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Nonexistence results for a time-fractional biharmonic diffusion equation

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Abstract

We consider weak solutions of the nonlinear time-fractional biharmonic diffusion equation $\partial_t^\alpha u + \partial_t^\beta u + u_{xxxx} = h(t,x)|u|^p$ in $(0,\infty) \times (0,1)$ subject to the initial conditions $u(0,x) = u_0(x)$, $u_t(0,x) = u_1(x)$ and the Navier boundary conditions $u(t,1) = u_{xx}(t,1) = 0$, where $\alpha \in (0,1)$, $\beta \in (1,2)$, ∂_t^α (resp. ∂_t^β) is the fractional derivative of order α (resp. β) with respect to the time-variable in the Caputo sense, $p > 1$ and h is a measurable positive weight function. Using nonlinear capacity estimates specifically adapted to the fourth-order differential operator $\frac{\partial^4}{\partial x^4}$, the domain, the initial conditions and the boundary condition, a general nonexistence result is established. Next, some special cases of weight functions h are discussed.

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Keywords: Time-fractional biharmonic diffusion equation; Weak solution; Nonexistence; Caputo fractional derivative

1 Introduction

In this paper, we study the nonexistence of weak solutions of the nonlinear time-fractional biharmonic diffusion equation

$$\partial_t^\alpha u + \partial_t^\beta u + u_{xxxx} = h(t,x)|u|^p, \quad t > 0, 0 < x < 1, \tag{1.1}$$

where $u = u(t,x)$, $\alpha \in (0,1)$, $\beta \in (1,2)$, ∂_t^α (resp. ∂_t^β) is the fractional derivative of order α (resp. β) with respect to the time-variable in the Caputo sense, $p > 1$ and h is a measurable weight function with $h(t,x) > 0$ almost everywhere in $(0,\infty) \times (0,1)$. Equation (1.1) is considered subject to the initial conditions

$$u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), \quad 0 < x < 1 \tag{1.2}$$

and the Navier boundary conditions

$$u(t,1) = u_{xx}(t,1) = 0, \quad t > 0, \tag{1.3}$$

where $u_0, u_1 \in L^1_{\text{loc}}((0,1])$. Namely, our goal is to establish sufficient conditions under which the considered problem admits no weak solution.

The topic of time-fractional evolution equations has gained considerable attention in recent decades due to its widespread applications in various fields of science, such as physics, chemistry and biology, see, e.g., [4, 9, 17, 18]. We can find in the literature several studies related to time-fractional evolution equations in both theoretical and numerical directions. In the theoretical point of view, several investigations have been made regarding well-posedness, inverse problems, asymptotic analysis, decay estimates, etc., see, e.g., [1, 8, 11, 15, 20, 23]. For some numerical contributions, we refer to [2, 5, 7].

The issue of nonexistence of solutions to time-fractional evolution equations was initiated by Kirane and his collaborators, see, e.g., Fino and Kirane [6], Kirane and Tatar [16], Kirane and Laskri [13] and Kirane and Malik [14]. Since then, this topic was developed by many authors, see, e.g., [3, 10, 19, 21, 22, 24] and the references therein. In particular, Tatar [22] considered the time-fractional diffusion equation

$$\partial_t^{1+\alpha} u + \partial_t^{\beta} u - \Delta u = h(t,x)|u|^p, \quad t > 0, x \in \mathbb{R}^N, \tag{1.4}$$

where $\alpha, \beta \in (0,1)$, $p > 1$ and h satisfies

$$h\big(R^2 t, R^{\beta} x\big) = R^{\rho} h(t,x)$$

for some $\rho > 0$ and large $R > 0$. Namely, it has been proven that if $u(0,\cdot), u_t(0,\cdot) \geq 0$ and

$$1 < p \leq 1 + \frac{2\beta + \rho}{1 + \beta N - 2\beta},$$

then (1.4) admits no weak solution.

The novelty of this work with respect to the above cited contributions (in particular [22]) lies in the following facts:

(a) Problem (1.1) is posed in a bounded domain;

(b) Problem (1.1) is governed by a fourth-order differential operator.

In this paper, our approach is based on nonlinear capacity estimates specifically adapted to the fourth-order differential operator $\frac{\partial^4}{\partial x^4}$, the domain, the initial conditions (1.2) and the boundary condition (1.3).

The rest of the paper is organized as follows. Section 2 is devoted to some preliminaries on fractional calculus. In Sect. 3, we define weak solutions to the considered problem and state our obtained results. In Sect. 4, we etsablish some useful lemmas. Finally, we prove our obtained results in Sect. 5.

Throughout this paper, we shall use the following notations. By C (or C_i), we mean a positive constant that is independent of the parameters T, R and the solution u. The value of this constant is not important and is not necessarily the same from one line to another. For a positive real number ℓ, the notation $\ell \gg 1$ means that ℓ is sufficiently large.

2 Preliminaries

In this section, we briefly recall some notions and results related to fractional operators and fix some notations. For more details, we refer to [12].

Let $T > 0$. The left-sided and right-sided Riemann–Liouville fractional integrals of order $\gamma > 0$ of $f \in C([0,T])$ are defined respectively by

$$I_0^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s)\, ds, \quad 0 < t \leq T$$

and

$$I_T^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_t^T (s-t)^{\gamma-1} f(s)\, ds, \quad 0 \le t < T,$$

where Γ is the Gamma function. It can be easily seen that

$$\lim_{t\to 0^+} I_0^{\gamma} f(t) = \lim_{t\to T^-} I_T^{\gamma} f(t) = 0. \tag{2.1}$$

We have the following integration by parts rule.

Lemma 2.1 *Let $\gamma > 0$ and $f, g \in C([0,T])$. We have*

$$\int_0^T g(t) I_0^{\gamma} f(t)\, dt = \int_0^T f(t) I_T^{\gamma} g(t)\, dt.$$

For $T > 0$ and $\ell \gg 1$, let

$$M(t) = T^{-\ell}(T-t)^{\ell}, \quad 0 \le t \le T. \tag{2.2}$$

For the proof of the following lemma, see [12, Property 2.1, p 71].

Lemma 2.2 *Let $0 < \kappa < 1$. For all $t \in [0,T]$, we have*

$$I_T^{\kappa} M(t) = \frac{\Gamma(\ell+1)}{\Gamma(\ell+1+\kappa)} T^{-\ell}(T-t)^{\ell+\kappa}, \tag{2.3}$$

$$\frac{d}{dt} I_T^{\kappa} M(t) = -\frac{\Gamma(\ell+1)}{\Gamma(\ell+\kappa)} T^{-\ell}(T-t)^{\ell+\kappa-1}, \tag{2.4}$$

$$\frac{d^2}{dt^2} I_T^{\kappa} M(t) = \frac{\Gamma(\ell+1)}{\Gamma(\ell+\kappa-1)} T^{-\ell}(T-t)^{\ell+\kappa-2}. \tag{2.5}$$

Let $\gamma \in (n-1, n)$, where n is positive integer and $f \in C^n([0,T])$. The Caputo fractional derivative of order γ of f is defined by

$$\begin{aligned} {}^{C}D_0^{\gamma} f(t) &= I_0^{n-\gamma} \frac{d^n f}{dt^n}(t) \\ &= \frac{1}{\Gamma(n-\gamma)} \int_0^t (t-s)^{n-\gamma-1} \frac{d^n f}{ds^n}(s)\, ds, \quad 0 < t < T. \end{aligned}$$

Let $F = F(t,x) : [0,T] \times J \to \mathbb{R}$, where $J \subset \mathbb{R}$. The left-sided and right-sided Riemann–Liouville fractional integrals of order $\gamma > 0$ of F with respect to the time-variable t are defined respectively by

$$\begin{aligned} I_0^{\gamma} F(t,x) &= I_0^{\gamma} F(\cdot,x)(t) \\ &= \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} F(s,x)\, ds \end{aligned}$$

and

$$\begin{aligned} I_T^\gamma F(t,x) &= I_T^\gamma F(\cdot,x)(t) \\ &= \frac{1}{\Gamma(\gamma)} \int_t^T (s-t)^{\gamma-1} F(s,x)\, ds. \end{aligned}$$

If $\gamma \in (n-1,n)$, where n is positive integer, the Caputo fractional derivative of order γ of F with respect to the time-variable t is denoted by $\partial_t^\gamma F(t,x)$ and is defined by

$$\begin{aligned} \partial_t^\gamma F(t,x) &= {}^C D_0^\gamma F(\cdot,x)(t) \\ &= I_0^{n-\gamma} \frac{\partial^n F}{\partial t^n}(t,x) \\ &= \frac{1}{\Gamma(n-\gamma)} \int_0^t (t-s)^{n-\gamma-1} \frac{\partial^n F}{\partial s^n}(s,x)\, ds. \end{aligned}$$

3 The results

Let us first define weak solutions to (1.1)–(1.3). For all $T > 0$, let

$$Q_T = [0,T] \times (0,1].$$

We introduce the set

$$\Phi_T = \left\{\varphi \in C^4(Q_T) : \varphi \geq 0, \operatorname{supp}(\varphi) \subset\subset Q_T, \varphi(T,\cdot) \equiv 0, \varphi(\cdot,1) = \varphi_{xx}(\cdot,1) \equiv 0\right\}.$$

Definition 3.1 We say that $u \in L^p_{\mathrm{loc}}([0,\infty) \times (0,1], h(t,x)\,dt\,dx) \cap L^1_{\mathrm{loc}}([0,\infty) \times (0,1])$ is a weak solution to (1.1)–(1.3), if

$$\begin{aligned} &\int_{Q_T} |u|^p h(t,x)\varphi\, dx\, dt + \int_0^1 u_0(x)\left(I_T^{1-\alpha}\varphi(0,x) - \left(I_T^{2-\beta}\varphi\right)_t(0,x)\right) dx \\ &\quad + \int_0^1 u_1(x) I_T^{2-\beta}\varphi(0,x)\, dx \\ &= \int_{Q_T} u\left(-\left(I_T^{1-\alpha}\varphi\right)_t + \left(I_T^{2-\beta}\varphi\right)_{tt} + \varphi_{xxxx}\right) dx\, dt \end{aligned} \tag{3.1}$$

for every $T > 0$ and $\varphi \in \Phi_T$.

Notice that if u is a classical solution to (1.1)–(1.3), then multiplying (1.1) by $\varphi \in \Phi_T$, integrating by parts, using property (2.1), the integration by parts rule given by Lemma 2.1, (1.2) and (1.3), we obtain (3.1).

We are now in position to state our main results. We first consider the case

$$u_1 \equiv 0, \qquad u_0 \in L^1\big((0,1)\big), \qquad \int_0^1 u_0(x)(1-x)\, dx > 0. \tag{3.2}$$

Theorem 3.1 *Let $0 < \alpha < 1 < \beta < 2$, $p > 1$, $h(t,x) > 0$ almost everywhere in $(0,\infty) \times (0,1)$ and $h^{\frac{-1}{p-1}} \in L^1_{\mathrm{loc}}([0,\infty) \times (0,1])$. Assume that the initial data satisfy (3.2). If there exists*

$\theta > 0$ such that

$$\liminf_{T\to\infty} T^{\alpha-1+\frac{4\theta p}{p-1}} \int_{t=0}^{T}\int_{x=T^{-\theta}}^{1} h^{\frac{-1}{p-1}}(t,x)\,dx\,dt = 0, \tag{3.3}$$

then (1.1)–(1.2)–(1.3) *admits no weak solution.*

We next consider the case

$$u_0 \equiv 0, \qquad u_1 \in L^1\big((0,1)\big), \qquad \int_0^1 u_1(x)(1-x)\,dx > 0. \tag{3.4}$$

Theorem 3.2 *Let $0 < \alpha < 1 < \beta < 2$, $p > 1$, $h(t,x) > 0$ almost everywhere in $(0,\infty) \times (0,1)$ and $h^{\frac{-1}{p-1}} \in L^1_{\mathrm{loc}}([0,\infty) \times (0,1])$. Assume that the initial data satisfy* (3.4). *If there exists $\theta > 0$ such that*

$$\liminf_{T\to\infty} T^{\beta-2+\frac{4\theta p}{p-1}} \int_{t=0}^{T}\int_{x=T^{-\theta}}^{1} h^{\frac{-1}{p-1}}(t,x)\,dx\,dt = 0, \tag{3.5}$$

then (1.1)–(1.2)–(1.3) *admits no weak solution.*

We finally consider the case

$$u_i \in L^1\big((0,1)\big), \qquad \int_0^1 u_i(x)(1-x)\,dx > 0, i = 0,1. \tag{3.6}$$

Theorem 3.3 *Let $0 < \alpha < 1 < \beta < 2$, $p > 1$, $h(t,x) > 0$ almost everywhere in $(0,\infty) \times (0,1)$ and $h^{\frac{-1}{p-1}} \in L^1_{\mathrm{loc}}([0,\infty) \times (0,1])$. Assume that the initial data satisfy* (3.6). *If there exists $\theta > 0$ such that*

$$\liminf_{T\to\infty} T^{-\max\{1-\alpha,2-\beta\}+\frac{4\theta p}{p-1}} \int_{t=0}^{T}\int_{x=T^{-\theta}}^{1} h^{\frac{-1}{p-1}}(t,x)\,dx\,dt = 0, \tag{3.7}$$

then (1.1)–(1.2)–(1.3) *admits no weak solution.*

We discuss below some particular cases of weight functions h. We first consider the case when $h^{\frac{-1}{p-1}}$ is a L^1-function.

Corollary 3.4 *Let $0 < \alpha < 1 < \beta < 2$, $p > 1$, $h(t,x) > 0$ almost everywhere in $(0,\infty) \times (0,1)$ and $h^{\frac{-1}{p-1}} \in L^1([0,\infty) \times (0,1))$. If the initial data satisfy* (3.2) *or* (3.4) *or* (3.6), *then* (1.1)–(1.3) *admit no weak solution.*

We now study the case when

$$h(t,x) = t^{\rho}x^{\sigma}, \quad t > 0, 0 < x < 1. \tag{3.8}$$

From Theorem 3.1, we deduce the following result.

Corollary 3.5 *Let $0 < \alpha < 1 < \beta < 2$ and h be the function defined by* (3.8), *where $\rho > 0$ and $\sigma \in \mathbb{R}$. Assume that the initial data satisfy* (3.2). *If*

$$1 + \rho < p < 1 + \frac{\rho}{\alpha}, \tag{3.9}$$

then (1.1)–(1.3) *admit no weak solution.*

From Theorem 3.2, we deduce the following result.

Corollary 3.6 *Let* $0 < \alpha < 1 < \beta < 2$ *and* h *be the function defined by* (3.8), *where* $\rho > 0$ *and* $\sigma \in \mathbb{R}$. *Assume that the initial data satisfy* (3.4). *If*

$$1 + \rho < p < 1 + \frac{\rho}{\beta - 1}, \tag{3.10}$$

then (1.1)–(1.3) *admit no weak solution.*

From Theorem 3.3, we deduce the following result.

Corollary 3.7 *Let* $0 < \alpha < 1 < \beta < 2$ *and* h *be the function defined by* (3.8), *where* $\rho > 0$ *and* $\sigma \in \mathbb{R}$. *Assume that the initial data satisfy* (3.6). *If*

$$1 + \rho < p < 1 + \frac{\rho}{1 - \max\{1 - \alpha, 2 - \beta\}}, \tag{3.11}$$

then (1.1)–(1.3) *admit no weak solution.*

4 Auxiliary results

Some useful estimates are established in this section. Throughout this section, we have $0 < \alpha < 1 < \beta < 2$, $p > 1$ and $h = h(t,x) > 0$ almost everywhere.

Let us consider a cut-off function $\xi \in C^\infty([0,\infty))$ satisfying

$$0 \le \xi \le 1, \quad \xi \equiv 0 \text{ in } \left[0, \frac{1}{2}\right], \xi \equiv 1 \text{ in } [1,\infty).$$

For $\ell, R \gg 1$, let

$$\xi_R(x) = (1 - x)\xi^\ell(Rx), \quad x \in (0,1],$$

that is,

$$\xi_R(x) = \begin{cases} 0 & \text{if } 0 < x \le \frac{1}{2R}, \\ (1 - x)\xi^\ell(Rx) & \text{if } \frac{1}{2R} \le x \le \frac{1}{R}, \\ 1 - x & \text{if } \frac{1}{R} \le x \le 1. \end{cases} \tag{4.1}$$

For $\ell, T, R \gg 1$, let

$$\varphi(t,x) = M(t)\xi_R(x), \quad (t,x) \in Q_T, \tag{4.2}$$

where M is the function defined by (2.2). The following lemma follows immediately from the properties of ξ, (4.1) and (4.2).

Lemma 4.1 *We have* $\varphi \in \Phi_T$.

We now introduce the nonlinear capacity terms

$$\mathrm{Cap}_1(\varphi)=\int_{Q_T}\varphi^{\frac{-1}{p-1}}\left|\left(I_T^{1-\alpha}\varphi\right)_t\right|^{\frac{p}{p-1}}h^{\frac{-1}{p-1}}\,dx\,dt \tag{4.3}$$

$$\mathrm{Cap}_2(\varphi)=\int_{Q_T}\varphi^{\frac{-1}{p-1}}\left|\left(I_T^{2-\beta}\varphi\right)_{tt}\right|^{\frac{p}{p-1}}h^{\frac{-1}{p-1}}\,dx\,dt, \tag{4.4}$$

$$\mathrm{Cap}_3(\varphi)=\int_{Q_T}\varphi^{\frac{-1}{p-1}}|\varphi_{xxxx}|^{\frac{p}{p-1}}h^{\frac{-1}{p-1}}\,dx\,dt. \tag{4.5}$$

Lemma 4.2 *Let* $h^{\frac{-1}{p-1}}\in L^1_{\mathrm{loc}}([0,\infty)\times(0,1])$. *We have*

$$\mathrm{Cap}_1(\varphi)\le CT^{\frac{-\alpha p}{p-1}}\int_{t=0}^{T}\int_{x=\frac{1}{2R}}^{1}h^{\frac{-1}{p-1}}(t,x)\,dx\,dt. \tag{4.6}$$

Proof By (2.2), (2.4) (with $\kappa=1-\alpha$) and (4.2), for all $(t,x)\in Q_T$, we have

$$\begin{aligned}\varphi^{\frac{-1}{p-1}}\left|\left(I_T^{1-\alpha}\varphi\right)_t\right|^{\frac{p}{p-1}}h^{\frac{-1}{p-1}}&=M^{\frac{-1}{p-1}}(t)\xi_R^{\frac{-1}{p-1}}(x)\xi_R^{\frac{p}{p-1}}(x)\left|\frac{dI_T^{1-\alpha}M}{dt}\right|^{\frac{p}{p-1}}h^{\frac{-1}{p-1}}(t,x)\\&=\xi_R(x)M^{\frac{-1}{p-1}}(t)\left|\frac{dI_T^{1-\alpha}M}{dt}\right|^{\frac{p}{p-1}}h^{\frac{-1}{p-1}}(t,x)\\&=C\xi_R(x)\left[T^{-\ell}(T-t)^{\ell}\right]^{\frac{-1}{p-1}}\left[T^{-\ell}(T-t)^{\ell-\alpha}\right]^{\frac{p}{p-1}}h^{\frac{-1}{p-1}}(t,x)\\&=CT^{-\ell}\xi_R(x)(T-t)^{\ell-\frac{\alpha p}{p-1}}h^{\frac{-1}{p-1}}(t,x).\end{aligned}$$

Integrating over Q_T, we get by the properties of ξ, (4.1) and (4.3) that

$$\begin{aligned}\mathrm{Cap}_1(\varphi)&\le CT^{-\ell}\int_{t=0}^{T}\int_{x=\frac{1}{2R}}^{1}\xi_R(x)(T-t)^{\ell-\frac{\alpha p}{p-1}}h^{\frac{-1}{p-1}}(t,x)\,dx\,dt\\&\le CT^{-\ell}\int_{t=0}^{T}\int_{x=\frac{1}{2R}}^{1}(T-t)^{\ell-\frac{\alpha p}{p-1}}h^{\frac{-1}{p-1}}(t,x)\,dx\,dt\\&\le CT^{\frac{-\alpha p}{p-1}}\int_{t=0}^{T}\int_{x=\frac{1}{2R}}^{1}h^{\frac{-1}{p-1}}(t,x)\,dx\,dt,\end{aligned}$$

which proves (4.6). □

Similarly, using (2.2), (2.5) (with $\kappa=2-\beta$), (4.2) and (4.4), we obtain the following estimate.

Lemma 4.3 *Let* $h^{\frac{-1}{p-1}}\in L^1_{\mathrm{loc}}([0,\infty)\times(0,1])$. *We have*

$$\mathrm{Cap}_2(\varphi)\le CT^{\frac{-\beta p}{p-1}}\int_{t=0}^{T}\int_{x=\frac{1}{2R}}^{1}h^{\frac{-1}{p-1}}(t,x)\,dx\,dt. \tag{4.7}$$

Lemma 4.4 *Let* $h^{\frac{-1}{p-1}}\in L^1_{\mathrm{loc}}([0,\infty)\times(0,1])$. *We have*

$$\mathrm{Cap}_3(\varphi)\le CR^{\frac{4p}{p-1}}\int_{t=0}^{T}\int_{x=\frac{1}{2R}}^{\frac{1}{R}}h^{\frac{-1}{p-1}}(t,x)\,dx\,dt. \tag{4.8}$$

Proof By (2.2) and (4.2), for all $(t,x) \in Q_T$, we have

$$\varphi^{\frac{-1}{p-1}} |\varphi_{xxxx}|^{\frac{p}{p-1}} h^{\frac{-1}{p-1}} = T^{-\ell}(T-t)^{\ell} \xi_R^{\frac{-1}{p-1}}(x) \left| \frac{d^4 \xi_R}{dx^4} \right|^{\frac{p}{p-1}} h^{\frac{-1}{p-1}}(t,x). \tag{4.9}$$

On the other hand, for all $x \in (0,1)$, we have by (4.1) that

$$\begin{aligned} \frac{d^4 \xi_R}{dx^4}(x) &= \frac{d^4}{dx^4}\big[(1-x)\xi^{\ell}(Rx)\big] \\ &= \frac{d^2}{dx^2}\left(\frac{d^2}{dx^2}\big[(1-x)\xi^{\ell}(Rx)\big]\right) \\ &= \frac{d^2}{dx^2}\left((1-x)\frac{d^2}{dx^2}\big[\xi^{\ell}(Rx)\big] - 2\frac{d}{dx}\big[\xi^{\ell}(Rx)\big]\right) \\ &= (1-x)\frac{d^4}{dx^4}\big[\xi^{\ell}(Rx)\big] - 4\frac{d^3}{dx^3}\big[\xi^{\ell}(Rx)\big], \end{aligned}$$

which implies by (4.1) that (recall that $0 \le \xi \le 1$)

$$\operatorname{supp}\left(\frac{d^4 \xi_R}{dx^4}\right) \subset \left[\frac{1}{2R}, \frac{1}{R}\right] \tag{4.10}$$

and for all $x \in \operatorname{supp}(\frac{d^4 \xi_R}{dx^4})$,

$$\begin{aligned} \left| \frac{d^4 \xi_R}{dx^4} \right| &\le C\left((1-x)\left|\frac{d^4}{dx^4}\big[\xi^{\ell}(Rx)\big]\right| + \left|\frac{d^3}{dx^3}\big[\xi^{\ell}(Rx)\big]\right|\right) \\ &\le C\big(R^4 \xi^{\ell-4}(Rx) + R^3 \xi^{\ell-3}(Rx)\big) \\ &\le C R^4 \xi^{\ell-4}(Rx). \end{aligned} \tag{4.11}$$

Then, from (4.5), (4.9), (4.10) and (4.11), we deduce that

$$\begin{aligned} \operatorname{Cap}_3(\varphi) &\le C R^{\frac{4p}{p-1}} T^{-\ell} \int_{t=0}^{T} \int_{x=\frac{1}{2R}}^{\frac{1}{R}} (T-t)^{\ell} \xi^{\ell - \frac{4p}{p-1}}(Rx) h^{\frac{-1}{p-1}}(t,x)\, dx\, dt \\ &\le C R^{\frac{4p}{p-1}} \int_{t=0}^{T} \int_{x=\frac{1}{2R}}^{\frac{1}{R}} h^{\frac{-1}{p-1}}(t,x)\, dx\, dt, \end{aligned}$$

which proves (4.8). □

5 Proofs of the obtained results

This section is devoted to the proofs of Theorems 3.1, 3.2, 3.3 and Corollaries 3.4, 3.5, 3.6 and 3.7.

Proof of Theorem 3.1 Let us suppose that

$$u \in L^p_{\text{loc}}\big([0,\infty) \times (0,1], h(t,x)\, dt\, dx\big) \cap L^1_{\text{loc}}\big([0,\infty) \times (0,1]\big)$$

is a weak solution to (1.1)–(1.3). By Lemma 4.1 and (3.1), for all $\ell, T, R \gg 1$, we have

$$\begin{aligned}
&\int_{Q_T} |u|^p h(t,x)\varphi\,dx\,dt + \int_0^1 u_0(x)\big(I_T^{1-\alpha}\varphi(0,x) - \big(I_T^{2-\beta}\varphi\big)_t(0,x)\big)\,dx \\
&\quad + \int_0^1 u_1(x) I_T^{2-\beta}\varphi(0,x)\,dx \\
&\leq \int_{Q_T} |u|\big(\big|\big(I_T^{1-\alpha}\varphi\big)_t\big| + \big|\big(I_T^{2-\beta}\varphi\big)_{tt}\big| + |\varphi_{xxxx}|\big)\,dx\,dt,
\end{aligned} \tag{5.1}$$

where φ is the function defined by (4.2). On the other hand, by Young's inequality, we have

$$\begin{aligned}
\int_{Q_T} |u|\big|\big(I_T^{1-\alpha}\varphi\big)_t\big|\,dx\,dt &= \int_{Q_T} \big(|u| h^{\frac{1}{p}}\varphi^{\frac{1}{p}}\big)\big(\varphi^{\frac{-1}{p}}\big|\big(I_T^{1-\alpha}\varphi\big)_t\big| h^{\frac{-1}{p}}\big)\,dx\,dt \\
&\leq \frac{1}{3}\int_{Q_T} |u|^p h(t,x)\varphi\,dx\,dt + C\,\mathrm{Cap}_1(\varphi),
\end{aligned} \tag{5.2}$$

where $\mathrm{Cap}_1(\varphi)$ is the integral term given by (4.3). Similarly, we have

$$\int_{Q_T} |u|\big|\big(I_T^{2-\beta}\varphi\big)_{tt}\big|\,dx\,dt \leq \frac{1}{3}\int_{Q_T} |u|^p h(t,x)\varphi\,dx\,dt + C\,\mathrm{Cap}_2(\varphi) \tag{5.3}$$

and

$$\int_{Q_T} |u||\varphi_{xxxx}|\,dx\,dt \leq \frac{1}{3}\int_{Q_T} |u|^p h(t,x)\varphi\,dx\,dt + C\,\mathrm{Cap}_3(\varphi), \tag{5.4}$$

where $\mathrm{Cap}_i(\varphi)$, $i = 2, 3$, are given by (4.4) and (4.5). In view of (5.1), (5.2), (5.3) and (5.4), we get

$$\begin{aligned}
&\int_0^1 u_0(x)\big(I_T^{1-\alpha}\varphi(0,x) - \big(I_T^{2-\beta}\varphi\big)_t(0,x)\big)\,dx + \int_0^1 u_1(x) I_T^{2-\beta}\varphi(0,x)\,dx \\
&\leq C\sum_{i=1}^{3}\mathrm{Cap}_i(\varphi).
\end{aligned} \tag{5.5}$$

On the other hand, by (4.2), (2.3) (with $\kappa \in \{1-\alpha, 2-\beta\}$) and (2.4) (with $\kappa = 2-\beta$), for all $x \in (0,1)$, we have

$$\begin{aligned}
&I_T^{1-\alpha}\varphi(0,x) = C_1 T^{1-\alpha}(1-x)\xi^{\ell}(Rx), \\
&\big(I_T^{2-\beta}\varphi\big)_t(0,x) = -C_2 T^{1-\beta}(1-x)\xi^{\ell}(Rx), \\
&I_T^{2-\beta}\varphi(0,x) = C_3 T^{2-\beta}(1-x)\xi^{\ell}(Rx).
\end{aligned}$$

Then, it holds that

$$\begin{aligned}
&\int_0^1 u_0(x)\big(I_T^{1-\alpha}\varphi(0,x) - \big(I_T^{2-\beta}\varphi\big)_t(0,x)\big)\,dx + \int_0^1 u_1(x) I_T^{2-\beta}\varphi(0,x)\,dx \\
&= \big(C_1 T^{1-\alpha} + C_2 T^{1-\beta}\big)\int_0^1 u_0(x)(1-x)\xi^{\ell}(Rx)\,dx
\end{aligned} \tag{5.6}$$

$$+ C_3 T^{2-\beta} \int_0^1 u_1(x)(1-x)\xi^\ell(Rx)\,dx.$$

Since $u_1 \equiv 0$, (5.6) reduces to

$$\begin{aligned}
&\int_0^1 u_0(x)\big(I_T^{1-\alpha}\varphi(0,x) - \big(I_T^{2-\beta}\varphi\big)_t(0,x)\big)\,dx + \int_0^1 u_1(x) I_T^{2-\beta}\varphi(0,x)\,dx \\
&\quad = \big(C_1 T^{1-\alpha} + C_2 T^{1-\beta}\big)\int_0^1 u_0(x)(1-x)\xi^\ell(Rx)\,dx.
\end{aligned} \tag{5.7}$$

Since $u_0 \in L^1((0,1))$, the properties of ξ and the dominated convergence theorem give us that

$$\lim_{R\to\infty}\int_0^1 u_0(x)(1-x)\xi^\ell(Rx)\,dx = \int_0^1 u_0(x)(1-x)\,dx.$$

Furthermore, the positivity of $\int_0^1 u_0(x)(1-x)\,dx$ (by (3.2)) and the definition of the limit imply that for $R \gg 1$,

$$\int_0^1 u_0(x)(1-x)\xi^\ell(Rx)\,dx \geq \frac{1}{2}\int_0^1 u_0(x)(1-x)\,dx,$$

which yields (for $T \gg 1$)

$$\big(C_1 T^{1-\alpha} + C_2 T^{1-\beta}\big)\int_0^1 u_0(x)(1-x)\xi^\ell(Rx)\,dx \geq C T^{1-\alpha}\int_0^1 u_0(x)(1-x)\,dx. \tag{5.8}$$

Now, it follows from (5.5), (5.7), (5.8), Lemmas 4.2, 4.3 and 4.4 that

$$\begin{aligned}
T^{1-\alpha}\int_0^1 u_0(x)(1-x)\,dx &\leq C\big(T^{\frac{-\alpha p}{p-1}} + T^{\frac{-\beta p}{p-1}} + R^{\frac{4p}{p-1}}\big)\int_{t=0}^T\int_{x=\frac{1}{2R}}^1 h^{\frac{-1}{p-1}}(t,x)\,dx\,dt \\
&\leq C\big(T^{\frac{-\alpha p}{p-1}} + R^{\frac{4p}{p-1}}\big)\int_{t=0}^T\int_{x=\frac{1}{2R}}^1 h^{\frac{-1}{p-1}}(t,x)\,dx\,dt,
\end{aligned}$$

that is,

$$\int_0^1 u_0(x)(1-x)\,dx \leq C\big(T^{-(\frac{\alpha}{p-1}+1)} + T^{\alpha-1}R^{\frac{4p}{p-1}}\big)\int_{t=0}^T\int_{x=\frac{1}{2R}}^1 h^{\frac{-1}{p-1}}(t,x)\,dx\,dt. \tag{5.9}$$

We now take $2R = T^\theta$, where $\theta > 0$, and (5.9) reduces to

$$\begin{aligned}
\int_0^1 u_0(x)(1-x)\,dx &\leq C\big(T^{-(\frac{\alpha}{p-1}+1)} + T^{\alpha-1+\frac{4\theta p}{p-1}}\big)\int_{t=0}^T\int_{x=T^{-\theta}}^1 h^{\frac{-1}{p-1}}(t,x)\,dx\,dt \\
&= C\big(T^{-(\frac{\alpha}{p-1}+\alpha+\frac{4\theta p}{p-1})} + 1\big)T^{\alpha-1+\frac{4\theta p}{p-1}}\int_{t=0}^T\int_{x=T^{-\theta}}^1 h^{\frac{-1}{p-1}}(t,x)\,dx\,dt,
\end{aligned}$$

which implies that

$$\int_0^1 u_0(x)(1-x)\,dx \leq C T^{\alpha-1+\frac{4\theta p}{p-1}}\int_{t=0}^T\int_{x=T^{-\theta}}^1 h^{\frac{-1}{p-1}}(t,x)\,dx\,dt. \tag{5.10}$$

Hence, if (3.3) is satisfied for some $\theta > 0$, then passing to the infimum limit as $T \to \infty$ in (5.10), we obtain $\int_0^1 u_0(x)(1-x)\,dx \le 0$, which contradicts (3.2). This completes the proof of Theorem 3.1. □

Proof of Theorem 3.2 We also use the contradiction argument supposing that u is a weak solution to (1.1)–(1.3). Following exactly the first steps of the proof of Theorem 3.1, we obtain (5.5) and (5.6). Since $u_0 \equiv 0$, (5.6) reduces to

$$\begin{aligned}&\int_0^1 u_0(x)\big(I_T^{1-\alpha}\varphi(0,x) - \big(I_T^{2-\beta}\varphi\big)_t(0,x)\big)\,dx + \int_0^1 u_1(x) I_T^{2-\beta}\varphi(0,x)\,dx \\ &\quad = C_3 T^{2-\beta}\int_0^1 u_1(x)(1-x)\xi^\ell(Rx)\,dx.\end{aligned} \tag{5.11}$$

As in the proof of Theorem 3.1, by (3.4), the properties of ξ and the dominated convergence theorem, we have

$$\lim_{R\to\infty}\int_0^1 u_1(x)(1-x)\xi^\ell(Rx)\,dx = \int_0^1 u_1(x)(1-x)\,dx > 0,$$

which implies that for $R \gg 1$,

$$\int_0^1 u_1(x)(1-x)\xi^\ell(Rx)\,dx \ge \frac{1}{2}\int_0^1 u_1(x)(1-x)\,dx. \tag{5.12}$$

Now, using (5.5), (5.11), (5.12), Lemmas 4.2, 4.3 and 4.4, we get

$$T^{2-\beta}\int_0^1 u_1(x)(1-x)\,dx \le C\big(T^{\frac{-\alpha p}{p-1}} + R^{\frac{4p}{p-1}}\big)\int_{t=0}^T\int_{x=\frac{1}{2R}}^1 h^{\frac{-1}{p-1}}(t,x)\,dx\,dt,$$

that is,

$$\int_0^1 u_1(x)(1-x)\,dx \le C\big(T^{\beta-2-\frac{\alpha p}{p-1}} + T^{\beta-2}R^{\frac{4p}{p-1}}\big)\int_{t=0}^T\int_{x=\frac{1}{2R}}^1 h^{\frac{-1}{p-1}}(t,x)\,dx\,dt. \tag{5.13}$$

We now take $2R = T^\theta$, where $\theta > 0$, and (5.13) reduces to

$$\begin{aligned}\int_0^1 u_1(x)(1-x)\,dx &\le C\big(T^{\beta-2-\frac{\alpha p}{p-1}} + T^{\beta-2+\frac{4\theta p}{p-1}}\big)\int_{t=0}^T\int_{x=T^{-\theta}}^1 h^{\frac{-1}{p-1}}(t,x)\,dx\,dt \\ &= C\big(T^{-(\frac{\alpha p}{p-1}+\frac{4\theta p}{p-1})} + 1\big)T^{\beta-2+\frac{4\theta p}{p-1}}\int_{t=0}^T\int_{x=T^{-\theta}}^1 h^{\frac{-1}{p-1}}(t,x)\,dx\,dt,\end{aligned}$$

which implies that

$$\int_0^1 u_1(x)(1-x)\,dx \le C T^{\beta-2+\frac{4\theta p}{p-1}}\int_{t=0}^T\int_{x=T^{-\theta}}^1 h^{\frac{-1}{p-1}}(t,x)\,dx\,dt. \tag{5.14}$$

Hence, if (3.5) is satisfied for some $\theta > 0$, then passing to the infimum limit as $T \to \infty$ in (5.14), we obtain $\int_0^1 u_1(x)(1-x)\,dx \le 0$, which contradicts (3.4). This completes the proof of Theorem 3.2. □

Proof of Theorem 3.3 Assuming that u is a weak solution to (1.1)–(1.3) and following exactly the first steps of the proof of Theorem 3.1, we obtain (5.5) and (5.6). On the other hand, by (3.6), the properties of ξ and the dominated convergence theorem, for all $i = 0, 1$, we get

$$\lim_{R\to\infty}\int_0^1 u_i(x)(1-x)\xi^\ell(Rx)\,dx = \int_0^1 u_i(x)(1-x)\,dx > 0,$$

which yields (for $T, R \gg 1$)

$$\begin{aligned}
&\left(C_1T^{1-\alpha}+C_2T^{1-\beta}\right)\int_0^1 u_0(x)(1-x)\xi^\ell(Rx)\,dx + C_3T^{2-\beta}\int_0^1 u_1(x)(1-x)\xi^\ell(Rx)\,dx\\
&\quad\geq C\left(T^{1-\beta}\int_0^1 u_0(x)(1-x)\,dx + T^{2-\beta}\int_0^1 u_1(x)(1-x)\,dx\right)\\
&\quad\geq CT^{\max\{1-\alpha,2-\beta\}}\int_0^1\big(u_0(x)+u_1(x)\big)(1-x)\,dx.
\end{aligned} \tag{5.15}$$

Then, by (5.5), (5.6), (5.15), Lemmas 4.2, 4.3 and 4.4, we obtain

$$\begin{aligned}
&T^{\max\{1-\alpha,2-\beta\}}\int_0^1\big(u_0(x)+u_1(x)\big)(1-x)\,dx\\
&\quad\leq C\left(T^{\frac{-\alpha p}{p-1}}+T^{\frac{-\beta p}{p-1}}+R^{\frac{4p}{p-1}}\right)\int_{t=0}^T\int_{x=\frac{1}{2R}}^1 h^{\frac{-1}{p-1}}(t,x)\,dx\,dt\\
&\quad\leq C\left(T^{\frac{-\alpha p}{p-1}}+R^{\frac{4p}{p-1}}\right)\int_{t=0}^T\int_{x=\frac{1}{2R}}^1 h^{\frac{-1}{p-1}}(t,x)\,dx\,dt,
\end{aligned}$$

that is,

$$\begin{aligned}
&\int_0^1\big(u_0(x)+u_1(x)\big)(1-x)\,dx\\
&\quad\leq C\left(T^{-\max\{1-\alpha,2-\beta\}-\frac{\alpha p}{p-1}}+T^{-\max\{1-\alpha,2-\beta\}}R^{\frac{4p}{p-1}}\right)\int_{t=0}^T\int_{x=\frac{1}{2R}}^1 h^{\frac{-1}{p-1}}(t,x)\,dx\,dt.
\end{aligned} \tag{5.16}$$

Taking $2R = T^\theta$, where $\theta > 0$, (5.16) reduces to

$$\begin{aligned}
&\int_0^1\big(u_0(x)+u_1(x)\big)(1-x)\,dx\\
&\quad\leq C\left(T^{-\max\{1-\alpha,2-\beta\}-\frac{\alpha p}{p-1}}+T^{-\max\{1-\alpha,2-\beta\}+\frac{4\theta p}{p-1}}\right)\int_{t=0}^T\int_{x=T^{-\theta}}^1 h^{\frac{-1}{p-1}}(t,x)\,dx\,dt\\
&\quad= C\left(T^{-(\frac{\alpha p}{p-1}+\frac{4\theta p}{p-1})}+1\right)T^{-\max\{1-\alpha,2-\beta\}+\frac{4\theta p}{p-1}}\int_{t=0}^T\int_{x=T^{-\theta}}^1 h^{\frac{-1}{p-1}}(t,x)\,dx\,dt,
\end{aligned}$$

which yields

$$\int_0^1\big(u_0(x)+u_1(x)\big)(1-x)\,dx \leq CT^{-\max\{1-\alpha,2-\beta\}+\frac{4\theta p}{p-1}}\int_{t=0}^T\int_{x=T^{-\theta}}^1 h^{\frac{-1}{p-1}}(t,x)\,dx\,dt. \tag{5.17}$$

Hence, if (3.7) is satisfied for some $\theta > 0$, then passing to the infimum limit as $T \to \infty$ in (5.17), we obtain $\int_0^1 (u_0(x) + u_1(x))(1-x)\,dx \le 0$, which contradicts (3.6). This completes the proof of Theorem 3.3. □

Proof of Corollary 3.4 If $h^{\frac{-1}{p-1}} \in L^1([0,\infty) \times (0,1))$, then for all $\theta > 0$, we have

$$\lim_{T\to\infty} \int_{t=0}^{T} \int_{x=T^{-\theta}}^{1} h^{\frac{-1}{p-1}}(t,x)\,dx\,dt = \int_{t=0}^{\infty} \int_{x=0}^{1} h^{\frac{-1}{p-1}}(t,x)\,dx\,dt \in (0,\infty).$$

Then, for $T \gg 1$,

$$0 < \int_{t=0}^{T} \int_{x=T^{-\theta}}^{1} h^{\frac{-1}{p-1}}(t,x)\,dx\,dt \le C. \tag{5.18}$$

(i) If the initial data satisfy (3.2), then for all $\theta > 0$, we have by (5.18) that

$$T^{\alpha-1+\frac{4\theta p}{p-1}} \int_{t=0}^{T} \int_{x=T^{-\theta}}^{1} h^{\frac{-1}{p-1}}(t,x)\,dx\,dt \le CT^{\alpha-1+\frac{4\theta p}{p-1}}. \tag{5.19}$$

In particular, for

$$0 < \theta < \frac{(p-1)(1-\alpha)}{4p},$$

we have $\alpha - 1 + \frac{4\theta p}{p-1} < 0$, which implies by (5.19) that

$$\lim_{T\to\infty} T^{\alpha-1+\frac{4\theta p}{p-1}} \int_{t=0}^{T} \int_{x=T^{-\theta}}^{1} h^{\frac{-1}{p-1}}(t,x)\,dx\,dt = 0.$$

Then Theorem 3.1 applies.

(ii) If the initial data satisfy (3.4), then for all $\theta > 0$, we have by (5.18) that

$$T^{\beta-2+\frac{4\theta p}{p-1}} \int_{t=0}^{T} \int_{x=T^{-\theta}}^{1} h^{\frac{-1}{p-1}}(t,x)\,dx\,dt \le CT^{\beta-2+\frac{4\theta p}{p-1}}. \tag{5.20}$$

In particular, for

$$0 < \theta < \frac{(p-1)(2-\beta)}{4p},$$

we have $\beta - 2 + \frac{4\theta p}{p-1} < 0$, which implies by (5.20) that

$$\lim_{T\to\infty} T^{\beta-2+\frac{4\theta p}{p-1}} \int_{t=0}^{T} \int_{x=T^{-\theta}}^{1} h^{\frac{-1}{p-1}}(t,x)\,dx\,dt = 0.$$

Then Theorem 3.2 applies.

(iii) If the initial data satisfy (3.6), then for all $\theta > 0$, we have by (5.18) that

$$T^{-\max\{1-\alpha,2-\beta\}+\frac{4\theta p}{p-1}} \int_{t=0}^{T} \int_{x=T^{-\theta}}^{1} h^{\frac{-1}{p-1}}(t,x)\,dx\,dt \le CT^{-\max\{1-\alpha,2-\beta\}+\frac{4\theta p}{p-1}}. \tag{5.21}$$

In particular, for

$$0 < \theta < \frac{(p-1)\max\{1-\alpha, 2-\beta\}}{4p},$$

we have $-\max\{1-\alpha, 2-\beta\} + \frac{4\theta p}{p-1} < 0$, which implies by (5.21) that

$$\lim_{T\to\infty} T^{-\max\{1-\alpha,2-\beta\}+\frac{4\theta p}{p-1}} \int_{t=0}^{T}\int_{x=T^{-\theta}}^{1} h^{\frac{-1}{p-1}}(t,x)\,dx\,dt = 0.$$

Then Theorem 3.3 applies. This completes the proof of Corollary 3.4. □

Proof of Corollary 3.5 For all $\theta > 0$, we have (since $p > 1+\rho$ by (3.9))

$$\begin{aligned} \int_{t=0}^{T}\int_{x=T^{-\theta}}^{1} h^{\frac{-1}{p-1}}(t,x)\,dx\,dt &= \int_{t=0}^{T}\int_{x=T^{-\theta}}^{1} t^{\frac{-\rho}{p-1}} x^{\frac{-\sigma}{p-1}}\,dx\,dt \\ &= CT^{1-\frac{\rho}{p-1}} \int_{T^{-\theta}}^{1} x^{\frac{-\sigma}{p-1}}\,dx. \end{aligned} \tag{5.22}$$

On the other hand, we have

$$\int_{T^{-\theta}}^{1} x^{\frac{-\sigma}{p-1}}\,dx \le \begin{cases} C & \text{if } 1-\frac{\sigma}{p-1} > 0, \\ C\ln T & \text{if } 1-\frac{\sigma}{p-1} = 0, \\ CT^{-\theta(1-\frac{\sigma}{p-1})} & \text{if } 1-\frac{\sigma}{p-1} < 0, \end{cases}$$

which implies that

$$\int_{T^{-\theta}}^{1} x^{\frac{-\sigma}{p-1}}\,dx \le C\big(\ln T + T^{-\theta(1-\frac{\sigma}{p-1})}\big). \tag{5.23}$$

Then, (5.22) and (5.23) yield

$$T^{\alpha-1+\frac{4\theta p}{p-1}} \int_{t=0}^{T}\int_{x=T^{-\theta}}^{1} h^{\frac{-1}{p-1}}(t,x)\,dx\,dt \le C\big(T^{\gamma_1(\theta)}\ln T + T^{\gamma_2(\theta)}\big), \tag{5.24}$$

where

$$\begin{aligned} \gamma_1(\theta) &= \frac{\alpha(p-1)-\rho+4\theta p}{p-1}, \\ \gamma_2(\theta) &= \frac{\alpha(p-1)-\rho+\theta(3p+\sigma+1)}{p-1}. \end{aligned}$$

We now take θ so that

$$\begin{cases} 0 < \theta < \frac{\rho-\alpha(p-1)}{4p}, \\ (3p+\sigma+1)\theta < \rho-\alpha(p-1). \end{cases} \tag{5.25}$$

Notice that due to (3.9), the set of θ satisfying (5.25) is nonempty. Namely, we know from (3.9) that $\rho - \alpha(p-1) > 0$. So, if $3p + \sigma + 1 \le 0$, then for all

$$0 < \theta < \frac{\rho - \alpha(p-1)}{4p},$$

(5.25) is satisfied. If $3p + \sigma + 1 > 0$, then for all

$$0 < \theta < \min\left\{\frac{\rho - \alpha(p-1)}{4p}, \frac{\rho - \alpha(p-1)}{3p + \sigma + 1}\right\},$$

(5.25) is satisfied. Remark also that by (5.25), we have $\gamma_i(\theta) < 0$ for all $i = 1, 2$. Then, passing to the limit as $T \to \infty$ in (5.24), we get

$$\lim_{T\to\infty} T^{\alpha-1+\frac{4\theta p}{p-1}} \int_{t=0}^{T}\int_{x=T^{-\theta}}^{1} h^{\frac{-1}{p-1}}(t,x)\,dx\,dt = 0,$$

so Theorem 3.1 applies. This completes the proof of Corollary 3.5. □

Proof of Corollary 3.6 Using (5.22) and (5.23), for all $\theta > 0$, we get

$$T^{\beta-2+\frac{4\theta p}{p-1}} \int_{t=0}^{T}\int_{x=T^{-\theta}}^{1} h^{\frac{-1}{p-1}}(t,x)\,dx\,dt \le C\left(T^{\mu_1(\theta)}\ln T + T^{\mu_2(\theta)}\right), \tag{5.26}$$

where

$$\mu_1(\theta) = \frac{(\beta-1)(p-1) - \rho + 4\theta p}{p-1},$$
$$\mu_2(\theta) = \frac{(\beta-1)(p-1) - \rho + \theta(3p + \sigma + 1)}{p-1}.$$

We now take θ so that

$$\begin{cases} 0 < \theta < \frac{\rho-(\beta-1)(p-1)}{4p}, \\ (3p + \sigma + 1)\theta < \rho - (\beta-1)(p-1). \end{cases} \tag{5.27}$$

Due to (3.10), the set of θ satisfying (5.27) is nonempty. Furthermore, we have $\mu_i(\theta) < 0$ for all $i = 1, 2$. Then, passing to the limit as $T \to \infty$ in (5.26), we get

$$\lim_{T\to\infty} T^{\beta-2+\frac{4\theta p}{p-1}} \int_{t=0}^{T}\int_{x=T^{-\theta}}^{1} h^{\frac{-1}{p-1}}(t,x)\,dx\,dt = 0,$$

so Theorem 3.2 applies. This completes the proof of Corollary 3.6. □

Proof of Corollary 3.7 Using (5.22) and (5.23), for all $\theta > 0$, we get

$$T^{-\max\{1-\alpha,2-\beta\}+\frac{4\theta p}{p-1}} \int_{t=0}^{T}\int_{x=T^{-\theta}}^{1} h^{\frac{-1}{p-1}}(t,x)\,dx\,dt \le C\left(T^{\lambda_1(\theta)}\ln T + T^{\lambda_2(\theta)}\right), \tag{5.28}$$

where

$$\lambda_1(\theta) = \frac{(1 - \max\{1-\alpha, 2-\beta\})(p-1) - \rho + 4\theta p}{p-1},$$

$$\lambda_2(\theta) = \frac{(1-\max\{1-\alpha,2-\beta\})(p-1)-\rho+\theta(3p+\sigma+1)}{p-1}.$$

We now take θ so that

$$\begin{cases} 0<\theta<\frac{\rho-(1-\max\{1-\alpha,2-\beta\})(p-1)}{4p}, \\ (3p+\sigma+1)\theta<\rho-(1-\max\{1-\alpha,2-\beta\})(p-1). \end{cases} \tag{5.29}$$

Due to (3.11), the set of θ satisfying (5.29) is nonempty. We also have by (5.29) that $\lambda_i(\theta)<0$ for all $i=1,2$. Then, passing to the limit as $T\to\infty$ in (5.28), we get

$$\lim_{T\to\infty} T^{-\max\{1-\alpha,2-\beta\}+\frac{4\theta p}{p-1}} \int_{t=0}^{T}\int_{x=T^{-\theta}}^{1} h^{\frac{-1}{p-1}}(t,x)\,dx\,dt=0,$$

so Theorem 3.3 applies. The proof of Corollary 3.7 is then completed. □

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