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New version of midpoint-type inequalities for co-ordinated convex functions via generalized conformable integrals

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Abstract
In the current research, some midpoint-type inequalities are generalized for co-ordinated convex functions with the help of generalized conformable fractional integrals. Moreover, some findings of this paper include results based on Riemann–Liouville fractional integrals and Riemann integrals.

Mathematics Subject Classification: 26B25; 26D07; 26D10; 26D15

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1 Introduction

Convex functions are a fundamental and widely-used mathematical concept in various fields of analysis and optimization. A function is considered convex if the line segment connecting any two points on its graph lies either below or on the graph itself, indicating a curve that is upward-curving. Convex functions have notable properties, including the fact that the slope between any two points is either increasing or constant, making them valuable in optimization problems to find minimum or maximum values. The definition known for convex functions is as follows:

Definition 1 [12] Let I be convex set on $\mathbb{R}$. The function $\chi : I \to \mathbb{R}$ is said to be convex on I if it satisfies the following inequality:

$$\chi\big(t\lambda_1 + (1-t)\lambda_2\big) \le t\chi(\lambda_1) + (1-t)\chi(\lambda_2) \tag{1.1}$$

for all $\lambda_1, \lambda_2 \in I$ and $t \in [0,1]$. The mapping χ is a concave on I if the inequality (1.1) holds in reversed direction for all $t \in [0,1]$ and $\lambda_1, \lambda_2 \in I$.

To define convexity on co-ordinates let us first consider a bidimensional interval $\Delta := [\lambda_1, \lambda_2] \times [\mu_1, \mu_2]$ in $\mathbb{R}^2$ with $\lambda_1 < \lambda_2$ and $\mu_1 < \mu_2$. A formal definition for co-ordinated convex function may be stated as follows:

Definition 2 [11] A function $\chi : \Delta \to \mathbb{R}$ will be called coordinated convex on Δ for all $(x_1, x_2), (y_1, y_2) \in \Delta$ and $t, s \in [0,1]$ if it satisfies the following inequality:

$$\begin{aligned} &\chi\big(tx_1 + (1-t)x_2, sy_1 + (1-s)y_2\big) \\ &\quad \le ts\chi(x_1,y_1) + t(1-s)\chi(x_1,y_2) + s(1-t)\chi(x_2,y_1) + (1-t)(1-s)\chi(x_2,y_2). \end{aligned}$$

It is clear that all convex functions are convex on co-ordinates. However, not every function that is a convex function in coordinates has to be convex (see, [11]).

In the realm of inequalities, one prominent result is the Hermite–Hadamard inequality, which holds for convex functions. This inequality gives upper and lower bounds for the average value of a convex function over an interval. It serves as a powerful tool in various mathematical analyzes and has applications in diverse scientific fields (see, e.g., [12], [25, p.137]). Hermite–Hadamard inequality is stated that if $\chi : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\lambda_1, \lambda_2 \in I$ with $\lambda_1 < \lambda_2$, then

$$\chi\left(\frac{\lambda_1+\lambda_2}{2}\right) \le \frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2} \chi(\delta)\,d\delta \le \frac{\chi(\lambda_1)+\chi(\lambda_2)}{2}. \tag{1.2}$$

If χ is concave, the inequality that is stated above is provided reversely. The references may be seen for the examples of Hermite–Hadamard's inequality for some convex function on the co-ordinates in mathematics literature [3–5, 7, 8, 10, 22, 23, 28, 31]. Recently, this inequality has been expanded by many researchers. The left side of the Hermite–Hadamard inequality, namely the midpoint type inequality, has been the focus of many studies. Midpoint type inequalities for convex functions were first derived by Kırmacıin [21]. In [32], Sarikaya et al. generalized the inequalities (1.2) for fractional integrals. Iqbal et al. proved corresponding midpoint type inequalities for Riemann–Liouville fractional integrals in [15].

In [11], Dragomir proved the Hermite–Hadamard inequality, which formed the basis of this article and is valid for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$.

Theorem 1 *Suppose that $\chi : \Delta \to \mathbb{R}$ is co-ordinated convex, then we have the following inequalities*:

$$\begin{aligned} \chi\left(\frac{\lambda_1+\lambda_2}{2}, \frac{\mu_1+\mu_2}{2}\right) &\le \frac{1}{2}\left[\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2} \chi\left(\delta, \frac{\mu_1+\mu_2}{2}\right) d\delta \right. \\ &\quad \left. + \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} \chi\left(\frac{\lambda_1+\lambda_2}{2}, \rho\right) d\rho\right] \\ &\le \frac{1}{(\lambda_2-\lambda_1)(\mu_2-\mu_1)}\int_{\lambda_1}^{\lambda_2}\int_{\mu_1}^{\mu_2} \chi(\delta,\rho)\,d\rho\,d\delta \\ &\le \frac{1}{4}\left[\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2} \chi(\delta,\mu_1)\,d\delta + \frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2} \chi(\delta,\mu_2)\,d\delta \right. \\ &\quad \left. + \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} \chi(\lambda_1,\rho)\,d\rho + \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} \chi(\lambda_2,\rho)\,d\rho\right] \\ &\le \frac{\chi(\lambda_1,\mu_1)+\chi(\lambda_1,\mu_2)+\chi(\lambda_2,\mu_1)+\chi(\lambda_2,\mu_2)}{4}. \end{aligned} \tag{1.3}$$

The inequalities in (1.3) hold in reverse direction if the mapping χ is a co-ordinated concave mapping.

The fractional calculus [16, 19, 24, 26, 27] is defined as any random real number or derivative and integral calculus in complex order. As a result of having various uses in other branches besides mathematics it is an updated study area. These definitions are the most notable definitions of Caputo, Riemann–Liouville, Grünwald–Letnikov play an important role in many fields such as physics, biology, and engineering. However, it is known that these definitions have some difficulties despite their availability. For instance, unless derivative of order in Riemann–Liouville fractional derivative definition is a natural number, derivative of fixed function is not 0. Likewise, the function f must be differentiable in Caputo fractional derivatives. Moreover, many definitions of fractional derivatives do not provide the quotient formula, the product of two functions, and the chain rule. In order to overcome these and similar difficulties, conformable fractional derivative was defined by Khalil et al. in [17]. Khalil et al. described the higher order ($\alpha > 1$) fractional derivative and the fractional integral of order ($0 < \alpha \leq 1$). They also proved important theorems such as the product rule, the fractional mean value theorem. They solved conformable fractional differential equations for fractional exponential functions (see, [2, 13, 17, 33]). Thus, conformable fractional integrals became an important field of study for many researchers. For some papers on conformable fractional integrals, please see [1, 6, 14, 17, 18, 29, 30].

The definitions and mathematical underpinnings of conformable fractional calculus principles that are used later in this study are provided below:

Definition 3 [19] For $\xi \in L_1[\eta_1, \eta_2]$, the Riemann–Liouville integrals of order $\alpha > 0$ are given by

$$J^{\alpha}_{\eta_1+}\xi(\delta) = \frac{1}{\Gamma(\alpha)}\int_{\eta_1}^{\delta} (\delta - t)^{\alpha-1}\xi(t)\,dt, \quad \delta > \eta_1 \tag{1.4}$$

and

$$J^{\alpha}_{\eta_2-}\xi(\delta) = \frac{1}{\Gamma(\alpha)}\int_{\delta}^{\eta_2} (t - \delta)^{\alpha-1}\xi(t)\,dt, \quad \delta < \eta_2, \tag{1.5}$$

respectively. Here Γ is the Gamma function. The Riemann–Liouville integrals will be equal to the classical Riemann integrals for the condition $\alpha = 1$.

Definition 4 [28] Let $\xi \in L_1([\eta_1, \eta_2] \times [\vartheta_1, \vartheta_2])$. The Riemann–Liouville integrals $J^{\alpha,\beta}_{\eta_1+,\vartheta_1^+}$, $J^{\alpha,\beta}_{\eta_1+,\vartheta_2^-}$, $J^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}$ and $J^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}$ of order $\alpha, \beta > 0$ with $\eta_1, \vartheta_1 \geq 0$ are defined by

$$\begin{aligned}
&J^{\alpha,\beta}_{\eta_1+,\vartheta_1^+}\xi(\delta,\rho) \\
&\quad = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\eta_1}^{\delta}\int_{\vartheta_1}^{\rho} (\delta - t)^{\alpha-1}(\rho - s)^{\beta-1}\xi(t,s)\,ds\,dt, \quad \delta > \eta_1, \rho > \vartheta_1, \qquad (1.6) \\
&J^{\alpha,\beta}_{\eta_1+,\vartheta_2^-}\xi(\delta,\rho) \\
&\quad = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\eta_1}^{\delta}\int_{\rho}^{\vartheta_2} (\delta - t)^{\alpha-1}(s - \rho)^{\beta-1}\xi(t,s)\,ds\,dt, \quad \delta > \eta_1, \rho < \vartheta_2, \qquad (1.7)
\end{aligned}$$

$$J^{\alpha,\beta}_{\eta_2-,\vartheta_1^+}\xi(\delta,\rho) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_\delta^{\eta_2}\int_{\vartheta_1}^{\rho}(t-\delta)^{\alpha-1}(\rho-s)^{\beta-1}\xi(t,s)\,ds\,dt, \delta<\eta_2, \quad \rho>\vartheta_1, \tag{1.8}$$

and

$$J^{\alpha,\beta}_{\eta_2-,\vartheta_2^-}\xi(\delta,\rho) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_\delta^{\eta_2}\int_{\rho}^{\vartheta_2}(t-\delta)^{\alpha-1}(s-\rho)^{\beta-1}\xi(t,s)\,ds\,dt, \quad \delta<\eta_2, \rho<\vartheta_2, \tag{1.9}$$

respectively.

Definition 5 [16] For $\xi\in L_1[\eta_1,\eta_2]$, the fractional conformable integral operator ${}^{\beta}I^{\alpha}_{\eta_1+}\xi$ and ${}^{\beta}I^{\alpha}_{\eta_2-}\xi$ of order $\beta>0$ and $\alpha\in(0,1]$ are presented by

$${}^{\beta}\mathcal{I}^{\alpha}_{\eta_1+}\xi(\delta) = \frac{1}{\Gamma(\beta)}\int_{\eta_1}^{\delta}\left(\frac{(\delta-\eta_1)^\alpha-(t-\eta_1)^\alpha}{\alpha}\right)^{\beta-1}\frac{\xi(t)}{(t-\eta_1)^{1-\alpha}}\,dt, \quad t>\eta_1 \tag{1.10}$$

and

$${}^{\beta}\mathcal{I}^{\alpha}_{\eta_2-}\xi(\delta) = \frac{1}{\Gamma(\beta)}\int_{\delta}^{\eta_2}\left(\frac{(\eta_2-\delta)^\alpha-(\eta_2-t)^\alpha}{\alpha}\right)^{\beta-1}\frac{\xi(t)}{(\eta_2-t)^{1-\alpha}}\,dt, \quad t<\eta_2, \tag{1.11}$$

respectively.

Definition 6 [9] Let $\xi\in L_1([\eta_1,\eta_2]\times[\vartheta_1,\vartheta_2])$ and let $\gamma_1\neq 0$, $\gamma_2\neq 0$, $\alpha,\beta\in\mathbf{C}$, $\mathrm{Re}(\alpha)>0$ and $\mathrm{Re}(\beta)>0$. The generalized conformable integral of order α,β of $\xi(\delta,\rho)$ are defined by

$$\begin{aligned}\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\right)(\delta,\rho) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\eta_1}^{\delta}\int_{\vartheta_1}^{\rho}\left(\frac{(\delta-\eta_1)^{\gamma_1}-(t-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1} \\ &\quad\times\left(\frac{(\rho-\vartheta_1)^{\gamma_2}-(s-\vartheta_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(t,s)}{(t-\eta_1)^{1-\gamma_1}(s-\vartheta_1)^{1-\gamma_2}}\,ds\,dt,\end{aligned} \tag{1.12}$$

$$\begin{aligned}\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\right)(\delta,\rho) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\delta}^{\eta_2}\int_{\vartheta_1}^{\rho}\left(\frac{(\eta_2-\delta)^{\gamma_1}-(\eta_2-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1} \\ &\quad\times\left(\frac{(\rho-\vartheta_1)^{\gamma_2}-(s-\vartheta_1)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(t,s)}{(\eta_2-t)^{1-\gamma_1}(s-\vartheta_1)^{1-\gamma_2}}\,ds\,dt,\end{aligned} \tag{1.13}$$

$$\begin{aligned}\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\right)(\delta,\rho) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\eta_1}^{\delta}\int_{\rho}^{\vartheta_2}\left(\frac{(\delta-\eta_1)^{\gamma_1}-(t-\eta_1)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1} \\ &\quad\times\left(\frac{(\vartheta_2-\rho)^{\gamma_2}-(\vartheta_2-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(t,s)}{(t-\eta_1)^{1-\gamma_1}(\vartheta_2-s)^{1-\gamma_2}}\,ds\,dt,\end{aligned} \tag{1.14}$$

and

$$\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\right)(\delta,\rho) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_\delta^{\eta_2}\int_\rho^{\vartheta_2}\left(\frac{(\eta_2-\delta)^{\gamma_1}-(\eta_2-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1} \times\left(\frac{(\vartheta_2-\rho)^{\gamma_2}-(\vartheta_2-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{\xi(t,s)}{(\eta_2-t)^{1-\gamma_1}(\vartheta_2-s)^{1-\gamma_2}}\,ds\,dt. \tag{1.15}$$

Remark 1 [9] If we choose $\gamma_1=\gamma_2=1$ in (1.12)–(1.15), then we have the fractional integrals (1.6)–(1.9), respectively.

Remark 2 [9] If we consider $\alpha=1$ and $\beta=1$ in (1.12)–(1.15), then we have

$$\left(I^{1,1}_{\eta_1^+,\vartheta_1^+}\xi\right)(\delta,\rho) = \int_{\eta_1}^{\delta}\int_{\vartheta_1}^{\rho}\frac{\xi(t,s)}{(t-\eta_1)^{1-\gamma_1}(s-\vartheta_1)^{1-\gamma_2}}\,ds\,dt, \tag{1.16}$$

$$\left(I^{1,1}_{\eta_2^-,\vartheta_1^+}\xi\right)(\delta,\rho) = \int_{\delta}^{\eta_2}\int_{\vartheta_1}^{\rho}\frac{\xi(t,s)}{(\eta_2-t)^{1-\gamma_1}(s-\vartheta_1)^{1-\gamma_2}}\,ds\,dt, \tag{1.17}$$

$$\left(I^{1,1}_{\eta_1^+,\vartheta_2^-}\xi\right)(\delta,\rho) = \int_{\eta_1}^{\delta}\int_{\rho}^{\vartheta_2}\frac{\xi(t,s)}{(t-\eta_1)^{1-\gamma_1}(\vartheta_2-s)^{1-\gamma_2}}\,ds\,dt, \tag{1.18}$$

and

$$\left(I^{1,1}_{\eta_2^-,\vartheta_2^-}\xi\right)(\delta,\rho) = \int_{\delta}^{\eta_2}\int_{\rho}^{\vartheta_2}\frac{\xi(t,s)}{(\eta_2-t)^{1-\gamma_1}(\vartheta_2-s)^{1-\gamma_2}}\,ds\,dt. \tag{1.19}$$

Theorem 2 [20] *Assume that $\xi:[\eta_1,\eta_2]\times[\vartheta_1,\vartheta_2]\to\mathbb{R}$ is a co-ordinated convex function and let $\gamma_1\neq 0$, $\gamma_2\neq 0$, $\gamma_1,\gamma_2\in(0,1]$, $\mathrm{Re}(\alpha)>0$ and $\mathrm{Re}(\beta)>0$. The following inequalities hold for generalized conformable fractional integrals.*

$$\begin{aligned}
&\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&\le \frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^\alpha\gamma_2^\beta}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\Bigg[{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&\quad+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&\quad+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\Bigg]\\
&\le \frac{\xi(\eta_1,\vartheta_1)+\xi(\eta_1,\vartheta_2)+\xi(\eta_2,\vartheta_1)+\xi(\eta_2,\vartheta_2)}{4}.
\end{aligned}\tag{1.20}$$

2 Midpoint type inequalities for co-ordinated convex functions

Lemma 1 *Let $\xi:\Delta:=[\eta_1,\eta_2]\times[\vartheta_1,\vartheta_2]\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $(\eta_1,\eta_2)]\times(\vartheta_1,\vartheta_2)$. If $\frac{\partial^2\xi(t,s)}{\partial t\partial s}\in L_1(\Delta)$, then the following identity holds*:

$$\begin{aligned}
&\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+\frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^\alpha\gamma_2^\beta}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\\
&\quad\times\Bigg[{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)
\end{aligned}\tag{2.1}$$

$$+^{\gamma_1\gamma_2} I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+^{\gamma_1\gamma_2} I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)$$
$$+^{\gamma_1\gamma_2} I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\Bigg]-A$$
$$=\frac{\gamma_1^{\alpha}\gamma_2^{\beta}(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}{16}$$
$$\times\left\{\int_0^1\int_0^1\left[\frac{1}{\gamma_1^{\alpha}}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\right]\left[\frac{1}{\gamma_2^{\beta}}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right]\right.$$
$$\times\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)ds\,dt$$
$$-\int_0^1\int_0^1\left[\frac{1}{\gamma_1^{\alpha}}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\right]\left[\frac{1}{\gamma_2^{\beta}}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right]$$
$$\times\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)ds\,dt$$
$$-\int_0^1\int_0^1\left[\frac{1}{\gamma_1^{\alpha}}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\right]\left[\frac{1}{\gamma_2^{\beta}}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right]$$
$$\times\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)ds\,dt$$
$$+\int_0^1\int_0^1\left[\frac{1}{\gamma_1^{\alpha}}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\right]\left[\frac{1}{\gamma_2^{\beta}}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right]$$
$$\left.\times\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)ds\,dt\right\},$$

where

$$A=\frac{2^{\gamma_2\beta-1}\gamma_2^{\beta}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\tag{2.2}$$
$$\times\left[{}^{\gamma_2}I^{\beta}_{\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_2}I^{\beta}_{\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right]$$
$$+.\frac{2^{\gamma_1\alpha-1}\gamma_1^{\alpha}\Gamma(\alpha+1)}{(\eta_2-\eta_1)^{\gamma_1\alpha}}\left[{}^{\gamma_1}I^{\alpha}_{\eta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1}I^{\alpha}_{\eta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right].$$

Proof By integration by parts, we get

$$I_1=\int_0^1\int_0^1\left[\frac{1}{\gamma_1^{\alpha}}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\right]\left[\frac{1}{\gamma_2^{\beta}}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right]\frac{\partial^2\xi}{\partial t\partial s}$$
$$\times\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)ds\,dt$$
$$=\int_0^1\left[\frac{1}{\gamma_2^{\beta}}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right]\left[\frac{1}{\gamma_1^{\alpha}}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\right]$$
$$\times\left\{\frac{-2}{(\eta_2-\eta_1)}\frac{\partial\xi}{\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)\Bigg|_0^1\right.$$
$$+\int_0^1\frac{2\alpha}{(\eta_2-\eta_1)}\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\frac{\partial\xi}{\partial s}$$

$$
\begin{aligned}
&\times \left(\frac{1+t}{2}\eta_1 + \frac{1-t}{2}\eta_2, \frac{1+s}{2}\vartheta_1 + \frac{1-s}{2}\vartheta_2\right) dt\Bigg\} ds \\
&= \int_0^1 \left[\frac{1}{\gamma_2^\beta} - \left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^\beta\right]\Bigg\{\left(\frac{2}{\eta_2-\eta_1}\right)\frac{1}{\gamma_1^\alpha}\frac{\partial\xi}{\partial s}\left(\frac{\eta_1+\eta_2}{2}, \frac{1+s}{2}\vartheta_1 + \frac{1-s}{2}\vartheta_2\right) \\
&\quad - \frac{2\alpha}{(\eta_2-\eta_1)}\int_0^1 \left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1}\frac{\partial\xi}{\partial s} \\
&\quad \times \left(\frac{1+t}{2}\eta_1 + \frac{1-t}{2}\eta_2, \frac{1+s}{2}\vartheta_1 + \frac{1-s}{2}\vartheta_2\right) dt\Bigg\} ds \\
&= \frac{2}{(\eta_2-\eta_1)\gamma_1^\alpha}\int_0^1 \left[\frac{1}{\gamma_2^\beta} - \left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^\beta\right]\frac{\partial\xi}{\partial s}\left(\frac{\eta_1+\eta_2}{2}, \frac{1+s}{2}\vartheta_1 + \frac{1-s}{2}\vartheta_2\right) ds \\
&\quad - \frac{2\alpha}{(\eta_2-\eta_1)}\Bigg[\int_0^1 \left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}(1-t)^{\gamma_1-1} \\
&\quad \times \Bigg\{\int_0^1 \left[\frac{1}{\gamma_2^\beta} - \left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^\beta\right] \\
&\quad \times \frac{\partial\xi}{\partial s}\left(\frac{1+t}{2}\eta_1 + \frac{1-t}{2}\eta_2, \frac{1+s}{2}\vartheta_1 + \frac{1-s}{2}\vartheta_2\right) ds\Bigg\} dt\Bigg] \\
&= \frac{2}{(\eta_2-\eta_1)}\left(\frac{1}{\gamma_1}\right)^\alpha \Bigg[\left(\frac{1}{\gamma_2}\right)^\beta \frac{2}{(\vartheta_2-\vartheta_1)}\xi\left(\frac{\eta_1+\eta_2}{2}, \frac{\vartheta_1+\vartheta_2}{2}\right) \\
&\quad - \frac{2\beta}{(\vartheta_2-\vartheta_1)}\int_0^1 \left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1}\xi\left(\frac{\eta_1+\eta_2}{2}, \frac{1+s}{2}\vartheta_1 + \frac{1-s}{2}\vartheta_2\right) ds\Bigg] \\
&\quad - \frac{2\alpha}{(\eta_2-\eta_1)}\int_0^1 \left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1} \\
&\quad \times (1-t)^{\gamma_1-1}\Bigg\{\left(\frac{1}{\gamma_2}\right)^\beta \frac{2}{(\vartheta_2-\vartheta_1)}\xi\left(\frac{1+t}{2}\eta_1 + \frac{1-t}{2}\eta_2, \frac{\vartheta_1+\vartheta_2}{2}\right) \\
&\quad - \frac{2\beta}{(\vartheta_2-\vartheta_1)}\int_0^1 \left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1} \\
&\quad \times (1-s)^{\gamma_2-1}\xi\left(\frac{1+t}{2}\eta_1 + \frac{1-t}{2}\eta_2, \frac{1+s}{2}\vartheta_1 + \frac{1-s}{2}\vartheta_2\right) ds\Bigg\} dt \\
&= \frac{4}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{1}{\gamma_1^\alpha\gamma_2^\beta}\xi\left(\frac{\eta_1+\eta_2}{2}, \frac{\vartheta_1+\vartheta_2}{2}\right) \\
&\quad - \frac{4\beta}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\left(\frac{1}{\gamma_1}\right)^\alpha \int_0^1 \left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1} \\
&\quad \times (1-s)^{\gamma_2-1}\xi\left(\frac{\eta_1+\eta_2}{2}, \frac{1+s}{2}\vartheta_1 + \frac{1-s}{2}\vartheta_2\right) ds \\
&\quad - \frac{4\alpha}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\left(\frac{1}{\gamma_2}\right)^\beta \int_0^1 \left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1} \\
&\quad \times (1-t)^{\gamma_1-1}\xi\left(\frac{1+t}{2}\eta_1 + \frac{1-t}{2}\eta_2, \frac{\vartheta_1+\vartheta_2}{2}\right) dt \\
&\quad + \frac{4\alpha\beta}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\Bigg[\int_0^1 \int_0^1 \left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}
\end{aligned}
$$

$$\times (1-t)^{\gamma_1-1}\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}(1-s)^{\gamma_2-1}$$
$$\times \xi\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2, \frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)ds\,dt\Bigg]. \tag{2.3}$$

In (2.3), using the change of the variables, we can write

$$\begin{aligned}
I_1 &= \frac{4}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{1}{\gamma_1^\alpha\gamma_2^\beta}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right) \\
&\quad - \frac{4\beta}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{1}{\gamma_1^\alpha}\left(\frac{2}{\vartheta_2-\vartheta_1}\right)^{\gamma_2\beta}\Gamma(\beta)\left({}^{\gamma_2}I^\beta_{\vartheta_1^+}\xi\right)\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right) \\
&\quad - \frac{4\alpha}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{1}{\gamma_2^\beta}\left(\frac{2}{\eta_2-\eta_1}\right)^{\gamma_1\alpha}\Gamma(\alpha)^{\gamma_1}I^\alpha_{\eta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right) \\
&\quad + \frac{4\alpha\beta}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{2^{\gamma_1\alpha}2^{\gamma_2\beta}\Gamma(\alpha)\Gamma(\beta)}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\right) \\
&\quad \times\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right).
\end{aligned} \tag{2.4}$$

Thus, similarly, by integration by parts it follows that

$$\begin{aligned}
I_2 &= \int_0^1\int_0^1\left[\frac{1}{\gamma_1^\alpha}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^\alpha\right]\left[\frac{1}{\gamma_2^\beta}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^\beta\right] \\
&\quad \times\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)ds\,dt \\
&= \frac{-4}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{1}{\gamma_1^\alpha\gamma_2^\beta}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right) \\
&\quad + \frac{4\beta}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{1}{\gamma_1^\alpha}\left(\frac{2}{\vartheta_2-\vartheta_1}\right)^{\gamma_2\beta}\Gamma(\beta)\left({}^{\gamma_2}I^\beta_{\vartheta_2^-}\xi\right)\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right) \\
&\quad + \frac{4\alpha}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{1}{\gamma_2^\beta}\left(\frac{2}{\eta_2-\eta_1}\right)^{\gamma_1\alpha}\Gamma(\alpha)\left({}^{\gamma_1}I^\alpha_{\eta_1^+}\xi\right)\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right) \\
&\quad - \frac{4\alpha\beta}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{2^{\gamma_1\alpha}2^{\gamma_2\beta}\Gamma(\alpha)\Gamma(\beta)}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left({}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\right) \\
&\quad \times\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right),
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
I_3 &= \int_0^1\int_0^1\left[\frac{1}{\gamma_1^\alpha}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^\alpha\right]\left[\frac{1}{\gamma_2^\beta}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^\beta\right] \\
&\quad \times\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)ds\,dt \\
&= \frac{-4}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{1}{\gamma_1^\alpha\gamma_2^\beta}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right) \\
&\quad + \frac{4\beta}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{1}{\gamma_1^\alpha}\left(\frac{2}{\vartheta_2-\vartheta_1}\right)^{\gamma_2\beta}\Gamma(\beta)\left({}^{\gamma_2}I^\beta_{\vartheta_1^+}\xi\right)\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{4\alpha}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{1}{\gamma_2^{\beta}}\left(\frac{2}{\eta_2-\eta_1}\right)^{\gamma_1\alpha}\Gamma(\alpha)\left({}^{\gamma_1}I_{\eta_2^-}^{\alpha}\xi\right)\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&-\frac{4\alpha\beta}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{2^{\gamma_1\alpha}2^{\gamma_2\beta}\Gamma(\alpha)\Gamma(\beta)}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left({}^{\gamma_1\gamma_2}I_{\eta_2^-,\vartheta_1^+}^{\alpha,\beta}\xi\right)\\
&\times\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right),
\end{aligned}
\tag{2.6}
$$

and

$$
\begin{aligned}
I_4&=\int_0^1\int_0^1\left[\frac{1}{\gamma_1^{\alpha}}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\right]\left[\frac{1}{\gamma_2^{\beta}}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right]\\
&\quad\times\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)ds\,dt\\
&=\frac{4}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{1}{\gamma_1^{\alpha}\gamma_2^{\beta}}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&\quad-\frac{4\beta}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{1}{\gamma_1^{\alpha}}\left(\frac{2}{\vartheta_2-\vartheta_1}\right)^{\gamma_2\beta}\Gamma(\beta)\left({}^{\gamma_2}I_{\vartheta_2^-}^{\beta}\xi\right)\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&\quad-\frac{4\alpha}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{1}{\gamma_2^{\beta}}\left(\frac{2}{\eta_2-\eta_1}\right)^{\gamma_1\alpha}\Gamma(\alpha)\left({}^{\gamma_1}I_{\eta_2^-}^{\alpha}\xi\right)\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&\quad+\frac{4\alpha\beta}{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}\frac{2^{\gamma_1\alpha}2^{\gamma_2\beta}\Gamma(\alpha)\Gamma(\beta)}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left({}^{\gamma_1\gamma_2}I_{\eta_2^-,\vartheta_2^-}^{\alpha,\beta}\xi\right)\\
&\quad\times\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right).
\end{aligned}
\tag{2.7}
$$

By the equalities (2.4)–(2.7), we obtain

$$
\begin{aligned}
&\frac{\gamma_1^{\alpha}\gamma_2^{\beta}(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}{16}[I_1-I_2-I_3+I_4]\\
&=\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+\frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^{\alpha}\gamma_2^{\beta}}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\\
&\quad\times\left[{}^{\gamma_1\gamma_2}I_{\eta_1^+,\vartheta_1^+}^{\alpha,\beta}f\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I_{\eta_1^+,\vartheta_2^-}^{\alpha,\beta}f\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right.\\
&\quad+{}^{\gamma_1\gamma_2}I_{\eta_2^-,\vartheta_1^+}^{\alpha,\beta}f\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&\quad\left.+{}^{\gamma_1\gamma_2}I_{\eta_2^-,\vartheta_2^-}^{\alpha,\beta}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right]-A.
\end{aligned}
$$

This completes the proof. □

Next, we start to state the first theorem containing the midpoint type inequality for generalized conformable fractional integrals.

Theorem 3 *Assume that the assumptions of Lemma 1 hold. If* $|\frac{\partial^2\xi(t,s)}{\partial t\partial s}|$ *is a co-ordinated convex function on* Δ, *then the following inequality holds.*

$$\begin{aligned}
&\Bigg|\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+\frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^\alpha\gamma_2^\beta}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\\
&\quad\times\Bigg[{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&\quad+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&\quad+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\Bigg]-A\Bigg|\\
&\le\frac{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}{16}\left[1-\frac{1}{\gamma_1}B\left(\alpha+1,\frac{1}{\gamma_1}\right)\right]\left[1-\frac{1}{\gamma_2}B\left(\beta+1,\frac{1}{\gamma_2}\right)\right]\\
&\quad\times\left[\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|\right],
\end{aligned}\tag{2.8}$$

where A is defined by (2.2) *and* $B(\cdot,\cdot)$ *refers to the Beta function.*

Proof From Lemma 1, we acquire

$$\begin{aligned}
&\Bigg|\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+\frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^\alpha\gamma_2^\beta}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\\
&\quad\times\Bigg[{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&\quad+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right){}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\Bigg]-A\Bigg|\\
&\le\frac{\gamma_1^\alpha\gamma_2^\beta(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}{16}\\
&\quad\times\Bigg\{\int_0^1\int_0^1\left[\frac{1}{\gamma_1^\alpha}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^\alpha\right]\left[\frac{1}{\gamma_2^\beta}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^\beta\right]\\
&\quad\times\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)\right|ds\,dt\\
&\quad+\int_0^1\int_0^1\left[\frac{1}{\gamma_1^\alpha}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^\alpha\right]\left[\frac{1}{\gamma_2^\beta}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^\beta\right]\\
&\quad\times\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)\right|ds\,dt\\
&\quad+\int_0^1\int_0^1\left[\frac{1}{\gamma_1^\alpha}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^\alpha\right]\left[\frac{1}{\gamma_2^\beta}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^\beta\right]\\
&\quad\times\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)\right|ds\,dt\\
&\quad+\int_0^1\int_0^1\left[\frac{1}{\gamma_1^\alpha}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^\alpha\right]\left[\frac{1}{\gamma_2^\beta}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^\beta\right]
\end{aligned}\tag{2.9}$$

$$\times \left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)\right| ds\,dt\Bigg\}.$$

Since $|\frac{\partial^2\xi}{\partial t\partial s}|$ is co-ordinated convex function on Δ, then one has:

$$\begin{aligned}
&\Bigg|\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+\frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^\alpha\gamma_2^\beta}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\\
&\quad\times\Bigg[{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&\quad+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&\quad+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\Bigg]-A\Bigg|\\
&\le\frac{\gamma_1^\alpha\gamma_2^\beta(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}{16}\\
&\quad\times\Bigg\{\int_0^1\int_0^1\left[\frac{1}{\gamma_1^\alpha}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^\alpha\right]\left[\frac{1}{\gamma_2^\beta}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^\beta\right]\\
&\quad\times\Bigg[\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|\\
&\quad+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|\Bigg]ds\,dt\\
&\quad+\int_0^1\int_0^1\left[\frac{1}{\gamma_1^\alpha}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^\alpha\right]\left[\frac{1}{\gamma_2^\beta}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^\beta\right]\\
&\quad\times\Bigg[\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|+\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|\\
&\quad+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|\Bigg]ds\,dt\\
&\quad+\int_0^1\int_0^1\left[\frac{1}{\gamma_1^\alpha}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^\alpha\right]\left[\frac{1}{\gamma_2^\beta}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^\beta\right]\\
&\quad\times\Bigg[\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|\\
&\quad+\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|\Bigg]ds\,dt\\
&\quad+\int_0^1\int_0^1\left[\frac{1}{\gamma_1^\alpha}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^\alpha\right]\left[\frac{1}{\gamma_2^\beta}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^\beta\right]\\
&\quad\times\Bigg[\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|\\
&\quad+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|+\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|\Bigg]ds\,dt\Bigg\}\\
&=\frac{\gamma_1^\alpha\gamma_2^\beta(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}{16}
\end{aligned}$$

$$\times\left(\int_0^1\int_0^1\left[\frac{1}{\gamma_1^{\alpha}}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\right]\left[\frac{1}{\gamma_2^{\beta}}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right]\right)$$
$$\times\left[\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|\right]$$
$$=\frac{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}{16}\left[1-\frac{1}{\gamma_1}B\left(\alpha+1,\frac{1}{\gamma_1}\right)\right]\left[1-\frac{1}{\gamma_2}B\left(\beta+1,\frac{1}{\gamma_2}\right)\right]$$
$$\times\left[\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|\right],$$

which finishes the proof. □

Remark 3 In Theorem 3, if we choose $\gamma_1 = 1$ and $\gamma_2 = 1$, then the following inequality for Riemann–Liouville fractional integrals is achieved

$$\begin{aligned}&\left|\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+\frac{2^{\alpha-1}2^{\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)}{(\eta_2-\eta_1)^{\alpha}(\vartheta_2-\vartheta_1)^{\beta}}\right.\\&\quad\times\left[J^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right.\\&\quad+J^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+J^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\\&\quad\left.\left.+J^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right]-D\right|\\&\le\frac{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}{16}\left(\frac{\alpha}{\alpha+1}\right)\left(\frac{\beta}{\beta+1}\right)\\&\quad\times\left[\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|\right],\end{aligned}\tag{2.10}$$

where

$$\begin{aligned}D={}&\frac{2^{\beta-1}\Gamma(\beta+1)}{(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left[J^{\beta}_{\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+J^{\beta}_{\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right]\\&+.\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\eta_2-\eta_1)^{\gamma_1\alpha}}\left[J^{\alpha}_{\eta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+J^{\alpha}_{\eta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right].\end{aligned}\tag{2.11}$$

The inequality (2.10) is the same of [10, Remark 5].

Remark 4 If we choose $\gamma_1 = \gamma_2 = \alpha = \beta = 1$ in Theorem 3, then Theorem 3 reduces to [23, Theorem 2].

Theorem 4 *Assume that the assumptions of Lemma 1 hold. If $|\frac{\partial^2\xi}{\partial t\partial s}|^q$, $q > 1$, is a co-ordinated convex function on Δ, then the following inequality holds.*

$$\begin{aligned}&\left|\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right.\\&\quad+\frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^{\alpha}\gamma_2^{\beta}}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}}\left[{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right.\end{aligned}\tag{2.12}$$

$$
\begin{aligned}
&+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\Bigg]-A\Bigg|\\
&\le\frac{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}{16}\left[\left(16-\frac{16}{\gamma_1}B\left(\alpha p+1,\frac{1}{\gamma_1}\right)\right)\left(16-\frac{16}{\gamma_2}B\left(\beta p+1,\frac{1}{\gamma_2}\right)\right)\right]^{\frac{1}{p}}\\
&\times\left[\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right]^{\frac{1}{q}},
\end{aligned}
$$

where A is defined by (2.2), $B(\cdot,\cdot)$ *refers to the Beta function and* $\frac{1}{p}=1-\frac{1}{q}$.

Proof By using the well-known Hölder's inequality for double integrals, since $|\frac{\partial^2\xi}{\partial t\partial s}|^q$ is convex functions on the co-ordinates on $\triangle$, we get

$$
\begin{aligned}
&\int_0^1\int_0^1\left|\frac{1}{\gamma_1^\alpha}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^\alpha\right|\left|\frac{1}{\gamma_2^\beta}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^\beta\right| \qquad (2.13)\\
&\quad\times\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)\right|ds\,dt\\
&\le\left(\int_0^1\int_0^1\left|\frac{1}{\gamma_1^\alpha}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^\alpha\right|^p\left|\frac{1}{\gamma_2^\beta}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^\beta\right|^p ds\,dt\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^1\int_0^1\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)\right|^q ds\,dt\right)^{\frac{1}{q}}\\
&\le\frac{1}{\gamma_1^\alpha}\frac{1}{\gamma_2^\beta}\left(\int_0^1\int_0^1\left(1-\left(1-(1-t)^{\gamma_1}\right)^{\alpha p}\right)\left(1-\left(1-(1-s)^{\gamma_2}\right)^{\beta p}\right)ds\,dt\right)^{\frac{1}{p}}\\
&\quad\times\Bigg\{\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q\\
&\quad+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q ds\,dt\Bigg\}^{\frac{1}{q}}\\
&\le\frac{1}{\gamma_1^\alpha}\frac{1}{\gamma_2^\beta}\left[\left(1-\frac{1}{\gamma_1}B\left(\alpha p+1,\frac{1}{\gamma_1}\right)\right)\left(1-\frac{1}{\gamma_2}B\left(\beta p+1,\frac{1}{\gamma_2}\right)\right)\right]^{\frac{1}{p}}\\
&\quad\times\left(\frac{9}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q+\frac{3}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q+\frac{3}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q\right.\\
&\quad\left.+\frac{1}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}
$$

Here, we take advantage of the fact that

$$(\varpi-\sigma)^j\le\varpi^j-\sigma^j,$$

for any $\varpi>\sigma\ge 0$ and $j\ge 1$.

Similarly, we have

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{1}{\gamma_1^{\alpha}}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\right|\left|\frac{1}{\gamma_2^{\beta}}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right| \\
&\quad\times\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)\right|ds\,dt\} \\
&\leq\frac{1}{\gamma_1^{\alpha}}\frac{1}{\gamma_2^{\beta}}\left[\left(1-\frac{1}{\gamma_1}B\left(\alpha p+1,\frac{1}{\gamma_1}\right)\right)\left(1-\frac{1}{\gamma_2}B\left(\beta p+1,\frac{1}{\gamma_2}\right)\right)\right]^{\frac{1}{p}} \\
&\quad\times\left(\frac{3}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q+\frac{9}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q+\frac{1}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q\right. \\
&\quad\left.+\frac{3}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right)^{\frac{1}{q}},
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{1}{\gamma_1^{\alpha}}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\right|\left|\frac{1}{\gamma_2^{\beta}}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right| \\
&\quad\times\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)\right|ds\,dt\} \\
&\leq\frac{1}{\gamma_1^{\alpha}}\frac{1}{\gamma_2^{\beta}}\left[\left(1-\frac{1}{\gamma_1}B\left(\alpha p+1,\frac{1}{\gamma_1}\right)\right)\left(1-\frac{1}{\gamma_2}B\left(\beta p+1,\frac{1}{\gamma_2}\right)\right)\right]^{\frac{1}{p}} \\
&\quad\times\left(\frac{3}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q+\frac{1}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q+\frac{9}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q\right. \\
&\quad\left.+\frac{3}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right)^{\frac{1}{q}},
\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{1}{\gamma_1^{\alpha}}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\right|\left|\frac{1}{\gamma_2^{\beta}}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right| \\
&\quad\times\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1-t}{2}\eta_1+\frac{1+t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)\right|ds\,dt \\
&\leq\frac{1}{\gamma_1^{\alpha}}\frac{1}{\gamma_2^{\beta}}\left[\left(1-\frac{1}{\gamma_1}B\left(\alpha p+1,\frac{1}{\gamma_1}\right)\right)\left(1-\frac{1}{\gamma_2}B\left(\beta p+1,\frac{1}{\gamma_2}\right)\right)\right]^{\frac{1}{p}} \\
&\quad\times\left(\frac{1}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q+\frac{3}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q+\frac{3}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q\right. \\
&\quad\left.+\frac{9}{16}\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.16}$$

If we substitute the inequalities (2.13)–(2.16) in (2.9), we obtain the desired inequality (2.12). □

Remark 5 If we take $\gamma_1 = 1$ and $\gamma_2 = 1$ in Theorem 4, then the following inequality for Riemann–Liouville fractional integrals is achieved

$$\begin{aligned}
&\Bigg|\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+\frac{2^{\alpha-1}2^{\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)}{(\eta_2-\eta_1)^{\alpha}(\vartheta_2-\vartheta_1)^{\beta}} \qquad (2.17)\\
&\quad\times\Bigg[J^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&\quad+J^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+J^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&\quad+J^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\Bigg]-D\Bigg|\\
&\leq\frac{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}{16}\left[\left(\frac{16\alpha p}{\alpha p+1}\right)\left(\frac{16\beta p}{\beta p+1}\right)\right]^{\frac{1}{p}}\\
&\quad\times\left[\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q+\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Theorem 5 *Assume that the assumptions of Lemma 1 hold. If* $|\frac{\partial^2\xi}{\partial t\partial s}|^q$, $q\geq 1$, *is a co-ordinated convex function on* Δ, *then we have the following inequality*:

$$\begin{aligned}
&\Bigg|\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+\frac{2^{\gamma_1\alpha-1}2^{\gamma_2\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)\gamma_1^{\alpha}\gamma_2^{\beta}}{(\eta_2-\eta_1)^{\gamma_1\alpha}(\vartheta_2-\vartheta_1)^{\gamma_2\beta}} \qquad (2.18)\\
&\quad\times\Bigg[{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&\quad+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\\
&\quad+{}^{\gamma_1\gamma_2}I^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\Bigg]-A\Bigg|\\
&\leq\frac{(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)}{\gamma_1\gamma_2}\left(\frac{1}{4}\right)^{2+\frac{1}{q}}\left[\left(1-\frac{1}{\gamma_1}B\left(\alpha+1,\frac{1}{\gamma_1}\right)\right)\left(1-\frac{1}{\gamma_2}B\left(\beta+1,\frac{1}{\gamma_2}\right)\right)\right]^{1-\frac{1}{q}}\\
&\quad\times\Bigg\{\Bigg(\left[\frac{3\gamma_1}{2}-2B\left(\alpha+1,\frac{1}{\gamma_1}\right)+B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\\
&\quad\times\left[\frac{3\gamma_2}{2}-2B\left(\beta+1,\frac{1}{\gamma_2}\right)+B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q\\
&\quad+\left[\frac{3\gamma_1}{2}-2B\left(\alpha+1,\frac{1}{\gamma_1}\right)+B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[\frac{\gamma_2}{2}-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q\\
&\quad+\left[\frac{\gamma_1}{2}-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[\frac{3\gamma_2}{2}-2B\left(\beta+1,\frac{1}{\gamma_2}\right)+B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q\\
&\quad+\left[\frac{\gamma_1}{2}-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[\frac{\gamma_2}{2}-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\Bigg)^{\frac{1}{q}}\\
&\quad+\Bigg(\left[\frac{3\gamma_1}{2}-2B\left(\alpha+1,\frac{1}{\gamma_1}\right)+B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[\frac{\gamma_2}{2}-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q\\
&\quad+\left[\frac{3\gamma_1}{2}-2B\left(\alpha+1,\frac{1}{\gamma_1}\right)+B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&\times\left[\frac{3\gamma_2}{2}-2B\left(\beta+1,\frac{1}{\gamma_2}\right)+B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q\\
&+\left[\frac{\gamma_1}{2}-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[\frac{\gamma_2}{2}-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q\\
&+\left[\frac{\gamma_1}{2}-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\\
&\times\left[\frac{3\gamma_2}{2}-2B\left(\beta+1,\frac{1}{\gamma_2}\right)+B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\Bigg)^{\frac{1}{q}}\\
&+\Bigg(\left[\frac{\gamma_1}{2}-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[\frac{3\gamma_2}{2}-2B\left(\beta+1,\frac{1}{\gamma_2}\right)+B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\\
&\times\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q\\
&+\left[\frac{\gamma_1}{2}-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[\frac{\gamma_2}{2}-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q\\
&+\left[\frac{3\gamma_1}{2}-2B\left(\alpha+1,\frac{1}{\gamma_1}\right)+B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\\
&\times\left[\frac{3\gamma_2}{2}-2B\left(\beta+1,\frac{1}{\gamma_2}\right)+B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q\\
&+\left[\frac{3\gamma_1}{2}-2B\left(\alpha+1,\frac{1}{\gamma_1}\right)+B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\\
&\times\left[\frac{\gamma_2}{2}-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\Bigg)^{\frac{1}{q}}\\
&+\Bigg(\left[\frac{\gamma_1}{2}-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[\frac{\gamma_2}{2}-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q\\
&+\left[\frac{\gamma_1}{2}-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[\frac{3\gamma_2}{2}-2B\left(\beta+1,\frac{1}{\gamma_2}\right)+B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q\\
&+\left[\frac{3\gamma_1}{2}-2B\left(\alpha+1,\frac{1}{\gamma_1}\right)+B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[\frac{\gamma_2}{2}-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q\\
&+\left[\frac{3\gamma_1}{2}-2B\left(\alpha+1,\frac{1}{\gamma_1}\right)+B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\\
&\times\left[\frac{3\gamma_2}{2}-2B\left(\beta+1,\frac{1}{\gamma_2}\right)+B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\Bigg)^{\frac{1}{q}}.
\end{aligned}
$$

Here, A is defined as in (2.2).

Proof By using power-mean inequality, we get

$$
\begin{aligned}
I_9 &= \int_0^1\int_0^1\left|\frac{1}{\gamma_1^\alpha}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^\alpha\right|\left|\frac{1}{\gamma_2^\beta}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^\beta\right|\\
&\quad\times\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)\right|ds\,dt\\
&\le\left(\int_0^1\int_0^1\left|\frac{1}{\gamma_1^\alpha}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^\alpha\right|\left|\frac{1}{\gamma_2^\beta}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^\beta\right|ds\,dt\right)^{1-\frac{1}{q}}
\end{aligned}
$$

$$\times\left(\int_0^1\int_0^1\left|\frac{1}{\gamma_1^{\alpha}}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\right|\left|\frac{1}{\gamma_2^{\beta}}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right|\right.$$
$$\left.\times\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)\right|^q ds\,dt\right)^{\frac{1}{q}}.$$

Taking into account co-ordinated convexity of $|\frac{\partial^2\xi}{\partial t\partial s}|^q$, we acquire

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{1}{\gamma_1^{\alpha}}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\right|\left|\frac{1}{\gamma_2^{\beta}}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right| \\
&\quad\times\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1+s}{2}\vartheta_1+\frac{1-s}{2}\vartheta_2\right)\right| ds\,dt \\
&\le\left(\int_0^1\int_0^1\left(\frac{1}{\gamma_1^{\alpha}}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\right)\left(\frac{1}{\gamma_2^{\beta}}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right)ds\,dt\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1\int_0^1\left(\frac{1}{\gamma_1^{\alpha}}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\right)\left(\frac{1}{\gamma_2^{\beta}}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right)\right. \\
&\quad\times\left\{\left(\frac{1+t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q+\left(\frac{1+t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q\right. \\
&\quad\left.+\left(\frac{1-t}{2}\right)\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q+\left(\frac{1-t}{2}\right)\left(\frac{1-s}{2}\right)\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q ds\,dt\right\}^{\frac{1}{q}} \\
&=\left[\frac{1}{\gamma_1^{\alpha}}\frac{1}{\gamma_2^{\beta}}\left(1-\frac{1}{\gamma_1}B\left(\alpha+1,\frac{1}{\gamma_1}\right)\right)\left(1-\frac{1}{\gamma_2}B\left(\beta+1,\frac{1}{\gamma_2}\right)\right)\right]^{1-\frac{1}{q}}\left\{\frac{1}{4}\frac{1}{\gamma_1^{\alpha+1}}\frac{1}{\gamma_2^{\beta+1}}\right. \\
&\quad\times\left(\left[\frac{3\gamma_1}{2}-2B\left(\alpha+1,\frac{1}{\gamma_1}\right)+B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\right. \\
&\quad\times\left[\frac{3\gamma_2}{2}-2B\left(\beta+1,\frac{1}{\gamma_2}\right)+B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q \\
&\quad+\left[\frac{3\gamma_1}{2}-2B\left(\alpha+1,\frac{1}{\gamma_1}\right)+B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[\frac{\gamma_2}{2}-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q \\
&\quad+\left[\frac{\gamma_1}{2}-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[\frac{3\gamma_2}{2}-2B\left(\beta+1,\frac{1}{\gamma_2}\right)+B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q \\
&\quad\left.\left.+\left[\frac{\gamma_1}{2}-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[\frac{\gamma_2}{2}-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right)\right\}^{\frac{1}{q}}.
\end{aligned}\tag{2.19}$$

Similarly, we have

$$\begin{aligned}
&\int_0^1\int_0^1\left(\frac{1}{\gamma_1^{\alpha}}-\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\alpha}\right)\left(\frac{1}{\gamma_2^{\beta}}-\left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^{\beta}\right) \\
&\quad\times\left|\frac{\partial^2\xi}{\partial t\partial s}\left(\frac{1+t}{2}\eta_1+\frac{1-t}{2}\eta_2,\frac{1-s}{2}\vartheta_1+\frac{1+s}{2}\vartheta_2\right)\right| ds\,dt \\
&\le\left[\frac{1}{\gamma_1^{\alpha}}\frac{1}{\gamma_2^{\beta}}\left(1-\frac{1}{\gamma_1}B\left(\alpha+1,\frac{1}{\gamma_1}\right)\right)\left(1-\frac{1}{\gamma_2}B\left(\beta+1,\frac{1}{\gamma_2}\right)\right)\right]^{1-\frac{1}{q}}\left\{\frac{1}{4}\frac{1}{\gamma_1^{\alpha+1}}\frac{1}{\gamma_2^{\beta+1}}\right. \\
&\quad\times\left(\left[\frac{3\gamma_1}{2}-2B\left(\alpha+1,\frac{1}{\gamma_1}\right)+B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\right.
\end{aligned}\tag{2.20}$$

$$\times \left[\frac{\gamma_2}{2} - B\left(\beta+1, \frac{2}{\gamma_2}\right)\right] \left|\frac{\partial^2 \xi}{\partial t \partial s}(\eta_1, \vartheta_1)\right|^q$$
$$+ \left[\frac{3\gamma_1}{2} - 2B\left(\alpha+1, \frac{1}{\gamma_1}\right) + B\left(\alpha+1, \frac{2}{\gamma_1}\right)\right]$$
$$\times \left[\frac{3\gamma_2}{2} - 2B\left(\beta+1, \frac{1}{\gamma_2}\right) + B\left(\beta+1, \frac{2}{\gamma_2}\right)\right] \left|\frac{\partial^2 \xi}{\partial t \partial s}(\eta_1, \vartheta_2)\right|^q$$
$$+ \left[\frac{\gamma_1}{2} - B\left(\alpha+1, \frac{2}{\gamma_1}\right)\right]\left[\frac{\gamma_2}{2} - B\left(\beta+1, \frac{2}{\gamma_2}\right)\right] \left|\frac{\partial^2 \xi}{\partial t \partial s}(\eta_2, \vartheta_1)\right|^q$$
$$+ \left[\frac{\gamma_1}{2} - B\left(\alpha+1, \frac{2}{\gamma_1}\right)\right]\left[\frac{3\gamma_2}{2} - 2B\left(\beta+1, \frac{1}{\gamma_2}\right) + B\left(\beta+1, \frac{2}{\gamma_2}\right)\right]$$
$$\times \left|\frac{\partial^2 \xi}{\partial t \partial s}(\eta_2, \vartheta_2)\right|^q\Bigg)\Bigg\}^{\frac{1}{q}},$$

$$\int_0^1 \int_0^1 \left(\frac{1}{\gamma_1^\alpha} - \left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^\alpha\right)\left(\frac{1}{\gamma_2^\beta} - \left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^\beta\right) \tag{2.21}$$
$$\times \left|\frac{\partial^2 \xi}{\partial t \partial s}\left(\frac{1-t}{2}\eta_1 + \frac{1+t}{2}\eta_2, \frac{1+s}{2}\vartheta_1 + \frac{1-s}{2}\vartheta_2\right)\right| ds\, dt$$
$$\leq \left[\frac{1}{\gamma_1^\alpha}\frac{1}{\gamma_2^\beta}\left(1 - \frac{1}{\gamma_1}B\left(\alpha+1, \frac{1}{\gamma_1}\right)\right)\left(1 - \frac{1}{\gamma_2}B\left(\beta+1, \frac{1}{\gamma_2}\right)\right)\right]^{1-\frac{1}{q}} \Bigg\{\frac{1}{4}\frac{1}{\gamma_1^{\alpha+1}}\frac{1}{\gamma_2^{\beta+1}}$$
$$\times \Bigg(\left[\frac{\gamma_1}{2} - B\left(\alpha+1, \frac{2}{\gamma_1}\right)\right]\left[\frac{3\gamma_2}{2} - 2B\left(\beta+1, \frac{1}{\gamma_2}\right) + B\left(\beta+1, \frac{2}{\gamma_2}\right)\right]$$
$$\times \left|\frac{\partial^2 \xi}{\partial t \partial s}(\eta_1, \vartheta_1)\right|^q$$
$$+ \left[\frac{\gamma_1}{2} - B\left(\alpha+1, \frac{2}{\gamma_1}\right)\right]\left[\frac{\gamma_2}{2} - B\left(\beta+1, \frac{2}{\gamma_2}\right)\right] \left|\frac{\partial^2 \xi}{\partial t \partial s}(\eta_1, \vartheta_2)\right|^q$$
$$+ \left[\frac{3\gamma_1}{2} - 2B\left(\alpha+1, \frac{1}{\gamma_1}\right) + B\left(\alpha+1, \frac{2}{\gamma_1}\right)\right]$$
$$\times \left[\frac{3\gamma_2}{2} - 2B\left(\beta+1, \frac{1}{\gamma_2}\right) + B\left(\beta+1, \frac{2}{\gamma_2}\right)\right] \left|\frac{\partial^2 \xi}{\partial t \partial s}(\eta_2, \vartheta_1)\right|^q$$
$$+ \left[\frac{3\gamma_1}{2} - 2B\left(\alpha+1, \frac{1}{\gamma_1}\right) + B\left(\alpha+1, \frac{2}{\gamma_1}\right)\right]$$
$$\times \left[\frac{\gamma_2}{2} - B\left(\beta+1, \frac{2}{\gamma_2}\right)\right] \left|\frac{\partial^2 \xi}{\partial t \partial s}(\eta_2, \vartheta_2)\right|^q\Bigg)\Bigg\}^{\frac{1}{q}}$$

and

$$\int_0^1 \int_0^1 \left(\frac{1}{\gamma_1^\alpha} - \left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^\alpha\right)\left(\frac{1}{\gamma_2^\beta} - \left(\frac{1-(1-s)^{\gamma_2}}{\gamma_2}\right)^\beta\right) \tag{2.22}$$
$$\times \left|\frac{\partial^2 f}{\partial t \partial s}\left(\frac{1-t}{2}a + \frac{1+t}{2}b, \frac{1-s}{2}c + \frac{1+s}{2}\, d\right)\right| ds\, dt$$
$$\leq \left[\frac{1}{\gamma_1^\alpha}\frac{1}{\gamma_2^\beta}\left(1 - \frac{1}{\gamma_1}B\left(\alpha+1, \frac{1}{\gamma_1}\right)\right)\left(1 - \frac{1}{\gamma_2}B\left(\beta+1, \frac{1}{\gamma_2}\right)\right)\right]^{1-\frac{1}{q}} \Bigg\{\frac{1}{4}\frac{1}{\gamma_1^{\alpha+1}}\frac{1}{\gamma_2^{\beta+1}}$$

$$\times\left(\left[\frac{\gamma_1}{2}-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[\frac{\gamma_2}{2}-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q\right.$$
$$+\left[\frac{\gamma_1}{2}-B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[\frac{3\gamma_2}{2}-2B\left(\beta+1,\frac{1}{\gamma_2}\right)+B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q$$
$$+\left[\frac{3\gamma_1}{2}-2B\left(\alpha+1,\frac{1}{\gamma_1}\right)+B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]\left[\frac{\gamma_2}{2}-B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q$$
$$+\left[\frac{3\gamma_1}{2}-2B\left(\alpha+1,\frac{1}{\gamma_1}\right)+B\left(\alpha+1,\frac{2}{\gamma_1}\right)\right]$$
$$\left.\times\left[\frac{3\gamma_2}{2}-2B\left(\beta+1,\frac{1}{\gamma_2}\right)+B\left(\beta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right)\Bigg\}^{\frac{1}{q}}.$$

By considering (2.19)–(2.22) in (2.9), we obtain the required inequality (2.18). □

Remark 6 If we take $\gamma_1 = 1$ and $\gamma_2 = 1$ in Theorem 5, then the following inequality for Riemann–Liouville fractional integrals is achieved

$$\left|\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+\frac{2^{\alpha-1}2^{\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)}{(\eta_2-\eta_1)^\alpha(\vartheta_2-\vartheta_1)^\beta}\right. \tag{2.23}$$
$$\times\left[J^{\alpha,\beta}_{\eta_1^+,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right.$$
$$+J^{\alpha,\beta}_{\eta_1^+,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)+J^{\alpha,\beta}_{\eta_2^-,\vartheta_1^+}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)$$
$$\left.\left.+J^{\alpha,\beta}_{\eta_2^-,\vartheta_2^-}\xi\left(\frac{\eta_1+\eta_2}{2},\frac{\vartheta_1+\vartheta_2}{2}\right)\right]-D\right|$$
$$\le(\eta_2-\eta_1)(\vartheta_2-\vartheta_1)\left(\frac{1}{4}\right)^{2+\frac{1}{q}}\left[\left(\frac{\alpha}{\alpha+1}\right)\left(\frac{\beta}{\beta+1}\right)\right]^{1-\frac{1}{q}}$$
$$\times\left\{\left(\left[\frac{3}{2}-\frac{2\alpha+3}{(\alpha+1)(\alpha+2)}\right]\left[\frac{3}{2}-\frac{2\beta+3}{(\beta+1)(\beta+2)}\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q\right.\right.$$
$$+\left[\frac{3}{2}-\frac{2\alpha+3}{(\alpha+1)(\alpha+2)}\right]\left[\frac{1}{2}-\frac{1}{(\beta+1)(\beta+2)}\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q$$
$$+\left[\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right]\left[\frac{3}{2}-\frac{2\beta+3}{(\beta+1)(\beta+2)}\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q$$
$$\left.+\left[\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right]\left[\frac{1}{2}-\frac{1}{(\beta+1)(\beta+2)}\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\left[\frac{3}{2}-\frac{2\alpha+3}{(\alpha+1)(\alpha+2)}\right]\left[\frac{1}{2}-\frac{1}{(\beta+1)(\beta+2)}\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q\right.$$
$$+\left[\frac{3}{2}-\frac{2\alpha+3}{(\alpha+1)(\alpha+2)}\right]\left[\frac{3}{2}-\frac{2\beta+3}{(\beta+1)(\beta+2)}\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q$$
$$+\left[\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right]\left[\frac{1}{2}-\frac{1}{(\beta+1)(\beta+2)}\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q$$
$$\left.+\left[\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right]\left[\frac{3}{2}-\frac{2\beta+3}{(\beta+1)(\beta+2)}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right)^{\frac{1}{q}}$$

$$
\begin{aligned}
&+\left(\left[\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right]\left[\frac{3}{2}-\frac{2\beta+3}{(\beta+1)(\beta+2)}\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q\right.\\
&+\left[\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right]\left[\frac{1}{2}-\frac{1}{(\beta+1)(\beta+2)}\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q\\
&+\left[\frac{3}{2}-\frac{2\alpha+3}{(\alpha+1)(\alpha+2)}\right]\left[\frac{3}{2}-\frac{2\beta+3}{(\beta+1)(\beta+2)}\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q\\
&\left.+\left[\frac{3}{2}-\frac{2\alpha+3}{(\alpha+1)(\alpha+2)}\right]\left[\frac{1}{2}-\frac{1}{(\beta+1)(\beta+2)}\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right)^{\frac{1}{q}}\\
&+\left(\left[\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right]\left[\frac{1}{2}-\frac{1}{(\beta+1)(\beta+2)}\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_1)\right|^q\right.\\
&+\left[\frac{1}{2}-\frac{1}{(\alpha+1)(\alpha+2)}\right]\left[\frac{3}{2}-\frac{2\beta+3}{(\beta+1)(\beta+2)}\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_1,\vartheta_2)\right|^q\\
&+\left[\frac{3}{2}-\frac{2\alpha+3}{(\alpha+1)(\alpha+2)}\right]\left[\frac{1}{2}-\frac{1}{(\beta+1)(\beta+2)}\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_1)\right|^q\\
&\left.+\left[\frac{3}{2}-\frac{2\alpha+3}{(\alpha+1)(\alpha+2)}\right]\left[\frac{3}{2}-\frac{2\beta+3}{(\beta+1)(\beta+2)}\right]\left|\frac{\partial^2\xi}{\partial t\partial s}(\eta_2,\vartheta_2)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}
$$

3 Conclusion

In this research, we acquired some inequality of midpoint type for co-ordinated convex functions by means of conformable fractional integrals. In the future studies, researchers can obtain some new inequalities with the aid of the different kinds of co-ordinated convex mappings or other types of fractional integral operators.

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T.Y.U and G. B. wrote the main sections. H. B and M. V. C. revised the paper. M. E. K supervised the paper.

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The authors declare no competing interests.

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References

1. Abdelhakim, A.A.: The flaw in the conformable calculus: it is conformable because it is not fractional. Fract. Calc. Appl. Anal. **22**, 242–254 (2019)
2. Abdeljawad, T.: On conformable fractional calculus. J. Comput. Appl. Math. **279**, 57–66 (2015)
3. Akkurt, A., Sarikaya, M.Z., Budak, H., Yildirim, H.: On the Hadamard's type inequalities for co-ordinated convex functions via fractional integrals. J. King Saud Univ., Sci. **29**, 380–387 (2017)
4. Akkurt, A., Sarikaya, M.Z., Budak, H., Yildirim, H.: On the Hermite-Hadamard type inequalities for co-ordinated convex functions. Appl. Comput. Math. **20**, 408–420 (2021)

5. Akkurt, A., Sarikaya, M.Z., Budak, H., Yildirim, H.: On the Hermite-Hadamard type inequalities for co-ordinated convex functions. Appl. Comput. Math. **20**(3), 408–420 (2021)
6. Akkurt, A., Yıldırım, M.E., Yıldırım, H.: A new generalized fractional derivative and integral. Konuralp J. Math. **5**(2), 248–259 (2017)
7. Alomari, M., Darus, M.: The hadamards inequality for s-convex function of 2-variables on the coordinates. Int. J. Math. Anal. **2**(13), 629–638 (2008)
8. Bakula, M.K.: An improvement of the Hermite-Hadamard inequality for functions convex on the coordinates. Aust. J. Math. Anal. Appl. **11**(1), 1–7 (2014)
9. Bozkurt, M., Akkurt, A., Yıldırım, H.: Conformable derivatives and integrals for the functions of two variables. Konuralp J. Math. **9**(1), 49–59 (2021)
10. Budak, H., Yildirim, S.K., Kara, H., Yildirim, H.: On new generalized inequalities with some parameters for coordinated convex functions via generalized fractional integrals. Math. Methods Appl. Sci. **44**(17), 13069–13098 (2021)
11. Dragomir, S.S.: On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane. Taiwan. J. Math. **4**, 775–788 (2001)
12. Dragomir, S.S., Pearce, C.E.M.: Selected Topics on Hermite-Hadamard Inequalities and Applications. RGMIA Monographs, Victoria University (2000)
13. Hyder, A., Soliman, A.H.: A new generalized θ-conformable calculus and its applications in mathematical physics. Phys. Scr. **96**, 015208 (2020)
14. Hyder, A.A., Almoneef, A.A., Budak, H., Barakat, M.A.: On new fractional version of generalized Hermite-Hadamard inequalities. Mathematics **10**(18), 3337 (2022)
15. Iqbal, M., Bhatti ve, M.I., Nazeer, K.: Generalization of inequalities analogous to Hermite-Hadamard inequality in fractional integrals. Bull. Korean Math. Soc. **52**(3), 707–716 (2015)
16. Jarad, F., Uğurlu, E., Abdeljawad, T., Baleanu, D.: On a new class of fractional operators. Adv. Differ. Equ. **2017**, 247 (2017)
17. Khalil, R., Al Horani, M., Yousef, A., Sababheh, M.: A new definition of fractional derivative. J. Comput. Appl. Math. **264**, 65–70 (2014)
18. Khan, T.U., Khan, M.A.: Generalized conformable fractional operators. J. Comput. Appl. Math. **346**, 378–389 (2019)
19. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
20. Kiris, M.E., Bayrak, G.: New version of Hermite-Hadamard inequality for co-ordinated convex function via generalized conformable integrals. Filomat (2024)
21. Kirmaci, U.S.: Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula. Appl. Math. Comput. **147**(1), 137–146 (2004)
22. Latif, M.A., Alomari, M.: Hadamard-type inequalities for product two convex functions on the co-ordinates. Int. Math. Forum **4**(47), 2327–2338 (2009)
23. Latif, M.A., Dragomir, S.S.: On some new inequalities for differentiable co-ordinated convex functions. J. Inequal. Appl. **2012**(1), 28 (2012)
24. Miller, S., Ross, B.: An Introduction to Thr Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
25. Pečarić, J.E., Proschan, F., Tong, Y.L.: Convex Functions, Partial Orderings and Statistical Applications. Academic Press, Boston (1992)
26. Podlubny, I.: Fractional Differantial Equations. Academic Press, San Diego (1999)
27. Samko, G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon & Breach, Yverdon (1993)
28. Sarikaya, M.Z.: On the Hermite-Hadamard-type inequalities for co-ordinated convex function via fractional integrals. Integral Transforms Spec. Funct. **25**(2), 134–147 (2014)
29. Sarikaya, M.Z., Akkurt, A., Budak, H., Yıldırım, M.E., Yıldırım, H.: Hermite-Hadamard's inequalities for conformable fractional integrals. Int. J. Optim. Control Theor. Appl. **9**(1), 49–59 (2019)
30. Sarikaya, M.Z., Budak, H., Usta, F.: On generalized the conformable fractional calculus. TWMS J. Appl. Eng. Math. **9**(4), 792–799 (2019)
31. Sarikaya, M.Z., Set, E., Ozdemir, M.E., Dragomir, S.S.: New some Hadamard's type inequalities for co-ordinated convex functions. Tamsui Oxf. J. Inf. Math. Sci. **28**(2), 137–152 (2012)
32. Sarikaya, M.Z., Set, E., Yaldiz, H., Basak, N.: Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. Math. Comput. Model. **57**(9–10), 2403–2407 (2013)
33. Zhao, D., Luo, M.: General conformable fractional derivative and its physical interpretation. Calcolo **54**, 903–917 (2017)