El-Gaber *Boundary Value Problems* (2024) 2024:67
https://doi.org/10.1186/s13661-024-01873-z

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

Open Access

Oscillatory criteria of noncanonical even-order differential equations with a superlinear neutral term

A.A. El-Gaber[1*]

*Correspondence: amina.aboal-nour@science.menofia.edu.eg
[1]Department of Mathematics and Computer Science, Faculty of Science, Menoufia University, Shebin El-Koom, Egypt

Abstract

The oscillatory behavior of solutions of an even-order differential equation with a superlinear neutral term is considered using Riccati and generalized Riccati transformations, the integral averaging technique, and the theory of comparison. New sufficient conditions are established in the noncanonical case. An example is given to support our results.

Mathematics Subject Classification: 34C10; 34K11

Keywords: Oscillation; Even-Order; Neutral differential equations

1 Introduction

In this article, we are concerned with the oscillation property of solutions of the half-linear even-order differential equation with a superlinear neutral term of the form

$$\left(r(t)\left(w^{(n-1)}(t)\right)^{\gamma}\right)' + \sum_{i=1}^{m} q_i(t)x^{\delta}\left(\eta_i(t)\right) = 0, \quad t \geq t_0 \geq 0, \tag{1.1}$$

where $w(t) = x(t) + p(t)x^{\beta}(\xi(t)), \beta, \gamma$, and δ are quotients of odd positive integers with $\beta \geq 1$, $\delta/\beta \leq \gamma$, and $n \geq 4$ is an even integer under the noncanonical condition

$$E(t_0) = \int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\gamma}}(t)}\, dt < \infty. \tag{1.2}$$

Throughout the paper, we assume that

(H_1) $r(t) \in C^1([t_0,\infty),(0,\infty))$, $r'(t) \geq 0$, $p(t) \in C([t_0,\infty),R)$, $p(t) \geq 1$, $p(t) \not\equiv 1$ for large t;

(H_2) $\xi(t), q_i(t) \in C([t_0,\infty),R)$, $\xi(t) \leq t$, ξ is strictly increasing, $\lim_{t\to\infty} \xi(t) = \infty$, $q_i(t) \geq 0$, and $q_i(t)$ are not equal to zero for large t, $i = 1,2,\ldots,m$;

(H_3) $\eta_i(t) \in C([t_0,\infty),R)$, there exists a function $\eta(t) \in C([t_0,\infty),R)$ such that $\eta(t) \leq \eta_i(t)$ for $i = 1,2,\ldots,m$, $\eta(t) < \xi(t)$ and $\lim_{t\to\infty} \eta(t) = \infty$;

(H_4) $h(t) = \xi^{-1}(\eta(t))$, where ξ^{-1} is the inverse function of ξ.

By a solution of (1.1), we mean a nontrivial function $x(t) \in C([t_x,\infty))$, $t_x \geq t_0$, which has the property $r(t)(w^{(n-1)}(t))^{\gamma} \in C^1([t_x,\infty))$ and satisfies (1.1). We consider only those solutions $x(t)$ of (1.1), which satisfy $\sup\{|x(t)| : t \geq t^*\} > 0$ for $t^* \geq t_x$. A solution $x(t)$ of (1.1) is termed oscillatory if it has arbitrarily large zeros on $[t_x,\infty)$; otherwise, it is said to be nonoscillatory. Equation (1.1) is termed oscillatory if all its solutions oscillate.

It is notable that in recent years, the oscillation property of solutions of differential equations and their applications have been and still are receiving intensive attention (see [2, 5–7, 15, 18–21]). In the natural sciences, technology, and population dynamics, differential equations find many application fields [1, 4, 8, 9, 13]. For particular applications of differential equations with neutral term they are often used for the study of distributed networks containing lossless transmission lines [10]. Moreover, for particular applications in superlinear wave equation [11]. Here, we mention some recent works concerned with special cases of (1.1), which motivated this work. Many authors have studied the oscillatory behavior of solutions of the differential equations

$$\left(\left(y^{(n-1)}(t)\right)^{\alpha}\right)' + q(t)y^{B}\left(g(t)\right) = 0, \tag{1.3}$$

where it is notable that some of their results can be extended to the following equations

$$\left(r(t)\left(y^{(n-1)}(t)\right)^{\alpha}\right)' + q(t)y^{B}\left(g(t)\right) = 0. \tag{1.4}$$

In [24], Zhang et al. studied (1.4) in a noncanoncail case as (1.2). They established new oscillation criteria claiming that it could not be applied in case $g(t) = t$. More recently, Zhang et al. [22] suggested some new oscillation criteria for the even-order delay differential equation (1.4) with the same noncanonical case for which they stressed that the study of oscillatory properties in this case brings in additional difficulties. Meanwhile, Li and Rogovchenko [14] discussed the oscillatory behavior of a class of even-order neutral differential equations of the form

$$w^{(n)}(t) + h(t)y\left(\xi(t)\right) = 0, \tag{1.5}$$

with $w(t) = y(t) + p(t)y(g(t))$. Their new theorems complement and improve a number of results reported in the literature.

In [7], Elabbasy et al. studied the even-order neutral differential equation with several delays

$$\left(r(t)\left(v^{(n-1)}\right)^{\alpha}\right)' + \sum_{i=1}^{k} q_i(t)f\left(u\left(\eta_i(t)\right)\right) = 0, \tag{1.6}$$

where $v(t) = u(t) + p(t)u(g(t))$, with $r'(t) \geq 0$, $p(t) \in [0,p_0]$, $\eta_i'(t) > 0$ in the canonical case $\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)}\,dt = \infty$. They used the Riccati substitution technique and comparison with delay equations of the first order to establish new oscillation criteria, which simplify and complement some related results in the literature.

In [9], Grace et al. studied the oscillation of the higher-order dynamic equation with superlinear neutral term

$$\left(a(\mu)y^{\Delta^{n-1}}(\mu)\right)^{\Delta} + q(\mu)x^{\beta}\left(\tau(\mu)\right) = 0, \tag{1.7}$$

where $y(t) = x(t) + p(t)x^{\alpha}(g(t))$, $a^{\Delta}(\mu) \geq 0$, $\beta \leq \alpha$, and $\alpha \geq 1$. Their proposed results provide a unified platform that adequately covers discrete and continuous equations and further sufficiently comments on the oscillation of a more general class of equations than the ones reported in the literature.

More recently, Dharuman et al. [3], were concerned with the oscillatory behavior of solutions of the even-order nonlinear differential equation with a superlinear neutral term

$$\left(b(t)z^{(n-1)}(t)\right)' + q(t)x^{\beta}\left(\sigma(t)\right) = 0, \tag{1.8}$$

where $z(t) = x(t) + p(t)x^{\alpha}(\tau(t))$, $\alpha > 1$, $b'(t) \geq 0$, in the canonical case $\int_{t_0}^{\infty} \frac{1}{b(t)}\,dt = \infty$. They established new comparison theorems that compare the higher-order equation (1.8) with a couple of first-order delay differential equations. Moreover, as many results are available in the literature on the oscillation of first-order delay differential equations, it would be possible to formulate many criteria for the oscillation of (1.8) based on their results.

In this article, we study the oscillatory behavior of solutions of Eq. (1.1) in the noncanonical case (1.2) by applying the Riccati and generalized Riccati transformations, the integral averaging technique, and comparison theory.

2 Auxiliary lemmas

In this section, we outline some lemmas needed for our results.

Lemma 1 [1] *Let $w(t) \in C^n([t_0,\infty),(0,\infty))$, $w^{(n)}(t)$ be of fixed sign, and $w^{(n-1)}(t)w^{(n)}(t) \leq 0$, for all $t \geq t_0$. If $\lim_{t\to\infty} w(t) \neq 0$, then for every $\epsilon \in (0,1)$ there may exist $t_\epsilon \geq t_0$ such that $w(t) \geq \frac{\epsilon}{(n-1)!}t^{n-1}|w^{(n-1)}(t)|$ for $t \geq t_\epsilon$.*

Lemma 2 [12] *Let the function $w(t)$ satisfy $w^{(i)}(t) > 0$, $i = 1,2,\dots,n-1$ and $w^{(n)}(t) \leq 0$, then*

$$\frac{w(t)}{t^{n-1}/(n-1)!} \geq \frac{w'(t)}{t^{n-2}/(n-2)!}.$$

Lemma 3 [22] *Let $w \in C^I([t_0,\infty),R^+)$. If $w^{(I)}(t)$ is eventually of one sign for all large t, then there exists $t_1 \geq t_0$ and an integer i, $0 \leq i \leq I$ with $I + i$ even for $w^{(I)}(t) \geq 0$, or $I + i$ odd for $w^{(I)}(t) \leq 0$ such that*

$$i > 0 \text{ yields} \quad w^{(j)}(t) > 0 \quad \text{for } t \geq t_1, j = 0,1,\dots,i-1 \quad \text{and}$$
$$i \leq I-1 \text{ yields} \quad (-1)^{i+j}w^{(j)}(\tau) > 0 \quad \text{for } t \geq t_1, j = i,i+1,\dots,I-1.$$

Lemma 4 [16] *Let $f(v) = Av - B(v-D)^{\frac{\gamma+1}{\gamma}}$, where $B > 0$, A, and D are constants. Then, the maximum value of f on R at $v^* = D + (\gamma A/(\gamma+1)B)^{\gamma}$ is*

$$\max_{v\in R} f(v) = f\left(v^*\right) = AD + \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{A^{\gamma+1}}{B^{\gamma}}.$$

Lemma 5 *Assume that $x(t)$ is an eventually positive solution of (1.1). Then, there exists $t_1 \geq t_0$ such that for $t \geq t_1$ the corresponding function w satisfies one of the following four cases*

$$c_1)\ w > 0, w' > 0, w^{(n-2)} > 0, w^{(n-1)} > 0, w^{(n)} \leq 0, \quad \left(r\left(w^{(n-1)}\right)^{\gamma}\right)' \leq 0,$$

c_2) $w > 0, w' > 0, w^{(n-2)} < 0, w^{(n-1)} > 0, w^{(n)} \leq 0, \quad (r(w^{(n-1)})^{\gamma})' \leq 0,$

c_3) $w > 0, w' > 0, w^{(n-2)} > 0, w^{(n-1)} < 0, \quad (r(w^{(n-1)})^{\gamma})' \leq 0,$

c_4) $w > 0, w^{(i)} < 0, w^{(i+1)} > 0$ *for every odd integer* $i \in \{1, 2, \dots, n-3\}$ *and*

$w^{(n-1)} < 0, \quad (r(w^{(n-1)})^{\gamma})' \leq 0.$

Proof Assume that $x(t)$ is an eventually positive solution of (1.1), then for any $t_1 \geq t_0$ such that $x(t) > 0$, $x(\xi(t)) > 0$, $x(\eta_i(t)) > 0$, $i = 1, 2, \dots, m$, for all $t \geq t_1$. Hence, using the definition of $w(t)$, we have $w(t) > 0$. It follows from (1.1) that $(r(w^{(n-1)})^{\gamma})' \leq 0$. Next following the proof of Theorem 2.1 of [22] with Lemma 3, we have three cases

1) $w > 0, w' > 0, w^{(n-1)} > 0, w^{(n)} \leq 0, \quad (r(w^{(n-1)})^{\gamma})' \leq 0,$

2) $w > 0, w' > 0, w^{(n-2)} > 0, w^{(n-1)} < 0, \quad (r(w^{(n-1)})^{\gamma})' \leq 0,$

3) $w > 0, w^{(i)} < 0, w^{(i+1)} > 0$ for every odd integer $i \in \{1, 2, \dots, n-3\}$ and

$w^{(n-1)} < 0, \quad (r(w^{(n-1)})^{\gamma})' \leq 0.$

Now going through as in [23], case 1 $w^{(n-2)}$ has two possibilities either $w^{(n-2)} > 0$ or $w^{(n-2)} < 0$. This completes the proof. □

3 Main results

Lemma 6 *Assume that* $x(t)$ *is an eventually positive solution of* (1.1) *with* w *satisfying case* 1 *of Lemma* 5, *then for all constants* $k_1 > 0$,

$$\left(r(t)\left(w^{(n-1)}(t)\right)^{\gamma}\right)' \leq -\sum_{i=1}^{m} q_i(t) P_1^{\frac{\delta}{\beta}}\left(\eta_i(t)\right) w^{\frac{\delta}{\beta}}\left(h(t)\right), \tag{3.1}$$

where $P_1(t) = \frac{1}{p(\xi^{-1}(t))}\left[1 - \left[\frac{\xi^{-1}(t)}{\xi^{-1}(\xi^{-1}(t))}\right]^{\frac{1-n}{\beta}} \frac{k_1^{\frac{1}{\beta}-1}}{p^{\frac{1}{\beta}}(\xi^{-1}(\xi^{-1}(t)))}\right] \geq 0.$

Proof Since $x(t)$ is an eventually positive solution of (1.1) such that $x(t) > 0$, $x(\xi(t)) > 0$, $x(\eta_i(t)) > 0$, $i = 1, 2, \dots, m$, for $t \geq t_1 \geq t_0$. From the definition of $w(t)$

$$x^{\beta}\left(\xi(t)\right) = \frac{1}{p(t)}\left[w(t) - x(t)\right] \leq \frac{w(t)}{p(t)},$$

which implies that

$$x\left(\xi^{-1}(t)\right) \leq \frac{w^{\frac{1}{\beta}}(\xi^{-1}(\xi^{-1}(t)))}{p^{\frac{1}{\beta}}(\xi^{-1}(\xi^{-1}(t)))}.$$

Using the above inequality in the definition of $w(t)$, we get

$$\begin{aligned} x^{\beta}(t) &= \frac{1}{p(\xi^{-1}(t))}\left[w\left(\xi^{-1}(t)\right) - x\left(\xi^{-1}(t)\right)\right] \\ &\geq \frac{1}{p(\xi^{-1}(t))}\left[w\left(\xi^{-1}(t)\right) - \frac{w^{\frac{1}{\beta}}(\xi^{-1}(\xi^{-1}(t)))}{p^{\frac{1}{\beta}}(\xi^{-1}(\xi^{-1}(t)))}\right]. \end{aligned} \tag{3.2}$$

Since $\xi(t) \le t$, $\xi(t)$ is strictly increasing and $t \le \xi^{-1}(t)$, then

$$\xi^{-1}(t) \le \xi^{-1}\bigl(\xi^{-1}(t)\bigr). \tag{3.3}$$

Since $w(t)$ satisfies case 1 of Lemma 5, so from Lemma 2, we obtain

$$\frac{w(t)}{t^{n-1}/(n-1)!} \ge \frac{w'(t)}{t^{n-2}/(n-2)!},$$

i.e.,

$$w(t) \ge \frac{tw'(t)}{n-1},$$

which implies

$$\bigl(t^{-n+1}w(t)\bigr)' = t^{-n+1}w'(t) + (1-n)t^{-n}w(t) = t^{-n}\bigl[tw'(t) + (1-n)w(t)\bigr] \le 0.$$

The function $t^{-n+1}w(t)$ is nonincreasing, which with (3.3) leads to

$$\bigl[\xi^{-1}(t)\bigr]^{1-n} w\bigl(\xi^{-1}(t)\bigr) \ge \bigl[\xi^{-1}\bigl(\xi^{-1}(t)\bigr)\bigr]^{1-n} w\bigl(\xi^{-1}\bigl(\xi^{-1}(t)\bigr)\bigr).$$

Thus, using (3.2), we obtain

$$x^{\beta}(t) \ge \frac{w(\xi^{-1}(t))}{p(\xi^{-1}(t))}\left[1 - \left[\frac{\xi^{-1}(t)}{\xi^{-1}(\xi^{-1}(t))}\right]^{\frac{1-n}{\beta}} \frac{w^{\frac{1}{\beta}-1}(\xi^{-1}(t))}{p^{\frac{1}{\beta}}(\xi^{-1}(\xi^{-1}(t)))}\right].$$

Hence since $w(t)$ is positive and increasing, there exists a positive constant k_1 such that $w(t) \ge k_1$. Thus, we have

$$x^{\beta}(t) \ge P_1(t)w\bigl(\xi^{-1}(t)\bigr).$$

Substituting into (1.1), we have

$$\bigl(r\bigl(w^{(n-1)}(t)\bigr)^{\gamma}\bigr)' \le -\sum_{i=1}^{m} q_i(t)P_1^{\frac{\delta}{\beta}}\bigl(\eta_i(t)\bigr)w^{\frac{\delta}{\beta}}\bigl(\xi^{-1}\bigl(\eta_i(t)\bigr)\bigr),$$

and since $\xi^{-1}(t)$, $w(t)$ are increasing functions, then we have

$$\bigl(r(t)\bigl(w^{(n-1)}(t)\bigr)^{\gamma}\bigr)' \le -\sum_{i=1}^{m} q_i(t)P_1^{\frac{\delta}{\beta}}\bigl(\eta_i(t)\bigr)w^{\frac{\delta}{\beta}}\bigl(h(t)\bigr).$$

□

Lemma 7 *Assume that $x(t)$ is an eventually positive solution of (1.1) with w satisfying case 2 of Lemma 5, then for all constants $k_2 > 0$,*

$$\bigl(r(t)\bigl(w^{(n-1)}(t)\bigr)^{\gamma}\bigr)' \le -\sum_{i=1}^{m} q_i(t)P_2^{\frac{\delta}{\beta}}\bigl(\eta_i(t)\bigr)w^{\frac{\delta}{\beta}}\bigl(h(t)\bigr), \tag{3.4}$$

where $P_2(t) = \frac{1}{p(\xi^{-1}(t))}\bigl[1 - \bigl[\frac{\xi^{-1}(t)}{\xi^{-1}(\xi^{-1}(t))}\bigr]^{\frac{3-n}{\beta}} \frac{k_2^{\frac{1}{\beta}-1}}{p^{\frac{1}{\beta}}(\xi^{-1}(\xi^{-1}(t)))}\bigr] \ge 0.$

Proof Since $x(t)$ is an eventually positive solution of (1.1) such that $x(t) > 0$, $x(\xi(t)) > 0$, $x(\eta_i(t)) > 0$, $i = 1, 2, \dots, m$, for $t \geq t_1 \geq t_0$. Suppose that $w(t)$ satisfies case 2 of Lemma 5, then by Lemma 2, we obtain

$$\frac{w(t)}{t^{n-3}/(n-3)!} \geq \frac{w'(t)}{t^{n-4}/(n-4)!},$$

i.e., $w(t) \geq \frac{tw'(t)}{n-3}$, which implies that

$$\left(t^{-n+3}w(t)\right)' = t^{-n+3}w'(t) + (3-n)t^{-n+2}w(t) = t^{-n+2}\left[tw'(t) + (3-n)w(t)\right] \leq 0,$$

and hence the function $t^{3-n}w(t)$ is nonincreasing and by (3.3)

$$\left[\xi^{-1}(t)\right]^{3-n} w\left(\xi^{-1}(t)\right) \geq \left[\xi^{-1}\left(\xi^{-1}(t)\right)\right]^{3-n} w\left(\xi^{-1}\left(\xi^{-1}(t)\right)\right).$$

Going through as in the proof of Lemma 6, we obtain (3.4). □

Lemma 8 *Suppose that $x(t)$ is an eventually positive solution of* (1.1) *with w satisfying case* 3 *of Lemma* 5, *then for all constants $k_3 > 0$,*

$$\left(r(t)\left(w^{(n-1)}(t)\right)^{\gamma}\right)' \leq -\sum_{i=1}^{m} q_i(t) P_3^{\frac{\delta}{\beta}}\left(\eta_i(t)\right) w^{\frac{\delta}{\beta}}\left(h(t)\right), \tag{3.5}$$

where $P_3(t) = \frac{1}{p(\xi^{-1}(t))}\left[1 - \left[\frac{\xi^{-1}(t)}{\xi^{-1}(\xi^{-1}(t))}\right]^{\frac{2-n}{\beta}} \frac{k_3^{\frac{1}{\beta}-1}}{p^{\frac{1}{\beta}}(\xi^{-1}(\xi^{-1}(t)))}\right] \geq 0.$

Proof Since $x(t)$ is an eventually positive solution of (1.1) such that $x(t) > 0$, $x(\xi(t)) > 0$, $x(\eta_i(t)) > 0$, $i = 1, 2, \dots, m$, for $t \geq t_1 \geq t_0$. Suppose that $w(t)$ satisfies case 3 of Lemma 5, then by Lemma 2, we obtain

$$\frac{w(t)}{t^{n-2}/(n-2)!} \geq \frac{w'(t)}{t^{n-3}/(n-3)!},$$

i.e., $w(t) \geq \frac{t}{n-2} w'(t)$, which implies that

$$\left(t^{-n+2}w(t)\right)' = t^{-n+2}w'(t) + (2-n)t^{-n+1}w(t) = t^{-n+1}\left[tw'(t) + (2-n)w(t)\right] \leq 0,$$

and hence the function $t^{2-n}w(t)$ is nonincreasing and by (3.3)

$$\left[\xi^{-1}(t)\right]^{2-n} w\left(\xi^{-1}(t)\right) \geq \left[\xi^{-1}\left(\xi^{-1}(t)\right)\right]^{2-n} w\left(\xi^{-1}\left(\xi^{-1}(t)\right)\right).$$

Going through as in the proof of Lemma 6, we obtain (3.5). □

Theorem 9 *Assume that there exist $\upsilon(\tau) \in C^1([\tau_0, \infty), (0, \infty))$, $c_1 > 0$, $k \in (0, 1)$ such that*

$$\int_{t_0}^{\infty} \left[\upsilon(s)\left[\frac{h(s)}{s}\right]^{(n-1)\delta/\beta} \sum_{i=1}^{m} q_i(s) P_1^{\frac{\delta}{\beta}}\left(\eta_i(s)\right) - \frac{L[\upsilon'(s)]^2 r(s)}{s^{(n-1)\delta/\beta-1}\upsilon(s)}\right] ds = \infty, \tag{3.6}$$

$L = [\frac{(n-1)!}{k}]^{\delta/\beta} \frac{\beta}{4\delta(n-1)c_1^{\delta/\beta-\gamma}}$. *Assume further that the equation*

$$z'(t) + \sum_{i=1}^{m} q_i(t) P_2^{\frac{\delta}{\beta}}\big(\eta_i(t)\big) \left[\frac{k h^{n-1}(t)}{(n-1)![r(h(t))]^{1/\gamma}}\right]^{\frac{\delta}{\beta}} z^{\delta/\beta\gamma}\big(h(t)\big) = 0 \tag{3.7}$$

is oscillatory if

$$\int_{t_0}^{\infty} \left[E^{\gamma}(s) c_2^{\frac{\delta}{\beta}-\gamma} \left[\frac{N h^{n-2}(s)}{(n-2)!}\right]^{\delta/\beta} \sum_{i=1}^{m} q_i(s) P_3^{\frac{\delta}{\beta}}\big(\eta_i(s)\big) - \frac{\gamma^{\gamma+1}}{(\gamma+1)^{\gamma+1} E(s) r^{1/\gamma}(s)}\right] ds$$
$$= \infty, \tag{3.8}$$

for some constants $c_2 > 0, N \in (0,1)$, *and*

$$\int_{t}^{\infty} E(s)\, ds = \infty. \tag{3.9}$$

Then, every solution of (1.1) *is oscillatory.*

Proof Without loss of generality, we may assume that $x(t)$ is an eventually positive solution of (1.1) such that $x(t) > 0$, $x(\xi(t)) > 0$, $x(\eta_i(t)) > 0$, $i = 1, 2, \dots, m$, for $t \geq t_1 \geq t_0$. Then, as in Lemma 5, there exist four possible cases. Suppose that $w(t)$ satisfies case 1, then, as in Lemma 6, the function $t^{1-n} w(t)$ is nonincreasing and since $h(t) < t$, we have

$$\big[h(t)\big]^{1-n} w\big(h(t)\big) \geq t^{1-n} w(t). \tag{3.10}$$

Define

$$\omega(t) = \upsilon(t) \frac{r(t)(w^{(n-1)}(t))^{\gamma}}{w^{\delta/\beta}(t)},$$

then $\omega(t) > 0$, and

$$\omega'(t) = \frac{\upsilon'(t)}{\upsilon(t)} \omega(t) + \upsilon(t) \frac{[r(t)(w^{(n-1)}(t))^{\gamma}]'}{w^{\delta/\beta}(t)} - \upsilon(t) \frac{\delta/\beta r(t)(w^{(n-1)}(t))^{\gamma} w'(t) w^{\delta/\beta-1}(t)}{w^{2\delta/\beta}(t)}.$$

Using (3.1), (3.10), we have

$$\begin{aligned}\omega'(t) \leq{}& \frac{\upsilon'(t)}{\upsilon(t)} \omega(t) - \upsilon(t) \left[\frac{h(t)}{t}\right]^{(n-1)\delta/\beta} \sum_{i=1}^{m} q_i(t) P_1^{\frac{\delta}{\beta}}\big(\eta_i(t)\big) \\ &- \upsilon(t) \frac{\delta/\beta r(t)(w^{(n-1)}(t))^{\gamma} w'(t) w^{\delta/\beta-1}(t)}{w^{2\delta/\beta}(t)}.\end{aligned}$$

Now from Lemma 1, we have

$$w(t) \geq \frac{k}{(n-1)!} t^{n-1} w^{(n-1)}(t), \quad k \in (0,1), \tag{3.11}$$

and $w'(t) \geq \frac{k}{(n-2)!} t^{n-2} w^{(n-1)}(t)$, then we have

$$\omega'(t) \leq \frac{\upsilon'(t)}{\upsilon(t)}\omega(t) - \upsilon(t)\left[\frac{h(t)}{t}\right]^{(n-1)\delta/\beta} \sum_{i=1}^{m} q_i(t) P_1^{\frac{\delta}{\beta}}\left(\eta_i(t)\right) - \left[\frac{k}{(n-1)!}\right]^{\delta/\beta} \frac{\delta/\beta(n-1) r(t)\upsilon(t) t^{(n-1)\delta/\beta-1}(w^{(n-1)}(t))^{2\gamma}(w^{(n-1)}(t))^{\delta/\beta-\gamma}}{w^{2\delta/\beta}(t)}.$$

Since $w^{(n-1)}(t)$ is positive and nonincreasing function, there exists a positive constant c_1 such that $w^{(n-1)}(t) \leq c_1$. Then,

$$\omega'(t) \leq \frac{\upsilon'(t)}{\upsilon(t)}\omega(t) - \upsilon(t)\left[\frac{h(t)}{t}\right]^{(n-1)\delta/\beta} \sum_{i=1}^{m} q_i(t) P_1^{\frac{\delta}{\beta}}\left(\eta_i(t)\right) - \left[\frac{k}{(n-1)!}\right]^{\delta/\beta} \frac{(n-1)\delta/\beta c_1^{\delta/\beta-\gamma} t^{(n-1)\delta/\beta-1}\omega^2(t)}{\upsilon(t) r(t)}.$$

By completing the squares

$$\omega'(t) \leq -\upsilon(t)\left[\frac{h(t)}{t}\right]^{(n-1)\delta/\beta} \sum_{i=1}^{m} q_i(t) P_1^{\frac{\delta}{\beta}}\left(\eta_i(t)\right) + L\frac{[\upsilon'(t)]^2 r(t)}{t^{(n-1)\delta/\beta-1}\upsilon(t)}.$$

Integrating from t_2 to t, we get

$$0 < \omega(t) \leq \omega(t_2) - \int_{t_2}^{t}\left(\upsilon(s)\left[\frac{h(s)}{s}\right]^{(n-1)\delta/\beta} \sum_{i=1}^{m} q_i(s) P_1^{\frac{\delta}{\beta}}\left(\eta_i(s)\right) - L\frac{[\upsilon'(s)]^2 r(s)}{s^{(n-1)\delta/\beta-1}\upsilon(s)}\right) ds.$$

This is a contradiction with (3.6). Assume that we have case 2, using Lemma 1, we find (3.11). Thus, using (3.4), we have

$$\left(r(t)\left(w^{(n-1)}(t)\right)^{\gamma}\right)' + \left[\frac{k}{(n-1)!} h^{n-1}(t) w^{(n-1)}\left(h(t)\right)\right]^{\frac{\delta}{\beta}} \sum_{i=1}^{m} q_i(t) P_2^{\frac{\delta}{\beta}}\left(\eta_i(t)\right) \leq 0,$$

then

$$\left(r(t)\left(w^{(n-1)}(t)\right)^{\gamma}\right)' + \left[\frac{k}{(n-1)![r(h(t))]^{\frac{1}{\gamma}}} h^{n-1}(t)\right]^{\frac{\delta}{\beta}} \sum_{i=1}^{m} q_i(t) P_2^{\frac{\delta}{\beta}}\left(\eta_i(t)\right)\left[\left[r\left(h(t)\right)\right]^{\frac{1}{\gamma}} w^{(n-1)}\left(h(t)\right)\right]^{\frac{\delta}{\beta}} \leq 0,$$

we see that $z(t) = r(t)(w^{(n-1)}(t))^{\gamma}$ is a positive solution of the differential inequality

$$z'(t) + \sum_{i=1}^{m} q_i(t) P_2^{\frac{\delta}{\beta}}\left(\eta_i(t)\right)\left[\frac{k h^{n-1}(t)}{(n-1)![r(h(t))]^{1/\gamma}}\right]^{\frac{\delta}{\beta}} z^{\delta/\beta\gamma}\left(h(t)\right) \leq 0,$$

using [[17], Corollary 1], we see that (3.7) has a positive solution, and this is a contradiction. Assume that we have case 3, then, as in Lemma 8, we have that $t^{2-n}w(t)$ is nonincreasing, and since $h(t) < t$, we have $[h(t)]^{2-n}w(h(t)) > t^{2-n}w(t)$. Now, we define

$$\phi(t) = \frac{r(t)[w^{(n-1)}(t)]^{\gamma}}{[w^{(n-2)}(t)]^{\gamma}},$$

then $\phi(t) < 0$, and by using (3.5),

$$\phi'(t) \leq -\frac{\sum_{i=1}^{m} q_i(t)P_3^{\frac{\delta}{\beta}}(\eta_i(t))w^{\frac{\delta}{\beta}}(h(t))}{[w^{(n-2)}(t)]^{\gamma}} - \gamma\frac{\phi^{\frac{\gamma+1}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(t)}.$$

On the other hand, using Lemma 1, we get

$$w(t) \geq \frac{N}{(n-2)!}t^{n-2}w^{(n-2)}(t) \tag{3.12}$$

for every $N \in (0,1)$, and all sufficiently large t, then

$$\phi'(t) \leq -\left[\frac{N}{(n-2)!}h^{n-2}(t)\right]^{\frac{\delta}{\beta}} \frac{\sum_{i=1}^{m} q_i(t)P_3^{\frac{\delta}{\beta}}(\eta_i(t))[w^{(n-2)}(h(t))]^{\frac{\delta}{\beta}}}{[w^{(n-2)}(t)]^{\gamma}} - \gamma\frac{\phi^{\frac{\gamma+1}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(t)},$$

since $w^{(n-2)}$ is positive and decreasing then $w^{(n-2)}(h(t)) \geq w^{(n-2)}(t)$. There exists a positive constant c_2 such that $w^{(n-2)}(t) \leq c_2$, then we have

$$\phi'(t) \leq -c_2^{\frac{\delta}{\beta}-\gamma}\left[\frac{N}{(n-2)!}h^{n-2}(t)\right]^{\frac{\delta}{\beta}} \sum_{i=1}^{m} q_i(t)P_3^{\frac{\delta}{\beta}}\left(\eta_i(t)\right) - \gamma\frac{\phi^{\frac{\gamma+1}{\gamma}}(t)}{r^{\frac{1}{\gamma}}(t)}. \tag{3.13}$$

Multiplying (3.13) by $E^{\gamma}(t)$ and integrating it from t_1 to t, we have

$$\begin{aligned} &E^{\gamma}(t)\phi(t) - E^{\gamma}(t_1)\phi(t_1) - \gamma\int_{t_1}^{t} E^{\gamma-1}(s)\phi(s)E'(s)\,ds \\ &\quad + \int_{t_1}^{t} E^{\gamma}(s)c_2^{\frac{\delta}{\beta}-\gamma}\left[\frac{N}{(n-2)!}h^{n-2}(s)\right]^{\frac{\delta}{\beta}} \sum_{i=1}^{m} q_i(s)P_3^{\frac{\delta}{\beta}}\left(\eta_i(s)\right)ds + \gamma\int_{t_1}^{t} E^{\gamma}(s)\frac{\phi^{\frac{\gamma+1}{\gamma}}(s)}{r^{\frac{1}{\gamma}}(s)}\,ds \\ &\quad \leq 0. \end{aligned}$$

Now using the inequality

$$Av^{\frac{\gamma+1}{\gamma}} - Bv \geq -\frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}}{A^{\gamma}}, \quad A, B > 0$$

with

$$A = \frac{E^{\gamma}(s)}{r^{\frac{1}{\gamma}}(s)}, \qquad B = r^{-\frac{1}{\gamma}}(s)E^{\gamma-1}(s) \quad \text{and} \quad v = -\phi(s),$$

we have

$$\begin{aligned} &\int_{t_1}^{t} E^{\gamma}(s)c_2^{\frac{\delta}{\beta}-\gamma}\left[\frac{N}{(n-2)!}h^{n-2}(s)\right]^{\frac{\delta}{\beta}} \sum_{i=1}^{m} q_i(s)P_3^{\frac{\delta}{\beta}}\left(\eta_i(s)\right) - \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}E(s)r^{\frac{1}{\gamma}}(s)}\,ds \\ &\quad \leq -E^{\gamma}(t)\phi(t) + E^{\gamma}(t_1)\phi(t_1). \end{aligned} \tag{3.14}$$

Noting that $r[w^{(n-1)}]^{\gamma}$ is nonincreasing, we have

$$r^{\frac{1}{\gamma}}(s)w^{(n-1)}(s) \leq r^{\frac{1}{\gamma}}(t)w^{(n-1)}(t), \quad s \geq t.$$

Dividing by $r^{\frac{1}{\gamma}}(s)$ and integrating from t to $\varkappa$ gives

$$w^{(n-2)}(\varkappa) \leq w^{(n-2)}(t) + r^{\frac{1}{\gamma}}(t)w^{(n-1)}(t)\int_t^{\varkappa} \frac{1}{r^{\frac{1}{\gamma}}(s)}\,ds.$$

Letting $\varkappa \to \infty$, we have

$$w^{(n-2)}(t) \geq -E(t)r^{\frac{1}{\gamma}}(t)w^{(n-1)}(t), \tag{3.15}$$

then

$$-\frac{E(t)r^{\frac{1}{\gamma}}(t)w^{(n-1)}(t)}{w^{(n-2)}(t)} \leq 1,$$

i.e.,

$$-E(t)\phi^{\frac{1}{\gamma}}(t) \leq 1.$$

By substituting into (3.14), we have

$$\begin{aligned}&\int_{t_1}^{t} E^{\gamma}(s)c_2^{\frac{\delta}{\beta}-\gamma}\left[\frac{N}{(n-2)!}h^{n-2}(s)\right]^{\frac{\delta}{\beta}}\sum_{i=1}^{m} q_i(s)P_3^{\frac{\delta}{\beta}}(\eta_i(s)) - \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}E(s)r^{\frac{1}{\gamma}}(s)}\,ds\\ &\quad\leq E^{\gamma}(t_1)\phi(t_1) + 1,\end{aligned}$$

which contradicts (3.8). Assume that we have case 4. Since $r[w^{(n-1)}]^{\gamma}$ is nonincreasing, as in the proof of case 3, we get (3.15). Hence, there exists a constant $c_3 > 0$, such that

$$w^{(n-2)}(t) \geq c_3 E(t).$$

Integrating from t_1 to t provides

$$-w^{(n-3)}(t_1) \geq c_3\int_{t_1}^{t} E(\varrho)\,d\varrho,$$

which contradicts (3.9), and this completes the proof. □

Theorem 10 *Assume that* (3.6), (3.7), *and* (3.9) *hold. If there exist* $d \in C^1([t_0,\infty], R^+)$, c_4 *is a positive constant such that*

$$\begin{aligned}&\limsup_{t\to\infty}\frac{E^{\gamma}(t)}{d(t)}\int_{t_0}^{t}\left[d(s)\left[\frac{N}{(n-2)!}h^{n-2}(s)\right]^{\frac{\delta}{\beta}}c_4^{\frac{\delta}{\beta}-\gamma}\sum_{i=1}^{m} q_i(s)P_3^{\frac{\delta}{\beta}}(\eta_i(s))\right.\\ &\quad\left.-\frac{r(s)(d'(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}d^{\gamma}(s)}\right]ds > 1,\end{aligned}\tag{3.16}$$

then every solution of (1.1) *oscillates.*

Proof Without loss of generality, we may assume that $x(t)$ is an eventually positive solution of (1.1) such that $x(t) > 0$, $x(\xi(t)) > 0$, $x(\eta_i(t)) > 0$, $i = 1, 2, \dots, m$, for $t \geq t_1 \geq t_0$. Then, as in Lemma 5, there exist four possible cases. The proofs in the three cases 1, 2, and 4 are the same as in Theorem 9. Now assume that case 3 holds. Since $r[w^{(n-1)}]^\gamma$ is nonincreasing, as in Theorem 9, we have (3.15). Define

$$\varphi(t) = d(t)\left(\frac{r(t)[w^{(n-1)}(t)]^\gamma}{[w^{(n-2)}(t)]^\gamma} + \frac{1}{E^\gamma(t)}\right). \tag{3.17}$$

From (3.15), $\varphi(t) > 0$ for $t \geq t_1$. Therefore, we have

$$\varphi'(t) = \frac{d'(t)}{d(t)}\varphi(t) + d(t)\frac{[r(t)[w^{(n-1)}(t)]^\gamma]'}{[w^{(n-2)}(t)]^\gamma} - d(t)\frac{\gamma r(t)[w^{(n-1)}(t)]^{\gamma+1}}{[w^{(n-2)}(t)]^{\gamma+1}} - \frac{\gamma d(t)E'(t)}{E^{\gamma+1}(t)},$$

by (3.5) and (3.17)

$$\varphi'(t) \leq \frac{d'(t)}{d(t)}\varphi(t) - d(t)\frac{\sum_{i=1}^m q_i(t)P_3^{\frac{\delta}{\beta}}(\eta_i(t))w^{\frac{\delta}{\beta}}(h(t))}{[w^{(n-2)}(t)]^\gamma} - \frac{\gamma}{d^{\frac{1}{\gamma}}(t)r^{\frac{1}{\gamma}}(t)}\left[\varphi(t) - \frac{d(t)}{E^\gamma(t)}\right]^{\frac{\gamma+1}{\gamma}} - \frac{\gamma d(t)E'(t)}{E^{\gamma+1}(t)}.$$

As in Theorem 2.2 of [16], using Lemma 4 with $A = \frac{d'(t)}{d(t)}$, $B = \frac{\gamma}{d^{\frac{1}{\gamma}}(t)r^{\frac{1}{\gamma}}(t)}$, $D = \frac{d(t)}{E^\gamma(t)}$ and $\nu = \varphi$, we have

$$\varphi'(t) \leq -d(t)\frac{\sum_{i=1}^m q_i(t)P_3^{\frac{\delta}{\beta}}(\eta_i(t))w^{\frac{\delta}{\beta}}(h(t))}{[w^{(n-2)}(t)]^\gamma} + \left[\frac{d(t)}{E^\gamma(t)}\right]' + \frac{r(t)(d'(t))^{\gamma+1}}{(\gamma+1)^{\gamma+1}d^\gamma(t)}.$$

Now using Lemma 1, we have (3.12). Since $w^{(n-2)}(t)$ is positive and decreasing $w^{(n-2)}(h(t)) \geq w^{(n-2)}(t)$, and there exists a positive constant c_4 such that $w^{(n-2)}(t) \leq c_4$, we have

$$\varphi'(t) \leq -d(t)c_4^{\frac{\delta}{\beta}-\gamma}\left[\frac{N}{(n-2)!}h^{n-2}(t)\right]^{\frac{\delta}{\beta}}\sum_{i=1}^m q_i(t)P_3^{\frac{\delta}{\beta}}(\eta_i(t)) + \left[\frac{d(t)}{E^\gamma(t)}\right]' + \frac{r(t)(d'(t))^{\gamma+1}}{(\gamma+1)^{\gamma+1}d^\gamma(t)}.$$

Integrating the above inequality from t_1 to t, we find

$$\begin{aligned}
&\varphi(t) - \varphi(t_1) \\
&\quad \leq -\int_{t_1}^t \left(d(s)\left[\frac{N}{(n-2)!}h^{n-2}(s)\right]^{\frac{\delta}{\beta}}c_4^{\frac{\delta}{\beta}-\gamma}\sum_{i=1}^m q_i(s)P_3^{\frac{\delta}{\beta}}(\eta_i(s)) - \frac{r(s)(d'(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}d^\gamma(s)}\right)ds \\
&\qquad + \frac{d(t)}{E^\gamma(t)} - \frac{d(t_1)}{E^\gamma(t_1)}.
\end{aligned}$$

From the definition of $\varphi(t)$, we see that

$$\int_{t_1}^t \left(d(s)\left[\frac{N}{(n-2)!}h^{n-2}(s)\right]^{\frac{\delta}{\beta}}c_4^{\frac{\delta}{\beta}-\gamma}\sum_{i=1}^m q_i(s)P_3^{\frac{\delta}{\beta}}(\eta_i(s)) - \frac{r(s)(d'(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}d^\gamma(s)}\right)ds$$

$$\le -\frac{d(t)r(t)[w^{(n-1)}(t)]^{\gamma}}{[w^{(n-2)}(t)]^{\gamma}} + \frac{d(t_1)r(t_1)[w^{(n-1)}(t_1)]^{\gamma}}{[w^{(n-2)}(t_1)]^{\gamma}}.$$

This leads to

$$\int_{t_1}^{t} d(s)\left[\frac{N}{(n-2)!}h^{n-2}(s)\right]^{\frac{\delta}{\beta}} c_4^{\frac{\delta}{\beta}-\gamma}\sum_{i=1}^{m} q_i(s)P_3^{\frac{\delta}{\beta}}\big(\eta_i(s)\big) - \frac{r(s)(d'(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}d^{\gamma}(s)}\,ds \le \frac{d(t)}{E^{\gamma}(t)}.$$

Hence

$$\frac{E^{\gamma}(t)}{d(t)}\int_{t_1}^{t} d(s)\left[\frac{N}{(n-2)!}h^{n-2}(s)\right]^{\frac{\delta}{\beta}} c_4^{\frac{\delta}{\beta}-\gamma}\sum_{i=1}^{m} q_i(s)P_3^{\frac{\delta}{\beta}}\big(\eta_i(s)\big) - \frac{r(s)(d'(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}d^{\gamma}(s)}\,ds \le 1,$$

which contradicts (3.16), and this completes the proof. □

Theorem 11 *Assume that* (3.6), (3.8), *and* (3.9) *hold,* $\frac{\delta}{\beta} = \gamma$. *If there exists* $\vartheta(t) \in C^1([t_0,\infty], R^+)$ *such that*

$$\int_{t_0}^{\infty}\left[\frac{\vartheta(s)}{(n-4)!}\int_{s}^{\infty}(s-\nu)^{n-4}\left(\frac{1}{r(\nu)}\int_{\nu}^{\infty}\sum_{i=1}^{m} q_i(\chi)P_2^{\frac{\delta}{\beta}}\big(\eta_i(\chi)\big)\left[\frac{h(\chi)}{\chi}\right]^{(n-3)\frac{\delta}{\beta}} d\chi\right)^{\frac{1}{\gamma}} d\nu \right.$$
$$\left. - \frac{(\vartheta'(s))^2}{4\vartheta(s)}\right] ds = \infty, \tag{3.18}$$

then (1.1) *is oscillatory.*

Proof Without loss of generality, we may assume that $x(t)$ is an eventually positive solution of (1.1) such that $x(t) > 0$, $x(\xi(t)) > 0$, $x(\eta_i(t)) > 0$, $i = 1, 2, \ldots, m$, for $t \ge t_1 \ge t_0$. Then, as in Lemma 5, there exist four possible cases. The proofs in the three cases 1, 3, and 4 are the same as in Theorem 9. Now assume that case 2 holds. Define

$$\varpi(t) = \vartheta(t)\frac{w'(t)}{w(t)},$$

then $\varpi > 0$,

$$\varpi'(t) = \vartheta(t)\frac{w''(t)}{w(t)} - \frac{\varpi^2(t)}{\vartheta(t)} + \frac{\vartheta'(t)\varpi(t)}{\vartheta(t)}. \tag{3.19}$$

Now by integrating (3.4) from t to ν

$$r(\nu)\big(w^{(n-1)}(\nu)\big)^{\gamma} - r(t)\big(w^{(n-1)}(t)\big)^{\gamma} \le -\int_{t}^{\nu}\sum_{i=1}^{m} q_i(s)P_2^{\frac{\delta}{\beta}}\big(\eta_i(s)\big)w^{\frac{\delta}{\beta}}\big(h(s)\big)\,ds. \tag{3.20}$$

As in the proof of Lemma 7, $t^{3-n}\, w(t)$ is nonincreasing since $h(t) < t$, then

$$w\big(h(t)\big) \ge \left[\frac{h(t)}{t}\right]^{n-3} w(t).$$

By substituting in (3.20) and letting $v \to \infty$, we have

$$r(t)\big(w^{(n-1)}(t)\big)^{\gamma} \geq w^{\frac{\delta}{\beta}}(t)\int_t^{\infty}\sum_{i=1}^{m} q_i(s)P_2^{\frac{\delta}{\beta}}\big(\eta_i(s)\big)\left[\frac{h(s)}{s}\right]^{(n-3)\frac{\delta}{\beta}} ds.$$

Integrating again from t to ∞, we get

$$w^{(n-2)}(t) \leq -w^{\frac{\delta}{\gamma\beta}}(t)\int_t^{\infty}\left(\frac{1}{r(v)}\int_v^{\infty}\sum_{i=1}^{m} q_i(s)P_2^{\frac{\delta}{\beta}}\big(\eta_i(s)\big)\left[\frac{h(s)}{s}\right]^{(n-3)\frac{\delta}{\beta}} ds\right)^{\frac{1}{\gamma}} dv. \tag{3.21}$$

Similarly, integrating (3.21) from t to ∞ a total $(n-4)$ times, we find

$$w''(t) \leq -\frac{w^{\frac{\delta}{\gamma\beta}}(t)}{(n-4)!}\int_t^{\infty}(t-v)^{n-4}\left(\frac{1}{r(v)}\int_v^{\infty}\sum_{i=1}^{m} q_i(s)P_2^{\frac{\delta}{\beta}}\big(\eta_i(s)\big)\left[\frac{h(s)}{s}\right]^{(n-3)\frac{\delta}{\beta}} ds\right)^{\frac{1}{\gamma}} dv.$$

Thus, by substituting in (3.19), we have

$$\begin{aligned}\varpi'(t) \leq{}& -\vartheta(t)\frac{w^{\frac{\delta}{\gamma\beta}}(t)}{(n-4)!w(t)}\\ &\times\int_t^{\infty}(t-v)^{n-4}\left(\frac{1}{r(v)}\int_v^{\infty}\sum_{i=1}^{m} q_i(s)P_2^{\frac{\delta}{\beta}}\big(\eta_i(s)\big)\left[\frac{h(s)}{s}\right]^{(n-3)\frac{\delta}{\beta}} ds\right)^{\frac{1}{\gamma}} dv\\ &-\frac{\varpi^2(t)}{\vartheta(t)}+\frac{\vartheta'(t)\varpi(t)}{\vartheta(t)}.\end{aligned}$$

By completing the squares, we have

$$\begin{aligned}\varpi'(t) \leq{}& -\vartheta(t)\frac{w^{\frac{\delta}{\gamma\beta}}(t)}{(n-4)!w(t)}\\ &\times\int_t^{\infty}(t-v)^{n-4}\left(\frac{1}{r(v)}\int_v^{\infty}\sum_{i=1}^{m} q_i(s)P_2^{\frac{\delta}{\beta}}\big(\eta_i(s)\big)\left[\frac{h(s)}{s}\right]^{(n-3)\frac{\delta}{\beta}} ds\right)^{\frac{1}{\gamma}} dv\\ &+\frac{(\vartheta'(t))^2}{4\vartheta(t)}.\end{aligned}$$

This yields

$$\begin{aligned}&\int_{t_1}^{t}\left[\frac{\vartheta(s)}{(n-4)!}\int_s^{\infty}(s-v)^{n-4}\left(\frac{1}{r(v)}\int_v^{\infty}\sum_{i=1}^{m} q_i(\chi)P_2^{\frac{\delta}{\beta}}\big(\eta_i(\chi)\big)\left[\frac{h(\chi)}{\chi}\right]^{(n-3)\frac{\delta}{\beta}} d\chi\right)^{\frac{1}{\gamma}} dv\right.\\ &\qquad\left.-\frac{(\vartheta'(s))^2}{4\vartheta(s)}\right]ds\\ &\leq \varpi(t_1),\end{aligned}$$

which contradicts (3.18), and this completes the proof. □

4 Examples and conclusion

Example 12 Consider the differential equation

$$\left(t^2\left(x(t)+tx^3\left(\frac{3t}{4}\right)\right)'''\right)'+\frac{q_0}{t}x^3(t)+\frac{q_0}{t^2}x^3(2t)=0,\quad q_0>0, t\geq 1. \tag{4.1}$$

Here $\delta=\beta=3$, $\gamma=1$, $n=4$, $r(t)=t^2$, $p(t)=t$, $\xi(t)=\frac{3t}{4}$, $q_1(t)=\frac{q_0}{t}$, $q_2(t)=\frac{q_0}{t^2}$, $\eta_1(t)=t$, $\eta_2(t)=2t$, $\xi^{-1}(t)=\frac{4}{3}t$, $\xi^{-1}(\xi^{-1}(t))=\frac{16}{9}t$. Taking $\eta(t)=\frac{t}{2}$, then $\eta(t)<\eta_i(t)$, $\lim_{t\to\infty}\eta(t)=\infty$, $\eta(t)<\xi(t)$, $h(t)=\frac{2}{3}t$,

$$P_1(t)=\frac{3}{4t}\left[1-\frac{4}{3k_1^{\frac{2}{3}}t^{\frac{1}{3}}}\sqrt[3]{\frac{9}{16}}\right],\qquad P_2(t)=\frac{3}{4t}\left[1-\left(\sqrt[3]{\frac{3}{4}}\right)\frac{1}{k_2^{\frac{2}{3}}t^{\frac{1}{3}}}\right],$$

$$P_3(t)=\frac{3}{4t}\left[1-\frac{1}{k_3^{\frac{2}{3}}t^{\frac{1}{3}}}\right].$$

Now let $\upsilon(t)=t$, $\vartheta(t)=t$, then

$$\int_{t_0}^{\infty}\left[\upsilon(s)\left[\frac{h(s)}{s}\right]^{(n-1)\delta/\beta}\sum_{i=1}^{m}q_i(s)P_1^{\frac{\delta}{\beta}}\left(\eta_i(s)\right)-\frac{L[\upsilon'(s)]^2r(s)}{s^{(n-1)\delta/\beta-1}\upsilon(s)}\right]ds$$
$$=\int_1^{\infty}\left(s\left(\frac{2}{3}\right)^3\left[\frac{3q_0}{4s^2}\left(1-\frac{4}{3k_1^{\frac{2}{3}}s^{\frac{1}{3}}}\sqrt[3]{\frac{9}{16}}\right)+\frac{3q_0}{8s^3}\left(1-\frac{4}{3k_1^{\frac{2}{3}}s^{\frac{1}{3}}}\sqrt[3]{\frac{9}{32}}\right)\right]-\frac{L}{s}\right)ds=\infty.$$

If $q_0>\frac{9L}{2}$, $L=[\frac{(n-1)!}{k}]^{\delta/\beta}\frac{\beta}{4\delta(n-1)c_1^{\delta/\beta-\gamma}}=\frac{1}{2k}$, i.e., $q_0>\frac{9}{4k}$, $k\in(0,1)$

$$\int_{t_0}^{\infty}\left[E^{\gamma}(s)c_2^{\frac{\delta}{\beta}-\gamma}\left[\frac{Nh^{n-2}(s)}{(n-2)!}\right]^{\delta/\beta}\sum_{i=1}^{m}q_i(s)P_3^{\frac{\delta}{\beta}}\left(\eta_i(s)\right)-\frac{\gamma^{\gamma+1}}{(\gamma+1)^{\gamma+1}E(s)r^{1/\gamma}(s)}\right]ds$$
$$=\int_{t_0}^{\infty}\left[\frac{2}{9}Ns\left[\frac{3q_0}{4s^2}\left(1-\frac{1}{k_3^{\frac{2}{3}}s^{\frac{1}{3}}}\right)+\frac{3q_0}{8s^3}\left(1-\frac{1}{k_3^{\frac{2}{3}}s^{\frac{1}{3}}}\sqrt[3]{\frac{1}{2}}\right)\right]-\frac{1}{4s}\right]ds=\infty.$$

If $q_0>\frac{3}{2N}$,

$$\int_t^{\infty}E(s)\,ds=\int_t^{\infty}\frac{1}{s}\,ds=\infty.$$

Moreover,

$$\int_{t_0}^{\infty}\left[\frac{\vartheta(s)}{(n-4)!}\int_s^{\infty}(s-\nu)^{n-4}\left(\frac{1}{r(\nu)}\int_{\nu}^{\infty}\sum_{i=1}^{m}q_i(\chi)P_2^{\frac{\delta}{\beta}}\left(\eta_i(\chi)\right)\left[\frac{h(\chi)}{\chi}\right]^{(n-3)\frac{\delta}{\beta}}d\chi\right)^{\frac{1}{\gamma}}d\nu\right.$$
$$\left.-\frac{(\vartheta'(s))^2}{4\vartheta(s)}\right]ds$$
$$=\int_1^{\infty}\left(q_0s\left[\frac{1}{4s^2}-\frac{9}{56}\frac{1}{k_2^{\frac{2}{3}}s^{\frac{7}{3}}}\sqrt[3]{\frac{3}{4}}+\frac{1}{24s^3}-\frac{9}{280}\frac{1}{k_2^{\frac{2}{3}}s^{\frac{10}{3}}}\sqrt[3]{\frac{3}{8}}\right]-\frac{1}{4s}\right)ds=\infty.$$

If $q_0 > 1$, then by Theorem 11, every solution of (4.1)oscillates for $q_0 > \max[1, \frac{3}{2N}, \frac{9}{4k}]$, $N, k \in (0,1)$.

Conclusion 13 *In this work, we use techniques of the Riccati and generalized Riccati transformations, integral averaging, and the method of comparison to establish some new oscillation criteria for the even-order differential equation with superlinear neutral term* (1.1) *in noncanonical case. The obtained results improve and complement some previous criteria in the literature. An example is provided to support the theoretical findings.*

Author contributions
the author reviewed the manuscript

Funding
Open access funding provided by The Science, Technology & Innovation Funding Authority (STDF) in cooperation with The Egyptian Knowledge Bank (EKB). This research was not support by any project.

Data availability
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Received: 31 October 2023 Accepted: 3 May 2024 Published online: 27 May 2024

References

1. Agarwal, R.P., Grace, S.R., Regan, D.O.: Oscillation Theory for Difference and Functional Differential Equations. Kluwer Academic, Dordrecht (2000)
2. Agarwal, R.P., Zhang, C., Li, T.: Some remarks on oscillation of second order neutral differential equations. Appl. Math. Comput. **274**, 178–181 (2016)
3. Dharuman, C., Prabaharan, N., Thandapani, E., Tunç, E.: Oscillatory behavior of even-order functional differential equations with a superlinear neutral term. Palest. J. Math. **12**(1), 722–731 (2023)
4. Džurina, J., Grace, S.R., Jadlovská, I., Li, T.: Oscillation criteria for second-order Emden-Fowler delay differential equations with a sublinear neutral term. Math. Nachr. **293**(5), 910–922 (2020)
5. El-Gaber, A.A., El-Sheikh, M.M.A.: Oscillation of fourth-order neutral differential equations with distributed deviating arguments. J. Math. Comput. Sci. **28**, 60–71 (2023)
6. El-Gaber, A.A., El-Sheikh, M.M.A., El-Saedy, E.I.: Oscillation of super-linear fourth-order differential equations with several sub-linear neutral terms. Bound. Value Probl. **41**, 1–14 (2022)
7. Elabbasy, E.M., Nabih, A., Nofal, T.A., Alharbi, W.R., Moaaz, O.: Neutral differential equations with noncanonical operator: oscillation behavior of solutions. AIMS Math. **6**(4), 3272–3287 (2021)
8. Grace, S.R., Abbas, S., Graef, J.R.: Oscillation of even order nonlinear dynamic equations on time-scales. Math. Morav. **26**(1), 47–55 (2022)
9. Grace, S.R., Alzabut, J., Abodayeh, K.: Oscillation theorems for higher order dynamic equations with superlinear neutral term. AIMS Math. **6**(6), 5493–5501 (2021)
10. Hale, J.K.: Theory of Functional Differential Equations. Springer, New York (1977)
11. Han, X., Wei, H.: Multiplicity of the large periodic solutions to a super-linear wave equation with general variable coefficient. Commun. Anal. Mech. **16**(2), 278–292 (2024)
12. Kiguradze, I.T., Chanturiya, T.A.: Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations. Kluwer Academic, Dordrecht (1993)
13. Li, T., Baculikova, B., Dzurina, J., Zhang, C.: Oscillation of fourth-order neutral differential equations with p-Laplacian like operators. Bound. Value Probl. **56**, 1–9 (2014)
14. Li, T., Rogovchenko, Y.V.: Oscillation criteria for even-order neutral differential equations. Appl. Math. Lett. **61**, 35–41 (2016)
15. Moaaz, O., Elabbasy, E.M., Muhib, A.: Oscillation criteria for even-order neutral differential equations with distributed deviating arguments. Adv. Differ. Equ. **297**, 1–10 (2019)
16. Moaaz, O., Muhip, A.: New oscillation criteria for nonlinear delay differential equations of fourth-order. Appl. Math. Comput. **377**, 1–11 (2020)
17. Philos, C.G.: On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations will positive delays. Arch. Math. **36**, 168–178 (1981)
18. Philos, C.G.: A new criterion for the oscillatory and asymptotic behavior of delay differential equations. Bull. Acad. Pol. Sci., Sér. Sci. Math. **39**, 61–64 (1981)
19. Tian, Y., Cail, Y., Fu, Y., Li, T.: Oscillation and asymptotic behavior of third-order neutral differential equations with distributed deviating arguments. Adv. Differ. Equ. **267**, 1–14 (2015)
20. Wang, H., Chen, G., Jiang, Y., Jiang, C., Li, T.: Asymptotic behavior of third-order neutral differential equations with distributed deviating arguments. J. Math. Comput. Sci. **17**, 194–199 (2017)

21. Wang, P.G.: Oscillation criteria for second-order neutral equations with distributed deviating arguments. Comput. Math. Appl. **47**, 1935–1946 (2004)
22. Zhang, C., Agarwal, R.P., Bohner, M., Li, T.: New results for oscillatory behavior of even-order half-linear delay differential equations. Appl. Math. Lett. **26**, 179–183 (2013)
23. Zhang, C., Li, T., Saker, S.H.: Oscillation of fourth-order delay differential equations. J. Math. Sci. **201**(3) (2014)
24. Zhang, C., Li, T., sun, B., Thandapani, E.: On oscillation of higher-order half-linear delay differential equations. Appl. Math. Lett. **24**, 1618–1621 (2011)

Publisher's Note