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Fredholm integral equation in composed-cone metric spaces

Anas A. Hijab[1,2], Laith K. Shaakir[1], Sarah Aljohani[3] and Nabil Mlaiki[3*]

*Correspondence:
nmlaiki@psu.edu.sa;
nmlaiki2012@gmail.com
[3]Department of Mathematics and Sciences, Prince Sultan University, Riyadh, 11586, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

The current paper introduces a novel generalization of cone metric spaces called type I and type II composed cone metric spaces. Therefore, examples of a type I and type II composed cone metric space, which is not a cone metric space, are given. We establish some results of fixed point precisely about Hardy–Rogers type contraction on C2CMS and provide examples. Finally, we present an application of our results and how our results solve the Fredholm integral equation of generalizing several existing and unique fixed point theorems.

Keywords: Cone metric space(CMS); Fixed point; Cone b-metric space(CbMS); Type I composed cone metric space(C1CMS); Type II composed cone metric space(C2CMS)

1 Introduction and preliminaries

In recent decades, an interesting area of functional analysis, known as a metric fixed point theory, has become increasingly attractive for research. The evolution of this theory focuses on two directions: either the generalization of metric spaces or the improvement of contractions, and sometimes it focuses on both of them. Here, an important generalization of metric space is (bMS) of Bakhtin [1]. At present, there are several generalizations of (bMS) as $b_v(s)$-metric spaces and b-rectangular metric spaces with some fixed point results (see [2–4]). In fact, the extended concepts of (bMS), known as controlled metric type spaces and double controlled metric type spaces, respectively, were introduced in 2018 with proving an analogue of Banach contraction principle (BCP) [5] on them [6–8]. In 2022, Karami et al. [9] gave an extension of a type of controlled metric spaces, defined to be expanded b-metric spaces. Thereafter, in 2023, Ayoobi et al. [10] gave off a new extension of the kinds of metric spaces known as double-composed metric spaces (DCMS), which represent the generalized expanded b-metric spaces. The first type depends on one controlled function (incomparable function), while the second one has two different controlled functions (see [11–13]).

In 2007, Huang et al. [14] replaced the real numbers by ordering Banach space and defined cone metric spaces. They proved some fixed point theorems of contractive mappings on cone metric spaces. Despite all of these studies on cone metric spaces [15–20], and Meng and Cho studying algebraic cone metric spaces [21, 22], there is much work concerning b-cone metric spaces, for instance, [23–25]

Hence, by utilizing the concepts in [9, 10], this study presents a generalization that represents type I composed cone metric spaces and type II composed cone metric spaces. The examples are for type I composed cone metric space, not cone metric space, and type II composed cone metric space, not for type I composed cone metric space. The study provides some fixed point results, involving Banach type, Kannan type, Reich type, and Hardy–Roger type contractions. The focus in on the Hardy–Rogers contraction of (C2CMS), go to corollary in (C1CMS). Finally, the study presents the application of the fixed point theorem to these new spaces as Fredholm integral equation.

Suppose that $\mathscr{E}$ is a real Banach space and $\mathscr{P} \subset \mathscr{E}$. The subset $\mathscr{P}$ is called a cone if the following conditions hold:

(P1) $\{0\} \neq \mathscr{P}$ is nonempty closed,

(P2) $\xi_1\mathfrak{f} + \xi_2\mathfrak{g} \in \mathscr{P}$ for all $\mathfrak{f}, \mathfrak{g} \in \mathscr{P}$, where $\xi_1, \xi_2 \geq 0$,

(P3) $\mathfrak{f} \in \mathscr{P}$ and $-\mathfrak{f} \in \mathscr{P}$ implies that $\mathfrak{f} = \{0\}$.

For a given cone $\mathscr{P}$, a partial ordering $\preceq$ can be defined on $\mathscr{E}$ with respect to $\mathscr{P}$ by $\mathfrak{f} \preceq \mathfrak{g}$ if and only if $\mathfrak{g} - \mathfrak{f} \in \mathscr{P}$. Also, $\mathfrak{f} \prec \mathfrak{g}$ indicates that $\mathfrak{f} \preceq \mathfrak{g}$ and $\mathfrak{f} \neq \mathfrak{g}$, while $\mathfrak{f} \ll \mathfrak{g}$ means that $\mathfrak{g} - \mathfrak{f} \in \operatorname{int}\mathscr{P}$ such that $\operatorname{int}\mathscr{P}$ represents the interior of $\mathscr{P}$.

$\mathscr{P}$ is a cone in a real Banach space $\mathscr{E}$ via $\operatorname{int}\mathscr{P}$ nonempty and $\preceq$ is partial ordering regarding $\mathscr{P}$. $\mathscr{P}$ is said to be a normal cone if there is a constant number $M > 0$ such that, for all $\mathfrak{f}, \mathfrak{g} \in \mathscr{E}$ and $0 \preceq \mathfrak{f} \preceq \mathfrak{g}$, it implies that $\|\mathfrak{f}\| \leq M\|\mathfrak{g}\|$ or, equivalently, if

$$\inf\Big\{\|\mathfrak{f} + \mathfrak{g}\| : \mathfrak{f}, \mathfrak{g} \in \mathscr{P}, \|\mathfrak{f}\| = \|\mathfrak{g}\| = 1\Big\} > 0,$$

it implies a non-normal cone (see, e.g., [22]). Also, if $\operatorname{int}\mathscr{P}$ is a nonempty set, it is called solid.

In the following explanations, some basic concepts of cone metric spaces with their properties are presented.

Definition 1.1 ([14] Cone metric space) Suppose that Γ is a nonempty set. A function $d_{\mathfrak{C}} : \Gamma \times \Gamma \to \mathscr{E}$ is called a cone metric on Γ if for each $\delta, \varrho, \gamma \in \Gamma$ the following conditions hold:

(CM1) $0 \preceq d_{\mathfrak{C}}(\delta, \varrho)$ and $d_{\mathfrak{C}}(\delta, \varrho) = 0$ if and only if $\delta = \varrho$,

(CM2) $d_{\mathfrak{C}}(\delta, \varrho) = d_{\mathfrak{C}}(\varrho, \delta)$,

(CM3) $d_{\mathfrak{C}}(\delta, \varrho) \preceq d_{\mathfrak{C}}(\delta, \gamma) + d_{\mathfrak{C}}(\gamma, \varrho)$.

Then $(\Gamma, d_{\mathfrak{C}})$ is called a cone metric space (CMS).

Clearly, CMS is a generalization of the metric spaces, but the reverse is not true (see [14]). Subsequently, the following explanations present the concepts of type I and type II composed cone metric spaces, inspired by Karami et al. [9] and Ayoobi et al. [10], which are special cases of Definitions 1.2 and 1.4 at $\mathscr{E} = \mathbb{R}$, as follows.

Definition 1.2 (Type I composed cone metric space) Presume $\Gamma \neq \phi$ set. A function $d_I : \Gamma \times \Gamma \to \mathscr{E}$ is a type I composed cone metric if there exists a $\psi : \mathscr{P} \to \mathscr{P}$ such that, for each $\delta, \varrho, \gamma \in \Gamma$, the following conditions hold:

(C1) $0 \preceq d_I(\delta, \varrho)$ and $d_I(\delta, \varrho) = 0$ if and only if $\delta = \varrho$,

(C2) $d_I(\delta, \varrho) = d_I(\varrho, \delta)$,

(C3) $d_I(\delta,\varrho) \preceq \psi(d_I(\delta,\gamma)) + \psi(d_I(\gamma,\varrho))$.
Then the triple (Γ, d_I, ψ) is called type I composed cone metric space (C1CMS).

Example 1.3 Let $\mathcal{E} = \mathbb{R}^2$, $\mathcal{P} = \{t = (t_1,t_2) \in \mathcal{E} : t_1, t_2 \geq 0\}$, and $\Gamma = \mathbb{R}$. Here, $d_I : \Gamma \times \Gamma \to \mathcal{E}$ is defined by

$$d_I(\delta,\varrho) = (\sinh\bigl(\beta_1 d(\delta,\varrho), \sinh\bigl(\beta_2 d(\delta,\varrho)\bigr)\bigr)$$

$\beta_1, \beta_2 \geq 0$, where (Γ, d) is a b-metric space with coefficient $s \geq 1$. Then d_I is a C1CMS with a controlled function,

$$\psi(t) = \bigl(\sinh(2st_1), \sinh(2st_2)\bigr), \quad t = (t_1,t_2) \in \mathcal{P}.$$

Obviously, d_I is not a CMS. Precisely, it is not a b-cone metric space, which is $d(\delta,\varrho) = (\delta - \varrho)^2$ a b-metric space at $b = 2$, but $d(\delta,\varrho) = \sinh(\delta - \varrho)^2$ is not a b-metric space (for more details, see [9, 16, 26, 27]).

Definition 1.4 (Type II composed cone metric space) Assume that Γ is a nonempty set and $\psi_1, \psi_2 : \mathcal{P} \to \mathcal{P}$ are two nonconstant functions. A mapping $d_c : \Gamma \times \Gamma \to \mathcal{E}$ is called a type II composed cone metric if for each $\delta, \varrho, \gamma \in \Gamma$ it satisfies the following conditions:
(CC1) $0 \preceq d_c(\delta,\varrho)$ and $d_c(\delta,\varrho) = 0$ if and only if $\delta = \varrho$,
(CC2) $d_c(\delta,\varrho) = d_c(\varrho,\delta)$,
(CC3) $d_c(\delta,\varrho) \preceq \psi_1(d_c(\delta,\gamma)) + \psi_2(d_c(\gamma,\varrho))$.
Then the pair (Γ, d_c) is called a type II composed cone metric space (C2CMS).

Each C1CMS is a C2CMS relating to $\psi_1(t) = \psi_2(t)$, $\forall t \in \mathcal{P}$.

Example 1.5 Let $\mathcal{E} = \mathbb{R}^2$, $\mathcal{P} = \{t = (t_1,t_2) \in \mathcal{E} : t_1, t_2 \geq 0\}$, and $\Gamma = \mathbb{R}$. Here, $d_c : \Gamma \times \Gamma \to \mathcal{E}$ is defined by $d_c(\delta,\varrho) = (\beta_1(\delta - \varrho)^2, \beta_2(\delta - \varrho)^2), \beta_1, \beta_2 \geq 0$. Then d_c is a C2CMS with two controlled functions, $\psi_1(t) = (e^{2t_1} - 1, e^{2t_2} - 1)$ and $\psi_2(t) = (2t_1, 2t_2)$, $t \in \mathcal{P}$. But notice that it is not a C1CMS.

Example 1.6 Let $\mathcal{E} = \mathbb{R}^2$, $\mathcal{P} = \{(t_1,t_2) \in \mathcal{E} : t_1, t_2 \geq 0\}$, and $\Gamma = \mathbb{R}$. Here, $d'_c : \Gamma \times \Gamma \to \mathcal{E}$ is defined by $d'_c(\delta,\varrho) = (\beta_1(\delta - \varrho)^2, \beta_2(\delta - \varrho)^2), \beta_1, \beta_2 \geq 0$. Then d'_c is a C2CMS with two controlled functions, $\psi_1(t) = (\frac{(1+2t_1^2)^{q_1}-1}{q_1}, \frac{(1+2t_2^2)^{q_1}-1}{q_1})$ and $\psi_2(t) = (\frac{(1+2t_1^2)^{q_2}-1}{q_2}, \frac{(1+2t_2^2)^{q_2}-1)}{q_2})$, $t \in \mathcal{P}$, $q_1 \geq q_2 \geq 1$. It is obvious that d'_c is a C1CMS at $q_2 = q_1$.

Remark 1.7 Notice that, in Example 1.6, which is $\psi_1(t) = ((1 + \frac{2t_1^2-1}{q_1})^{q_1}, (1 + \frac{2t_2^2-1}{q_1})^{q_1})$ and $\psi_2(t) = ((1 + \frac{2t_1^2-1}{q_2})^{q_2}, (1 + \frac{2t_2^2-1}{q_2})^{q_2})$, $t \in \mathcal{P}$, $q_1 \geq q_2 \geq 1$ also represent Γ regarding $\mathcal{E}$ as a C2CMS, but a C1CMS at $q_2 = q_1$.

Example 1.8 In the space $\Gamma = \ell_p(\mathbb{R})$, $\Delta : \Gamma \times \Gamma \to [0,\infty)$ is defined as

$$\Delta(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{1/p},$$

$p \in (0,1)$ and $\delta = \{x_n\}$, $\varrho = \{y_n\}$. It is given in [10], (Γ, Δ) is a (DCMS) with functions $\alpha(t) = \beta(t) = 2^{1/p}t$.

Now, presuppose that $\mathcal{E} = \mathbb{R}^2$, $\mathcal{P} = \{(t_1,t_2) \in \mathcal{E} : t_1, t_2 \geq 0\}$. Define $\Delta_c : \Gamma \times \Gamma \to \mathcal{E}$ as

$$\Delta_c(\delta,\varrho) = \big(M_1\Delta(\delta,\varrho), M_2\Delta(\delta,\varrho)\big),$$

wherever M_1 and M_2 are nonnegative constants. Then (Γ, Δ_c) is a C2CMS with functions $\psi_1(t) = (2^{1/p}t_1, 0)$ and $\psi_2(t) = (0, 2^{1/p}t_2)$.

In the same way, $\Gamma = L_p[0,1]$ can be defined in Example 1.8, wherever

$$\Lambda(g,f) = \left(\int_0^1 \left|g(t) - f(t)\right|^p dt\right)^{1/p}$$

for each $g(t), f(t) \in \Gamma$. Then (Γ, Δ_Λ) is a C2CMS.

Now, since each C1CMS is a C2CMS, the focus is going to be on type II composed cone metric spaces and the definition of their topology of C2CMS.

Definition 1.9 Let (Γ, d_c) be a C2CMS with respect to ψ_1 and ψ_2.

(1) The sequence $\{x_n\}$ converges to any η in Γ if for each $c \in \mathcal{E}$ via $0 \ll c$, $\exists N$ so as $d_c(x_n, \eta) \ll c$ for all $n > N$. It is written as $\lim_{n\to\infty} x_n = \eta$.

(2) The sequence $\{x_n\}$ is said to be Cauchy if for every $c \in \mathcal{E}$ via $0 \ll c$ $\exists N$ so as $d_c(x_n, x_m) \ll c$ for all $n, m > N$.

(3) (Γ, d_c) is said to be complete if every Cauchy sequence is convergent.

Lemma 1.10 *Let (Γ, d_c) be a C2CMS respecting ψ_1 and ψ_2, $\mathcal{P}$ be a normal cone with normal constant M. Let $\{x_n\}$ be a sequence in Γ, then $\{x_n\}$ converges to ν if and only if* $\lim_{n\to\infty} d_c(x_n, \nu) = 0$.

Proof The proof is obvious, therefore we omit it. □

Lemma 1.11 *Let (Γ, d_c) be a C2CMS respecting ψ_1 and ψ_2, and let M be a normal constant for a normal cone $\mathcal{P}$. Let $\{x_n\}$ be a sequence in Γ so that $\{x_n\}$ converges to η and ϑ. If ψ_1 and ψ_2 are bounded, then $\eta = \vartheta$. That is, the limit of $\{x_n\}$ is unique.*

Proof For any $c \in \mathcal{E}$ with $0 \ll c$, there is N such that, for all $n > N$, $d_c(x_n, \eta) \ll c$ and $d_c(x_n, y) \ll c$, the result is $d_c(\eta, \vartheta) \preceq \psi_1(d_c(\eta, x_n)) + \psi_2(d_c(x_n, \vartheta))$.

Hence,

$\|d_c(\eta,\vartheta)\| \leq M\|\psi_1(d_c(\eta,x_n)) + \psi_2(d_c(x_n,\vartheta))\| \leq M(\|\psi_1(d_c(\eta,x_n))\| + \|\psi_2(d_c(x_n,\vartheta))\|)$ because ψ_1, ψ_2 are bounded. So there are constants $k_1, k_2 > 0$ such that

$\|d_c(\eta,\vartheta)\| \leq M(k_1c + k_2c)$. Since M, k_1, and k_2 are finite and c is arbitrary, $\|d_c(\eta,\vartheta)\| = 0$, therefore $\eta = \vartheta$. □

Definition 1.12 [26] The function $\phi : \mathcal{P} \to \mathcal{P}$, wherever $\mathcal{P}$ is a cone in a Banach space $\mathcal{E}$, is known as a comparison function if it satisfies the following conditions:

(1) For all $t \in \mathcal{E}$, $\phi(t) \prec t$,

(2) For all $t_1, t_2 \in \mathcal{P}$ and $t_1 \prec t_2$, it yields $\phi(t_1) \prec \phi(t_2)$,
(3) $\lim_{n\to\infty} \|\phi^n(t)\| = 0$ for each $t \in \mathcal{P}$.

Definition 1.13 The function $\psi : \mathcal{P} \to \mathcal{P}$, where $\mathcal{P}$ is a cone in $\mathcal{E}$, is called an incomparison function if it satisfies these conditions:
(1) For all $t \in \mathcal{P}$, $t \prec \psi(t)$;
(2) For all $t_1, t_2 \in \mathcal{P}$ and $t_1 \prec t_2$, it yields $\psi(t_1) \prec \psi(t_2)$;
(3) $\lim_{n\to\infty} \|\psi^n(t)\| = 0$ for each $t \in \mathcal{P}$

2 The main results

In this section, we present some fixed point results in C2CMS. It is assumed that (Γ, d_c) represents a complete C2CMS with respect to the functions $\psi_1, \psi_2 : \mathcal{P} \to \mathcal{P}$, where $\mathcal{P}$ is a normal cone with normal constant M.

Theorem 2.1 *Suppose that* (Γ, d_c) *is a complete C2CMS with* $\mathcal{P}$ *as a normal cone via normal constant* M. *Let* $T : \Gamma \to \Gamma$ *be a mapping satisfying the following condition*:

$$\begin{aligned} d_c(T\delta, T\varrho) \preceq{}& \mathcal{K}_1 d_c(\delta, \varrho) + \mathcal{K}_2 d_c(\delta, T\delta) + \mathcal{K}_3 d_c(\varrho, T\varrho) + \mathcal{K}_4 d_c(\delta, T\varrho) \\ &+ \mathcal{K}_5 d_c(\varrho, T\delta), \quad \forall \delta, \varrho \in \Gamma, \end{aligned}$$

where $\mathcal{K}_i \in (0,1)$, $i = 1, 2, \ldots, 5$, *and* $\sum_{i=1}^{5} \mathcal{K}_i < 1$. *For any* $w_0 \in \Gamma$, *choose* $w_n = T^n w_0$. *So,*
(1) *Let* ψ_1, ψ_2 *be bounded, nondecreasing,* ψ_2 *be a subadditive comparison function, and* ψ_1 *be an incomparison function.*
(2) $\| \sum_{i=m}^{n-2} \psi_2^{i-m} \psi_1(\mathcal{R}^i \psi_1^i(d_c(w_0, w_1))) + \psi_2^{n-m-1}(\mathcal{R}^{n-1}\psi_1^{n-1}(d_c(w_0, w_1)))\| \to 0$ *(as* $n, m \to \infty$*), wherever* $\mathcal{R} = \frac{\mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_4}{1 - \mathcal{K}_3 - \mathcal{K}_4}$.

Then T *has a unique fixed point.*

Proof Presuppose that $w_0 \in \Gamma$. Define a sequence $\{w_n\}$ in Γ with $w_n = T^n w_0$ or $w_{n+1} = Tw_n \forall n \in \mathbb{N}$. Letting $\delta = w_{n-1}$ and $\varrho = w_n$ in the Hardy–Rogers contraction, the result is as follows:

$$\begin{aligned} d_c(w_n, w_{n+1}) &= d_c(Tw_{n-1}, Tw_n) \\ &\preceq \mathcal{K}_1 d_c(w_{n-1}, w_n) + \mathcal{K}_2 d_c(w_{n-1}, Tw_{n-1}) + \mathcal{K}_3 d_c(w_n, Tw_n) \\ &\quad + \mathcal{K}_4 d_c(w_{n-1}, Tw_n) + \mathcal{K}_5 d_c(w_n, Tw_{n-1}) \\ &\preceq \mathcal{K}_1 d_c(w_{n-1}, w_n) + \mathcal{K}_2 d_c(w_{n-1}, w_n) \\ &\quad + \mathcal{K}_3 d_c(w_n, w_{n+1}) + \mathcal{K}_4 d_c(w_{n-1}, w_{n+1}) + \mathcal{K}_5 d_c(w_n, w_n). \end{aligned} \tag{2.1}$$

Note that in C2CMS, $d_c(w_n, w_n) = 0$. By utilizing CC3 in C2CMS, the result is as follows:

$$d_c(w_{n-1}, w_{n+1}) \preceq \psi_1\big(d_c(w_{n-1}, w_n)\big) + \psi_2\big(d_c(w_n, w_{n+1})\big).$$

Hence,

$$(1 - \mathcal{K}_3) d_c(w_n, w_{n+1}) - \mathcal{K}_4 \psi_2\big(d_c(w_n, w_{n+1})\big)$$

$$\preceq (\mathcal{K}_1 + \mathcal{K}_2)d_c(w_{n-1}, w_n) + \mathcal{K}_4\psi_1\big(d_c(w_{n-1}, w_n)\big),$$

which yields with condition 1

$$\begin{aligned}(1 - \mathcal{K}_3)d_c(w_n, w_{n+1}) - \mathcal{K}_4 d_c(w_n, w_{n+1}) &\preceq (1 - \mathcal{K}_3)d_c(w_n, w_{n+1}) - \mathcal{K}_4\psi_2\big(d_c(w_n, w_{n+1})\big)\\ &\preceq (\mathcal{K}_1 + \mathcal{K}_2)d_c(w_{n-1}, w_n) + \mathcal{K}_4\psi_1\big(d_c(w_{n-1}, w_n)\big)\\ &\preceq (\mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_4)\psi_1\big(d_c(w_{n-1}, w_n)\big).\end{aligned}$$

Thus, ψ_2 is a subadditive comparison function, that is, $\psi_2(\mathcal{K}t) \preceq \mathcal{K}t \prec t$, $\mathcal{K} \in (0,1)$. Also, by considering ψ_1 an incomparison function, that is, $\psi_1(t) \succeq t$, the result is $d_c(w_n, w_{n+1}) \preceq \mathcal{R}\psi_1(d_c(w_{n-1}, w_n))$, where $\mathcal{R} = \frac{\mathcal{K}_1+\mathcal{K}_2+\mathcal{K}_4}{1-\mathcal{K}_3-\mathcal{K}_4}$. By repeating this process, the following formula is obtained:

$$d_c^n \preceq \mathcal{R}^n\psi_1^n\big(d_c^0\big) = \mathcal{R}^n\psi_1^n\big(d_c(w_0, w_1)\big), \tag{2.2}$$

where $d_c^n = d_c(w_n, w_{n+1})$ and $\mathcal{R}^n\psi_1^n(d_c^0)$ is the iterative sequence of order n. Therefore, for all $m > n$, there is

$$\begin{aligned}d_c(w_m, w_n) &\preceq \psi_1\big(d_c(w_m, w_{m+1})\big) + \psi_2\big(d_c(w_{m+1}, w_n)\big),\\ &\vdots\\ &\preceq \sum_{i=m}^{n-2}\psi_2^{i-m}\psi_1\big(\big(d_c(w_i, w_{i+1})\big)\big) + \psi_2^{n-m-1}\big(\big(d_c(w_{n-1}, w_n)\big)\big).\end{aligned} \tag{2.3}$$

By considering conditions 1 and 2 to establish inequalities (2.2) and (2.3), we obtain

$$d_c(w_m, w_n) \preceq \sum_{i=m}^{n-2}\psi_2^{i-m}\psi_1\big(\mathcal{R}^i\psi_1^i\big(d_c(w_0, w_1)\big)\big) + \psi_2^{n-m-1}\big(\mathcal{R}^{n-1}\psi_1^{n-1}\big(d_c(w_0, w_1)\big)\big),$$

hence

$$\begin{aligned}&\big\|d_c(w_m, w_n)\big\|\\ &\quad\le M\left\|\sum_{i=m}^{n-2}\psi_2^{i-m}\psi_1\big(\mathcal{R}^i\psi_1^i\big(d_c(w_0, w_1)\big)\big) + \psi_2^{n-m-1}\big(\mathcal{R}^{n-1}\psi_1^{n-1}\big(d_c(w_0, w_1)\big)\big)\right\|.\end{aligned} \tag{2.4}$$

Letting $m, n \to \infty$ and condition 2 of Theorem 2.1 hold, the result is $\|d_c(w_m, w_n)\| = 0$ for each $n, m \in \mathbb{N}$. Therefore, the sequence $\{w_n\}$ is Cauchy in Γ. Since Γ is a complete C2CMS, there is an element $x^* \in \Gamma$ such that $\{w_n\} \to x^*$. Let $Tx^* \neq x^*$, it is assumed that

$$\begin{aligned}0 &\preceq d_c\big(x^*, Tx^*\big)\\ &\preceq \psi_1\big(d_c\big(x^*, w_n\big)\big) + \psi_2\big(d_c\big(w_n, Tx^*\big)\big) = \psi_1\big(d_c\big(x^*, w_n\big)\big) + \psi_2\big(d_c\big(Tw_{n-1}, Tx^*\big)\big).\end{aligned}$$

By using x^* instead of w_n in Eq. (2.1), we have

$$d_c\big(x^*, Tx^*\big) \preceq \psi_1\big(d_c\big(x^*, w_n\big)\big) + \psi_2\big(\mathcal{K}_1 d_c\big(w_{n-1}, x^*\big)$$

$$
\begin{aligned}
&+ \mathcal{K}_2 d_c(w_{n-1}, Tw_{n-1}) + \mathcal{K}_3 d_c(x^*, Tx^*) \\
&+ \mathcal{K}_4 d_c(w_{n-1}, Tx^*) + \mathcal{K}_5 d_c(x^*, Tw_{n-1})\big).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\preceq \psi_1\big(d_c(x^*, w_n)\big) + \psi_2\big(\mathcal{K}_1 d_c(w_{n-1}, x^*) + \mathcal{K}_2 d_c(w_{n-1}, w_n) + \mathcal{K}_3 d_c(x^*, Tx^*) \\
&\quad + \mathcal{K}_4 d_c(w_{n-1}, Tx^*) + \mathcal{K}_5 d_c(x^*, w_n)\big)
\end{aligned}
$$

by condition 1 and $\{w_n\}$ convergent to x^* via Lemma 1.10, hence

$$
\begin{aligned}
&\preceq \psi_2\big(\mathcal{K}_3 d_c(x^*, Tx^*) + \mathcal{K}_4 d_c(w_{n-1}, Tx^*)\big) \\
&\preceq \psi_2\big(\mathcal{K}_3 d_c(x^*, Tx^*) + \mathcal{K}_4\big(\psi_1\big(d_c(w_{n-1}, x^*)\big) + \psi_2\big(d_c(x^*, Tx^*)\big)\big)\big).
\end{aligned}
$$

Again by Lemma 1.10 and since ψ_2 in condition 1 is a subadditive comparison function, the result is as follows:

$$
\begin{aligned}
&\preceq \psi_2\big(\mathcal{K}_3 d_c(x^*, Tx^*)\big) + \psi_2\big(\mathcal{K}_4\big(\psi_2\big(d_c(x^*, Tx^*)\big)\big)\big) \\
&\preceq \mathcal{K}_3 d_c(x^*, Tx^*) + \mathcal{K}_4 d_c(x^*, Tx^*) \\
&= (\mathcal{K}_3 + \mathcal{K}_4) d_c(x^*, Tx^*).
\end{aligned}
$$

Since $\mathcal{K}_3 + \mathcal{K}_4 < 1$, we obtain that

$$
0 \preceq d_c(x^*, Tx^*) \preceq d_c(x^*, Tx^*). \tag{2.5}
$$

So $0 \preceq \|d_c(x^*, Tx^*)\| \preceq \|d_c(x^*, Tx^*)\|$ and this is inconsistency. Therefore, $Tx^* = x^*$.

Assume that T has a different fixed point q, so

$$
\begin{aligned}
0 &\preceq d_c(x^*, q) \\
&= d_c(Tx^*, Tq) \\
&\preceq \mathcal{K}_1 d_c(x^*, q) + \mathcal{K}_2 d_c(x^*, Tx^*) + \mathcal{K}_3 d_c(q, Tq) + \mathcal{K}_4 d_c(x^*, Tq) + \mathcal{K}_5 d_c(q, Tx^*) \\
&= \mathcal{K}_1 d_c(x^*, q) + \mathcal{K}_2 d_c(x^*, x^*) + \mathcal{K}_3 d_c(q, q) + \mathcal{K}_4 d_c(x^*, q) + \mathcal{K}_5 d_c(q, x^*) \\
&= (\mathcal{K}_1 + \mathcal{K}_4 + \mathcal{K}_5) d_c(x^*, q).
\end{aligned}
$$

Indeed $\mathcal{K}_1 + \mathcal{K}_4 + \mathcal{K}_5 < 1$, which implies that $\|d_c(x^*, q)\| = 0$. This further implies that $x^* = q$ and T has a unique fixed point. □

Corollary 2.2 *Assume* (Γ, Λ) *to be a complete* $(DCMS)$ *and* $\Lambda : \Gamma \times \Gamma \to \mathbb{R}^+$. *Let* $T : \Gamma \to \Gamma$ *be a mapping satisfying the following condition*:

$$
\begin{aligned}
\Lambda(T\delta, T\varrho) \leq{}& K_1 \Lambda(\delta, \varrho) + K_2 \Lambda(\delta, T\delta) + K_3 \Lambda(\varrho, T\varrho) + K_4 \Lambda(\delta, T\varrho) \\
&+ K_5 \Lambda(\varrho, T\delta), \quad \forall \delta, \varrho \in \Gamma,
\end{aligned}
$$

where $K_i \in [0, 1)$, $i = 1, 2, \ldots, 5$, *and* $\sum_{i=1}^{5} K_i < 1$. *For any* $x_0 \in \Gamma$, *choose* $x_n = T^n x_0$. *So*,

(1) *Let ψ_1, ψ_2 be continuous, nondecreasing, ψ_2 be a subadditive and comparison function, and ψ_1 be an incomparison function.*
(2) $\lim_{n,m\to\infty}\sum_{i=m}^{n-2}\psi_2^{i-m}\psi_1(R^i\psi_1^i(\Lambda(x_0,x_1)))+\psi_2^{n-m-1}(R^{n-1}\psi_1^{n-1}(\Lambda(x_0,x_1)))\to 0$ *(as $n,m\to\infty$), where $R=\frac{K_1+K_2+K_4}{1-K_3-K_4}$. Then T has a unique fixed point.*

In the following results, special cases of Chatterjee type and Reich type contractions are presented by considering the cone $\mathcal{P}=\mathbb{R}^+$.

Corollary 2.3 *When $\mathcal{K}_1=\mathcal{K}_2=\mathcal{K}_3=0$, in Theorem 2.1, the Chatterjee contraction is obtained.*

Note that, from inequality (2.1),

$$\begin{aligned}d_c(x_n,x_{n+1})&\preceq\mathcal{K}_4 d_c(x_{n-1},x_{n+1})\preceq\mathcal{K}_4^2 d_c(x_{n-2},x_{n+1})\preceq\cdots\preceq\mathcal{K}_4^n d_c(x_0,x_{n+1})\\&=\mathcal{K}_4^n d_c(x_{n+1},x_0)\preceq\cdots\preceq\mathcal{K}_4^{2n}d_c(x_1,x_0).\end{aligned}$$

So, the proof just takes $\mathcal{R}^n=\mathcal{K}_4^{2n}<1$.

Corollary 2.4 *When $\mathcal{K}_4=\mathcal{K}_5=0$, in Theorem 2.1, the Reich type contraction is obtained.*

Moreover, as consequences of Theorem 2.1, we present the following two corollaries.

Corollary 2.5 *Suppose that (Γ,d_c) is a complete C2CMS, and let M be a normal constant for a normal cone $\mathcal{P}$. Assume that $T:\Gamma\to\Gamma$ satisfies the (BCP)*

$$d_c(T\delta,T\varrho)\preceq\mathcal{K}d_c(\delta,\varrho),\quad\forall\delta,\varrho\in\Gamma,\mathcal{K}\in(0,1).\tag{2.6}$$

For any $w_0\in\Gamma$, choose $w_n=T^n w_0$. Suppose that
(1) *ψ_1, ψ_2 are bounded, nondecreasing and ψ_2 is a subadditive function;*
(2) $\|\sum_{i=m}^{n-2}\psi_2^{i-m}\psi_1(\mathcal{K}^i d_c(w_0,w_1))+\psi_2^{n-m-1}(\mathcal{K}^{n-1}d_c(w_0,w_1))\|\to 0$ *(as $n,m\to\infty$), where $\psi_2^{i-m}\psi_1(\mathcal{K}^i d_c(w_0,w_1))$ and $\psi_2^{n-m-1}(\mathcal{K}^{n-1}d_c(w_0,w_1))$ denote the iterative functions.*
Then T has a unique fixed point.

Proof Note that this corollary is a direct consequence of Theorem 2.1 by taking $\mathcal{K}_2=\mathcal{K}_3=\mathcal{K}_4=\mathcal{K}_5=0$. □

Theorem 2.6 *Assume (Γ,d_c) to be a complete C2CMS with $\mathcal{P}$ as a normal cone via the normal constant M. Let $T:\Gamma\to\Gamma$ be a mapping satisfying the Kannan contraction condition*

$$\begin{aligned}d_c(T\delta,T\varrho)&\preceq\mathcal{K}_1 d_c(\delta,T\delta)\\&\quad+\mathcal{K}_2 d_c(\varrho,T\varrho),\quad\forall\delta,\varrho\in\Gamma,\text{where }\mathcal{K}_1,\mathcal{K}_2\in(0,1),\mathcal{K}_1+\mathcal{K}_2<1.\end{aligned}$$

For any $w_0\in\Gamma$, choose $w_n=T^n w_0$. Suppose that
(1) *ψ_1, ψ_2 are bounded, nondecreasing and ψ_2 is a subadditive comparison function.*
(2) $\|\sum_{i=m}^{n-2}\psi_2^{i-m}\psi_1(\mathcal{R}^i d_c(w_0,w_1))+\psi_2^{n-m-1}(\mathcal{R}^{n-1}d_c(w_0,w_1))\|\to 0$ *(as $n,m\to\infty$), where $\mathcal{R}=\frac{\mathcal{K}_1}{1-\mathcal{K}_2}$. Then T has a unique fixed point.*

Proof Note that this corollary is a direct consequence of Theorem 2.1 by taking $\mathcal{K}_1 = \mathcal{K}_4 = \mathcal{K}_5 = 0$. □

The following result is analogous to the Hardy–Rogers type fixed point theorem, discussed as an open problem in [10], when $\mathbb{R}^+ = \mathcal{P} \subset \mathcal{E} = \mathbb{R}$ as a special case in this theorem.

3 Applications

This section provides a few applications of already proven results.

Example 3.1 Consider the space of all continuous real-valued functions $\Gamma = C[a,b]$ and $d_c(f(\tau),g(\tau)) : \Gamma \times \Gamma \to \mathcal{E}$, where $\mathcal{E} = \mathbb{R}^2$, $\mathcal{P} = \{\nu = (\nu_1,\nu_2) \in \mathcal{E} : \nu_1,\nu_2 \geq 0\}$ is defined as

$$d_c\big(f(\tau),g(\tau)\big) = \Big(\beta_1 \sup_{\tau\in[a,b]} \big|f(\tau) - g(\tau)\big|, \beta_2 \sup_{\tau\in[a,b]} \big|f(\tau) - g(\tau)\big|\Big),$$

$\beta_1, \beta_2 \geq 0$, and $\psi_1, \psi_2 : \mathcal{P} \to \mathcal{P}$ by $\psi_1(\nu) = (e^{\nu_1} - 1, e^{\nu_2} - 1)$, $\psi_2(\nu) = (\nu_1, \nu_2)$. Then (Γ, d_c) is a complete C2CMS with two functions ψ_1, ψ_2.

Proof It is only proving the triangle inequality. By using $\eta \leq e^{\eta} - 1$ and $|\eta - \vartheta| \leq |\eta| + |\vartheta|, \forall \eta, \vartheta \in \mathbb{R}$, the result is as follows:

$$\begin{aligned} \big|f(\tau) - g(\tau)\big| &= \big|f(\tau) - h(\tau) + h(\tau) - g(\tau)\big| \\ &\leq \big|f(\tau) - h(\tau)\big| + \big|h(\tau) - g(\tau)\big| \\ &\leq \big[e^{|f(\tau)-h(\tau)|} - 1\big] + \big|h(\tau) - g(\tau)\big|. \end{aligned}$$

Hence,

$$\sup_{\tau\in[a,b]} \big|f(\tau) - g(\tau)\big| \leq \big[e^{\sup_{\tau\in[a,b]} |f(\tau)-h(\tau)|} - 1\big] + \sup_{\tau\in[a,b]} \big|h(\tau) - g(\tau)\big|,$$

scalar constant $\beta_1, \beta_2 \geq 0$. So, the result is

$$d_c\big(f(\tau),g(\tau)\big) \preceq \psi_1\big(d_c\big(f(\tau),h(\tau)\big)\big) + \psi_2\big(d_c\big(h(\tau),g(\tau)\big)\big).$$

Therefore, (Γ, d_c) is a C2CMS via ψ_1, ψ_2. It is not difficult to show the completeness of Γ. □

Remark 3.2 $\psi_1(\nu) = (\sin^{-1}(\nu_1), \sin^{-1}(\nu_2))$ in Example 3.1 can be considered, so (Γ, d_c) is a C2CMS. While considering $\psi_1(\nu) = \psi_2(\nu)$, (Γ, d_c) is a C1CMS.

Theorem 3.3 *Let $\Gamma = C[a,b]$ be the C2CMS given in Example 3.1. Consider the following Fredholm integral equation*:

$$f(\tau) = \Theta(\tau) + \int_a^b \Im\big(\tau, \varsigma, f(\tau)\big)\, d\varsigma, \tag{3.1}$$

wherever $\Im(\tau, \varsigma, f(\tau)) : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ is a given continuous function satisfying the following condition for all $f(\tau), g(\tau) \in \Gamma$, $\tau, \varsigma \in [a,b]$:

$$\big|\Im\big(\tau, \varsigma, f(\tau)\big) - \Im\big(\tau, \varsigma, g(\tau)\big)\big| \leq \frac{\Omega(\tau)}{|b - a|}, \tag{3.2}$$

where

$$\Omega(\tau) = K_1\, d\big(f(\tau), g(\tau)\big) + K_2\, d\big(f(\tau), Tf(\tau)\big) + K_3\, d\big(g(\tau), Tg(\tau)\big) + K_4\, d\big(f(\tau), Tg(\tau)\big) + K_5\, d\big(g(\tau), Tf(\tau)\big),$$

where (Γ, d) *is any completeness metric space, where* $K_i \in (0,1)$, $i = 1,2,\dots,5$, *and* $\sum_{i=1}^{5} K_i < 1$.

$$Tf(\tau) = \Theta(\tau) + \int_a^b \Im\big(\tau, \varsigma, f(\tau)\big)\, d\varsigma. \tag{3.3}$$

Then the integral Eq. (3.1) *has a unique solution in* Γ.

Proof Let $T : \Gamma \to \Gamma$ be defined by Eq. (3.3), and then $d_c(f(\tau), g(\tau))$ be defined in Example 3.1. Hence,

$$d_c\big(Tf(\tau), Tg(\tau)\big) = \Big(\beta_1 \sup_{\tau\in[a,b]} \big|Tf(\tau) - Tg(\tau)\big|, \beta_2 \sup_{\tau\in[a,b]} \big|Tf(\tau) - Tg(\tau)\big|\Big)$$

since

$$\begin{aligned}
&\sup_{\tau\in[a,b]} \big|Tf(\tau) - Tg(\tau)\big| \\
&\quad= \sup_{\tau\in[a,b]} \left|\Theta(\tau) + \int_a^b \Im\big(\tau, \varsigma, f(\tau)\big)\, d\varsigma - \Theta(\tau) - \int_a^b \Im\big(\tau, \varsigma, g(\tau)\big)\, d\varsigma\right| \\
&\quad\preceq \sup_{\tau\in[a,b]} \int_a^b \big|\Im\big(\tau, \varsigma, f(\tau)\big) - \Im\big(\tau, \varsigma, g(\tau)\big)\big|\, d\varsigma \\
&\quad\preceq \sup_{\tau\in[a,b]} \int_a^b \frac{\Omega(\tau)}{|b-a|}\, d\varsigma \\
&\quad\preceq \sup_{\tau\in[a,b]} \frac{\Omega(\tau)}{|b-a|} \int_a^b d\varsigma \\
&\quad\preceq K_1\, d\big(f(\tau), g(\tau)\big) + K_2\, d\big(f(\tau), Tf(\tau)\big) \\
&\qquad + K_3\, d\big(g(\tau), Tg(\tau)\big) + K_4\, d\big(f(\tau), Tg(\tau)\big) + K_5\, d\big(g(\tau), Tf(\tau)\big).
\end{aligned}$$

Thus,

$$d_c\big(Tf(\tau), Tg(\tau)\big) \preceq K_1\, d\big(f(\tau), g(\tau)\big) + K_2\, d\big(f(\tau), Tf(\tau)\big) + K_3\, d\big(g(\tau), Tg(\tau)\big) + K_4\, d\big(f(\tau), Tg(\tau)\big) + K_5\, d\big(g(\tau), Tf(\tau)\big).$$

Then ψ_1, ψ_2 are bounded and ψ_1 is an incomparison function, but ψ_2 is a sub-additive comparison function and subadditive. That is, all the conditions of Theorem 2.1 are satisfied, so there is $h \in \Gamma$ such that $Th = h$. This implies that the integral Eq. (3.1) has a unique solution. □

4 Conclusions

Based on the theoretical and practical aspects explained above, the study presents several concepts of generalized cone metric spaces, namely type I and type II composed cone metric spaces, with some examples. It provides some results for the Banach, Kannan, and Hardy–Rogers type fixed point theorems in C2CMS. Moreover, it illustrates the application of Hardy–Rogers type fixed point theorem for Fredholm integral equation. The results provide answers to the open problem extension in a generalized C2CMS of Hardy–Rogers contraction in Ayoobi et al. [9]. Finally, the study recommends the following issues for future studies: the strongly composed cone metric space. Also, this concept can be applied for high order structures in generalized contractions and established for non-normal or nonsolid with some new applications with nonlinear (or fractional) differential equations.

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Competing interests
The authors declare no competing interests.

Author details
[1]Department of Mathematics, College of Computer Sciences and Mathematics, Tikrit University, Tikrit, Iraq. [2]Department of Mathematics, College of Education for Pure Sciences, Tikrit University, Tikrit, Iraq. [3]Department of Mathematics and Sciences, Prince Sultan University, Riyadh, 11586, Saudi Arabia.

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