence, $|\mathbb{T}(u)| \le \digamma_r$ for all $u \in \mathbf{H}$, where $\digamma_r$ is given by (12). This demonstrates that $\mathbb{T}$ is uniformly bounded on $\mathbf{H}$.

Step 2: We prove that the operator $\mathbb{T}$ is continuous. Consider a sequence in $\mathbf{H}$, u_n, that converges to $u \in \mathbf{H}$. From the Lebesgue-dominated convergence theorem, we now obtain:

$$
\begin{aligned}
&\lim_{n\to\infty} \mathbb{T}(u_n)(\zeta) \\
&\quad = \lim_{n\to\infty}\big[g\big(\zeta,u_n(\zeta),{}^{C}D_{c^+}^{\varsigma,\psi}u_n(\zeta)\big)\big]\Big[hu_n(\zeta) \\
&\qquad - \frac{(\psi(\zeta)-\psi(c))g(d,u_n(d),{}^{C}D_{c^+}^{\varsigma,\psi}u_n(d))hu_n(d)-\xi g(\delta,u_n(\delta),{}^{C}D_{c^+}^{\varsigma,\psi}u_n(\delta))hu_n(\delta)}{[g(d,u(d),{}^{C}D_{c^+}^{\varsigma,\psi}u_n(d))(\psi(d)-\psi(c))-\xi g(\delta,u(\delta),{}^{C}D_{c^+}^{\varsigma,\psi}u_n(\delta))(\psi(\delta)-\psi(c))}\Big] \\
&\quad = g\big(\zeta,u(\zeta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\zeta)\big)\Big[hu(\zeta) \\
&\qquad - \frac{(\psi(\zeta)-\psi(c))g(d,u(d),{}^{C}D_{c^+}^{\varsigma,\psi}u(d))hu(d)-\xi g(\delta,u(\delta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\delta))hu(\delta)}{[g(d,u(d),{}^{C}D_{c^+}^{\varsigma,\psi}u(d))(\psi(d)-\psi(c))-\xi g(\delta,u(\delta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\delta))(\psi(\delta)-\psi(c))}\Big] \\
&\quad = \mathbb{T}(u)(\zeta)
\end{aligned}
$$

for every $\zeta \in \mathbf{J}$. Therefore, on $\mathbf{H}$, $\mathbb{T}$ is a continuous operator.

Step 3: Let $\zeta_1, \zeta_2 \in \mathbf{J}$ such that $\zeta_1 < \zeta_2$. Then, for any $u \in \mathbf{H}$, according to (10), we obtain:

$$
\begin{aligned}
&\left|hu(\zeta_2) - hu(\zeta_1)\right| \\
&\leq \int_{\zeta_1}^{\zeta_2} \frac{\psi'(s)}{\Gamma((\sigma+\varsigma))}\left[\left(\psi(\zeta_2)-\psi(s)\right)^{\sigma+\varsigma-1} - \left(\psi(\zeta_1)-\psi(s)\right)^{\sigma+\varsigma-1}\right] \\
&\quad \times f\left|\left(s, u(s), {}^{C}D_{c^+}^{\varsigma,\psi} u(s)\right)\right| ds \\
&\quad + \int_{\zeta_1}^{\zeta_2} \frac{\psi'(s)}{\Gamma((\sigma+\varsigma))}[\left(\psi(\zeta_2)-\psi(s)\right)^{\sigma+\varsigma-1} f\left|\left(s, u(s), {}^{C}D_{c^+}^{\varsigma,\psi} u(s)\right)\right| ds \\
&\quad + |\mu| \int_{a}^{\zeta_1} \frac{\psi'(s)}{\Gamma((\varsigma))}\left[\left(\psi(\zeta_2)-\psi(s)\right)^{\varsigma-1} - \left(\psi(\zeta_1)-\psi(s)\right)^{\sigma+\varsigma-1}\right]\left|u(s)\right| ds \\
&\quad + \int_{\zeta_1}^{\zeta_2} \frac{\psi'(s)}{\Gamma((\varsigma))}[\left(\psi(\zeta_2)-\psi(s)\right)^{\varsigma-1}\left|\left(u(s)\right)\right| ds \\
&\leq \frac{[2(\psi(\zeta_2)-\psi(\zeta_1))^{\sigma+\varsigma} + (\psi(\zeta_2)-\psi(c))^{\sigma+\varsigma} - (\psi(\zeta_1)-\psi(c))^{\sigma+\varsigma}]\|p\|\Upsilon(r)}{\Gamma(\sigma+\varsigma+1)} \\
&\quad + \frac{|\mu|[2(\psi(\zeta_2)-\psi(\zeta_1))^{\varsigma} + (\psi(\zeta_2)-\psi(c))^{\varsigma} - (\psi(\zeta_1)-\psi(c))^{\varsigma}]r}{\Gamma(\varsigma+1)}.
\end{aligned}
$$

Using similar arguments as in (13), we obtain

$$
\begin{aligned}
&\left|\frac{g(d,u(d),{}^{C}D_{c^+}^{\varsigma,\psi}u(d))hu(d) - \xi g(\delta,u(\delta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\delta))hu(\delta)}{[g(d,u(d),{}^{C}D_{c^+}^{\varsigma,\psi}u(d))(\psi(d)-\psi(c)) - \xi g(\delta,u(\delta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\delta))(\psi(\delta)-\psi(c))}\right| \\
&\leq \frac{|g(d,u(d),{}^{C}D_{c^+}^{\varsigma,\psi}u(d))|}{\theta}\left(\frac{(\psi(d)-\psi(c))^{\sigma+\varsigma}\|p\|\Upsilon(r)}{\Gamma(\sigma+\varsigma+1)} + \frac{|\mu|(\psi(d)-\psi(c))^{\varsigma}r}{\Gamma(\varsigma+1)}\right) \\
&\quad + \frac{|\xi||g(\delta,u(\delta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\delta))|}{\theta}\left(\frac{(\psi(\delta)-\psi(c))^{\sigma+\varsigma}\|p\|\Upsilon(r)}{\Gamma(\sigma+\varsigma+1)} + \frac{|\mu|(\psi(\delta)-\psi(c))^{\varsigma}r}{\Gamma(\varsigma+1)}\right) \\
&\leq \frac{g_0\|p\|(\psi(d)-\psi(c))^{\sigma+\varsigma}\Upsilon(r)}{\theta\Gamma(\sigma+\varsigma+1)} + \frac{r\Upsilon(r)(\psi(d)-\psi(c))^{\sigma+\varsigma+1}\|p\|\|\varphi\|}{\theta\Gamma(\sigma+\varsigma+1)} \\
&\quad + \frac{|\mu|r^2(\psi(d)-\psi(c))^{\varsigma}\|\varphi\|}{\theta\Gamma(\varsigma+1)} + \frac{g_0|\mu|r(\psi(d)-\psi(c))^{\varsigma}}{\theta\Gamma(\varsigma+1)}.
\end{aligned}
$$

Let

$$
\begin{aligned}
\mathbf{B} = {} & \frac{g_0\|p\|(\psi(d)-\psi(c))^{\sigma+\varsigma}\Upsilon(r)}{\theta\Gamma(\sigma+\varsigma+1)} + \frac{r\Upsilon(r)(\psi(d)-\psi(c))^{\sigma+\varsigma+1}\|p\|\|\varphi\|}{\theta\Gamma(\sigma+\varsigma+1)} \\
& + \frac{|\mu|r^2(\psi(d)-\psi(c))^{\varsigma}\|\varphi\|}{\theta\Gamma(\varsigma+1)} + \frac{g_0|\mu|r(\psi(d)-\psi(c))^{\varsigma}}{\theta\Gamma(\varsigma+1)}.
\end{aligned}
$$

This implies that

$$
\left|\frac{g(d,u(d),{}^{C}D_{c^+}^{\varsigma,\psi}u(d))hu(d) - \xi g(\delta,u(\delta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\delta))hu(\delta)}{[g(d,u(d),{}^{C}D_{c^+}^{\varsigma,\psi}u(d))(\psi(d)-\psi(c)) - \xi g(\delta,u(\delta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\delta))(\psi(\delta)-\psi(c))}\right| \leq \mathbf{B}.
$$

Therefore,

$$\begin{aligned}
&\left|\mathbb{T}(u)(\zeta_2) - \mathbb{T}(u)(\zeta_1)\right| \\
&\quad \le \left|hu(\zeta_2) - hu(\zeta_1)\right| + \left|\psi(\zeta_2) - \psi(\zeta_1)\right| \\
&\qquad \times \left|\frac{g(d,u(d),{}^{C}D_{c^+}^{\varsigma,\psi}u(d))hu(d) - \xi g(\delta,u(\delta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\delta))hu(\delta)}{g(d,u(d),{}^{C}D_{c^+}^{\varsigma,\psi}u(d))(\psi(d)-\psi(c)) - \xi g(\delta,u(\delta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\delta))(\psi(\delta)-\psi(c))}\right| \\
&\quad \le \frac{[2(\psi(\zeta_2)-\psi(\zeta_1))^{\sigma+\varsigma} + (\psi(\zeta_2)-\psi(c))^{\sigma+\varsigma} - (\psi(\zeta_1)-\psi(c))^{\sigma+\varsigma}]\|p\|\Upsilon(r)}{\Gamma(\sigma+\varsigma+1)} \\
&\qquad + \frac{[2(\psi(\zeta_2)-\psi(\zeta_1))^{\varsigma} + (\psi(\zeta_2)-\psi(c))^{\varsigma} - (\psi(\zeta_1)-\psi(c))^{\varsigma}]r}{\Gamma(\varsigma+1)} \\
&\qquad + \mathbf{B}\left|\psi(\zeta_2) - \psi(\zeta_1)\right|.
\end{aligned}$$

This implies

$$\left|\mathbb{T}(u)(\zeta_2) - \mathbb{T}(u)(\zeta_1)\right| \to 0 \quad \text{as } \zeta_1 \to \zeta_2.$$

Therefore, $\mathbb{T}$ fulfils the equicontinuity criterion within the Banach space $\mathbf{C}$. Consequently, $\mathbb{T}$ is relatively compact, thus satisfying the Arzelà-Ascoli theorem, which implies complete continuity of $\mathbb{T}$. Consequently, $\mathbb{T}$ is compact on $\mathbf{H}$, thereby fulfilling Theorem (2). Consequently, a mild solution exists on $\mathbf{J}$ to problem (1), thus establishing the desired result. □

Remark 3.3 Let $\xi = \frac{g(d,u(d),{}^{C}D_{c^+}^{\varsigma,\psi}u(d))}{g(\delta,u(\delta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\delta))}$. The integral solution (10), therefore, has the following form when reduced:

$$\begin{aligned}
u(\zeta) &= g\big(\zeta,u(\zeta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\zeta)\big) \\
&\quad \times \bigg(h\big(\zeta,u(\zeta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\zeta)\big) \\
&\quad - \frac{\psi(\zeta)-\psi(c)}{\psi(d)-\psi(\delta)}\Big[h\big(d,u(d),{}^{C}D_{c^+}^{\varsigma,\psi}u(d)\big) - h\big(\delta,u(\delta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\delta)\big)\Big]\bigg),
\end{aligned}$$

where h is defined by (9).

The following outcome depends on the application of the Banach fixed-point theorem. We additionally presume the following conditions for the forthcoming outcome:

($\mathbf{S}_1$): The function $g : \mathbf{J} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R} \setminus \{0\}$ is continuous if there exists a function $\varphi \in C(\mathbf{J}, \mathbf{R}^+)$, with the supremum $\|\varphi\|$ such that

$$\left|g(\zeta,u_1,u_2) - g(\zeta,v_1,v_2)\right| \le \varphi(\zeta)\big(|u_1 - v_1| + |u_2 - v_2|\big)$$

for all $(\zeta,u_1,u_2), (\zeta,v_1,v_2) \in \mathbf{J} \times \mathbf{R} \times \mathbf{R}$. Furthermore, let exist a constant $k_g > 0$ such that:

$$\left|g(\zeta,u_1,u_2)\right| \le k_g$$

for all $(\zeta,u_1,u_2) \in \mathbf{J} \times \mathbf{R} \times \mathbf{R}$.

($\mathbf{S}_2$): The function f defined on $\mathbf{J}\times\mathbf{R}\times\mathbf{R}$ and mapping to $\mathbf{R}$ is continuous. Furthermore, $p\in C(\mathbf{J},\mathbf{R}^+)$ is a continuous function, and its supremum is $\|p\|$

$$\left|f(\zeta,u_1,u_2)-f(\zeta,v_1,v_2)\right|\le p(\zeta)\big(|u_1-v_1|+|u_2-v_2|\big)$$

for all $(\zeta,u_1,u_2),(\zeta,v_1,v_2)\in\mathbf{J}\times\mathbf{R}\times\mathbf{R}$. Furthermore, there exists a positive constant k_f such that

$$\left|f(\zeta,u_1,u_2)\right|\le k_f\quad\text{for all }(\zeta,u_1,u_2)\in\mathbf{J}\times\mathbf{R}\times\mathbf{R}.$$

($\mathbf{S}_3$): Assume that $\Xi<1$, where

$$\begin{aligned}\Xi={}&\frac{|\mu|\|\varphi\|(\psi(d)-\psi(c))}{\Gamma(\varsigma+1)}\\&\times\left(\big(\psi(d)-\psi(c)\big)^{\varsigma-1}+\frac{(\psi(d)-\psi(c))^{\varsigma}+(\psi(\delta)-\psi(c))^{\varsigma}}{(\psi(d)-\psi(\delta))}\right)\\&+\frac{(k_g(\|p\|+|\mu|)+k_f\|\varphi\|)(\psi(d)-\psi(c))}{\Gamma(\sigma+\varsigma+1)}\\&\times\left(\big(\psi(d)-\psi(c)\big)^{\sigma+\varsigma-1}+\frac{(\psi(d)-\psi(c))^{\sigma+\varsigma}+(\psi(\delta)-\psi(c))^{\sigma+\varsigma}}{(\psi(\delta)-\psi(c))}\right).\end{aligned}$$

Theorem 4 *Given that conditions* ($\mathbf{S}_1$)–($\mathbf{S}_3$) *are satisfied, it follows that there exists a unique mild solution to the problem* (1) *over the interval* $\mathbf{J}$.

Proof Suppose the operator $\mathbb{T}:\mathbf{C}\to\mathbf{C}$ is defined as follows:

$$\mathbb{T}u(\zeta)=g\big(\zeta,u(\zeta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\zeta)\big)\left(hu(\zeta)-\frac{\psi(\zeta)-\psi(c)}{\psi(d)-\psi(\delta)}\big[hu(d)-hu(\delta)\big]\right),\quad\zeta\in\mathbf{J}.$$

As a result, $\mathbb{T}$ is well-defined and continuous because both g and h are continuous. Given $u,v\in\mathbf{C}$, condition $\mathbf{S}_2$ implies:

$$\begin{aligned}\left|hu(\zeta)-hv(\zeta)\right|&\le I_{c^+}^{\sigma+\varsigma,\psi}\left|f\big(\zeta,u(\zeta),{}^{C}D_{a^+}^{\varsigma+\psi}u(\zeta)\big)-f\big(\zeta,v(\zeta),{}^{C}D_{a^+}^{\varsigma+\psi}v(\zeta)\big)\right|\\&\quad+\mu I_{c^+}^{\varsigma,\psi}\left|u(\zeta)-v(\zeta)\right|\\&\le\left(\frac{\psi(\zeta)-\psi(c))^{\sigma+\varsigma}\|p\|}{\Gamma(\sigma+\varsigma+1)}+\frac{|\mu|(\psi(\zeta)-\psi(c))^{\varsigma}}{\Gamma(\varsigma+1)}\right)\|u-v\|\end{aligned}\tag{14}$$

and

$$\left|hu(\zeta)\right|\le\frac{k_f(\psi(\zeta)-\psi(c))^{\sigma+\varsigma}}{\Gamma(\sigma+\varsigma+1)}+\frac{|\mu|(\psi(\zeta)-\psi(c))^{\varsigma}}{\Gamma(\varsigma+1)}.\tag{15}$$

Utilizing the triangle inequality, we obtain:

$$\begin{aligned}&\left|\mathbb{T}u(\zeta)-\mathbb{T}v(\zeta)\right|\\&\quad\le\left|g\big(\zeta,u(\zeta),{}^{C}D_{c^+}^{\varsigma,\psi}u(\zeta)\big)hu(\zeta)-G\big(\zeta,v(\zeta),{}^{C}D_{c^+}^{\varsigma,\psi}v(\zeta)\big)hv(\zeta)\right|\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\psi(\zeta)-\psi(c)}{\psi(d)-\psi(\delta)}\left|g\left(\zeta, v(\zeta), {}^{C}D_{c^+}^{\varsigma,\psi} v(\zeta)\right)hv(b) - g\left(\zeta, u(\zeta), {}^{C}D_{c^+}^{\varsigma,\psi} u(\zeta)\right)hu(d)\right| \\
&+ \frac{\psi(\zeta)-\psi(c)}{\psi(d)-\psi(\delta)}\left|g\left(\zeta, u(\zeta) {}^{C}D_{c^+}^{\varsigma,\psi} u(\zeta)\right)hu(\delta) - g\left(\zeta, v(\zeta), {}^{C}D_{c^+}^{\varsigma,\psi} v(\zeta)\right)hv(\delta)\right| \\
&\leq \left|g\left(\zeta, u(\zeta), {}^{C}D_{c^+}^{\varsigma,\psi} u(\zeta)\right)\right|\left|hu(\zeta) - hv(\zeta)\right| \\
&+ \left|hv(\zeta)\right|\left|g\left(\zeta, u(\zeta), {}^{C}D_{c^+}^{\varsigma,\psi} u(\zeta)\right) - \left|g\left(\zeta, v(\zeta), {}^{C}D_{c^+}^{\varsigma,\psi} v(\zeta)\right)\right|\right. \\
&+ \frac{\psi(\zeta)-\psi(c)}{\psi(d)-\psi(\delta)}\left|g\left(\zeta, v(\zeta), {}^{C}D_{c^+}^{\varsigma,\psi} v(\zeta)\right)\right|\left|hv(b) - hu(d)\right| \\
&+ \frac{\psi(\zeta)-\psi(c)}{\psi(d)-\psi(\delta)}\left|hu(d)\right|\left|g\left(\zeta, v(\zeta), {}^{C}D_{c^+}^{\varsigma,\psi} v(\zeta)\right) - g\left(\zeta, u(\zeta), {}^{C}D_{c^+}^{\varsigma,\psi} u(\zeta)\right)\right| \\
&+ \frac{\psi(\zeta)-\psi(c)}{\psi(d)-\psi(\delta)}\left|g\left(\zeta, u(\zeta), {}^{C}D_{c^+}^{\varsigma,\psi} u(\zeta)\right)\right|\left|hu(\delta) - hv(\delta)\right| \\
&+ \frac{\psi(\zeta)-\psi(c)}{\psi(d)-\psi(\delta)}\left|hv(\delta)\right|\left|g\left(\zeta, u(\zeta), {}^{C}D_{c^+}^{\varsigma,\psi} u(\zeta)\right) - g\left(\zeta, v(\zeta), {}^{C}D_{c^+}^{\varsigma,\psi} v(\zeta)\right)\right|.
\end{aligned}
$$

From equations (14) and (15), under assumptions ($\mathbf{S}_1$)–($\mathbf{S}_2$), we conclude that

$$
\begin{aligned}
&\left|\mathbb{T}u(\zeta) - \mathbb{T}v(\zeta)\right| \\
&\leq k_g\left(\frac{\psi(\zeta)-\psi(c)^{\sigma+\varsigma}\|p\|}{\Gamma(\sigma+\varsigma+1)} + \frac{|\mu|(\psi(\zeta)-\psi(c))^{\varsigma}}{\Gamma(\varsigma+1)}\right)\|u-v\| \\
&+ \|\varphi\|\left(\frac{k_f(\psi(\zeta)-\psi(c))^{\sigma+\varsigma}}{\Gamma(\sigma+\varsigma+1)} + \frac{|\mu|(\psi(\zeta)-\psi(c))^{\varsigma}}{\Gamma(\varsigma+1)}\right)\|u-v\| \\
&+ \frac{k_f(\psi(\zeta)-\psi(c))}{\psi(d)-\psi(\delta)}\left(\frac{\psi(\zeta)-\psi(c))^{\sigma+\varsigma}\|p\|}{\Gamma(\sigma+\varsigma+1)} + \frac{|\mu|(\psi(\zeta)-\psi(c))^{\varsigma}}{\Gamma(\varsigma+1)}\right)\|u-v\| \\
&+ \frac{\|\varphi\|[\psi(\zeta)-\psi(c)]}{\psi(d)-\psi(\delta)}\left(\frac{k_f(\psi(\zeta)-\psi(c))^{\sigma+\varsigma}}{\Gamma(\sigma+\varsigma+1)} + \frac{|\mu|(\psi(\zeta)-\psi(c))^{\varsigma}}{\Gamma(\varsigma+1)}\right)\|u-v\| \\
&+ \frac{k_g(\psi(\zeta)-\psi(c))}{\psi(d)-\psi(\delta)}\left(\frac{\psi(\delta)-\psi(c))^{\sigma+\varsigma}}{\Gamma(\sigma+\varsigma+1)\|p\|} + \frac{|\mu|(\psi(\zeta)-\psi(c))^{\varsigma}}{\Gamma(\varsigma+1)}\right)\|u-v\| \\
&+ \frac{\|\varphi\|[\psi(\zeta)-\psi(c)]}{\psi(d)-\psi(\delta)}\left(\frac{k_f(\psi(\zeta)-\psi(c))^{\sigma+\varsigma}}{\Gamma(\sigma+\varsigma+1)} + \frac{|\mu|(\psi(\zeta)-\psi(c))^{\varsigma}}{\Gamma(\varsigma+1)}\right)\|u-v\|.
\end{aligned}
$$

Upon taking the supremum over $\mathbf{J}$ and simplifying, we obtain:

$$
\|\mathbb{T}u - \mathbb{T}v\| \leq \Xi\|u-v\|.
$$

The proof is completed by applying the Banach fixed-point theorem (Theorem 1), which is made possible by the validity of hypothesis ($\mathbf{S}_3$). □

3.1 Stability results

Let $\varphi : \mathbf{J} \to \mathbf{R}^+$ be a continuous function and $\epsilon > 0$. We will examine the inequality below:

$$
\left|{}^{C}D_{a^+}^{\sigma,\psi}\left({}^{C}D_{c^+}^{\varsigma,\psi}\left[\frac{u(\zeta)}{G(\zeta, u(\zeta), {}^{C}D_{c^+}^{\varsigma,\psi} u(\zeta))}\right] - \mu u(\zeta)\right) - F\left(\zeta, u(\zeta), {}^{C}D_{c^+}^{\varsigma,\psi} u(\zeta)\right)\right| \leq \epsilon \qquad (16)
$$

for $\zeta \in \mathbf{J}$.

Definition 3.4 [13] The problem (1) demonstrates the Ulam-Hyers stability if there exists a $\lambda > 0$. Consequently, for any $\epsilon > 0$ and solution $u \in \mathbf{C}$ to equations (16), there exists a solution $v \in \mathbf{C}$ to (1) such that

$$\|u - v\| \le \lambda\epsilon.$$

Definition 3.5 [13] The problem (1) is said to be generalized Ulam-Hyers stable if there exists a function $\varpi \in C(\mathbf{R}^+, \mathbf{R}^+)$ with $\varpi(0) = 0$, such that for any $\epsilon > 0$ and every solution $u \in \mathbf{C}$ to equations (16), there exists a solution $v \in \mathbf{C}$ of (1) such that:

$$\|u - v\| \le \varpi(\epsilon).$$

Remark 3.6 Definition 3.4, where $\varpi(\epsilon) = c\epsilon$, evidently implies Definition 3.5. However, the reverse implication does not hold universally.

Remark 3.7 [13] A function $u \in C$ adheres to inequality (16) if and only if there exists a function $q \in C(\mathbf{J}, \mathbf{R})$ contingent upon u, such that:

- $|q(\zeta)| \le \epsilon$, $\zeta \in \mathbf{J}$,
- ${}^{C}D^{\sigma,\psi}_{c^+}[{}^{C}D^{\varsigma;\psi}_{c^+}[\frac{u(\zeta)}{g(\zeta,u(\zeta),{}^{C}D^{\varsigma;\psi}_{c^+}u(u))}] - \mu u(\zeta)] = f(\zeta, u(\zeta), {}^{C}D^{\varsigma;\psi}_{c^+}u(\zeta)) + q(\zeta)$, $\zeta \in \mathbf{J}$.

To simplify the equations in the following result, we indicate

$$\begin{aligned}
\mathbf{K}_1 &= \frac{\|\varphi\|}{\Gamma(\sigma+\varsigma+1)}\frac{(\psi(d)-\psi(c))^{\sigma+\varsigma+1}}{(\psi(d)-\psi(\delta))},\\
\mathbf{K}_2 &= \frac{k_g(\psi(d)-\psi(c))}{\Gamma(\sigma+\varsigma+1)}\\
&\quad\times\left[\big(\psi(d)-\psi(c)\big)^{\sigma+\varsigma-1}+\frac{(\psi(d)-\psi(c))^{\sigma+\varsigma}+(\psi(\delta)-\psi(c))^{\sigma+\varsigma}}{\psi(d)-\psi(\delta)}\right].
\end{aligned}$$

Theorem 5 *If* $\mathbf{K}_1 < \frac{1}{2}$ *and* $\mathbf{K}_2 < \frac{1}{2}$, *and assuming that hypotheses* $(\mathbf{S}_1)$–$(\mathbf{S}_3)$ *hold, then the problem* (1) *exhibits the generalized Ulam-Hyers stability.*

Proof Let $\epsilon > 0$ and $u \in \mathbf{C}$ be solutions to (16). Given Lemma (3.1) and Remark (3.7), there must exist a function $q \in C(\mathbf{J}, \mathbf{R})$ such that $|q(\zeta)| \le \epsilon$

$$\begin{aligned}
u(\zeta) &= g\big(\zeta, u_1(\zeta), {}^{C}D^{\varsigma,\psi}_{c^+}u_2(\zeta), {}^{C}D^{\varsigma,\psi}_{c^+}u(\zeta)\big)\\
&\quad\times\left(F(u_1,u_2)(\zeta) - \frac{\psi(\zeta)-\psi(c)}{\psi(d)-\psi(\delta)}\big[F(u_1,u_2)(d) - F(u_1,u_2)(\delta)\big]\right),
\end{aligned}$$

where

$$F(u_1,u_2)(\zeta) = I^{\sigma+\varsigma,\psi}_{c^+}\big[F\big(\zeta, u_1(\zeta), {}^{C}D^{\varsigma,\psi}_{c^+}u_2(\zeta)\big) + g(\zeta)\big] + \mu I^{\varsigma,\psi}_{c^+}u(\zeta).$$

Let $u \in \mathbf{C}$ be a solution to the problem (1). Then it satisfies the integral equation. Using $(\mathbf{S}_2)$, we have

$$\begin{aligned}
&\left|F(u_1,u_2)(\zeta)-h(v_1,v_2)(\zeta)\right| \\
&\quad\le I_{c^+}^{\sigma+\varsigma;\psi}\left|F\left(\zeta,u_1(\zeta),{}^{C}D_{c^+}^{\varsigma,\psi}u_2(\zeta)\right)-F\left(\zeta,v_1(\zeta),{}^{C}D_{c^+}^{\varsigma,\psi}v_2(\zeta)\right)\right|+I_{c^+}^{\sigma+\varsigma,\psi}\left|g(\zeta)\right| \\
&\qquad+|\mu|I_{c^+}^{\varsigma,\psi}\left|u(\zeta)-v(\zeta)\right| \\
&\quad\le\frac{\|p\|_{(\psi(\zeta)-\psi(c))^{\sigma+\varsigma}}}{\Gamma(\sigma+\varsigma+1)}\|u-v\|+\frac{\epsilon(\psi(\zeta)-\psi(c))^{\sigma+\varsigma}}{\Gamma(\sigma+\varsigma+1)} \\
&\qquad+\frac{|\mu|(\psi(\zeta)-\psi(c))^{\sigma+\varsigma}}{\Gamma)(\sigma+\varsigma+1)}\|u-v\|,
\end{aligned}\tag{17}$$

and

$$\begin{aligned}
&\left|h(u_1,u_2)(\zeta)\right|\le\frac{k_f(\psi(\zeta)-\psi(c))^{\sigma+\varsigma}}{\Gamma(\sigma+\varsigma+1)}+\frac{|\mu|(\psi(\zeta)-\psi(c))^{\varsigma}}{\Gamma(\varsigma+1)} \\
&\left|F(u_1,u_2)(\zeta)\right|\le\frac{(k_f+\epsilon)(\psi(\zeta)-\psi(c))^{\sigma+\varsigma}}{\Gamma(\sigma+\varsigma+1)}+\frac{|\mu|(\psi(\zeta)-\psi(c))^{\varsigma}}{\Gamma(\varsigma+1)}.
\end{aligned}\tag{18}$$

Applying the triangle inequality, we obtain

$$\begin{aligned}
&\left|u(\zeta)-v(\zeta)\right| \\
&\quad\le\left|g\zeta,u_1(\zeta),{}^{C}D_{a^+}^{\varsigma+\psi}u_2(\zeta)\right|\left|F(u_1,u_2)(\zeta)-h(v_1,v_2)(\zeta)\right| \\
&\qquad+\left|h(v_1,v_2)(\zeta)\right|\left|g\left(\zeta,u_1(\zeta),{}^{C}D_{c^+}^{\varsigma,\psi}u_2(\zeta)\right)-g\left(\zeta,v_1(\zeta),{}^{C}D_{c^+}^{\varsigma,\psi}v_2(\zeta)\right)\right| \\
&\qquad+\left|\frac{\psi(\zeta)-\psi(c)}{\psi(d)-\psi(\delta)}\right|h(v_1,v_2)(b)-F(u_1,u_2)(b)\left|g\left(\zeta,v_1(\zeta),{}^{C}D_{c^+}^{\varsigma,\psi}v_2(\zeta)\right)\right. \\
&\qquad+\left|\frac{\psi(\zeta)-\psi(c)}{\psi(d)-\psi(\delta)}\right|g\left(\zeta,v_2,{}^{C}D_{c^+}^{\varsigma,\psi}v_2(\zeta)\right)-g\left(\zeta,u_1(\zeta),{}^{C}D_{c^+}^{\varsigma,\psi}u_2(\zeta)\right)\left|F(u_1,u_2)(b))\right| \\
&\qquad+\left|\frac{\psi(\zeta)-\psi(c)}{\psi(d)-\psi(\delta)}\right|F(u_1,u_2)(\delta)-h(v_1,v_2)(\delta)\left|g\left(\zeta,u_1(\zeta),{}^{C}D_{c^+}^{\varsigma,\psi}u_2(\zeta)\right)\right. \\
&\qquad+\left|\frac{\psi(\zeta)-\psi(c)}{\psi(d)-\psi(\delta)}\right|g\left(\zeta,u_1,{}^{C}D_{c^+}^{\varsigma,\psi}u_2(\zeta)\right)(\zeta) \\
&\qquad-g\left(\zeta,u_1(\zeta),{}^{C}D_{c^+}^{\varsigma,\psi}u_2(\zeta)\right)(\zeta)\left|h\left(\zeta,v_1(\zeta),{}^{C}D_{c^+}^{\varsigma,\psi}v_2(\zeta)\right)\right..
\end{aligned}$$

By $(\mathbf{S}_1)$, $(\mathbf{S}_2)$, (17), and (18), we have

$$\begin{aligned}
\left|u(\zeta)-v(\zeta)\right|\le{}&k_g\left(\frac{\epsilon(\psi(\zeta)-\psi(c))^{\sigma+\varsigma}}{\Gamma(\sigma+\varsigma+1)}+\frac{\|p\|(\psi(\zeta)-\psi(c))^{\sigma+\varsigma}}{\Gamma(\sigma+\varsigma+1)}\|u-v\|\right. \\
&\left.+\frac{|\mu|(\psi(\zeta)-\psi(c))^{\sigma+\varsigma}}{\Gamma(\sigma+\varsigma+1)}\|u-v\|\right) \\
&+\|\varphi\|\left(\frac{k_f(\psi(\zeta)-\psi(c))^{\sigma+\varsigma}}{\Gamma(\sigma+\varsigma+1)}+\frac{|\mu|(\psi(\zeta)-\psi(c))^{\varsigma}}{\Gamma(\varsigma+1)}\right)\|u-v\| \\
&+k_g\frac{\psi(\zeta)-\psi(c)}{\psi(d)-\psi(\delta)}\left(\frac{\epsilon(\psi(d)-\psi(c))^{\sigma+\varsigma}}{\Gamma(\sigma+\varsigma+1)}\right. \\
&\left.+\left(\frac{\|p_i\|(\psi(\zeta)-\psi(c))^{\sigma+\varsigma}}{\Gamma(\sigma+\varsigma+1)}+\frac{|\mu|(\psi(\zeta)-\psi(c))^{\varsigma}}{\Gamma(\varsigma+1)}\right)\|u-v\|\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \|\varphi\| \frac{\psi(\zeta)-\psi(c)}{\psi(d)-\psi(\delta)} \\
&\times \left(\frac{(k_f+\epsilon)(\psi(\zeta)-\psi(c))^{\sigma+\varsigma}}{\Gamma(\sigma+\varsigma+1)} + \frac{|\mu|(\psi(\zeta)-\psi(c))^{\varsigma}}{\Gamma(\varsigma+1)} \|u-v\| \right. \\
&+ k_g \frac{\psi(\zeta)-\psi(c)}{\psi(d)-\psi(\delta)} \left(\frac{\epsilon(\psi(d)-\psi(c))^{\sigma+\varsigma}}{\Gamma(\sigma+\varsigma+1)} \right. \\
&+ \left. \left(\frac{\|p\|(\psi(\zeta)-\psi(c))^{\sigma+\varsigma}}{\Gamma(\sigma+\varsigma+1)} + \frac{|\mu|(\psi(\zeta)-\psi(c))^{\varsigma}}{\Gamma(\varsigma+1)} \right) \|u-v\| \right) \\
&+ \|\varphi\| \frac{\psi(\zeta)-\psi(c)}{\psi(d)-\psi(\delta)} \\
&+ \left(\frac{k_f(\psi(\zeta)-\psi(c))^{\sigma+\varsigma}}{\Gamma(\sigma+\varsigma+1)} + \frac{|\mu|(\psi(\zeta)-\psi(c))^{\varsigma}}{\Gamma(\varsigma+1)} \right) \|u-v\|.
\end{aligned}
$$

Simplifying yields

$$\|u-v\| \le 2(\mathbf{K}_1+\mathbf{K}_2)\frac{\epsilon}{1-\epsilon}, \quad \epsilon < 1.$$

If we assume that $\varpi(\epsilon) = 2(\mathbf{K}_1+\mathbf{K}_2)(\frac{\epsilon}{1-\epsilon})$ and $\varpi(0) = 0$, then the generalized Ulam-Hyers stability condition is satisfied. The proof is now complete. □

Remark 3.8 The term $|F(u_1,u_2)(\zeta)|$ estimated in (18) is the reason why ϵ appears in the denominator. This suggests that applying the criteria of Theorem (5) does not guarantee that the problem (1) is Ulam-Hyers stable.

4 Application

To implement and validate the conclusions drawn in the preceding sections, this section provides an illustrative example with specific parameter settings. We construct a practical scenario with clearly defined parameters to demonstrate how the theoretical insights presented in this work can be applied in real-world situations.

4.1 Example

Consider the problem

$$
\begin{cases}
{}^{C}D_{1^+}^{0.4,\psi}\left[{}^{C}D_{1^+}^{1.25,\psi}\left(\frac{u(\zeta)}{2+0.02\zeta\cos u(\zeta)+0.02\zeta\cos {}^{C}D_{1^+}^{\varsigma+\psi}u(\zeta)}\right)-u(\zeta)\right] = \frac{e^t}{2+\frac{1}{7}u(t)+\frac{1}{7}{}^{C}D_{1^+}^{\varsigma,\psi}u(t)} \\
u(0)=0, \qquad {}^{C}D_{1^+}^{1.5,\psi}\left(\frac{u(\zeta)}{1+0.02\zeta\cos u(\zeta)+0.02\zeta\,{}^{C}D_{1^+}^{\varsigma+\psi}u(\zeta)}\right)\Big|_{\zeta=0}=0, \qquad u(e)=3u(2).
\end{cases}
\tag{19}
$$

Clearly the function $g(\zeta,u_1,u_2) = 0.05 + 0.02\zeta^2\cos u_1 + 0.02\zeta^2\cos u_2$ is continuous, and lipschitz such that $\|\varphi\| = 0.02e^2 \approx 0.014778$, $k_g = 0.05+0.002e^2+0.002e^2 \approx 0.0795562244$, and $\Upsilon(r) = \sqrt{r}+1$, is nondecreasing, $\mathbf{J} = [1,e]$.

$$\left|g(d,u_1,u_2)\big(\psi(d)-\psi(c)\big) - \xi g(\delta,u_,u_2)\big(\psi(\delta)-\psi(c)\big)\right| \ge \theta = 0.03745 > 0$$

$F_1 = 4.091133$

$$\frac{g_0 F_1}{1-\|\varphi\| F_1} = 0.2177195378 < 1$$

and

$$\|\varphi\|_{\mathcal{F}_1} = 0.0604587635 < 1. \tag{20}$$

The hypotheses in $\mathbb{Z}_1$–$\mathbb{Z}_3$ are met. Next, Theorem (3) guarantees that problem (19) has at least one nonzero mild solution.

The functions $f(\zeta, u_1, u_2) = \frac{e^{\zeta}}{2+\frac{1}{40}u_1(\zeta)+\frac{1}{40}{}^{C}D^{\varsigma,\psi}u_2(\zeta)}$ are Lipschitzian with common constants $\|p\| = 0.378856556$ and $k_f = 0.378856556$. The fractional derivative becomes the Hadamard derivative if $\psi(\zeta) = \ln\zeta$. In this case, we obtain:

$$\Xi = 0.5908822224.$$

This suggests that all of the hypotheses $\mathbb{S}_1$–$\mathbb{S}_3$ in Theorem (4) are satisfied since $\Xi < 1$. For problem (19), there is a unique nonzero mild solution. Additionally, we have:

$$\mathbf{K}_1 = 0.0324259456, \qquad \mathbf{K}_2 = 0.3490735522.$$

Therefore, we conclude that problem (19) is generalized Ulam-Hyers stable based on Theorem (5).

5 Conclusion

In this paper, we considered solutions' existence, uniqueness, and Ulam-Hyers stability for a new class of hybrid Langevin fractional differential systems subject to three-point boundary conditions in view of the ψ-Caputo derivatives. The problem was solved using Schauder's and Banach's fixed-point theorems. Additionally, we provided an illustrative scenario to support our theoretical findings, which were novel and extended the scope of numerous previous studies in this area. As research in this field continued to evolve, we advocated for further exploration using generalized fractional derivatives and qualitative analysis of comparable systems. Future studies could also explore different fractional models, including multipoint boundary conditions and a range of fractional derivatives.

Acknowledgements
The authors A. Aloqaily and N. Mlaiki would like to thank Prince Sultan University for paying the APC and for the support through the TAS research lab.

Author contributions
G.A, R.A.K and K. wrote the main manuscript text, A.A and N.M verified the results. All authors reviewed the manuscript.

Funding
This research is supported by Prince Sultan University, Saudi Arabia.

Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
There does not exist any ethical issue regarding this work.

Competing interests
The authors declare no competing interests.

Author details
[1]Department of Mathematics, Islamia College Peshawar, Peshawar, Khyber Pakhtoon Khwa, Pakistan. [2]Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan. [3]Department of Mathematics and Sciences, Prince Sultan University, Riyadh, Saudi Arabia.

Received: 5 March 2024 Accepted: 3 May 2024 Published online: 17 May 2024

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