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Fractional double-phase nonlocal equation in Musielak-Orlicz Sobolev space

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Abstract

In this paper, we analyze the existence of solutions to a double-phase fractional equation of the Kirchhoff type in Musielak-Orlicz Sobolev space with variable exponents. Our approach is mainly based on the sub-supersolution method and the mountain pass theorem.

Mathematics Subject Classification: 35J60; 35B30; 35B40

Keywords: p(x)-Laplacian; Sub-supersolutions; Variable exponents; Existence; Mountain pass theory; Musielak-Orlicz Sobolev; Nonlinear Equations

1 Introduction and background

In recent years, partial differential equations and variational problems using a double-phase operator have the attracted attention of many researchers; see, for example [14–16] and the references therein. It sheds light on various fields of applications, including but not limited to anisotropic materials, Lavrentiev's phenomenon, and elasticity theory. The study of mathematical problems involving variable exponents lies in the modeling of many physical applications, for example, image processing [1, 7, 9, 17, 21, 24, 35], space technology, the field of robotics, and electrorheological fluids. Winslow [32] studied the electrofluids, which were noted at the beginning of the last century, and they possess a very important property, namely, the electric field affects the viscosity of theses liquids. Furthermore, it was discovered that viscosity is inversely proportional to the strength of the fluid when the electric field occurs. In this case, it is called the Winslow effect, for more benefit, see Halsey [8]. Radulescu's [21] work on electrorheological fluids and image restoration via Gaussian smoothing can also be found in the work by Chen et al. [4].

This paper deals with the $p(\xi)$-Laplacian fractional Kirchhoff double-phase equation

$$\begin{cases} \mathcal{K}(J(\xi))\mathcal{R}^{\kappa(.)}_{p(.),q(.)} = \sigma(x)\xi^{\lambda(x)-1} + g(x,\xi), & \text{in } \Lambda \\ \xi > 0, & \text{in } \Lambda \\ \xi = 0, & \text{on } \partial\Lambda \end{cases} \tag{1.1}$$

with $\Lambda = [0,T]\times[0,T] \subset \mathbb{R}^2$, $0 < \sigma \in L^\infty(\Lambda)$, $\lambda \in C(\overline{\Lambda})$ with $\lambda > 1$; and $\mathcal{R}^{\kappa(.)}_{p(.),q(.)}$ denotes the double-phase operator given by

$$\mathcal{R}^{\kappa(x)}_{p(x),q(x)}\xi := {}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{T}\left(\left|{}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}\xi\right|^{p(x)-2}.{}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}\xi + \kappa(x)\left|{}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}\xi\right|^{q(x)-2}.{}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}\xi\right),$$

where ${}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{T}(.)$ and ${}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}(.)$ are the ψ-Hilfer fractional operator of order $\frac{1}{p(x)} < \gamma(x) < 1$ and type δ $(0 \le \delta \le 1)$, and

$$J(\xi) = \int_\Lambda \left(\frac{1}{p(x)}\left|{}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}\xi\right|^{p(x)} dx + \frac{\kappa(x)}{q(x)}\left|{}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}\xi\right|^{q(x)} dx\right) \tag{1.2}$$

In addition, we assume the following:

(a) The functions $p(.), q(.) \in C(\overline{\Lambda})$ verify the following assumptions:

$$p(x) < 2, \qquad 1 < p(x) \le q(x) < p^\star = \frac{2p(x)}{2-p(x)}, \qquad \frac{q(x)}{p(x)} < \frac{3}{2}, \quad \text{for any } x \in \overline{\Lambda}.$$

Equation (1.1) is a generalization of the nonlocal problem

$$\rho\frac{\partial^2\xi}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial\xi}{\partial x}\right|^2 dx\right)\frac{\partial^2\xi}{\partial x^2} = 0,$$

which represents a general case of D'Alembert's vibration equation Provided by Kirchhoff [12]. Additionally, in [33], a time-related equation was given in the following form:

$$\xi_{tt} + \Delta^2\xi - \mathcal{K}\left(\|\nabla\xi\|_2^2\right)\Delta\xi = f(x,\xi). \tag{1.3}$$

Many authors have worked on problems related to double-phase operators and have obtained several results, including the following:

In [15], Liu and Dai proved the existence and multiplicity of solutions to the double-phase problem of the form

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) = f(x,u), & \text{in } \Delta, \\ u = 0, & \text{on } \partial\Delta, \end{cases}$$

where Δ is a bounded domain with smooth boundary, $N \ge 2$, $1 < p < q$, $\frac{q}{p} < 1 + \frac{1}{N}$, $a. : \overline{\Delta} \to [0;+\infty)$ is Lipschitz continuous, and f fulfills certain conditions. For more information, one can also see the works by Ragusa and Tachikawa [22] and Wulong et al. [34].

In [19], the existence of positive solutions to a class of double-phase Dirichlet equations that have combined effects of the singular term and the parametric linear term is studied. The reader can be referred to many other papers that discuss double-phase problems, including but not limited to [2, 5] and the references therein.

It is worth noting that fractional differential equations have led to the modeling of many phenomena in many fields of science [11, 26], and applications of the latter have appeared in engineering, medicine, and mechanics, which increased the researchers' interest in these equations, especially in mathematical aspect; see, for example, [15, 36]. In [26], the authors were able to construct the ψ-Hilfer fractional operator with several examples. See also [30], where the space $\mathbb{H}_p^{\alpha,\beta,\psi}([0,T],\mathbb{R})$ is created, allowing the study of many of these equations involving the ψ-Hilfer fractional in the appropriate spaces.

In [31], using the Nehari manifolds technique and combining it with fiber maps, the authors presented an analysis of weak solutions by studying a fractional problem of the following form:

$$\begin{cases} {}^{\mathbb{H}}\mathbb{D}_T^{\alpha,\beta,\psi}(|\mathbb{D}_{0^+}^{\alpha,\beta,\psi}\xi(t)|^{p-2}\mathbb{D}_{0^+}^{\alpha,\beta,\psi}\xi(t)) = \lambda|\xi(t)|^{p-2}\xi(t) + b(x)|\xi(t)|^{q-2}\xi(t), \\ I_{0^+}^{\beta(\beta-1),\psi}\xi(0) = I_T^{\beta(\beta-1),\psi}\xi(T) \end{cases} \tag{1.4}$$

where $\frac{1}{p} < \alpha < 1$, $0 \le \beta \le 1$, $1 < q < p-1 < \infty$, $b \in L^\infty([0,T])$, and $\lambda > 0$. In [31], the result of bifurcation from infinity to equation (1.4) is also given.

In [23], the authors present the existence and multiplicity of solutions of the Kirchhoff ψ-Hilfer fractional p-Laplacian equation using critical point theory.

Researchers worked on many models of fractional differential equations using variational problems that include fractional operators, for example, Nyamoradi and Tayyebi [18], Ghanmi and Zhang [6], Kamache et al. [10], Sousa et al. [27, 30]. For example, in [10], Kamache et al. discussed a class of perturbed nonlinear fractional p-Laplacian differential systems and proved the existence of three weak solutions using the variational method and Ricceri's critical points theorems. On the other hand, in [29], the existence and multiplicity of solutions of the following $\kappa(\xi)$-Kirchhoff equation are proven using the variational method

$$\begin{cases} \mathfrak{R}(\int_\Lambda \frac{1}{\kappa(\xi)}|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\mu,\upsilon;\psi}\phi|^{\kappa(\xi)}\,d\xi)\boldsymbol{L}_{\kappa(\xi)}^{\mu,\upsilon,\psi}\phi = g(x,\xi), \quad \text{in } \Lambda = [0,T]\times[0,T], \\ \phi = 0, \quad \text{on } \partial\Lambda, \end{cases} \tag{1.5}$$

where

$$\boldsymbol{L}_{\kappa(x)}^{\mu,\upsilon,\psi}\phi := \mathcal{R}_{p(.),q(.)}^{\kappa(.)}\nu,$$

$g(x,\xi) : \Lambda \times \mathbb{R} \to \mathbb{R}$ is the Caratheodory function, satisfying some conditions, and $\mathfrak{R}(t)$ is a continuous function.

In [28], the author discusses the multiplicity nontrivial solution for a new class of fractional differential equations of the Kirchhoff type in the ψ-fractional space $\mathrm{S}_{\mathcal{H},0}^{\alpha,\beta,\psi}$ via critical point result and variational methods.

In [3], Tahar et al. studied the existence and multiplicity of solutions for problem (1.1) with $\kappa(x) = 0$,and $\lambda(x) = q(x)$ proving their results using the mountain pass theorem combined with the sub-supersolution method.

Motivated by these works, we study the existence and multiplicity of solutions for class of fractional fractional Kirchhoff double-phase problem involving a ψ-Hilfer fractional operator with variable exponent using the sub-supersolution method and mountain pass theorem.

Here, we take the Kirchhoff function $\mathcal{K}$ and the source term g with the following conditions:

(K_0) Let $\mathcal{K} : [0,+\infty) \to [k_0,+\infty)$ be a continuous function, $k_0 > 0$, and nondecreasing;

(K_1) Let $\theta \in (0,1)$ such that

$$\widehat{\mathcal{K}}(t) := \int_0^t \mathcal{K}(\tau)\,d\tau \geq (1-\theta)\mathcal{K}(t)t \quad \text{for all } t \geq 0;$$

(g_1) $g \in C(\Lambda \times [0,+\infty), \mathbb{R})$ and $\exists l > 0$ such that

$$g(x,t) \geq \sigma(x)\big(1 - t^{\lambda(x)-1}\big) \quad \text{for all } (x,t) \in \Lambda \times [0,l];$$

(g_2) There is a function $\gamma : \overline{\Lambda} \to (1,+\infty)$, which fulfills

$$\big|g(x,t)\big| \leq \sigma(x)\big(1 + t^{\gamma(x)-1}\big) \quad \text{for all } (x,t) \in \Lambda \times [0,+\infty);$$

(g_3) There is $\mu > \frac{q^+}{1-\theta}$ such that

$$0 < \mu G(x,t) := \mu \int_0^t g(x,\tau)\,d\tau \leq g(x,t)t \quad \text{a.e. } x \in \Lambda \text{ and for all } 0 < T < t.$$

We now give our results as follows:

Theorem 1.1 *Let us consider that* (K_0) *and* (g_1)–(g_2) *are satisfied. Then, for some* $\alpha_\star > 0$, *the problem* (1.1) *has at least one solution with condition* $\|\sigma\|_\infty < \alpha_\star$.

Theorem 1.2 *Let us consider that* (K_0)–(K_1) *and* (g_1)–(g_3) *are satisfied. If* $\lambda^+ < p^- < (p^\star)$ *or* ($\lambda^- > \frac{q^+}{1-\theta}$ *or*), *then for some* $\alpha^\star > 0$, *the problem* (1.1) *accepts two solutions under the condition* $\|\sigma\|_\infty < \alpha^\star$.

We arrange our paper in the following manner: In Sect. 2, we give some definitions and lemmas for the Lebesgue and Musielak-Orlicz Sobolev spaces. In Sect. 3, we present some results that will be needed in our study of the problem (1.1). Sections 4 and 5 deal with the main proofs of the Theorems 1.1 and 1.2, respectively.

2 Preliminaries

In this section, the basic concepts and ideas on Lebesgue and Musielak-Orlicz Sobolev spaces that we will need in arriving at the results will be presented (see [25]).

Let $\nu \in C(\overline{\Lambda})$, with $\nu > 1$, and denote

$$\nu^+ := \max_{\overline{\Lambda}} \nu(x) \quad \text{and} \quad \nu^- := \min_{\overline{\Lambda}} \nu(x).$$

The variable exponent Lebesgue space

$$L^{\nu(x)}(\Lambda) = \left\{ u : \Lambda \to \mathbb{R} \text{ measurable} : \int_\Lambda |u|^{\nu(x)}\,dx < \infty \right\},$$

with the norm

$$\|u\|_{\nu(x)} = \inf\left\{ \tau > 0 : \int_\Delta \left|\frac{u}{\tau}\right|^{\nu(x)} dx \leq 1 \right\},$$

is a reflexive and separable Banach space, whose conjugate space is $L^{\nu'(x)}(\Lambda)$, where $\nu'(x) = \frac{\nu(x)}{\nu(x)-1}$.

Lemma 2.1 ([28]) *Let* $(u,v) \in L^{\nu(x)}(\Lambda) \times L^{\nu'(x)}(\Lambda)$, *then*

$$\int_\Lambda |uv|\,dz \le 2\|u\|_{\nu(x)}\|v\|_{\nu'(x)} \quad (\textit{Holder-type inequality}).$$

Lemma 2.2 ([28]) *For* $u \in L^{\nu(x)}$, *then*

$$\min\bigl(\|u\|_{\nu(x)}^{\nu^-}, \|u\|_{\nu'(x)}^{\nu^+}\bigr) \le \int_\Lambda |u|^{\nu(x)}\,dx \le \max\bigl(\|u\|_{\nu(x)}^{\nu^-}, \|u\|_{\nu'(x)}^{\nu^+}\bigr).$$

Assume that (a) is achieved, and let $\mathcal{A} : \Lambda \times [0,+\infty) \to [0,+\infty)$ defined by

$$\mathcal{A}(x,t) = t^{p(x)} + \kappa(x)t^{q(x)},$$

the modular associated with $\mathcal{A}$ is given by

$$\rho_{\mathcal{A}}(u) = \int_\Lambda \mathcal{A}\bigl(x,|u|\bigr)\,dx = \int_\Lambda \bigl(u^{p(x)} + \kappa(x)u^{q(x)}\bigr)\,dx, \quad \text{for all } u \in \Theta(\Lambda),$$

where $\Theta(\Lambda)$ is a measurable functions space. Let $L^{\mathcal{A}}$ be the Musielak-Orlicz space defined by

$$L^{\mathcal{A}} = \bigl\{u \in \Theta(\Lambda) : \rho_{\mathcal{A}}(u) < \infty\bigr\},$$

endowed with the norm

$$\|u\|_{\mathcal{A}} = \inf\left\{\tau > 0 : \rho_{\mathcal{A}}\left(\frac{u}{\tau}\right) \le 1\right\}.$$

Lemma 2.3 ([28]) *Assuming that* (a) *is achieved, we confirm that the following is true*:
(1) *If* $\|u\|_{\mathcal{A}} \le 1$, *then* $\|u\|_{\mathcal{A}}^{q^+} \le \rho_{q(\xi)} \le \|u\|_{\mathcal{A}}^{p^-}$;
(2) *If* $\|u\|_{\mathcal{A}} > 1$, *then* $\|u\|_{\mathcal{A}}^{p^-} \le \rho_{q(\xi)} \le \|u\|_{\mathcal{A}}^{q^+}$.

Define the Musielak-Orlicz Sobolev space $\mathcal{H}^{1,\mathcal{A}}(\Lambda)$ as follows:

$$\mathcal{H}^{1,\mathcal{A}}(\Lambda) = \bigl\{u \in L^{\mathcal{A}}(\Lambda) : \bigl|\mathbb{D}_{0^+}^{\alpha,\beta,\psi}u\bigr| \in L^{\mathcal{A}}(\Lambda)\bigr\}$$

equipped by

$$\|u\|_{1,\mathcal{A}} = \|u\|_{\mathcal{A}} + \bigl\|\mathbb{D}_{0^+}^{\alpha,\beta,\psi}u\bigr\|_{\mathcal{A}},$$

where $\|\mathbb{D}_{0^+}^{\alpha,\beta,\psi}u\|_{\mathcal{A}} = \||\mathbb{D}_{0^+}^{\alpha,\beta,\psi}u|\|$.

Let us denote $\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)$ the closure of $C_0^\infty(\Lambda)$ in $\mathcal{H}^{1,\mathcal{A}}(\Lambda)$. From [28], $L^{\mathcal{A}}(\Lambda)$, $\mathcal{H}^{1,\mathcal{A}}(\Lambda)$ and $\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)$ are reflexive Banach spaces.

Lemma 2.4 ([28]) *Let (a) be verified and* $\nu \in (\overline{\Lambda} \times (1,+\infty))$. *Then,*

(*j*) $\mathcal{H}_0^{1,\mathcal{A}}(\Lambda) \hookrightarrow L^{\nu(x)}(\Lambda)$ *is continuous when* $\nu(x) \leq p^\star(x)$ *for any* $x \in \Lambda$;

(*jj*) $\mathcal{H}_0^{1,\mathcal{A}}(\Lambda) \hookrightarrow L^{\nu(x)}(\Lambda)$ *is compact when* $\nu(x) < p^\star(x)$ *for any* $x \in \Lambda$;

(*jjj*) *There is* $c > 0$ *fulfilling* $\|u\|_{\mathcal{A}} \leq c\|\mathbb{D}_{0^+}^{\alpha,\beta,\psi} u\|_{\mathcal{A}}$ *for all* $u \in \mathcal{H}_0^{1,\mathcal{A}}(\Lambda)$ (*Poincaré-type inequality*).

3 Auxiliary results

We present some important definitions and concepts to create appropriate sub-supersolutions to our problem.

Definition 3.1 Let $\omega_1, \omega_2 \in \mathcal{H}_0^{1,\mathcal{A}}(\Lambda)$. We say that

$$\mathcal{K}\big(J(\omega_1)\big)\mathcal{R}_{p(x),q(x)}^{\kappa(x)}\omega_1 \leq \mathcal{K}\big(J(\omega_2)\big)\mathcal{R}_{p(x),q(x)}^{\kappa(x)}\omega_2$$

if for all nonnegative function $\varphi \in \mathcal{H}_0^{1,\mathcal{A}}(\Lambda)$,

$$\begin{aligned}
&\mathcal{K}\big(J(\omega_1)\big)\int_\Lambda \big(\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\big|^{p(x)-2}\,{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1 \\
&\qquad + \kappa(x)\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\big|^{q(x)-2}\,{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\big).{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\varphi\,dx \\
&\quad\leq \mathcal{K}\big(J(\omega_2)\big)\int_\Lambda \big(\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\big|^{p(x)-2}\,{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2 \\
&\qquad + \kappa(x)\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\big|^{q(x)-2}\,{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\big).{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\varphi\,dx.
\end{aligned}$$

Lemma 3.2 *Let* (K_0) *be satisfied. Then,* φ: $\mathcal{H}_0^{1,\mathcal{A}}(\Lambda) \to (\mathcal{H}_0^{1,\mathcal{A}}(\Lambda))^*$ *of the form*

$$\begin{aligned}
\langle\varphi(\xi),\chi\rangle = \mathcal{K}\big(J(\xi)\big)\int_\Lambda &\big(\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\xi\big|^{p(x)-2}\,{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\xi + \kappa(x)\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\xi\big|^{q(x)-2}\,{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\xi\big) \\
&\times {}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\chi\,dx, \quad \xi,\chi \in \mathcal{H}_0^{1,\mathcal{A}}(\Lambda)
\end{aligned} \tag{3.1}$$

is strictly monotone and continuous.

Proof φ is clearly continuous. We are concerned here with monotonicity completely. Let $\omega_1 \neq \omega_2 \in \mathcal{H}_0^{1,\mathcal{A}}(\Lambda)$, and let us suppose that $J(\omega_1) \geq J(\omega_2)$, and $\mathcal{K}$ is nondecreasing, implying that

$$\mathcal{K}\big(J(\omega_1)\big)) \geq \mathcal{K}\big(J(\omega_2)\big)). \tag{3.2}$$

Further, we have

$${}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2 \leq \frac{1}{2}\big(\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\big|^2 + \big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\big|^2\big). \tag{3.3}$$

Thus,

$$\begin{aligned}
&\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\big|^{p(x)} - \big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\big|^{p(x)-2}\,{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2 \\
&\quad\geq \frac{1}{2}\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\big|^{p(x)-2}\big(\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\big|^2 - \big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\big|^2\big)
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{p(x)} - \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{p(x)-2}{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1 \\ &\quad \geq \frac{1}{2}\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{p(x)-2}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^2 - \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^2\right). \end{aligned} \tag{3.5}$$

Similarly, we have

$$\begin{aligned} &\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{q(x)} - \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{q(x)-2}{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2 \\ &\quad \geq \frac{1}{2}\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{q(x)-2}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^2 - \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^2\right) \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{q(x)} - \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{q(x)-2}{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1 \\ &\quad \geq \frac{1}{2}\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{q(x)-2}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^2 - \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^2\right). \end{aligned} \tag{3.7}$$

We put

$$\Lambda_1 = \left\{x \in \Lambda : \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right| \geq \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|\right\}$$

and

$$\Lambda_1^c = \left\{x \in \Lambda : \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right| < \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|\right\}.$$

By (3.2), (3.4)–(3.7) and (K_0), we get

$$\begin{aligned} A :=\ &\mathcal{K}\big(J(\omega_1)\big)\bigg[\int_{\Lambda_1}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{p(x)} - \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{p(x)-2}{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right)dx \\ &+ \int_{\Lambda_1}\kappa(x)\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{q(x)} - \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{q(x)-2}{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right)dx\bigg] \\ &\times \mathcal{K}\big(J(\omega_2)\big)\bigg[\int_{\Lambda_1}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{p(x)} - \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{p(x)-2}{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right)dx \\ &+ \int_{\Lambda_1}\kappa(x)\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{q(x)} - \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{q(x)-2}{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right)dx\bigg] \\ \geq\ &\frac{1}{2}\mathcal{K}\big(J(\omega_1)\big)\bigg[\int_{\Lambda_1}\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{p(x)-2}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^2 - \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^2\right)dx \qquad (3.8) \\ &+ \int_{\Lambda_1}\kappa(x)\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{q(x)-2} - \left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^2 - \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^2\right)dx] \\ &- \frac{1}{2}\mathcal{K}\big(J(\omega_2)\big)\bigg[\int_{\Lambda_1}\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{p(x)-2}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^2 - \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^2\right)dx \\ &+ \int_{\Lambda_1}\kappa(x)\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{q(x)-2} - \left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^2 - \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^2\right)dx\bigg] \\ \geq\ &\frac{1}{2}\mathcal{K}\big(J(\omega_2)\big)\bigg[\int_{\Lambda_1}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{p(x)-2} - \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{p(x)-2}\right) \end{aligned}$$

$$
\begin{aligned}
&\times \left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^2-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^2\right)dx\\
&+\int_{\Lambda_1}\kappa(x)\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{q(x)-2}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{q(x)-2}\right)\\
&\times \left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^2-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^2\right)dx\Bigg]\\
\geq{}& \frac{k_0}{2}\Bigg[\int_{\Lambda_1}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{p(x)-2}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{p(x)-2}\right)\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^2-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^2\right)dx\\
&+\int_{\Lambda_1}\kappa(x)\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{q(x)-2}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{q(x)-2}\right)\\
&\times \left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^2-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^2\right)dx\Bigg]\\
\geq{}& 0.
\end{aligned}
$$

So, we have

$$
\begin{aligned}
B :={}& \mathcal{K}\big(J(\omega_1)\big)\Bigg[\int_{\Lambda_1^c}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{p(x)}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{p(x)-2}{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right)dx\\
&+\int_{\Lambda_1^c}\kappa(x)\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{q(x)}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{q(x)-2}{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right)dx\Bigg] \qquad (3.9)\\
&\times \mathcal{K}\big(J(\omega_2)\big)\Bigg[\int_{\Lambda_1^c}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{p(x)}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{p(x)-2}{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right)dx\\
&+\int_{\Lambda_1^c}\kappa(x)\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{q(x)}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{q(x)-2}{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right)dx\Bigg]\\
\geq{}& \frac{1}{2}\mathcal{K}\big(J(\omega_1)\big)\Bigg[\int_{\Lambda_1^c}\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{p(x)-2}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^2-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^2\right)dx\\
&+\int_{\Lambda_1^c}\kappa(x)\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{q(x)-2}-\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^2-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^2\right)dx\Bigg]\\
&-\frac{1}{2}\mathcal{K}\big(J(\omega_2)\big)\Bigg[\int_{\Lambda_1^c}\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{p(x)-2}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^2-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^2\right)dx\\
&+\int_{\Lambda_1^c}\kappa(x)\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{q(x)-2}-\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^2-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^2\right)dx\Bigg]\\
\geq{}& \frac{1}{2}\mathcal{K}\big(J(\omega_2)\big)\Bigg[\int_{\Lambda_1^c}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{p(x)-2}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{p(x)-2}\right)\\
&\times \left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^2-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^2\right)dx\\
&+\int_{\Lambda_1^c}\kappa(x)\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{q(x)-2}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{q(x)-2}\right)\\
&\times \left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^2-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^2\right)dx\Bigg]\\
\geq{}& \frac{k_0}{2}\Bigg[\int_{\Lambda_1^c}\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{p(x)-2}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{p(x)-2}\right)\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^2-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^2\right)dx\\
&+\int_{\Lambda_1^c}\kappa(x)\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{q(x)-2}-\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{q(x)-2}\right)
\end{aligned}
$$

$$\times \left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^2 - \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^2\right) dx\Bigg]$$
$$\geq 0.$$

Then, we get

$$\begin{aligned}
&\langle\varphi(\omega_1) - \varphi(\omega_2), \omega_1 - \omega_2\rangle \\
&= \langle\varphi(\omega_1), \omega_1\rangle - \langle\varphi(\omega_1), \omega_2\rangle \\
&\quad + \langle\varphi(\omega_2), \omega_2\rangle - \langle\varphi(\omega_2), \omega_1\rangle \\
&= \mathcal{K}(J(\omega_1))\Bigg[\int_\Lambda \left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{p(x)} - \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{p(x)-2}{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right) dx \qquad (3.10)\\
&\quad + \int_\Lambda \kappa(x)\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{q(x)} - \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{q(x)-2}{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right) dx\Bigg] \\
&\quad + \mathcal{K}(J(\omega_2)) \\
&\quad \times \Bigg[\int_{\Lambda_1^c} \left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{p(x)} - \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{p(x)-2}{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right) dx \\
&\quad + \int_{\Lambda_1} \kappa(x)\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{q(x)} - \left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{q(x)-2} - {}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right) dx\Bigg] \\
&= A + B \geq 0.
\end{aligned}$$

Hence, $\langle\varphi(\omega_1) - \varphi(\omega_2), \omega_1 - \omega_2\rangle > 0$. On the other hand, using (3.8)–(3.10), we have

$$\begin{aligned}
0 &= \langle\varphi(\omega_1) - \varphi(\omega_2), \omega_1 - \omega_2\rangle \\
&\geq \frac{k_0}{2}\Bigg[\int_\Lambda \left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{p(x)-2} - \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{p(x)-2}\right)\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^2 - \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^2\right) dx \\
&\quad + \int_\Lambda \kappa(x)\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{q(x)-2} - \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^{q(x)-2}\right) \\
&\quad \times \left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^2 - \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^2\right) dx\Bigg] \\
&\geq 0. \qquad (3.11)
\end{aligned}$$

Taking this into consideration $\kappa(x) \geq 0$ in Λ yields $|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1| = |{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2|$ in a.e. Λ. Hence

$$\mathcal{K}(J(\omega_1)) = \mathcal{K}(J(\omega_2))$$

and from (3.10)–(3.11), we get

$$\begin{aligned}
0 &= \langle\varphi(\omega_1) - \varphi(\omega_2), \omega_1 - \omega_2\rangle \\
&= \Bigg[\int_\Lambda \left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{p(x)-2} + \kappa(x)\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^{q(x)-2}\right)\left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_1\right|^2 - \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\omega_2\right|^2\right) dx,
\end{aligned}$$

which leads to a contradiction $\omega_1 = \omega_2$ in $\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)$. This ensures that φ is strictly monotonic. □

Lemma 3.3 *Let* (K_0) *be satisfied, and* $\omega_1, \omega_2 \in \mathcal{H}_0^{1,\mathcal{A}}(\Lambda)$ *verify*

$$\mathcal{K}\big(J(\omega_1)\big)\mathcal{R}^{\kappa(.)}_{p(.),q(.)}\omega_1 \leq \mathcal{K}\big(J(\omega_2)\big)\mathcal{R}^{\kappa(.)}_{p(.),q(.)}\omega_2 \tag{3.12}$$

and $\omega_1 \leq \omega_2$ *on* $\partial\Lambda$, *i.e.,* $(\omega_1 - \omega_2)^+ \in \mathcal{H}_0^{1,\mathcal{A}}(\Lambda)$. *Then,* $\omega_1 \leq \omega_2$ *a.e., in* Λ.

Proof We choose the test function $\chi = (\omega_1 - \omega_2)^+$ in (3.12), then, by (3.1), the following is obtained

$$\begin{aligned}
&\big\langle \varphi(\omega_1) - \varphi(\omega_2)), (\omega_1 - \omega_2)^+\big\rangle \\
&= \mathcal{K}\big(J(\omega_1)\big)\int_{\Lambda\cap[\omega_1>\omega_2]} \big(\big|{}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}\omega_1\big|^{p(x)-2}\,{}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}\omega_1 \\
&\quad + \kappa(x)\big|{}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}\omega_1\big|^{q(x)-2}\,{}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}\omega_1\big) \\
&\quad \times {}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}(\omega_1 - \omega_2)^+\,dx \\
&\quad - \mathcal{K}\big(J(\omega_2)\big)\int_{\Lambda\cap[\omega_1>\omega_2]} \big(\big|{}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}\omega_2\big|^{p(x)-2}\,{}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}\omega_2 \\
&\quad + \kappa(x)\big|{}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}\omega_2\big|^{q(x)-2}\,{}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}\omega_2\big) \\
&\quad \times {}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}(\omega_1 - \omega_2)^+\,dx \\
&\leq 0.
\end{aligned}$$

By monotonicity of φ, we get

$$\big\langle \varphi(\omega_1) - \varphi(\omega_2)), (\omega_1 - \omega_2)^+\big\rangle = \big\langle \varphi(\omega_1) - \varphi(\omega_2)), (\omega_1 - \omega_2)^+\big\rangle_{\Lambda\cap[\omega_1>\omega_2]} \geq 0.$$

Hence, $\langle \varphi(\omega_1) - \varphi(\omega_2)), (\omega_1 - \omega_2)^+\rangle = 0$. Through Lemma 3.2, we conclude that $(\omega_1 - \omega_2)^+ = 0$, and thus we complete the proof. □

Lemma 3.4 *Assuming that* (K_0) *is satisfied, and* $\sigma \in L^\infty(\Lambda)$, *the problem*

$$\begin{cases} \mathcal{K}(J(\xi))\mathcal{R}^{\kappa(.)}_{p(.),q(.)}\xi = \sigma(x) & \text{in } \Lambda \\ \xi = 0 & \text{on } \partial\Lambda \end{cases} \tag{3.13}$$

accepts a unique solution in $\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)$.

Proof According to (K_0) and Lemma 2.2 for all $\|\xi\|_{\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)} > 1$, we have

$$\begin{aligned}
\lim_{n\to+\infty}\frac{\langle\varphi(\xi),\xi\rangle}{\|\xi\|_{\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)}} &\geq \lim_{n\to+\infty}\frac{k_0\rho_{\mathcal{A}}(|{}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}\xi|)}{\|\xi\|_{\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)}} \\
&\geq k_0\|\xi\|^{p^-}_{\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)}.
\end{aligned} \tag{3.14}$$

Therefore,

$$\lim_{n\to+\infty}\frac{\langle\varphi(\xi),\xi\rangle}{\|\xi\|_{\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)}} = +\infty,$$

that is, φ is coercive; thus, φ is a surjection. By applying the theorem of Minty-Browder [37], the equation $\varphi(\xi) = \sigma$ is uniquely solvable in $\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)$. □

In the following lemma, l_0 indicates the best constant of $\mathcal{H}_0^{1,\mathcal{A}}(\Lambda). \hookrightarrow L^2(\Lambda)$ by l_0. Then,

$$\|\xi\|_{L^2(\Lambda)} \leq l_0 \|\xi\|_{\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)} \quad \text{for all } \xi \in \mathcal{H}_0^{1,\mathcal{A}}(\Lambda).$$

Lemma 3.5 *We suppose that* (K_0) *is satisfied. Let* $\varrho > 0$ *and* ξ_ϱ *be the unique solution of the following*

$$\begin{cases} \mathcal{K}(J(\xi_\varrho))\mathcal{R}^{\kappa(.)}_{p(.),q(.)}\xi_\varrho = \varrho & \text{in } \Lambda \\ \xi_\varrho = 0 & \text{on } \partial\Lambda. \end{cases} \tag{3.15}$$

Put $\delta = \frac{k_0 p^-}{2C_0|\Lambda|^{\frac{1}{2}}}$. *Then, when* $\varrho \geq \delta$, $\xi_\varrho \in L^\infty(\Lambda)$ *with*

$$\|\xi_\varrho\|_\infty \leq l_1^\star \mathcal{K}\big(l_2^\star \varrho^{(p^-)'}\big)\varrho^{\frac{1}{p^--1}},$$

and when $\varrho < \delta$,

$$\||\xi_\varrho\||_\infty \leq l_\star \varrho^{\frac{1}{p^+-1}},$$

in which $l_1^\star$, $l_2^\star$, *and* $l_\star > 0$ *are dependent on* Λ, k_0 *and* p.

Proof Let $\zeta \geq 0$ be fixed, and put $\Lambda_\zeta = \{x \in \Lambda : \xi_\varrho(x) > \zeta\}$ and $\xi_\varrho \geq 0$ using comparison principle. Testing equation (3.15) with $(\xi_\varrho - \zeta)^+$ and from the Young inequality

$$\begin{aligned} &\int_\Lambda \big(\big|{}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}\xi_\varrho\big|^{p(x)} + \kappa(x)\big|{}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}\xi_\varrho\big|^{q(x)}\big)\,dx \\ &\quad = \frac{\varrho}{\mathcal{K}(J(\xi_\varrho))}\int_{\Lambda_\zeta}(\xi_\varrho - \zeta)^+\,dx \qquad (3.16) \\ &\quad \leq \frac{\varrho|\Lambda_\zeta|^{\frac{1}{2}}}{\mathcal{K}(J(\xi_\varrho))}\big\|(\xi_\varrho - \zeta)^+\big\|_{L^2(\Lambda)} \\ &\quad \leq \frac{\varrho|\Lambda_\zeta|^{\frac{1}{2}} l_0}{k_0}\int_{\Lambda_\zeta}\big|{}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}\xi_\varrho\big|\,dz \\ &\quad \leq \frac{\varrho|\Lambda_\zeta|^{\frac{1}{2}} l_0}{k_0}\int_{\Lambda_\zeta}\big|{}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}\xi_\varrho\big|^{p(x)}\,dz \\ &\quad \leq \frac{\varrho|\Lambda_\zeta|^{\frac{1}{2}} l_0}{k_0}\bigg(\int_{\Lambda_\zeta}\frac{\varepsilon^{p(x)}\big|{}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}\xi_\varrho\big|^{p(z)}}{p(x)}\,dx + \int_{\Lambda_\zeta}\frac{\varepsilon^{-p'(x)}}{p'(x)}\,dx\bigg). \qquad (3.17) \end{aligned}$$

For $\varrho \geq \delta$, taking

$$\epsilon = \bigg(\frac{k_0 p^-}{2\varrho|\Lambda|^{\frac{1}{2}} l_0}\bigg)^{\frac{1}{p^-}} = \bigg(\frac{\delta}{\varrho}\bigg)^{\frac{1}{p^-}}, \tag{3.18}$$

one has $\epsilon \leq 1$. Thus,

$$\begin{aligned} &\frac{\varrho|\Lambda_\zeta|^{\frac{1}{2}} l_0}{k_0} \int_{\Lambda_\zeta} \frac{\varepsilon^{p(x)} |{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi} \xi_\varrho|^{p(x)}}{p(x)} \, dx \\ &\quad \leq \frac{\varrho|\Lambda_\zeta|^{\frac{1}{2}} l_0 \epsilon^{p^-}}{k_0 p^-} \int_{\Lambda_\zeta} \left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi} \xi_\varrho\right|^{p(x)} dx, \\ &\quad \leq \frac{\varrho|\Lambda_\zeta|^{\frac{1}{2}} l_0 \epsilon^{p^-}}{k_0 p^-} \int_{\Lambda} \left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi} \xi_\varrho\right|^{p(x)} dx + \kappa(x)\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi} \xi_\varrho\right|^{q(x)}\right) dx \\ &\quad = \frac{1}{2} \int_{\Lambda_\zeta} \left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi} \xi_\varrho\right|^{p(x)} dx + \kappa(x)\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi} \xi_\varrho\right|^{q(x)}\right) dx. \end{aligned} \tag{3.19}$$

From (3.16) and (3.19), we get

$$\begin{aligned} &\int_{\Lambda_\zeta} \left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi} \xi_\varrho\right|^{p(x)} + \kappa(x)\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi} \xi_\varrho\right|^{q(x)}\right) dx \\ &\quad \leq \frac{2\varrho|\Lambda_\zeta|^{\frac{1}{2}} l_0}{k_0 (p^+)'} \int_{\Lambda_\zeta} \epsilon^{-(p^-)'} \, dx = \frac{2\varrho l_0 \epsilon^{-(p^-)'}}{k_0 (p^+)'} |\Lambda_\zeta|^{\frac{3}{2}}. \end{aligned} \tag{3.20}$$

Similarly, with ξ_ϱ as test function in (3.15), we also obtain

$$\int_{\Lambda} \left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi} \xi_\varrho\right|^{p(x)} + \kappa(x)\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi} \xi_\varrho\right|^{q(x)}\right) dx \leq \frac{2\varrho C_0 \epsilon^{-(p^-)'}}{k_0 (p^+)'} |\Lambda|^{\frac{3}{2}}. \tag{3.21}$$

From (3.17), (3.21), and with $\mathcal{K}$ being monotonic, we find

$$\begin{aligned} \int_{\Lambda_\zeta} (\xi_\varrho - \zeta)^+ \, dx &= \frac{\mathcal{K}(J(\xi_\varrho))}{\varrho} \int_{\Lambda_\zeta} \left(\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi} \xi_\varrho\right|^{p(x)} + \kappa(x)\left|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi} \xi_\varrho\right|^{q(x)}\right) dx \\ &\leq \mathcal{K}\left(\frac{2\varrho l_0 \epsilon^{-(p^-)'}}{k_0 p^- (p^+)'} |\Lambda|^{\frac{3}{2}}\right) \frac{2 l_0 \epsilon^{-(p^-)'}}{k_0 (p^+)} |\Lambda_\zeta|^{\frac{3}{2}}. \end{aligned}$$

Through Lemma 5.1 in [13], the following is achieved

$$\|\xi_\varrho\|_\infty \leq \mathcal{K}\left(\frac{2\varrho l_0 \epsilon^{-(p^-)'}}{k_0 p^- (p^+)'} |\Lambda|^{\frac{3}{2}}\right) \frac{6 l_0 \epsilon^{-(p^-)'}}{k_0 (p^+)'} |\Lambda|^{\frac{1}{2}}. \tag{3.22}$$

It follows from (3.18) and (3.22) that

$$\|\nu_\varrho\|_\infty \leq l_1^\star \mathcal{K}\left(l_2^\star \varrho^{(p^-)'}\right) \varrho^{\frac{1}{p^- - 1}},$$

where

$$l_1^\star := \frac{3(2 l_0)^{(p^-)'}}{(p^+) k_0^{(p^-)'} (p^-)^{\frac{1}{p^- - 1}}} |\Lambda|^{\frac{(p^-)'}{2}},$$

and

$$l_2^\star := \frac{(2 l_0)^{(p^-)'}}{(p^+)' k_0^{(p^-)'} (p^-)^{(p^-)'}} |\Lambda|^{1 + \frac{(p^-)'}{2}}.$$

When $\varrho < \delta$, taking

$$\varepsilon = \left(\frac{k_0 p^-}{2\varrho|\Lambda|^{\frac{1}{2}} l_0}\right)^{\frac{1}{p^+}} = \left(\frac{\delta}{\varrho}\right)^{\frac{1}{p^+}},$$

we have $\epsilon < 1$. By the same approach, we get

$$\|\xi_\varrho\|_\infty \le l_\star \varrho^{\frac{1}{p^+-1}},$$

where

$$l_* = \frac{3(2l_0)^{(p^+)'}}{(p^+)' k_0^{(p^+)'} (p^-)^{\frac{1}{p^+-1}}} |\Lambda|^{\frac{(p^+)'}{2}} \mathcal{K}\left(\frac{(2\delta l_0)^{(p^+)'}}{(p^+)' k_0^{(p^+)'} (p^-)^{(p^+)'}} |\Lambda|^{1+\frac{(p^+)'}{2}}\right). \qquad \square$$

4 Proof of Theorem 1.1

We say that $(\underline{\xi}, \overline{\xi})$ are sub-supersolution of problem (1.1) if $\underline{\xi}, \overline{\xi} \in L^\infty(\Lambda)$, $\underline{\xi} \le \overline{\xi}$ a.e., in Λ and

$$\begin{cases} \mathcal{K}(J(\underline{\xi})) \int_\Lambda (|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi} \underline{\xi}|^{p(x)-2}.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi} \underline{\xi} \\ \quad + \kappa(x)|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi} \underline{\xi}|^{q(x)-2}.|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi} \underline{\xi}|^{q(x)}).{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi} \eta\, dx \\ \le \int_\Lambda \sigma(x) \underline{\xi}^{\lambda(x)-1} \eta\, dx + \int_\Lambda g(x, \underline{\xi}) \eta\, dx \end{cases} \tag{4.1}$$

$$\begin{cases} \mathcal{K}(J(\overline{\xi})) \int_\Lambda (|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi} \overline{\xi}|^{p(x)-2}.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi} \overline{\xi} \\ \quad + \kappa(x)|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi} \overline{\xi}|^{q(x)-2}.|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi} \overline{\xi}|^{q(x)}).{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi} \eta\, dx \\ \ge \int_\Lambda \sigma(x) \overline{\xi}^{\lambda(x)-1} \eta\, dx + \int_\Lambda g(x, \overline{\xi}) \eta\, dx, \end{cases} \tag{4.2}$$

for all arbitrary nonnegative function $\eta \in \mathcal{H}_0^{1,\mathcal{A}}(\Lambda)$.

Lemma 4.1 *Let (K_0) and (g_1)–(g_2) be fulfilled. Then, there is $\sigma_\star > 0$ such that (1.1) has sub-supersolution $(\underline{\xi}, \overline{\xi}) \in (\mathcal{H}_0^{1,\mathcal{A}}(\Lambda) \cap L^\infty(\Lambda) \times (\mathcal{H}_0^{1,\mathcal{A}}(\Lambda) \cap L^\infty(\Lambda))$ with $\|\xi\|_\infty \le l$, provided that $\|\sigma\|_\infty < \alpha$, where l is defined in (g_1).*

Proof Using the Lemmas 3.2, 3.3, and 3.4, there exists a unique solution $(0,0) \le (\underline{\xi}, \overline{\xi}) \in (\mathcal{H}_0^{1,\mathcal{A}}(\Lambda) \cap L^\infty(\Lambda))$ of the following problems

$$\begin{cases} \mathcal{K}(J(\underline{\xi})) \mathcal{R}_{p(.),q(.)}^{\kappa(.)} = \sigma(x) & \text{in } \Lambda \\ \underline{\xi} = 0 & \text{on } \partial\Lambda \end{cases} \tag{4.3}$$

and

$$\begin{cases} \mathcal{K}(J(\overline{\xi})) \mathcal{R}_{p(.),q(.)}^{\kappa(.)} = 1 + \sigma(x) & \text{in } \Lambda \\ \overline{\xi} = 0 & \text{on } \partial\Lambda \end{cases} \tag{4.4}$$

such that

$$\|\underline{\xi}\|_\infty \le \max\left(l_1^\star \mathcal{K}\left(C_2^\star \|\sigma\|_\infty^{(p^-)'}\right) \|\sigma\|^{\frac{1}{p^--1}}, l_\star \|\sigma\|^{\frac{1}{p^+-1}}\right),$$

where $l_1^\star$, $l_2^\star$, and $l_\star$ are given in Lemma 3.5. Next, consider that $\mathcal{K}$ is nondecreasing and there exits $\alpha > 0$ relying only on $l_1^\star$, $l_2^\star$, and $l_\star$ such that $\|\underline{\xi}\|_\infty \leq l$, provided that $\|\sigma\|_\infty < \alpha$. Moreover, by Lemma 3.2, $\underline{\xi} \leq \overline{\xi}$.

Let ξ in $\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)$. By (4.1) and ($g_1$), we get

$$\begin{aligned}
&\mathcal{K}\big(J(\underline{\xi})\big)\int_\Lambda \big(\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\underline{\xi}\big|^{p(x)-2}.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\underline{\xi}\\
&\qquad + \kappa(x)\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\underline{\xi}\big|^{q(x)-2}.\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\underline{\xi}\big|^{q(x)}\big).{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\eta\,dx\\
&\qquad - \int_\Lambda \sigma(x)\underline{\xi}^{\lambda(x)-1}\eta\,dx + \int_\Lambda g(x,\underline{\xi})\eta\,dx\\
&\quad \leq \int_\Lambda \sigma(x)\eta\,dx - \int_\Lambda \sigma(x)\underline{\xi}^{\lambda(x)-1}\eta\,dx - \int_\Lambda \sigma(x)\big(1-\underline{\xi}^{\lambda(x)-1}\big)\eta\,dx = 0.
\end{aligned}$$

From (4.2) and (g_2), we have

$$\begin{aligned}
&\mathcal{K}\big(J(\overline{\xi})\big)\int_\Lambda \big(\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\overline{\xi}\big|^{p(x)-2}.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\overline{\xi}\\
&\qquad + \kappa(x)\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\overline{\xi}\big|^{q(x)-2}.\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\overline{\xi}\big|^{q(x)}\big).{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\eta\,dx\\
&\qquad - \int_\Lambda \sigma(x)\overline{\xi}^{\lambda(x)-1}\eta\,dx - \int_\Lambda g(x,\overline{\xi})\eta\,dx\\
&\quad \geq \int_\Lambda \big(1 - A_\infty\|\sigma\|_\infty\big)\eta\,dx,
\end{aligned}$$

where

$$A_\infty := \max\big(\|\overline{\xi}\|_\infty^{\lambda^+-1}, \|\overline{\xi}\|_\infty^{\lambda^--1}\big) + \max\big(\|\overline{\xi}\|_\infty^{\gamma^+-1}, \|\overline{\xi}\|_\infty^{\gamma^--1}\big).$$

Thus, choosing $\alpha_\star = \min(\alpha, \frac{1}{A_\infty})$ yields

$$\int_\Lambda \big(1 - A_\infty\|\gamma\|_\infty\big)\eta\,dx \geq 0 \quad \text{for } \|\sigma\|_\infty < \alpha_\star.$$

Hence, we get the expected result.

We now highlight the proof of Theorem 1.1:

Let $\underline{\xi}, \overline{\xi} \in \mathcal{H}_0^{1,\mathcal{A}}(\Lambda). \cap L^\infty(\Lambda)$. According to the previous lemma, we can write

$$h(x,t) = \begin{cases} \sigma(x)\overline{\xi}(x)^{\lambda(x)-1} + g(x,\overline{\xi}(x)) & \text{if } t > \overline{\xi}(x)\\ \sigma(x)\overline{\xi}t^{\lambda(x)-1} + g(x,t) & \text{if } \underline{\xi}(x) \leq t \leq \overline{\xi}(x)\\ \sigma(x)\underline{\xi}(x)^{\lambda(x)-1} + g(x,\underline{\xi}(x)) & \text{if } t < \underline{\xi}(x). \end{cases}$$

We define the problem

$$\begin{cases} \mathcal{K}(J(\xi))\mathcal{R}_{p(.),q(.)}^{\kappa(.)} = h(x,\xi) & \text{in } \Lambda\\ \xi = 0 & \text{on } \partial\Lambda \end{cases} \tag{4.5}$$

and the energy functional attached to it $\mathcal{I}: \mathcal{H}_0^{1,\mathcal{A}}(\Lambda) \to \mathbb{R}$ given by

$$\mathcal{I}(\xi) = \widehat{\mathcal{K}}\big(J(\xi)\big) - \int_\Lambda H(x,\xi)\,dx,$$

where $H(x,t) = \int_0^t h(x,\tau)\,d\tau$. Then, $\mathcal{I} \in C^1$, it is clear that the critical points for $\mathcal{I}$ are solutions to (4.5). According to (K_0), $\mathcal{I}$ is coercive and sequentially weakly lower semicontinuous. Hence, $\mathcal{I}$ attains its minimum in the weakly closed subset $[\underline{\xi}, \overline{\xi}] \cap \mathcal{H}_0^{1,\mathcal{A}}(\Lambda)$ at some ξ_0, which represents a critical point in $\mathcal{I}$. The proof of Theorem 1.1 is complete. □

5 Proof of Theorem 1.2

Let the function f be defined as

$$f(x,t) = \begin{cases} \sigma(x)t^{\lambda(x)-1} + g(x,t) & \text{if } t \geq \underline{\xi}(x), \\ \sigma(x)\underline{\xi}(x)^{\lambda(x)-1} + g(x,\underline{\xi}(x)) & \text{if } t \leq \underline{\xi}(x). \end{cases}$$

Also, one can look at the problem

$$\begin{cases} \mathcal{K}(J(\xi))\mathcal{R}^{\kappa(.)}_{p(.),q(.)} = f(x,\xi) & \text{in } \Lambda \\ \xi = 0 & \text{on } \partial\Lambda. \end{cases} \tag{5.1}$$

To find solutions for (4.5), we follow the approach of identifying critical points of the C^1-functional $\mathcal{J}: \mathcal{H}_0^{1,\mathcal{A}}(\Lambda) \to \mathbb{R}$, defined as:

$$\mathcal{J}(\xi) = \widehat{\mathcal{K}}\big(J(\xi)\big) - \int_\Lambda F(x,\xi)\,dx,$$

where $F(x,t) = \int_0^t f(x,\tau)\,d\tau$.

Lemma 5.1 *Assuming that the conditions of Theorem 1.2 are satisfied, the functional $\mathcal{J}$ satisfies the Palais-Smale condition.*

Proof Assume that $\{\xi_n\} \subset \mathcal{H}_0^{1,\mathcal{A}}(\Lambda)$ is a sequence such that

$$\mathcal{J}(\xi_n) \to c \in \mathbb{R} \quad \text{and} \quad \mathcal{J}(\xi_n) \to 0 \quad \text{in } \big(\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)\big)^*. \tag{5.2}$$

Here, we prove that $\{\xi_n\}$ is bounded in $\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)$.

Case 1: $\lambda^- > \frac{p^+}{1-\theta}$. Let $\mu_0 \in (\frac{p^+}{1-\theta}, \min(\mu, \lambda^-))$. By $(K_0)-(K_1)$, (g_3), and Lemmas 2.3 and 2.4, for sufficiently large n, we obtain

$$\begin{aligned} 1 + c + \|\xi_n\|_{\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)} &\geq \mathcal{J}(\xi_n) - \frac{1}{\mu_0}\big\langle \mathcal{J}'(\xi_n), \xi_n \big\rangle \\ &\geq (1-\theta)\mathcal{K}\big(J(\xi_n)\big)J(\xi_n) \\ &\quad - \frac{1}{\mu_0}\mathcal{K}\big(J(\xi_n)\big)\int_\Lambda \Big(\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\xi_n\big|^{p(x)}\,dx + \kappa(x)\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\xi_n\big|^{q(x)}\Big)\,dx \\ &\quad + \int_\Lambda \bigg(\frac{1}{\mu_0}f(x,\xi_n)\xi_n - F(x,\xi_n)\bigg)\,dx \end{aligned}$$

$$
\begin{aligned}
&\geq k_0\left(\frac{1-\theta}{p^+}-\frac{1}{\mu_0}\right)\left(\|\xi_n\|^{p^-}_{\mathcal{H}^{1,\mathcal{A}}_0(\Lambda)}-1\right)\\
&\quad+\int_{[\xi_n\geq\underline{\xi}]}\left(\frac{1}{\mu_0}g(x,\xi_n)\xi_n-H(x,\xi_n)\right)dx\\
&\quad+\int_{[\xi_n>\underline{\xi}]}\left(\frac{1}{\mu_0}-\frac{1}{q(x)}\right)\sigma(x)\xi_n^{\lambda(x)}\,dx-C_1\|\xi_n\|_{\mathcal{H}^{1,\mathcal{A}}_0(\Lambda)}-C_2\\
&\geq k_0\left(\frac{1-\theta}{p^+_M}-\frac{1}{\mu_0}\right)\left(\|\xi_n\|^{p^-}_{\mathcal{H}^{1,\mathcal{A}}_0(\Lambda)}-1\right)-C_1\|\xi_n\|_{\mathcal{H}^{1,\mathcal{A}}_0(\Lambda)}-C_2,
\end{aligned}
$$

where $C_1, C_2 > 0$. Thus, the sequence $\{\xi_n\}$ is bounded in $\mathcal{H}^{1,\mathcal{A}}_0(\Lambda)$ as $p^- > 1$.

Case 2: $p > \lambda^{+-}$. Using (K_0)–(K_1), (g_3), and Lemmas 2.3 and 2.4, we get

$$
\begin{aligned}
1+c+\|\xi_n\|_{\mathcal{H}^{1,\mathcal{A}}_0(\Lambda)}&\geq\mathcal{J}(\xi_n)-\frac{1}{\mu}\left\langle\mathcal{J}'(\xi_n),\xi_n\right\rangle\\
&\geq k_0\left(\frac{1-\theta}{\lambda^+}-\frac{1}{\mu}\right)\left(\|\xi_n\|^{p^-}_{\mathcal{H}^{1,\mathcal{A}}_0(\Lambda)}-1\right)\\
&\quad+\int_{[\xi_n>\underline{\xi}]}\left(\frac{1}{\mu}g(x,\xi_n)\xi_n-H(x,\xi_n)\right)dx\\
&\quad+\int_{[\xi_n>\underline{\xi}]}\left(\frac{1}{\mu}-\frac{1}{\lambda(x)}\right)\sigma(x)\xi_n^{\lambda(x)}\,dx-C_3\|\xi_n\|_{\mathcal{H}^{1,\mathcal{A}}_0(\Lambda)}-C_4\\
&\geq k_0\left(\frac{1-\theta}{\lambda^+}-\frac{1}{\mu}\right)\left(\|\xi_n\|^{p^-}_{\mathcal{H}^{1,\mathcal{A}}_0(\Lambda)}-1\right)\\
&\quad-C_5\left(\|\xi_n\|^{\lambda^+}_{\mathcal{H}^{1,\mathcal{A}}_0(\Lambda)}+\|\xi_n\|^{\lambda^-}_{\mathcal{H}^{1,\mathcal{A}}_0(\Lambda)}\right)\\
&\quad-C_3\|\xi_n\|_{\mathcal{H}^{1,\mathcal{A}}_0(\Lambda)}-C_4,
\end{aligned}
$$

where C_3, C_4, and $C_5 > 0$. Thus, $\{\xi_n\}$ is bounded in $\mathcal{H}^{1,\mathcal{A}}_0(\Lambda)$ and is proved because $p^- > \lambda^+$. Further, we have

$$
\begin{cases}
\xi_n\rightharpoonup\xi & \text{in } \mathcal{H}^{1,\mathcal{A}}_0(\Lambda)\\
\xi_n\to\xi & \text{a.e. in } \Lambda\\
\xi_n\to\xi & \text{in } L^{w(x)}(\Lambda) \text{ with } 1<w^-\leq w^+<(p^\star)^-.
\end{cases}
\tag{5.3}
$$

Thus,

$$
\begin{aligned}
o_n(1)&=\left\langle\mathcal{J}'(\xi_n),\xi_n-\xi\right\rangle\\
&=\mathcal{K}\left(J(\xi_n)\right)\int_\Lambda\left(\left|{}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}\xi_n\right|^{p(x)-2}+\kappa(x)\left|{}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}\xi_n\right|^{q(x)-2}.{}^{\mathbb{H}}\mathbb{D}^{\gamma,\delta;\psi}_{0^+}(\xi_n-\xi)\right)dx\\
&\quad-\int_\Lambda f(x,\xi_n)(\xi_n-\xi)\,dx.
\end{aligned}
$$

From the Holder inequality (g_2), (5.3), we get

$$
\int_\Lambda f(x,\xi_n)(\xi_n-\xi)\to 0
$$

so that

$$\mathcal{K}\big(J(\xi_n)\big)\int_\Lambda \big(\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\xi_n\big|^{p(x)-2}+\kappa(x)\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\xi_n\big|^{q(x)-2}.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}(\xi_n-\xi)\big)\,dx\to 0.$$

From the hypothesis (K_0), we get

$$\int_\Lambda \big(\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\xi_n\big|^{p(x)-2}+\kappa(x)\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\xi_n\big|^{q(x)-2}.{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}(\xi_n-\xi)\big)\,dx\to 0.$$

Thus, $\xi_n\to\xi$ in $\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)$. From the (S_+) property, we finish the proof.

Combining with Lemma 2.3, we have $\nu_n\to\nu$ in $\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)$. □

Lemma 5.2 *Assuming that the conditions of Theorem 1.2 are satisfied, for $\|\sigma\|_\infty$ sufficiently small, we have*

(i) $\exists\varsigma>0$ *and* $\vartheta>\|\underline{\xi}\|_{\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)}$ *such that*

$$\mathcal{J}(\underline{\xi})<0<\varsigma\le\inf_{\xi\in\partial B_\vartheta(0)}\mathcal{J}(\xi);$$

(ii) $\exists e\in\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)$ *such that* $\|e\|_{\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)}>2\vartheta$ *and* $\mathcal{J}(e)<\varsigma$.

Proof (i) We choose $\eta=\underline{\xi}$ in the first inequality of (4.1). By applying the nondecreasing property to $\mathcal{K}$, we have

$$\begin{aligned}\mathcal{J}(\underline{\xi}) &= \mathcal{K}\big(J(\underline{\xi})\big)-\int_\Lambda F(x,\xi)\,dx\\ &\le \mathcal{K}\big(J(\underline{\xi})\big)J(\underline{\xi})-\int_\Lambda \sigma(x)\underline{\xi}^{\lambda(x)}\,dx-\int_\Lambda g(x,\underline{\xi})\underline{\xi}\,dx\\ &< \mathcal{K}\big(J(\underline{\xi})\big)\int_\Lambda\big(\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\alpha,\beta;\psi}\underline{\xi}\big|^{p(x)}+\kappa(x)\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\underline{\xi}\big|^{q(x)}\big)\,dx\\ &\quad -\int_\Lambda\sigma(x)\underline{\xi}^{\lambda(z)}\,dx-\int_\Lambda g(x,\underline{\xi})\underline{\xi}\,dx\\ &\le 0.\end{aligned}$$

Therefore, $\mathcal{J}(\underline{\xi})<0$. Further, assume that $\xi\in\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)$ with $\|\xi\|_{\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)}\ge 1$. By (K_0), (g_2), (3.12) Lemmas 2.3 and 2.4, we infer

$$\mathcal{J}(\xi)\ge\frac{k_0}{q^+}\big(\|\xi\|_{X_0}^{p^-}-1\big)-C_6\|\sigma\|_\infty\big(\|\xi\|_{\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)}+\|\xi\|_{\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)}^{\lambda^+}+\|\nu\|_{\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)}^{\gamma^+}\big)-C_7,$$

where $C_6,C_7>0$. Observe that one can choose $\varsigma>0$ and $\vartheta>\|\underline{\xi}\|_{\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)}$ such that

$$\frac{k_0}{q^+}\big(\|\vartheta\|_{\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)}^{p^-}-1\big)-C_7\ge 2\varsigma.$$

Then, letting $\|\sigma\|_\infty\le\frac{\varsigma}{C_6(\vartheta+\vartheta^{\lambda^+}+\vartheta^{\gamma^+})}$, this implies that $\mathcal{J}(\xi)\ge\varsigma$ for $\|\xi\|_{\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)}=\vartheta$.

(ii) By (K_1), there is $C_8>0$ such that

$$\widehat{\mathcal{K}}(t)\le C_8t^{\frac{1}{1-\theta}}\quad\text{for all } t>1. \tag{5.4}$$

From (5.2) and (g_3), for all $t > 1$, we have

$$\begin{aligned}\mathcal{J}(t\underline{\xi}) &= \mathcal{K}\big(J(t\underline{\xi})\big) - \int_\Lambda F(x,t\underline{\xi})\,dx \\ &\leq C_7 t^{\frac{q^+}{1-\theta}}\big(J(\underline{\xi})\big)^{\frac{1}{1-\theta}} - t^{\lambda^-}\int_\Lambda \sigma(x)\underline{\xi}^{\lambda(x)}\,dx - C_8 t^{\mu}\int_\Delta \underline{\xi}^{\mu}\,dz + C_9.\end{aligned}$$

Then, for some $t_0 > 1$ large enough, $\mathcal{J}(t_0\underline{\xi}) < 0$ and $\|t_0\underline{\xi}\|_{\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)} > 2\vartheta$, due to $\frac{q^+}{1-\theta} < \mu$. Thus, we take $e = t_0\underline{\xi}$, the proof is complete.

We currently prove Theorem 1.2.

Let $\xi_0 \in [\underline{\xi},\overline{\xi}] \cap \mathcal{H}_0^{1,\mathcal{A}}(\Lambda)$ be the previous solution of (1.1) obtained from Theorem 1.1, which satisfies

$$\mathcal{I}(\xi_0) = \inf_{\xi\in\Lambda}\mathcal{I}(\xi),$$

with $\xi_0 \in \Lambda := [\underline{\xi},\overline{\xi}] \cap \mathcal{H}_0^{1,\mathcal{A}}(\Lambda)$. Using mountain pass theorem [20] and Lemmas 5.1 and 5.2, we determine the value

$$d^* := \inf_{\zeta\in\Xi}\max_{t\in[0,1]}\mathcal{J}\big(\zeta(t)\big),$$

with

$$\Xi := \big\{\zeta \in C\big([0,1],\mathcal{H}_0^{1,\mathcal{A}}(\Lambda)\big); \zeta(0) = \underline{\xi}, \zeta(1) = e\big\}$$

being a critical value of $\mathcal{J}$. Then, there exist $\xi_1 \in \mathcal{H}_0^{1,\mathcal{A}}(\Lambda)$ fulfilling $\mathcal{J}'(\xi_1) = 0$ and $\mathcal{J}'(\xi_1) = d^*$. Taking into consideration that $\mathcal{I}(\xi) = \mathcal{J}(\xi)$ for all $\xi \in [0,\overline{\xi}] \cap \mathcal{H}_0^{1,\mathcal{A}}(\Lambda)$, it follows that $\mathcal{J}(\xi_0) \leq \mathcal{J}(\underline{\xi})$. Now, we prove that $\xi_1 \geq \underline{\xi}$ a.e. in Λ. Utilizing $(\underline{\xi} - \xi_1)^+$ as a test function in $\mathcal{J}'(\xi_1) = 0$ and from the first inequality of (4.1), we have

$$\begin{aligned}&\mathcal{K}\big(J(\xi_1)\big)\int_\Lambda \big(\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;;\psi}\underline{\xi}\big|^{p(x)-2} + \kappa(x)\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\underline{\xi}\big|^{q(x)-2}\big).{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}(\underline{\xi} - \xi_1)^+\,dx \\ &\quad= \int_\Lambda f(x,\xi_1)(\underline{\xi} - \xi_1)^+\,dx \\ &\quad= \int_\Lambda \sigma(x)\underline{\xi}(x)^{\lambda(x)-1} + g(x,\underline{\xi}))(\underline{\xi} - \xi_1)^+\,dx \\ &\quad\geq \mathcal{K}\big(J(\underline{\xi})\big) \times \int_\Lambda \big(\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;;\psi}\underline{\xi}\big|^{p(x)-2} + \kappa(x)\big|{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}\underline{\xi}\big|^{q(x)-2}\big).{}^{\mathbb{H}}\mathbb{D}_{0^+}^{\gamma,\delta;\psi}(\underline{\xi} - \xi_1)^+\,dx\end{aligned}$$

so that

$$\big\langle\varphi(\underline{\xi}) - \varphi(\xi_1), (\underline{\xi} - \xi_1)^+\big\rangle \leq 0.$$

So, because φ is strictly monotone, then $(\underline{\xi} - \xi_1)^+ = 0$ a.e. in Λ. This leads to $\xi_1 \geq \underline{\xi}$ a.e. in Λ. Therefore, ξ_0 and ξ_1 are nonnegative solutions to the problem with

$$\mathcal{J}(\xi_0) \leq \mathcal{J}(\underline{\xi}) < 0 < \varsigma \leq d^* = \mathcal{J}(\xi_1).$$

We finished the proof of Theorem 1.2. □

6 Conclusion

In this work, we have analyzed the existence of solutions to a double-phase fractional equation of the Kirchhoff type in Musielak-Orlicz Sobolev space with variable exponents. Our approach is mainly based on the sub-supersolution method and the mountain pass theorem. In future work, we will follow the current study with general source terms.

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