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Exploring solutions to specific class of fractional differential equations of order $3 < \hat{u} \leq 4$

Saleh Fahad Aljurbua[1*]

*Correspondence:
s.aljurbua@qu.edu.sa
[1] Department of Mathematics, College of Science, Qassim University, P.O. Box 6644, Buraydah, 51452, Saudi Arabia

Abstract

This paper focuses on exploring the existence of solutions for a specific class of FDEs by leveraging fixed point theorem. The equation in question features the Caputo fractional derivative of order $3 < \hat{u} \leq 4$ and includes a term $\Theta(\beta, \mathscr{Z}(\beta))$ alongside boundary conditions. Through the application of a fixed point theorem in appropriate function spaces, we consider nonlocal conditions along with necessary assumptions under which solutions to the given FDE exist. Furthermore, we offer an example to illustrate the results.

Keywords: Fractional derivatives; Differential equations; Fractional differential equations; Antiperiodic; Nonlocal boundary conditions; Existence

1 Introduction

The focus of this research delves into the existence of solutions of the following problems:

$$\begin{cases} {}^cD^{\hat{u}}\mathscr{Z}(\beta) = \Theta(\beta, \mathscr{Z}(\beta)), & 3 < \hat{u} \leq 4, 0 \leq \hat{c} < \sigma, \beta \in [0, \sigma], \\ \mathscr{Z}^{(i)}(\hat{c}) = -\mathscr{Z}^{(i)}(\sigma), & i = 0, 1, 2, 3, \end{cases} \tag{1.1}$$

where ${}^cD^{\hat{u}}$ represents the Caputo fractional derivative with a specific order denoted by the symbol $\hat{u}$, and $\Theta : [0, \sigma] \times \mathbb{R} \longrightarrow \mathbb{R}$.

In the last few decades, noninteger-order calculus became very interesting to mathematicians and modelers. Fractional differential equations play a major role in applied sciences. The fractional derivatives provide an excellent description of the ecological models, anomalous diffusion, turbulent flow in a porous medium, synchronization of chaotic systems, and disease models; see [1–6].

The investigation of differential equations involving fractional calculus has attracted considerable attention and importance in the realm of mathematics research due to their capability to describe nonlinear and nonlocal phenomena in diverse scientific disciplines. These equations incorporate fractional derivatives and generalizations of integer-order derivatives employed in classical calculus. Nevertheless, solving fractional differential equations explicitly can be a difficult task, mainly when the order of the equation lies between 3 and 4.

Fractional differential equations (FDEs) have emerged as a subject of considerable interest from researchers due to their ability to capture hereditary effects and long-term memory. The use of fractional calculus has proven to be a potent tool in modeling dynamic phenomena that exhibit such characteristics. This makes fractional calculus a suitable methodology for analyzing anomalous transport in complex heterogeneous aquifers and for exploring various applications in disciplines like biology, chemistry, physics, and economics. It is crucial to solve FDEs with antiperiodic boundary conditions, which are observed in many situations, including local or midpoint conditions, as demonstrated by several studies in the literature [7–12].

In fractional calculus, several researchers have explored the existence of solutions for antiperiodic boundary value problems in many methods theoretically and numerically. Ahmad and Nieto [13] used Leray–Schauder degree theory to investigate the existence results for

$$\begin{cases} {}^{c}D^{\hat{u}}\mathscr{Z}(\beta) = \Theta(\beta, \mathscr{Z}(\beta)), & 1 < \hat{u} \leq 2, \beta \in [0,\sigma], \\ \mathscr{Z}^{(i)}(0) = -\mathscr{Z}^{(i)}(\sigma), & i = 0,1, \end{cases}$$

where ${}^{c}D^{\hat{u}}$ represents the Caputo fractional derivative of order $\hat{u}$, and $\Theta : [0,\sigma] \times \mathbb{R} \longrightarrow \mathbb{R}$.

Agarwal and Bashir [14] expanded the analysis to include fractional differential equations and inclusions. They utilized the nonlinear alternative degree and Leray–Schauder theory to obtain their results. The research contributes to the theoretical understanding of antiperiodic boundary conditions and provides insights into the behavior of solutions under such conditions.

Ahmed [15] explored the existence of solutions to FDEs of order $\hat{u} \in (2,3]$ for

$$\begin{cases} {}^{c}D^{\hat{u}}\mathscr{Z}(\beta) = \Theta(\beta, \mathscr{Z}(\beta)), & 2 < \hat{u} \leq 3, \beta \in [0,\sigma], \\ \mathscr{Z}^{(i)}(0) = -\mathscr{Z}^{(i)}(\sigma), & i = 0,1,3, \end{cases}$$

where ${}^{c}D^{\hat{u}}$ represents the Caputo fractional derivative of order $\hat{u}$, and $\Theta : [0,\sigma] \times U \longrightarrow U$ for a Banach space $(U, \|\cdot\|)$. The paper contributes to understanding fractional calculus and its application to boundary value problems, where the existence results were obtained via the contraction mapping principle and Krasnoselskii's fixed point theorem.

Furthermore, in [16], scholars acquired the existence of solutions to

$$\begin{cases} {}^{c}D^{\hat{u}}\mathscr{Z}(\beta) = \Theta(\beta, \mathscr{Z}(\beta)), & 1 < \hat{u} \leq 2, 0 \leq \hat{c} < \sigma, \beta \in [0,\sigma], \\ \mathscr{Z}^{(i)}(\hat{c}) = -\mathscr{Z}^{(i)}(\sigma), & i = 0,1, \end{cases}$$

where ${}^{c}D^{\hat{u}}$ represents the Caputo fractional derivative of order $\hat{u}$, and $\Theta : [0,\sigma] \times \mathbb{R} \longrightarrow \mathbb{R}$. This research expands the understanding of fractional differential equations with non-standard boundary conditions and provides insights into the behavior of solutions under parametric antiperiodic constraints.

We structured this paper as follows. The following section provides a background information, including definitions and theorems. Section 3 presents the results. Finally, Sect. 4 concludes the paper.

2 Preliminaries

Definition 1 For $\mathscr{Z}(\beta) \in C^n([0,\infty],\mathbb{R})$, the *Caputo fractional derivative* of order $\hat{u} > 0$, denoted by ${}^cD^{\hat{u}}$, is defined by

$$ {}^cD^{\hat{u}}\mathscr{Z}(\beta) = \frac{1}{\Gamma(n-\hat{u})}\int_0^\beta (\beta-\delta)^{n-\hat{u}-1}\mathscr{Z}^{(n)}(\delta)\,d\delta, $$

where $n = [\hat{u}] + 1$.

Definition 2 For any order $\hat{u} > 0$, the *Riemann–Liouville fractional integral* of a function $\mathscr{Z}(\beta)$, denoted $I^{\hat{u}}$, is defined by

$$ I^{\hat{u}}\mathscr{Z}(\beta) = \frac{1}{\Gamma(\hat{u})}\int_0^\beta (\beta-\delta)^{\hat{u}-1}\mathscr{Z}(\delta)\,d\delta. $$

Lemma 2.1 [17, 18]

For $\hat{u} > 0$, the general solution of ${}^cD^{\hat{u}}\mathscr{Z}(\beta) = 0$ is given by

$$ \mathscr{Z}(\beta) = \eta_1 + \eta_2\beta + \eta_2\beta^2 + \cdots + \eta_n\beta^{n-1}, \tag{2.1} $$

where $\eta_\kappa \in \mathbb{R}, \tau = 1,2,\dots,n$.

Lemma 2.2 *The unique solution of*

$$ \begin{cases} {}^cD^{\hat{u}}\mathscr{Z}(\beta) = \mathfrak{Z}(\beta), & \beta \in [0,\sigma], 3 < \hat{u} \le 4, 0 \le \hat{c} < \sigma, \\ \mathscr{Z}^{(i)}(\hat{c}) = -\mathscr{Z}^{(i)}(\sigma), & i = 0,1,2,3, \end{cases} \tag{2.2} $$

where $\mathfrak{Z} \in C[0,\sigma]$, is given by

$$ \begin{aligned} \mathscr{Z}(\beta) &= \int_0^\beta \frac{(\beta-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\mathfrak{Z}(\delta)\,d\delta - \frac{1}{2}\left[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\mathfrak{Z}(\delta)\,d\delta + \int_0^\sigma \frac{(\sigma-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\mathfrak{Z}(\delta)\,d\delta\right] \\ &\quad + \frac{(\hat{c}+\sigma)-2\beta}{4}\left[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\mathfrak{Z}(\delta)\,d\delta + \int_0^\sigma \frac{(\sigma-\delta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\mathfrak{Z}(\delta)\,d\delta\right] \\ &\quad + \frac{\beta((\hat{c}+\sigma)-\beta)-\hat{c}\sigma}{4}\left[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\mathfrak{Z}(\delta)\,d\delta + \int_0^\sigma \frac{(\sigma-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\mathfrak{Z}(\delta)\,d\delta\right] \\ &\quad + \frac{6\beta^2(\hat{c}+\sigma)-4\beta^3-12\beta\hat{c}\sigma-(\hat{c}+\sigma)[\sigma^2-4\hat{c}\sigma+\hat{c}^3]}{48}\left[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\mathfrak{Z}(\delta)\,d\delta \right. \\ &\quad \left. + \int_0^\sigma \frac{(\sigma-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\mathfrak{Z}(\delta)\,d\delta\right]. \end{aligned} \tag{2.3} $$

Proof By utilizing equation (2.1) for the constants $\eta_\kappa \in \mathbb{R}$, $\kappa = 1,2,3,4$, we have

$$ \begin{aligned} \mathscr{Z}(\beta) &= I^{\hat{u}}\mathfrak{Z}(\beta) - \eta_1 - \eta_2\beta - \eta_3\beta^2 - \eta_4\beta^3 \\ &= \int_0^\beta \frac{(\beta-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\mathfrak{Z}(\delta)\,d\delta - \eta_1 - \eta_2\beta - \eta_3\beta^2 - \eta_4\beta^3, \end{aligned} \tag{2.4} $$

where

$$
\begin{aligned}
\eta_1 &= \frac{1}{2}\left[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\mathfrak{Z}(\delta)\,d\delta + \int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\mathfrak{Z}(\delta)\,d\delta\right] \\
&\quad - \frac{(\hat{c}+\sigma)}{4}\left[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\mathfrak{Z}(\delta)\,d\delta + \int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\mathfrak{Z}(\delta)\,d\delta\right] \\
&\quad + \frac{(\hat{c}\sigma)}{4}\left[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-3)}\mathfrak{Z}(\delta)\,d\delta + \int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\mathfrak{Z}(\delta)\,d\delta\right] \\
&\quad + \frac{(\hat{c}+\sigma)[\sigma^2-4\hat{c}\sigma+\hat{c}^2]}{48}\left[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\mathfrak{Z}(\delta)\,d\delta + \int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\mathfrak{Z}(\delta)\,d\delta\right], \\
\eta_2 &= \frac{1}{2}\left[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\mathfrak{Z}(\delta)\,d\delta + \int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\mathfrak{Z}(\delta)\,d\delta\right] \\
&\quad - \frac{(\hat{c}+\sigma)}{4}\left[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\mathfrak{Z}(\delta)\,d\delta + \int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\mathfrak{Z}(\delta)\,d\delta\right] \\
&\quad + \frac{(\hat{c}\sigma)}{4}\left[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\mathfrak{Z}(\delta)\,d\delta + \int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\mathfrak{Z}(\delta)\,d\delta\right], \\
\eta_3 &= \frac{1}{4}\left[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\mathfrak{Z}(\delta)\,d\delta + \int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\mathfrak{Z}(\delta)\,d\delta\right] \\
&\quad - \frac{(\hat{c}+\sigma)}{8}\left[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\mathfrak{Z}(\delta)\,d\delta + \int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\mathfrak{Z}(\delta)\,d\delta\right], \\
\eta_4 &= \frac{1}{12}\left[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\mathfrak{Z}(\delta)\,d\delta + \int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\mathfrak{Z}(\delta)\,d\delta\right].
\end{aligned}
$$

Substituting the values of η_1, η_2, η_3, and η_4 into equation (2.4) leads to the solution to equation (2.3). □

Remark 2.1 It is worth noting that the expressions in (2.3) pertain to the nonlocal FDE, which is characterized by an order $\hat{u} \in (1,2]$ as per the findings of [16]. Similarly, the first four terms correspond to the nonlocal FDE with order $\hat{u} \in (2,3]$. Upon raising the order to $\hat{u} \in (3,4]$, the solution to the problem will incorporate an additional term, as indicated in Lemma 2.2.

Remark 2.2 The solution of the classical problem

$$
\begin{cases}
{}^{c}D^{\hat{u}}\mathscr{Z}(\beta) = \mathfrak{Z}(\beta), & \beta \in [0,\sigma], 3 < \hat{u} \leq 4, \\
\mathscr{Z}^{(i)}(0) = -\mathscr{Z}^{(i)}(\sigma), & i = 0,1,2,3,
\end{cases}
$$

is

$$
\begin{aligned}
\mathscr{Z}(\beta) &= \int_0^{\beta} \frac{(\beta-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\mathfrak{Z}(\delta)\,d\delta - \frac{1}{2}\int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\mathfrak{Z}(\delta)\,d\delta \\
&\quad + \frac{(\sigma-2\beta)}{4}\int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\mathfrak{Z}(\delta)\,d\delta + \frac{(\sigma\beta-\beta^2)}{4}\int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\mathfrak{Z}(\delta)\,d\delta \\
&\quad + \frac{6\beta^2\sigma-4\beta^3-\sigma^3}{48}\left[\int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\mathfrak{Z}(\delta)\,d\delta\right].
\end{aligned}
\tag{2.5}
$$

Upon careful examination of equations (2.3) and (2.5), it becomes apparent that four supplementary terms have been introduced in comparison to [14]. Furthermore, the first two components in (2.5) correspond to the problem characterized by an order of $\hat{u} \in (1,2]$, as observed in [13]. Similarly, the first three terms in (2.5) dovetail to the problem of order $\hat{u} \in (2,3]$, as reported in [15].

Theorem 2.1 *Let $\mathscr{B}$ be a Banach space. Let $\mathscr{P} \in \mathscr{B}$ be an open bounded subset with $0 \in \mathscr{P}$, and let $\mathcal{K} : \overline{\mathscr{P}} \longrightarrow \mathscr{B}$ be a completely continuous operator such that $\|\mathcal{K}\omega\| \leq \|\omega\|$ for every $\omega \in \partial\mathscr{P}$. Then $\mathcal{K}$ has a fixed point in $\partial\mathscr{P}$.*

Theorem 2.2 [19, 20] *Let $\mathscr{B}$ be a Banach space, and let $\omega \in \mathscr{B}$ be a nonempty closed convex subset. Let $\mathscr{K}_1$ and $\mathscr{K}_2$ be e operators such that $\mathscr{K}_1\beta_1 + \mathscr{K}_2\beta_2 \in \omega$ for $\beta_1, \beta_2 \in \omega$. Suppose that $\mathscr{K}_1$ is continuous and compact and $\mathscr{K}_2$ is a contraction mapping. Then there exists $\delta \in \omega$ such that $\delta = \mathscr{K}_1\delta + \mathscr{K}_2\delta$.*

3 Results

Let $\mathscr{W} = C([0,\sigma],\mathbb{R})$ with $\|\mathscr{Z}\| = \sup_{\beta\in[0,\sigma]} |\mathscr{Z}(\beta)|$. Define $\mathcal{K} : \mathscr{W} \longrightarrow \mathscr{W}$ by

$$\begin{aligned}
&(\mathcal{K}\mathscr{Z})(\beta) \\
&\quad= \int_0^\beta \frac{(\beta-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\Theta\big(\delta,\mathscr{Z}(\delta)\big)\,d\delta \\
&\quad\quad - \frac{1}{2}\left[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\Theta\big(\delta,\mathscr{Z}(\delta)\big)\,d\delta + \int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\Theta\big(\delta,\mathscr{Z}(\delta)\big)\,d\delta\right] \\
&\quad\quad + \frac{\hat{c}+\sigma-2\beta}{4}\left[\int_0^{\hat{c}} \frac{(\hat{c}-\beta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\Theta\big(\delta,\mathscr{Z}(\delta)\big)\,d\delta + \int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\Theta\big(\delta,\mathscr{Z}(\delta)\big)\,d\delta\right] \\
&\quad\quad + \frac{\beta[(\hat{c}+\sigma)-\beta]-\hat{c}\sigma}{4}\left[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\Theta\big(\delta,\mathscr{Z}(\delta)\big)\,d\delta\right. \\
&\quad\quad \left. + \int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\Theta\big(\delta,\mathscr{Z}(\delta)\big)\,d\delta\right] \\
&\quad\quad + \frac{6\beta^2(\hat{c}+\sigma)-4\beta^3-12\beta\hat{c}\sigma-(\hat{c}+\sigma)[\sigma^2-4\hat{c}\sigma+\hat{c}^2]}{48} \\
&\quad\quad \times\left[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\Theta\big(\delta,\mathscr{Z}(\delta)\big)\,d\delta\right. \\
&\quad\quad \left. + \int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\Theta\big(\delta,\mathscr{Z}(\delta)\big)\,d\delta\right].
\end{aligned} \tag{3.1}$$

Theorem 3.1 *Let $\Theta : [0,\sigma] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function satisfying $\|\Theta(\beta,\mathscr{Z}_1) - \Theta(\beta,\mathscr{Z}_2)\| \leq L\|\mathscr{Z}_1 - \mathscr{Z}_2\|$ for all $\beta \in [0,\sigma]$ and $\mathscr{Z}_1, \mathscr{Z}_2 \in \mathbb{R}$, with $LM < 1$, where*

$$\begin{aligned}
M = \max_{\beta\in[0,\sigma]}\bigg\{&\frac{3\sigma^{\hat{u}}+\hat{c}^{\hat{u}}}{\Gamma(\hat{u}+1)} + \frac{|(\sigma+\hat{c})-2\beta|(\sigma^{\hat{u}-1}+\hat{c}^{\hat{u}-1})}{2\Gamma(\hat{u})} + \frac{|\beta[(\hat{c}+\sigma)-\beta]-\hat{c}\sigma|(\sigma^{\hat{u}-2}+\hat{c}^{\hat{u}-2})}{2\Gamma(\hat{u}-1)} \\
&+ \frac{|6\beta^2(\hat{c}+\sigma)-4\beta^3-12\beta\hat{c}\sigma-(\hat{c}+\sigma)[\sigma^2-4\beta\sigma+\hat{c}^2]|(\sigma^{\hat{u}-3}+\hat{c}^{\hat{u}-3})}{24\Gamma(\hat{u}-2)}\bigg\}.
\end{aligned}$$

Then problem (1.1) has a unique solution.

Proof Define $\mathcal{K}:\mathscr{W}\longrightarrow\mathscr{W}$ as in (3.1). For a fixed point, setting $\max_{\beta\in[0,\sigma]}\|\Theta(\beta,0)\| = \mu < \infty$ and taking $r\geq\mu M$, we will show that $\mathcal{K}\mathcal{B}_r\subset\mathcal{B}_r := \{\mathscr{Z}\in\mathscr{W}:\|\mathscr{Z}\|\leq r\}$. For $\beta\in\mathcal{B}_r$, we have

$$
\begin{aligned}
&\|(\mathcal{K}\mathscr{Z})(\beta)\|\\
&\leq\int_0^\beta\frac{(\beta-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\big[\|\Theta(\delta,\mathscr{Z}(\delta))-\Theta(\delta,0)\|+\|\Theta(\delta,0)\|\big]\,d\delta\\
&\quad+\frac{1}{2}[\int_0^{\hat{c}}\frac{(\hat{c}-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\big[\|\Theta(\delta,\mathscr{Z}(\delta))-\Theta(\delta,0)\|+\|\Theta(\delta,0)\|\big]\,d\delta\\
&\quad+\int_0^{\sigma}\frac{(\sigma-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\big[\|\Theta(\delta,\mathscr{Z}(\delta))-\Theta(\delta,0)\|+\|\Theta(\delta,0)\|\big]\,d\delta\\
&\quad+\frac{|(\hat{c}+\sigma)-2\beta|}{4}\bigg[\int_0^{\hat{c}}\frac{(\hat{c}-\delta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\big[\|\Theta(\delta,\mathscr{Z}(\delta))-\Theta(\delta,0)\|+\|\Theta(\delta,0)\|\big]\,d\delta\\
&\quad+\int_0^{\sigma}\frac{(\sigma-\delta^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\big[\|\Theta(\delta,\mathscr{Z}(\delta))-\Theta(\delta,0)\|+\|\Theta(\delta,0)\|\big]\,d\delta\bigg]\\
&\quad+\frac{|\beta[(\hat{c}+\sigma)-\beta]-\hat{c}\sigma|}{4}\bigg[\int_0^{\hat{c}}\frac{(\hat{c}-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\big[\|\Theta(\delta,\mathscr{Z}(\delta))-\Theta(\delta,0)\|+\|\Theta(\delta,0)\|\big]\,d\delta\\
&\quad+\int_0^{\sigma}\frac{(\sigma-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\big[\|\Theta(\delta,\mathscr{Z}(\delta))-\Theta(\delta,0)\|+\|\Theta(\delta,0)\|\big]\,d\delta\bigg]\\
&\quad+\frac{|6\beta^2(\hat{c}+\sigma)-4\beta^3-12\beta\hat{c}\sigma-(\hat{c}+\sigma)[\sigma^2-4\hat{c}\sigma+\hat{c}^2]|}{48}\\
&\quad\times\bigg[\int_0^{\hat{c}}\frac{(\hat{c}-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\big[\|\Theta(\delta,\mathscr{Z}(\delta))-\Theta(\delta,0)\|+\|\Theta(\delta,0)\|\big]\,d\delta\\
&\quad+\int_0^{\sigma}\frac{(\sigma-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\big[\|\Theta(\delta,\mathscr{Z}(\delta))-\Theta(\delta,0)\|+\|\Theta(\delta,0)\|\big]\,d\delta\bigg]\\
&\leq(Lr+\mu)[\int_0^\beta\frac{(\beta-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\,d\delta+\frac{1}{2}\bigg[\int_0^{\hat{c}}\frac{(\hat{c}-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\,d\delta+\int_0^{\sigma}\frac{(\sigma-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\,d\delta\bigg]\\
&\quad+\frac{|(\hat{c}+\sigma)-2\beta|}{4}\bigg[\int_0^{\hat{c}}\frac{(\hat{c}-\delta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\,d\delta+\int_0^{\sigma}\frac{(\sigma-\delta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\,d\delta\bigg]\\
&\quad+\frac{|\beta[(\hat{c}+\sigma)-\beta]-\hat{c}\sigma|}{4}\bigg[\int_0^{\hat{c}}\frac{(\hat{c}-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\,d\delta+\int_0^{\sigma}\frac{(\sigma-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\,d\delta\bigg]\\
&\quad+\frac{|6\beta^2(\hat{c}+\sigma)-4\beta^3-12\beta\hat{c}\sigma-(\hat{c}+\sigma)[\sigma^2-4\beta\sigma+\hat{c}^2]|}{48}\bigg[\int_0^{\hat{c}}\frac{(\hat{c}-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\,d\delta\\
&\quad+\int_0^{\sigma}\frac{(\sigma-\delta)^{\hat{u}-4}}{\delta(\hat{u}-3)}\,d\delta\bigg]\\
&\leq(Lr+\mu)\frac{1}{2}\bigg[\frac{2|\beta^{\hat{u}}|+\sigma^{\hat{u}}+\hat{c}^{\hat{u}}}{\Gamma(\hat{u}+1)}+\frac{|(\hat{c}+\sigma)-2\beta|(\sigma^{\hat{u}-1}+\hat{c}^{\hat{u}-1})}{2\Gamma(\hat{u})}\\
&\quad+\frac{|\beta[(\hat{c}+\sigma)-\beta]-\hat{c}\sigma|(\sigma^{\hat{u}-2}+\hat{c}^{\hat{u}-2})}{2\Gamma(\hat{u}-1)}\\
&\quad+\frac{|6\beta^2(\hat{c}+\sigma)-4\beta^3-12\beta\hat{c}\sigma-(\hat{c}+\sigma)[\beta^2-4\hat{c}\sigma+\hat{c}^2]|(\sigma^{\hat{u}-3}+\hat{c}^{\hat{u}-3})}{24\Gamma(\hat{u}-2)}\bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left(\frac{Lr}{2}+\frac{\mu}{2}\right)M \leq r,\\
&\times \|(\mathcal{K}\mathscr{Z}_1(\beta)-(\mathcal{K}\mathscr{Z}_2)(\beta)\| \leq \int_0^\beta \frac{(\beta-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\big\|\Theta\big(\delta,\mathscr{Z}_1(\delta)\big)-\Theta\big(\delta,\mathscr{Z}_2(\delta)\big)\big\|\,d\delta\\
&+\frac{1}{2}\bigg[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\big\|\Theta\big(\delta,\mathscr{Z}_1(\delta)\big)-\Theta\big(\delta,\mathscr{Z}_2(\delta)\big)\big\|\,d\delta\\
&+\int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\big\|\Theta\big(\delta,\mathscr{Z}_1(\delta)\big)-\Theta\big(\delta,\mathscr{Z}_2(\delta)\big)\big\|\,d\delta\bigg]\\
&+\frac{|(\hat{c}+\sigma)-2\beta|}{4}\bigg[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\big\|\Theta\big(\delta,\mathscr{Z}_1(\delta)\big)-\Theta\big(\delta,\mathscr{Z}_2(\delta)\big)\big\|\,d\delta\\
&+\int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\big\|\Theta\big(\delta,\mathscr{Z}_1(\delta)\big)-\Theta\big(\delta,\mathscr{Z}_2(\delta)\big)\big\|\,d\delta\bigg]\\
&+\frac{|\beta[(\hat{c}+\sigma)-\beta]-\hat{c}\sigma|}{4}\bigg[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\big\|\Theta\big(\delta,\mathscr{Z}_1(\delta)\big)-\Theta\big(\delta,\mathscr{Z}_2(\delta)\big)\big\|\,d\delta\\
&+\int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\big\|\Theta\big(\delta,\mathscr{Z}_1(\delta)\big)-\Theta\big(\delta,\mathscr{Z}_2(\delta)\big)\big\|\,d\delta\bigg]\\
&+\frac{6\beta^2(\beta+\sigma)-4\beta^3-12\beta\hat{c}\sigma-(\hat{c}+\sigma)[\sigma^2-4\hat{c}\sigma+\hat{c}^2]}{48}\\
&\times [\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}[\big\|\Theta\big(\delta,\mathscr{Z}_1(\delta)\big)-\Theta\big(\delta,\mathscr{Z}_2(\delta)\big)\big\|\,d\delta\\
&+\int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}[\big\|\Theta\big(\delta,\mathscr{Z}_1(\delta)\big)-\Theta\big(\delta,\mathscr{Z}_2(\delta)\big)\big\|\,d\delta]\\
&\leq L\|\mathscr{Z}_1-\mathscr{Z}_2\|[\int_0^\beta \frac{(\beta-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\,d\delta+\frac{1}{2}\bigg[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\,d\delta+\int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\,d\delta\bigg]\\
&+\frac{|(\hat{c}+\sigma)-2\beta|}{4}\bigg[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\,d\delta+\int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\,d\delta\bigg]\\
&+\frac{|\beta[(\hat{c}+\sigma)-\beta]-\hat{c}\sigma|}{4}\bigg[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\,d\delta+\int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\,d\delta\bigg]\\
&+\frac{|6\beta^2(\hat{c}+\sigma)-4\beta^3-12\beta\hat{c}\sigma-(\hat{c}+\sigma)[\sigma^2-4\hat{c}\sigma+\hat{c}^2]|}{48}\bigg[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\,d\delta\\
&+\int_0^{\sigma} \frac{(\sigma-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\,d\delta\bigg]\\
&\leq \frac{M}{2}L\|\mathscr{Z}_1-\mathscr{Z}_2\|,
\end{aligned}
$$

so that $\|(\mathcal{K}\mathscr{Z}_1)(\beta)-(\mathcal{K}\mathscr{Z}_2)(\beta)\| \leq LM\|\mathscr{Z}_1-\mathscr{Z}_2\|$.

By the assumption $LM < 1$ it follows by the Banach contraction principle that $\mathcal{K}$ is a contraction, which implies that there exists a unique solution for the antiperiodic problem (1.1). □

Lemma 3.1 *The operator $\mathcal{K}$ defined in* (3.1) *is completely continuous.*

Proof Let $\mathscr{U} \subset \mathscr{W}$. Then there exists $A_1 > 0$ such that $|\Theta(\mathscr{Z}_1,\mathscr{Z}_2)| \leq A_1$ for all $\beta \in [0,\sigma]$ and $\mathscr{Z} \in \mathscr{U}$.

Define the operator $\mathcal{K}$ as in (3.1). Then

$$\begin{aligned}
&\bigl|(\mathcal{K}\mathscr{Z})(\beta)\bigr| \\
&\quad\le \int_0^\beta \frac{(\beta-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\bigl|\Theta\bigl(\delta,\mathscr{Z}(\delta)\bigr)\bigr|\,d\delta \\
&\qquad+\frac{1}{2}\biggl[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\bigl|\Theta\bigl(\delta,\mathscr{Z}(\delta)\bigr)\bigr|\,d\delta+\int_0^\sigma \frac{(\sigma-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\bigl|\Theta\bigl(\delta,\mathscr{Z}(\delta)\bigr)\bigr|\,d\delta\biggr] \\
&\qquad+\frac{|(\hat{c}+\sigma)-2\beta|}{4}\biggl[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\bigl|\Theta\bigl(\delta,\mathscr{Z}(\delta)\bigr)\bigr|\,d\delta+\int_0^\sigma \frac{(\sigma-\delta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\bigl|\Theta\bigl(\delta,\mathscr{Z}(\delta)\bigr)\bigr|\,d\delta\biggr] \\
&\qquad+\frac{|\beta[(\hat{c}+\sigma)-\beta]-\hat{c}\sigma|}{4}\biggl[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\bigl|\Theta\bigl(\delta,\mathscr{Z}(\delta)\bigr)\bigr|\,d\delta \\
&\qquad+\int_0^\sigma \frac{(\sigma-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\bigl|\Theta\bigl(\delta,\mathscr{Z}(\delta)\bigr)\bigr|\,d\delta\biggr] \\
&\qquad+\frac{|6\beta^2(\hat{c}+\sigma)-4\beta^3-12\beta\hat{c}\sigma-(\hat{c}+\sigma)[\sigma^2-4\hat{c}\sigma+\hat{c}^2]|}{48} \\
&\qquad\times\biggl[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\bigl|\Theta\bigl(\delta,\mathscr{Z}(\delta)\bigr)\bigr|\,d\delta \\
&\qquad+\int_0^\sigma \frac{(\sigma-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\bigl|\Theta\bigl(\delta,\mathscr{Z}(\delta)\bigr)\bigr|\,d\delta\biggr] \\
&\quad\le A_1\biggl[\max_{\beta\in[0,\sigma]}\biggl\{\frac{2|\beta^{\hat{u}}|+\sigma^{\hat{u}}+\hat{c}^{\hat{u}}}{2\Gamma(\hat{u}+1)}+\frac{|(\hat{u}+\sigma)-2\beta|(\sigma^{\hat{u}-1}+\hat{c}^{\hat{u}-1})}{4\Gamma(\hat{u})} \\
&\qquad+\frac{|\beta[(\hat{c}+\sigma)-\beta]-\hat{c}\sigma|(\sigma^{\hat{u}-2}+\hat{c}^{\hat{u}-2})}{4\Gamma(\hat{u}-1)} \\
&\qquad+\frac{|6\beta^2(\hat{c}+\sigma)-4\beta^3-12\beta\hat{c}\sigma-(\hat{c}+\sigma)[\sigma^2-4\hat{c}\sigma+\hat{c}^2]|(\sigma^{\hat{u}-3}+\hat{c}^{\hat{u}-3})}{48\Gamma(\hat{u}-2)}\biggr\}\biggr]=A_2,
\end{aligned}$$

which implies that $\|(\mathcal{K}\mathscr{Z})\|\le A_2$. Further,

$$\begin{aligned}
&\bigl|(\mathcal{K}\mathscr{Z})'(\beta)\bigr| \\
&\quad\le \int_0^\beta \frac{(\beta-\delta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\bigl|\Theta\bigl(\delta,\mathscr{Z}(\delta)\bigr)\bigr|\,d\delta \\
&\qquad+\frac{1}{2}\biggl[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\bigl|\Theta\bigl(\delta,\mathscr{Z}(\delta)\bigr)\bigr|\,d\delta+\int_0^\sigma \frac{(\sigma-\delta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\bigl|\Theta\bigl(\delta,\mathscr{Z}(\delta)\bigr)\bigr|\,d\delta\biggr] \\
&\qquad+\frac{|(\hat{c}+\sigma)-2\beta|}{4}\biggl[\int_0^{\hat{c}} \frac{(\hat{c}-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\bigl|\Theta\bigl(\delta,\mathscr{Z}(\delta)\bigr)\bigr|\,d\delta+\int_0^\sigma \frac{(\sigma-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\bigl|\Theta\bigl(\delta,\mathscr{Z}(\delta)\bigr)\bigr|\,d\delta\biggr] \\
&\qquad+\frac{|\beta[(\hat{c}+\sigma)-\beta]-\hat{c}\sigma|}{4}\biggl[\int_0^{\hat{c}} \frac{(\hat{c}-\sigma)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\bigl|\Theta\bigl(\sigma,\mathscr{Z}(\sigma)\bigr)\bigr|\,d\sigma \\
&\qquad+\int_0^\sigma \frac{(\sigma-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\bigl|\Theta\bigl(\delta,\mathscr{Z}(\delta)\bigr)\bigr|\,d\delta\biggr] \\
&\quad\le A_1\biggl[\max_{\beta\in[0,\sigma]}\biggl\{\frac{2|\beta^{\hat{u}-1}|+\sigma^{\hat{u}-1}+\hat{c}^{\hat{u}-1}}{2\Gamma(\hat{u})}+\frac{|(\hat{c}+\sigma)-2\beta|(\sigma^{\hat{u}-2}+\hat{c}^{\hat{u}-2})}{4\Gamma(\hat{u}-1)}
\end{aligned}$$

$$+\frac{|\beta[(\hat{c}+\sigma)-\beta]-\hat{c}\sigma|}{4\Gamma(\hat{u}-2)}\Bigg\}\Bigg]=A_3.$$

Hence, for all $\beta_1,\beta_2\in[0,\sigma]$, we have

$$\big|(\mathcal{K}\mathscr{Z})(\beta_2)-(\mathcal{K}\mathscr{Z})(\beta_1)\big|\le\int_{\beta_1}^{\beta_2}\big|(\mathcal{K}\mathscr{Z})'(\delta)\big|\,d\delta\le A_3(\beta_2-\beta_1),$$

so that $\mathcal{K}$ is equicontinuous on $[0,\sigma]$ and thus, by the Arzelá–Ascoli theorem, is completely continuous. □

Theorem 3.2 *Let* $\Theta:[0,\sigma]\times\mathbb{R}\longrightarrow\mathbb{R}$ *be continuous, and let the following assumptions hold*:

1. $\|\Theta(\beta,\mathscr{Z}_1)-\Theta(\beta,\mathscr{Z}_2)\|\le L\|\Theta(\beta,\mathscr{Z}_1)-\Theta(\beta,\mathscr{Z}_2)\|$ $\forall\beta\in[0,\sigma]$ *and* $\mathscr{Z}_1,\mathscr{Z}_2\in\mathbb{R}$;
2. $|\Theta(\beta,\mathscr{Z})|\le g(\beta)$, $\forall(\beta,g)\in[0,1]\times\mathbb{R}$, *where* $g\in L_1([0,\sigma],R^+)$;

Then problem (1.1) *has at least one solution on* $[0,\sigma]$ *if* $LK<1$, *where*

$$K=\max_{\beta\in[0,\sigma]}\Bigg\{\frac{|\sigma^{\hat{u}}+\hat{c}^{\hat{u}}}{\Gamma(\hat{u}+1)}+\frac{|(\hat{c}+\sigma)-2\beta|(\sigma^{\hat{u}-1}+\hat{c}^{\hat{u}-1})}{2\Gamma(\hat{u})}+\frac{|\beta[(\hat{c}+\sigma)-\beta]-\hat{c}\sigma|(\sigma^{\hat{u}-2}+\hat{c}^{\hat{u}-2})}{2\Gamma(\hat{u}-1)}$$
$$+\frac{|6\beta^2(\hat{c}+\sigma)-4\beta^2-12\beta\hat{c}\sigma-(\hat{c}+\sigma)[\sigma^2-4\hat{c}\sigma+\hat{c}^2]|(\sigma^{\hat{u}-3}+\hat{c}^{\hat{u}-3})}{24\Gamma(\hat{u}-2)}\Bigg\}.$$

Proof Fix $r\ge\frac{\|g\|_{L_1}}{2}M$, where M is defined in Theorem 3.1, and consider $\mathcal{B}_r=\{\mathscr{Z}\in\mathscr{W}:\|\mathscr{Z}\|\le r\}$. Define the operators $\mathscr{K}_1$ and $\mathscr{K}_2$ on $\mathcal{B}_r$ by

$$(\mathscr{K}_1\mathscr{Z})(\beta)=\int_0^\beta\frac{(\beta-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\Theta\big(\delta,\mathscr{Z}(\delta)\big)\,d\delta,$$

$$\begin{aligned}
&\big(\mathscr{K}_2(\beta)\big)(\beta)\\
&=-\frac{1}{2}\bigg[\int_0^{\hat{c}}\frac{(\hat{c}-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\Theta\big(\delta,\mathscr{Z}(\delta)\big)\,d\delta+\int_0^{\sigma}\frac{(\sigma-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\Theta\big(\delta,\mathscr{Z}(\delta)\big)\,d\delta\bigg]\\
&\quad+\frac{(\hat{c}+\sigma)-2\beta}{4}\bigg[\int_0^{\hat{c}}\frac{(\hat{c}-\delta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\Theta\big(\delta,\mathscr{Z}(\delta)\big)\,d\delta+\int_0^{\sigma}\frac{(\sigma-\delta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\Theta\big(\delta,\mathscr{Z}(\delta)\big)\,d\delta\bigg]\\
&\quad+\frac{\beta[(\hat{c}+\sigma)-\beta]-\hat{c}\sigma}{4}\bigg[\int_0^{\hat{c}}\frac{(\hat{c}-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\Theta\big(\delta,\mathscr{Z}(\delta)\big)\,d\delta\\
&\quad+\int_0^{\sigma}\frac{(\sigma-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\Theta\big(\delta,\mathscr{Z}(\delta)\big)\,d\delta\bigg]\\
&\quad+\frac{6\beta^2(\hat{c}+\sigma)-4\beta^2-12\beta\hat{c}\sigma-(\hat{c}+\sigma)[\sigma^2-4\hat{c}\sigma+\hat{c}^2]}{48}\\
&\quad\times\bigg[\int_0^{\hat{c}}\frac{(\hat{c}-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\Theta\big(\delta,\mathscr{Z}(\delta)\big)\,d\delta+\int_0^{\sigma}\frac{(\sigma-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\Theta\big(\beta,\mathscr{Z}(\delta)\big)\,d\delta\bigg].
\end{aligned}$$

For $\mathscr{Z}_1,\mathscr{Z}_2\in B_r$, we have

$$\|\mathscr{K}_1\mathscr{Z}_1+\mathscr{K}_2\mathscr{Z}_2\|\le\frac{\|g\|_{L_1}}{2}M\le r.$$

Thus $\mathscr{K}_1\mathscr{Z}_1 + \mathscr{K}_2\mathscr{Z}_2 \in \mathcal{B}_r$. By assumption 1, $\mathscr{K}_2$ is a contraction mapping if $LK < 1$. Also, the operator $\mathscr{K}_1$ is continuous due to the continuity of Θ and is uniformly bounded on $\mathcal{B}_r$:

$$\left\|(\mathscr{K}_1\mathscr{Z})(\beta)\right\| \leq \int_0^\beta \frac{(\beta-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\left|\Theta\big(\delta,\mathscr{Z}(\delta)\big)\right|d\delta \leq \frac{\|g\|_{L_1}|\beta^{\hat{u}}|}{\Gamma(\hat{u}+1)}.$$

Moreover, in view of assumption 1, we define $\sup_{(\mathscr{Z}_1,\mathscr{Z}_2)\in[0,\sigma]\times\mathcal{B}_r}\|\Theta(\beta,\mathscr{Z})\| = \Theta_{\max}$, and we have

$$\begin{aligned}
&\left\|(\mathscr{K}_1\mathscr{Z})(\beta_1)-(\mathscr{K}_2\mathscr{Z})(\beta_2)\right\| \\
&\quad= \frac{1}{\Gamma(\hat{u})}\left\|\int_0^{\beta_1}\left[(\beta_1-\delta)^{\hat{u}-1}-(\beta_2-\delta)^{\hat{u}-1}\right]\Theta\big(\delta,\mathscr{Z}(\delta)\big)\,d\delta\right. \\
&\qquad\left.+\int_{\beta_1}^{\beta_2}(\beta_2-\delta)^{\hat{u}-1}\Theta\big(\delta,\mathscr{Z}(\delta)\big)\,d\delta\right\| \leq \frac{\|\Theta\|_\infty}{\Gamma(\hat{u}+1)}\left|2(\beta_2-\beta_1)^{\hat{u}}+\beta_1^{\hat{u}}-\beta_2^{\hat{u}}\right|.
\end{aligned}$$

According to [21], the norm value is does not depend of $\mathscr{Z}$ and approaches zero as β_2 tends to β_1. This implies that $\mathscr{K}_1$ is relatively compact on $\mathcal{B}_r$. By the Arzelà–Ascoli theorem, $\mathscr{K}_1$ is compact on $\mathcal{B}_r$. Therefore problem (1.1) has at least one solution on $[0,\sigma]$. □

Theorem 3.3 *Let* $\Theta : [0,\sigma]\times\mathbb{R}\longrightarrow\mathbb{R}$ *be a given continuous function satisfying* $|\Theta(\beta,\mathscr{Z}(\beta))| \leq \mathscr{Q}(\beta)+\mathscr{E}|\mathscr{Z}|$ *for* $\beta\in[0,\sigma]$, $\mathscr{Z}\in\mathbb{R}$, *and* $\mathscr{E}>0$, *and let* $\mathscr{Q}\in L_\infty([0,\sigma],\mathbb{R}^+)$ *with* $\mathscr{E}<\frac{1}{\nu}$, *where*

$$\begin{aligned}
\nu = \max_{\beta\in[0,\sigma]}\Bigg\{&\frac{3\sigma^{\hat{u}}+\hat{c}^{\hat{u}}}{2\Gamma(\hat{u}+1)}+\frac{|(\sigma+\hat{c})-2\beta|(\sigma^{\hat{u}-1}+\hat{c}^{\hat{u}-1})}{4\Gamma(\hat{u})} \\
&+\frac{|\beta[(\hat{c}+\sigma)-\beta]-\hat{c}\sigma|(\sigma^{\hat{u}-2}+\hat{c}^{\hat{u}-2})}{4\Gamma(\hat{u}-1)} \\
&+\frac{|6\beta^2(\hat{c}+\sigma)-4\beta^3-12\beta\hat{c}\sigma-(\hat{c}+\sigma)[\sigma^2-4\beta\sigma+\hat{c}^2]|(\sigma^{\hat{u}-3}+\hat{c}^{\hat{u}-3})}{48\Gamma(\hat{u}-2)}\Bigg\}.
\end{aligned}$$

Then problem (1.1) *has at least one solution.*

Proof Define the operator $\mathcal{K}$ as in (3.1) with $\mathscr{Z}=\mathcal{K}\mathscr{Z}$, and let $\mathcal{B}_r=\{\mathscr{Z}\in C[0,\sigma]|\|\mathscr{Z}(\beta)<r\|\}$. Setting $\mathcal{D}(\rho,\mathscr{Z})=\rho\mathcal{K}\mathscr{Z}$, $\rho\in[0,1]$ and $\mathscr{Z}\in C(\mathbb{R})$, we have that $d_\rho(\mathscr{Z})=\mathscr{Z}-\mathcal{D}(\rho,\mathscr{Z})=\mathscr{Z}-\rho\mathcal{K}\mathscr{Z}$ is completely continuous.

If $\mathscr{Z}\neq\rho\mathcal{K}\mathscr{Z}$ for all $\mathscr{Z}\in\partial\mathcal{B}_r$, then for all $\rho\in[0,\sigma]$, $\deg(d_\rho,\mathcal{B}_r,0)=\deg(I-\rho\mathcal{K}\mathscr{Z},\mathcal{B}_r,0)=\deg(d_1,\mathcal{B}_r,0)=\deg(d_0,\mathcal{B}_r,0)=1\neq 0\in\mathcal{B}_r$.

For at least one $\mathscr{Z}\in\mathcal{B}_r$, we have $\mathscr{Z}-\rho\mathcal{K}\mathscr{Z}=0$. Then

$$\begin{aligned}
\left|\mathscr{Z}(\beta)\right| &= \left|\rho\mathcal{K}\mathscr{Z}(\beta)\right| \\
&\leq \int_0^\beta\frac{(\beta-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\left|\Theta\big(\delta,\mathscr{Z}(\delta)\big)\right|d\delta+\frac{1}{2}\left[\int_0^{\hat{c}}\frac{(\hat{c}-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\left|\Theta\big(\delta,\mathscr{Z}(\delta)\big)\right|d\delta\right. \\
&\quad\left.+\int_0^\sigma\frac{(\sigma-\delta)^{\hat{u}-1}}{\Gamma(\hat{u})}\left|\Theta\big(\delta,\mathscr{Z}(\delta)\big)\right|d\delta\right] \\
&\quad+\frac{|\hat{c}+\sigma-2\beta|}{4}\left[\int_0^{\hat{c}}\frac{(\hat{c}-\beta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\left|\Theta\big(\delta,\mathscr{Z}(\delta)\big)\right|d\delta\right.
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{\sigma}\frac{(\sigma-\delta)^{\hat{u}-2}}{\Gamma(\hat{u}-1)}\big|\Theta\big(\delta,\mathscr{Z}(\delta)\big)\big|\,d\delta\Big]\\
&+\frac{|\beta[(\hat{c}+\sigma)-\beta]-\hat{c}\sigma|}{4}\Big[\int_0^{\hat{c}}\frac{(\hat{c}-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\big|\Theta\big(\delta,\mathscr{Z}(\delta)\big)\big|\,d\delta\\
&+\int_0^{\sigma}\frac{(\sigma-\delta)^{\hat{u}-3}}{\Gamma(\hat{u}-2)}\big|\Theta\big(\delta,\mathscr{Z}(\delta)\big)\big|\,d\delta\Big]\\
&+\frac{|6\beta^2(\hat{c}+\sigma)-4\beta^3-12\beta\hat{c}\sigma-(\hat{c}+\sigma)[\sigma^2-4\hat{c}\sigma+\hat{c}^2]|}{48}\\
&\times\Big[\int_0^{\hat{c}}\frac{(\hat{c}-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\big|\Theta\big(\delta,\mathscr{Z}(\delta)\big)\big|\,d\delta\\
&+\int_0^{\sigma}\frac{(\sigma-\delta)^{\hat{u}-4}}{\Gamma(\hat{u}-3)}\big|\Theta\big(\delta,\mathscr{Z}(\delta)\big)\big|\,d\delta\Big]\\
&\le\big[\|\mathscr{Q}\|+\mathscr{E}|\mathscr{Z}|\big]\max_{\beta\in[0,\sigma]}\Big\{\frac{3\sigma^{\hat{u}}+\hat{c}^{\hat{u}}}{2\delta(\hat{u}+1)}+\frac{|(\sigma+\hat{c})-2\beta|(\sigma^{\hat{u}-1}+\hat{c}^{\hat{u}-1})}{4\Gamma(\hat{u})}\\
&+\frac{|\beta[(\hat{c}+\sigma)-\beta]-\hat{c}\sigma|(\sigma^{\hat{u}-2}+\hat{c}^{\hat{u}-2})}{4\Gamma(\hat{u}-1)}\\
&+\frac{|6\beta^2(\hat{c}+\sigma)-4\beta^3-12\beta\hat{c}\sigma-(\hat{c}+\sigma)[\sigma^2-4\beta\sigma+\hat{c}^2]|(\sigma^{\hat{u}-3}+\hat{c}^{\hat{u}-3})}{48\Gamma(\hat{u}-2)}\Big\},
\end{aligned}
$$

$$
|\mathscr{Z}|\le\big[\|\mathscr{Q}\|+\mathscr{E}|\mathscr{Z}|\big]\nu
$$

Therefore

$$
\|\mathscr{Z}\|\le\frac{\|\mathscr{Q}\|\nu}{1-\mathscr{E}\nu}.
$$

The proof is complete by choosing $r>\frac{\|\mathscr{Q}\|\nu}{1-\mathscr{E}\nu}$. □

Example 3.1 Consider the following FDE problem:

$$
\begin{cases}
{}^cD^{\frac{7}{2}}\mathscr{Z}(\beta)=\frac{1}{(\beta+3)^4}\frac{\|\mathscr{Z}\|}{1+\|\mathscr{Z}\|}, & \beta\in[0,2.01],\\
\mathscr{Z}^{(i)}(0.\hat{0}1)=-\mathscr{Z}^{(i)}(2.01), & i=0,1,2,3,
\end{cases}
\tag{3.2}
$$

All the assumptions of Theorem 3.1 are satisfied, since $\|\Theta(\beta,\mathscr{Z}_1)-\Theta(\beta,\mathscr{Z}_2)\|\le\frac{1}{81}\|\mathscr{Z}_1-\mathscr{Z}_2\|$, $\hat{u}=\frac{7}{2}$, $\hat{c}=0.01$, $\sigma=2.01$ with $M\approx 6.721799$, and $LM\approx 0.082882<1$ Therefore the solution to (3.2) exists.

4 Conclusions

The nonlocal antiperiodic boundary conditions in a fractional differential equation result in additional terms in the integral solutions, as discussed in this paper. The outcomes of the study are in agreement with the situation where the classical antiperiodic problem of the interval $[0,\sigma]$ is shifted. As $\hat{c}$ approaches zero from the positive side, the results obtained in this paper are consistent with the results of the classical antiperiodic problem, as reported in [14].

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Material preparation and analysis were performed by S.A.

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
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