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On the fractional perturbed neutral integro-differential systems via deformable derivatives: an existence study

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Abstract

In this paper, we provide some appropriate conditions for the existence of solutions for a perturbed fractional neutral integro-differential system under the deformable derivative in a Banach space. Using the Banach contraction principle and Krasnoselskii's fixed point theorem, we establish some new existence theorems. Moreover, we provide two numerical examples to demonstrate the applicability of the theoretical results

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1 Introduction

On 30 September 1695, L'Hôpital said to Leibniz about a particular notation that he had used in his manuscripts for the n-derivative of the function $f(x) = x$, i.e., $\frac{d^n f}{dx^n}$. The question posed to Leibniz by L'Hôpital: What would happen when $n = \frac{1}{2}$? Leibniz [1] answered as follows: "An apparent paradox, this will have useful effects one day." As predicted by Leibniz, fractional calculus was born with this simple idea.

Fractional calculus was primarily a spark for the best creative minds in mathematics. There have already been several forms of fractional derivative presented. Interestingly, the majority of fractional derivative formulations have an integral form. The readers of this research are advised to read some well-known monographs such as [2–4] for a complete knowledge of fractional calculus and also are asked to see the related research publications on fractional differential (or integro-differential) equations (FDEs) in [5–17]. By reading these papers we can easily understand the continual contribution of fractional calculus in the field of dynamic systems and mathematical modeling. These papers, and many others, show the role of fractional derivatives in describing relationships between differential equations. Nowadays, it is clear that the results of fractional models are more accurate and flexible in most cases than the results of classical models, which are based on integer-order derivatives.

As was already mentioned, the integral form is used in the majority of definitions of fractional derivatives. In contrast, Khalil et al. [18] proposed a limit-based definition of a fractional derivative, which characterizes it as a conformable fractional derivative that is comparable to a traditional one. Then Zulfeqarr, Ahuja, and Ujlayan came up with the new concept of the deformable derivative [$\mathcal{DD}$] [19]. This new derivative makes use of the same limit method as the conventional derivative. They called it "deformable" as it possesses the inherent quality of constantly deforming functions into derivatives. This idea opens up new research possibilities, such as analyzing qualitative and quantitative behavior in diverse systems. A definition, an integral operator, properties, and an inverse property of deformable derivatives are also provided by Zulfeqarr et al. [19], who then go on to explain how a deformable derivative is used with homogeneous linear FDEs. Recently, Ahuja et al. [20] developed their studies on the deformable Laplace transform and solved some deformable differential equations based on this technique.

A review of the previous works will serve as an introduction to this research. Especially, the authors in [21] discussed the existence of mild solutions of the model

$$\begin{aligned} &{}^{\mathcal{DD}}D^{\vartheta} w(\delta) = Aw(\delta) + f\big(\delta, w(\delta)\big), \quad \delta \in [0,T], 0 < \vartheta < 1, \\ &w(0) = w_0, \end{aligned}$$

where deformable fractional-order derivatives are used. The conclusions are made by the Banach contraction principle and Schauder fixed point theorem in the semigroup theory. Later, Mebrat et al. [22] examined the same properties including the existence of unique solutions for two fractional deformable differential (integro-differential) equations

$$\begin{aligned} &{}^{\mathcal{DD}}D^{\vartheta} w(\delta) = f\big(\delta, w(\delta)\big), \quad \delta \in [0,T], 0 < \vartheta < 1, \\ &w(0) + g(w) = w_0, \end{aligned}$$

and

$$\begin{aligned} &{}^{\mathcal{DD}}D^{\vartheta} w(\delta) = h\big(w(\delta)\big) + f\big(\delta, w(\delta)\big) + \int_0^{\delta} K\big(\delta, \sigma, w(\sigma)\big)\, d\sigma, \quad \delta \in [0,T], \\ &w(0) = w_0. \end{aligned}$$

These existence results were proved by Krasnoselskii's fixed point theorem. Recently, another existence study was conducted by Etefa et al. [23] for the fractional impulsive deformable system

$$\begin{aligned} &{}^{\mathcal{DD}}D^{\vartheta} w(\delta) = f\big(\delta, w(\delta)\big), \quad \delta \in I = [0,T], \delta \neq \delta_k, k = 1, 2, \dots, m, \\ &\Delta w|_{\delta=\delta_k} = I_k\big(\big(w\big(\delta_k^-\big)\big), \quad w(0) = w_0. \end{aligned}$$

By the above ideas the main aim of the present work is to investigate some qualitative properties like the existence of a unique solution for the following fractional perturbed neutral integro-differential system under the deformable derivative:

$${}^{\mathcal{DD}}D^{\vartheta} \big[w(\delta) - H\big(\delta, w(\delta)\big)\big]$$

$$
\begin{aligned}
&= F\left(\delta, w(\delta), \int_0^\delta h\big(\delta,\sigma,w(\sigma)\big)\,d\sigma\right) + G\left(\delta, w(\delta), \int_0^\delta h\big(\delta,\sigma,w(\sigma)\big)\,d\sigma\right),\\
&w(0) = w_0, \quad \delta \in [0,\zeta], 0 < \vartheta < 1,
\end{aligned}
\tag{1.1}
$$

where ${}^{\mathcal{DD}}D^{\vartheta}$ is the deformable fractional derivative, $F, G : [0,\zeta] \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ are continuous functions, $h : Z \times \mathbb{X} \to \mathbb{X}$ is a continuous function with $Z = \{(\delta,\phi) : 0 \le \phi \le \delta \le \zeta\}$, $H : [0,\zeta] \times \mathbb{X} \to \mathbb{X}$ is a continuously differentiable function, and $w_0 \in \mathbb{X}$, where $\mathbb{X}$ is a Banach space (real or complex).

The significant findings of the present study are as follows:

1. This is the first effort, to the best of our knowledge, to address the structure of the fractional perturbed neutral integro-differential system that includes the fractional deformable operator in the context of the system introduced in (1.1).
2. By using the fractional derivative of deformable type and its properties we provide the exponential-based solution to the given peturbed system (1.1); see Lemma 2.6.
3. The alternative fixed point criteria attributed to Banach and Krasnoselskii are used to obtain the main existence theorems. We provide few instances to highlight how our main results may be used in the final section of our presentation.
4. In addition, the results of this work generalized and also upgraded the previous studies published in the literature, including those cited in [22–24].

This research is organized as follows. The equivalent solution to the mentioned deformable perturbed system is built under the fractional deformable integral equation in Sect. 2. The initial result is based on the Banach contraction principle, and the next one is based on Krasnoselskii's fixed point theorem, which will be proved in Sect. 3. Moreover, we consider a nonlocal perturbed integro-differential version of the system and extend our results in this case. Two numerical examples, as some applications of the given deformable perturbed systems, are provided in Sect. 4. The conclusion of this paper is given in Sect. 5 and suggests some future ideas whether we can use these results under artificial intelligence algorithms to simulate dynamics of the models.

2 Preliminaries

This section provides an overview of the essential concepts and properties of the fractional deformable derivatives, which will be used in establishing our main theorems.

Let $\mathbb{X}$ be a Banach space with norm $\|\cdot\|$, and let $\mathcal{C}([0,\xi],\mathbb{X})$ be the Banach space of all continuous functions from $[0,\xi]$ into $\mathbb{X}$ endowed with the supremum norm $\|p\|_{\mathcal{C}} = \sup_{\delta\in[0,\xi]} \|p(\delta)\|$.

Definition 2.1 [20] The fractional deformable derivative of order $\vartheta \in [0,1]$ for a function $w : (a,b) \to \mathbb{R}$ is defined by

$$
{}^{\mathcal{DD}}D^{\vartheta} w(\delta) = \lim_{\epsilon\to 0} \frac{(1+\epsilon\rho)w(\delta+\epsilon\vartheta) - w(\delta)}{\epsilon}
$$

with $\vartheta + \rho = 1$, provided that the limit exists.

Remark 2.2 If $\vartheta = 0$, then ${}^{\mathcal{DD}}D^{0} w(\delta) = w(\delta)$, and if $\vartheta = 1$, then ${}^{\mathcal{DD}}D w(\delta) = w'(\delta)$.

Definition 2.3 [20] Let $\vartheta + \rho = 1$ and $\vartheta \in (0,1]$. The fractional deformable integral of order ϑ for the continuous function w on $[a,b]$ is given by

$$^{\mathcal{D}}I_a^{\vartheta} w(\delta) = \frac{1}{\vartheta} e^{\frac{-\rho}{\vartheta}\delta} \int_a^{\delta} e^{\frac{\rho}{\vartheta}\phi} w(\phi)\, d\phi. \tag{2.1}$$

Note that if $a = 0$ in (2.1), then

$$^{\mathcal{D}}I^{\vartheta} w(\delta) = \frac{1}{\vartheta} e^{\frac{-\rho}{\vartheta}\delta} \int_0^{\delta} e^{\frac{\rho}{\vartheta}\phi} w(\phi)\, d\phi.$$

Theorem 2.4 [19] *Let a function w be differentiable at a point $\delta \in (a,b)$. Then w is always ϑ-differentiable in the sense of deformable derivative at that point for every ϑ. Furthermore, in such a case,*

$$^{\mathcal{D}\mathcal{D}}D^{\vartheta} w(\delta) = \rho w(\delta) + \vartheta D w(\delta),$$

where $Dw = \frac{\mathrm{d}}{\mathrm{d}\delta} w$ is an ordinary derivative.

Theorem 2.5 [19] *Assume that w is continuous on $[a,b]$. Then $^{\mathcal{D}}I_a^{\vartheta} w$ is ϑ-differentiable in the sense of deformable derivative in (a,b). Moreover,*

$$^{\mathcal{D}\mathcal{D}}D^{\vartheta}\big(^{\mathcal{D}}I_a^{\vartheta} w\big)(\delta) = w(\delta) \quad \textit{and} \quad ^{\mathcal{D}}I_a^{\vartheta}\big(^{\mathcal{D}\mathcal{D}}D^{\vartheta} w\big)(\delta) = w(\delta) - e^{\frac{\rho}{\vartheta}(a-\delta)} w(a).$$

For more properties and characteristics of deformable operators, we invite the readers to carefully read [19, 20].

Before we go on to define the solution structure of system (1.1), let us consider the following linear system and find its solution structure:

$$\begin{aligned} &^{\mathcal{D}\mathcal{D}}D^{\vartheta}\big[w(\delta) - h(\delta)\big] = f(\delta) + g(\delta), \quad \delta \in [0,\zeta], 0 < \vartheta < 1, \\ &w(0) = w_0. \end{aligned} \tag{2.2}$$

Here $^{\mathcal{D}\mathcal{D}}D^{\vartheta}$ is the deformable fractional derivative.

Lemma 2.6 *Let $f, g : [0,\zeta] \to \mathbb{X}$ be continuous functions, and let h be a continuously differentiable function. A function w is the solution of the integral equation*

$$w(\delta) = e^{\frac{-\rho}{\vartheta}\delta}\big[w_0 - h(0)\big] + h(\delta) + \frac{1}{\vartheta} e^{\frac{-\rho}{\vartheta}\delta} \int_0^{\delta} e^{\frac{\rho}{\vartheta}\phi}\big[f(\phi) + g(\phi)\big]\, d\phi \tag{2.3}$$

if and only if w is a solution of the linear system (2.2).

Proof Applying $^{\mathcal{D}}I_0^{\vartheta}$ to both sides of (2.2), we have

$$^{\mathcal{D}}I_0^{\vartheta}\big(^{\mathcal{D}\mathcal{D}}D^{\vartheta}\big[w(\delta) - h(\delta)\big]\big) = {}^{\mathcal{D}}I_0^{\vartheta}\big(f(\delta) + g(\delta)\big).$$

Using the linearity of the fractional deformable derivative $^{\mathcal{D}\mathcal{D}}D^{\vartheta}$, we have

$$^{\mathcal{D}}I_0^{\vartheta}\big(^{\mathcal{D}\mathcal{D}}D^{\vartheta} w(\delta)\big) - {}^{\mathcal{D}}I_0^{\vartheta}\big(^{\mathcal{D}\mathcal{D}}D^{\vartheta} h(\delta)\big) = {}^{\mathcal{D}}I_0^{\vartheta}\big(f(\delta)\big) + {}^{\mathcal{D}}I_0^{\vartheta}\big(g(\delta)\big).$$

By using Theorem 2.5 and $w(0) = w_0$ we get

$$w(\delta) - e^{\frac{-\rho}{\vartheta}\delta} w_0 - h(\delta) + e^{\frac{-\rho}{\vartheta}\delta} h(0) = \frac{1}{\vartheta} e^{\frac{-\rho}{\vartheta}\delta} \int_0^\delta e^{\frac{\rho}{\vartheta}\phi} \bigl[f(\phi) + g(\phi)\bigr]\, d\phi.$$

Therefore

$$w(\delta) = e^{\frac{-\rho}{\vartheta}\delta} \bigl[w_0 - h(0)\bigr] + h(\delta) + \frac{1}{\vartheta} e^{\frac{-\rho}{\vartheta}\delta} \int_0^\delta e^{\frac{\rho}{\vartheta}s} \bigl[f(s) + g(s)\bigr]\, ds.$$

On the other hand, assume that w satisfies (2.3). Applying ${}^{\mathcal{DD}}D^{\vartheta}$ to both sides of (2.3), by Theorem 2.4 we get

$$\begin{aligned}
&{}^{\mathcal{DD}}D^{\vartheta} \bigl[w(\delta) - h(\delta)\bigr] \\
&\quad = \rho \left(e^{\frac{-\rho}{\vartheta}\delta} \bigl[w_0 - h(0)\bigr] + \frac{1}{\vartheta} e^{\frac{-\rho}{\vartheta}\delta} \int_0^\delta e^{\frac{\rho}{\vartheta}\phi} \bigl[f(\phi) + g(\phi)\bigr]\, d\phi \right) \\
&\qquad + \vartheta \frac{d}{d\delta} \left(e^{\frac{-\rho}{\vartheta}\delta} \bigl[w_0 - h(0)\bigr] + \frac{1}{\vartheta} e^{\frac{-\rho}{\vartheta}\delta} \int_0^\delta e^{\frac{\rho}{\vartheta}\phi} \bigl[f(\phi) + g(\phi)\bigr]\, d\phi \right) \\
&\quad = \rho e^{\frac{-\rho}{\vartheta}\delta} \bigl[w_0 - h(0)\bigr] + \frac{\rho}{\vartheta} e^{\frac{-\rho}{\vartheta}\delta} \int_0^\delta e^{\frac{\rho}{\vartheta}\phi} \bigl[f(\phi) + g(\phi)\bigr]\, d\phi \\
&\qquad + \vartheta \frac{d}{d\delta} \bigl(e^{\frac{-\rho}{\vartheta}\delta} \bigl[w_0 - h(0)\bigr]\bigr) + \frac{d}{d\delta} \left(e^{\frac{-\rho}{\vartheta}\delta} \int_0^\delta e^{\frac{\rho}{\vartheta}\phi} \bigl[f(\phi) + g(\phi)\bigr]\, d\phi \right).
\end{aligned} \tag{2.4}$$

Since

$$\vartheta \frac{d}{d\delta} \bigl(e^{\frac{-\rho}{\vartheta}\delta} \bigl[w_0 - h(0)\bigr]\bigr) = \vartheta \bigl[w_0 - h(0)\bigr] \left(\frac{-\rho}{\vartheta} \right) e^{\frac{-\rho}{\vartheta}\delta} = -\rho e^{\frac{-\rho}{\vartheta}\delta} \bigl[w_0 - h(0)\bigr]$$

and

$$\begin{aligned}
&\frac{d}{d\delta} \left(e^{\frac{-\rho}{\vartheta}\delta} \int_0^\delta e^{\frac{\rho}{\vartheta}\phi} \bigl[f(\phi) + g(\phi)\bigr]\, d\phi \right) \\
&\quad = \left(\frac{-\rho}{\vartheta} \right) e^{\frac{-\rho}{\vartheta}\delta} \int_0^\delta e^{\frac{\rho}{\vartheta}\phi} \bigl[f(\phi) + g(\phi)\bigr]\, d\phi \\
&\qquad + e^{\frac{-\rho}{\vartheta}\delta} \frac{d}{d\delta} \left[\int_0^\delta e^{\frac{\rho}{\vartheta}\phi} \bigl[f(\phi) + g(\phi)\bigr]\, d\phi \right] \\
&\quad = \frac{-\rho}{\vartheta} e^{\frac{-\rho}{\vartheta}\delta} \int_0^\delta e^{\frac{\rho}{\vartheta}\phi} \bigl[f(\phi) + g(\phi)\bigr]\, d\phi + e^{\frac{-\rho}{\vartheta}\delta} e^{\frac{\rho}{\vartheta}\delta} \bigl[f(\delta) + h(\delta)\bigr],
\end{aligned}$$

(2.4) becomes

$${}^{\mathcal{DD}}D^{\vartheta} \bigl[w(\delta) - h(\delta)\bigr] = f(\delta) + h(\delta).$$

The proof is complete. □

We now can define the solution to the nonlinear fractional deformable perturbed neutral integro-differential system (1.1) using the information provided in the previous lemma. For our convenience, we denote $Ew(\delta) = \int_0^\delta h(\delta, \phi, w(\phi))\, d\phi$.

Definition 2.7 A function w is said to be a solution to the fractional deformable perturbed neutral integro-differential system (1.1) if

$$w(\delta) = e^{\frac{-\rho}{\vartheta}\delta}\big[w_0 - H(0,w_0)\big] + H\big(\delta,w(\delta)\big) + \frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}F\big(\phi,w(\phi),Ew(\phi)\big)\,d\phi \\ + \frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}G\big(\phi,w(\phi),Ew(\phi)\big)\,d\phi, \quad \delta\in[0,\zeta], \tag{2.5}$$

provided that the above integral is finite.

3 Existence theorems

In this section, we present and establish the existence results for the given deformable system (1.1) under the Banach contraction principle and Krasnoselskii's fixed point theorems. To apply these fixed point theorems, we need the following hypotheses:

(M0) The function $h: Z\times\mathbb{X}\to\mathbb{X}$, where $Z=\{(\delta,\phi): 0\le\phi\le\delta\le\zeta\}$, is continuous, and there is a constant $\mathcal{K}_h>0$ such that

$$\left\|\int_0^\delta h\big(\delta,\phi,u(\phi)\big)\,d\phi - \int_0^\delta h\big(\delta,\phi,v(\phi)\big)\,d\phi\right\| \le \mathcal{K}_h\|u-v\|, \\ (\delta,\phi)\in[0,\zeta]\times[0,\zeta], u,v\in\mathbb{X},$$

and $\overline{\mathcal{K}}_h = \sup_{(\phi,\tau)\in[0,\zeta]\times[0,\zeta]}\|\int_0^\phi h(\phi,\tau,0)\,d\tau\|$.

(M1) The functions $F,G:[0,\zeta]\times\mathbb{X}\times\mathbb{X}\to\mathbb{X}$ are continuous, $H:[0,\zeta]\times\mathbb{X}\to\mathbb{X}$ is continuously differentiable, and there are $\mathcal{K}_F,\mathcal{K}_G,\mathcal{K}_H>0$ such that

(i)

$$\big\|F(\delta,u,v)-F(\delta,\bar u,\bar v)\big\| \le \mathcal{K}_F\big[\|u-\bar u\|+\|v-\bar v\|\big], \quad \delta\in[0,\zeta], u,\bar u,v,\bar v\in\mathbb{X},$$

and $\overline{\mathcal{K}}_F = \sup_{\delta\in[0,\zeta]}\|F(\delta,0,0)\|$;

(ii)

$$\big\|G(\delta,u,v)-G(\delta,\bar u,\bar v)\big\| \le \mathcal{K}_G\big[\|u-\bar u\|+\|v-\bar v\|\big], \quad \delta\in[0,\zeta], u,\bar u,v,\bar v\in\mathbb{X},$$

and $\overline{\mathcal{K}}_G = \sup_{\delta\in[0,\zeta]}\|G(\delta,0,0)\|$;

(iii)

$$\big\|H(\delta,u)-H(\delta,\bar u)\big\| \le \mathcal{K}_H\|u-\bar u\|, \quad \delta\in[0,\zeta], u,\bar u\in\mathbb{X},$$

and $\overline{\mathcal{K}}_H = \sup_{\delta\in[0,\zeta]}\|H(\delta,0)\|$.

(M2) There are nonnegative constants $\widehat{\mathcal{K}}_F,\widehat{\mathcal{K}}_G,\widehat{\mathcal{K}}_H,\widetilde{\mathcal{K}}_F,\widetilde{\mathcal{K}}_G,\widetilde{\mathcal{K}}_H$ such that

(i)

$$\big\|F(\delta,u,v)\big\| \le \widehat{\mathcal{K}}_F + \widetilde{\mathcal{K}}_F\big[\|u\|+\|v\|\big], \quad \delta\in[0,\zeta], u,v\in\mathbb{X}.$$

(ii)

$$\big\|G(\delta,u,v)\big\| \le \widehat{\mathcal{K}}_G + \widetilde{\mathcal{K}}_G\big[\|u\|+\|v\|\big], \quad \delta\in[0,\zeta], u,v\in\mathbb{X}.$$

(iii)

$$\big\|H(\delta,w)\big\| \le \widehat{\mathcal{K}}_H + \widetilde{\mathcal{K}}_H\|w\|, \quad \delta\in[0,\zeta], w\in\mathbb{X}.$$

(M3) (i) There is $\widetilde{\mathcal{K}}_F \in L^1([0,\zeta],\mathbb{X})$ such that

$$\big\|F(\delta,u,v)\big\| \le \widetilde{\mathcal{K}}_F(\delta), \quad (\delta,u,v)\in[0,\zeta]\times\mathbb{X}\times\mathbb{X}.$$

(ii) There is $\widetilde{\mathcal{K}}_G \in L^1([0,\zeta],\mathbb{X})$ such that

$$\big\|G(\delta,u,v)\big\| \le \widetilde{\mathcal{K}}_G(\delta), \quad (\delta,u,v)\in[0,\zeta]\times\mathbb{X}\times\mathbb{X}.$$

(iii) There is $\widetilde{\mathcal{K}}_H \in L^1([0,\zeta],\mathbb{X})$ such that

$$\big\|H(\delta,u)\big\| \le \widetilde{\mathcal{K}}_H(\delta), \quad (\delta,u)\in[0,\zeta]\times\mathbb{X}.$$

Moreover, we consider the Banach space $Q := \mathcal{C}([0,\zeta],\mathbb{X})$ of all continuous functions with values in $\mathbb{X}$ with sup-norm $\|\cdot\|_Q$. In this theorem, we prove that a unique solution exists for the mentioned perturbed problem.

Theorem 3.1 *Let F, G, H, h satisfy* (M0)–(M1). *If*

$$\Lambda = \left[\mathcal{K}_H + \frac{1}{\rho}\big[(\mathcal{K}_F+\mathcal{K}_G)(1+\mathcal{K}_h)\big]\right] < 1, \tag{3.1}$$

then the fractional deformable perturbed neutral integro-differential system (1.1) *has a unique solution on* $[0,\zeta]$.

Proof To use the well-known fixed point theorem, we define $\psi : Q \to Q$ by

$$\begin{aligned}(\psi w)(\delta) &= e^{\frac{-\rho}{\vartheta}\delta}\big[w_0 - H(0,w_0)\big] + H\big(\delta,w(\delta)\big) + \frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}F\big(\phi,w(\phi),Ew(\phi)\big)\,d\phi \\ &\quad + \frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}G\big(\phi,w(\phi),Ew(\phi)\big)\,d\phi, \quad \delta\in[0,\zeta].\end{aligned} \tag{3.2}$$

We now show that $\psi B_O \subset B_O$, where B_O is the ball $B(0,O) = \{w\in\mathcal{C}([0,\zeta],\mathbb{X}) = Q : \|w\|_Q \le O\}$ with radius

$$O > \frac{\|\Omega_1\|}{1-\widetilde{\mu}},$$

so that $\|\Omega_1\| = \|\Omega\| + (\overline{\mathcal{K}}_H + \frac{1}{\rho}(\overline{\mathcal{K}}_h(\mathcal{K}_F+\mathcal{K}_G) + \overline{\mathcal{K}}_F + \overline{\mathcal{K}}_G))$, $\|\Omega\| = [\|w_0\| + \|H(0,w_0)\|]$, and $\widetilde{\mu} = \mathcal{K}_H + \frac{1}{\rho}[(\mathcal{K}_F+\mathcal{K}_G)(1+\mathcal{K}_h)]$. Indeed, let $w\in B_O$ and consider (M0)–(M1). We have

$$\begin{aligned}&\big\|(\psi w)(\delta)\big\| \\ &\quad = \left\|e^{\frac{-\rho}{\vartheta}\delta}\big[w_0 - H(0,w_0)\big] + H\big(\delta,w(\delta)\big) + \frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}F\big(\phi,w(\phi),Ew(\phi)\big)\,d\phi\right.\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}G\big(\phi,w(\phi),Ew(\phi)\big)\,d\phi\Big\|\\
&\le\|\Omega\|+\big\|H\big(\delta,w(\delta)\big)-H(\delta,0)\big\|+\big\|H(\delta,0)\big\|\\
&+\frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}\bigg(\bigg\|F\bigg(\phi,w(\phi),\int_0^\phi h\big(\phi,\tau,w(\tau)\big)\,d\tau\bigg)-F(\phi,0,0)\bigg\|\\
&+\big\|F(\phi,0,0)\big\|\bigg)\,d\phi\\
&+\frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}\bigg(\bigg\|G\bigg(\phi,w(\phi),\int_0^\phi h\big(\phi,\tau,w(\tau)\big)\,d\tau\bigg)-G(\phi,0,0)\bigg\|\\
&+\big\|G(\phi,0,0)\big\|\bigg)\,d\phi\\
&\le\|\Omega\|+\mathcal{K}_HO+\overline{\mathcal{K}}_H\\
&+\frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}\bigg[\mathcal{K}_F\bigg\{\|w\|+\bigg\|\int_0^\phi h\big(\phi,\tau,w(\tau)\big)\,d\tau\bigg\|\bigg\}+\overline{\mathcal{K}}_F\bigg]\,d\phi\\
&+\frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}\bigg[\mathcal{K}_G\bigg\{\|w\|+\bigg\|\int_0^\phi h\big(\phi,\tau,w(\tau)\big)\,d\tau\bigg\|\bigg\}+\overline{\mathcal{K}}_G\bigg]\,d\phi\\
&\le\|\Omega\|+\mathcal{K}_HO+\overline{\mathcal{K}}_H+\big[\mathcal{K}_F\big[(1+\mathcal{K}_h)O+\overline{\mathcal{K}}_h\big]+\overline{\mathcal{K}}_F\big]\frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}\,d\phi\\
&+\big[\mathcal{K}_G\big[(1+\mathcal{K}_h)O+\overline{\mathcal{K}}_h\big]+\overline{\mathcal{K}}_G\big]\frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}\,d\phi\\
&\le\|\Omega\|+\mathcal{K}_HO+\overline{\mathcal{K}}_H+\big[\mathcal{K}_F\big[(1+\mathcal{K}_h)O+\overline{\mathcal{K}}_h\big]+\overline{\mathcal{K}}_F\big]\frac{1}{\rho}\big(1-e^{\frac{-\rho}{\vartheta}\delta}\big)\\
&+\big[\mathcal{K}_G\big[(1+\mathcal{K}_h)O+\overline{\mathcal{K}}_h\big]+\overline{\mathcal{K}}_G\big]\frac{1}{\rho}\big(1-e^{\frac{-\rho}{\vartheta}\delta}\big)\\
&=\|\Omega\|+\mathcal{K}_HO+\overline{\mathcal{K}}_H+\frac{1}{\rho}\big[(\mathcal{K}_F+\mathcal{K}_G)(1+\mathcal{K}_h)O+\overline{\mathcal{K}}_h(\mathcal{K}_F+\mathcal{K}_G)\\
&+\overline{\mathcal{K}}_F+\overline{\mathcal{K}}_G\big]\big(1-e^{\frac{-\rho}{\vartheta}\delta}\big)\\
&\le\|\Omega_1\|+\bigg[\mathcal{K}_H+\frac{1}{\rho}(\mathcal{K}_F+\mathcal{K}_G)(1+\mathcal{K}_h)\bigg]O.
\end{aligned}
$$

Thus, for $\delta\in[0,\zeta]$ and $w\in B_O$, we have

$$
\big\|\psi(w)\big\|_Q\le\|\Omega_1\|+\bigg[\mathcal{K}_H+\frac{1}{\rho}(\mathcal{K}_F+\mathcal{K}_G)(1+\mathcal{K}_h)\bigg]O<O.
$$

This shows that the ball B_O is mapped into itself under the operator ψ, that is, $\psi B_O\subset B_O$.

Following the proof, for each $w,\overline{w}\in B_O$, we estimate

$$
\begin{aligned}
&\big\|(\psi w)(\delta)-(\psi\overline{w})(\delta)\big\|\\
&\le\big\|H\big(\delta,w(\delta)\big)-H\big(\delta,\overline{w}(\delta)\big)\big\|\\
&+\frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}\big\|F\big(\phi,w(\phi),Ew(\phi)\big)-F\big(\phi,\overline{w}(\phi),E\overline{w}(\phi)\big)\big\|\,d\phi\\
&+\frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}\big\|G\big(\phi,w(\phi),Ew(\phi)\big)-G\big(\phi,\overline{w}(\phi),E\overline{w}(\phi)\big)\big\|\,d\phi
\end{aligned}
$$

$$\leq \mathcal{K}_H \|w-\overline{w}\|_Q + \frac{1}{\rho}\big[(\mathcal{K}_F+\mathcal{K}_G)(1+\mathcal{K}_h)\big]\|w-\overline{w}\|_Q\big(1-e^{\frac{-\rho}{\vartheta}\delta}\big)$$

$$\leq \left[\mathcal{K}_H + \frac{1}{\rho}\big[(\mathcal{K}_F+\mathcal{K}_G)(1+\mathcal{K}_h)\big]\right]\|w-\overline{w}\|_Q.$$

Hence, for each $\delta \in [0,\zeta]$, we obtain

$$\big\|\psi(w)-\psi(\overline{w})\big\|_Q \leq \Lambda\|w-\overline{w}\|_Q,$$

where $\Lambda = \Lambda(\mathcal{K}_F,\mathcal{K}_G,\mathcal{K}_H,\mathcal{K}_h,\rho)$, defined in (3.1), depends on the system parameters. Now, as $\Lambda < 1$, ψ is a contraction. Hence the fractional deformable perturbed neutral integro-differential system (1.1) has a unique solution on $[0,\zeta]$. □

Now we will prove that there are solutions for the fractional deformable perturbed neutral integro-differential system (1.1) under the hypotheses of Krasnoselskii's fixed point theorem.

Theorem 3.2 *Suppose that hypotheses* (M0), (M1)(ii)(iii) *and* (M2) *are satisfied and that* $[\mathcal{K}_H + \frac{1}{\rho}\mathcal{K}_G(1+\mathcal{K}_h)] < 1$. *Then the fractional deformable perturbed neutral integro-differential system* (1.1) *has at least one solution on* $[0,\zeta]$.

Proof Consider two operators on Q participating in (2.5),

$$(\psi_1 w)(\delta) = e^{\frac{-\rho}{\vartheta}\delta}\big[w_0 - H(0,w_0)\big] + H\big(\delta,w(\delta)\big) + \frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}G\big(\phi,w(\phi),Ew(\phi)\big)\,d\phi \tag{3.3}$$

and

$$(\psi_2 w)(\delta) = \frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}F\big(\phi,w(\phi),Ew(\phi)\big)\,d\phi. \tag{3.4}$$

We now show that $\psi B_O \subset B_O$, where B_O is the ball $B(0,O) = \{w \in \mathcal{C}([0,\zeta],\mathbb{X}) = Q : \|w\|_Q \leq O\}$ with radius $O > \frac{\|\Omega_1^*\|}{1-\widetilde{\mu}_1}$, so that $\|\Omega_1^*\| = \|\Omega\| + \widehat{\mathcal{K}}_H + \frac{1}{\rho}[(\widehat{\mathcal{K}}_G+\widehat{\mathcal{K}}_F)+\overline{\mathcal{K}}_h(\widetilde{\mathcal{K}}_G+\widetilde{\mathcal{K}}_F)]$, $\|\Omega\| = [\|w_0\| + \|H(0,w_0)\|]$, and $\widetilde{\mu}_1 = \widetilde{\mathcal{K}}_H + \frac{1}{\rho}[(\widetilde{\mathcal{K}}_G+\widetilde{\mathcal{K}}_F)(1+\mathcal{K}_h)]$.

For all $w, w_1 \in B_O$, we have

$$\begin{aligned}
&\big\|\psi_1 w(\delta) + \psi_2 w_1(\delta)\big\| \\
&\leq \left\| e^{\frac{-\rho}{\vartheta}\delta}\big[w_0 - H(0,w_0)\big] + H\big(\delta,w(\delta)\big) + \frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}G\big(\phi,w(\phi),Ew(\phi)\big)\,d\phi \right.\\
&\quad \left. + \frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}F\big(\phi,w_1(\phi),Ew_1(\phi)\big)\,d\phi \right\| \\
&\leq \|\Omega\| + \widehat{\mathcal{K}}_H + \widetilde{\mathcal{K}}_H O + \frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}\big[\widehat{\mathcal{K}}_G + \widetilde{\mathcal{K}}_G\big[(1+\mathcal{K}_h)O + \overline{\mathcal{K}}_h\big]\big]\,d\phi \\
&\quad + \frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}\big[\widehat{\mathcal{K}}_F + \widetilde{\mathcal{K}}_F\big[(1+\mathcal{K}_h)O + \overline{\mathcal{K}}_h\big]\big]\,d\phi \\
&\leq \|\Omega\| + \widehat{\mathcal{K}}_H + \frac{1}{\rho}\big[(\widehat{\mathcal{K}}_G+\widehat{\mathcal{K}}_F) + \overline{\mathcal{K}}_h(\widetilde{\mathcal{K}}_G+\widetilde{\mathcal{K}}_F)\big] + \left[\widetilde{\mathcal{K}}_H + \frac{1}{\rho}(\widetilde{\mathcal{K}}_G+\widetilde{\mathcal{K}}_F)(1+\mathcal{K}_h)\right]O
\end{aligned}$$

$$\leq \|\Omega_1^*\| + \left[\widetilde{\mathcal{K}}_H + \frac{1}{\rho}(\widetilde{\mathcal{K}}_G + \widetilde{\mathcal{K}}_F)(1 + \mathcal{K}_h)\right]O.$$

Thus, for all $\delta \in [0, \zeta]$ and $w \in B_O$, it becomes

$$\|\psi_1(w) + \psi_2(w_1)\|_Q \leq \|\Omega_1^*\| + \left[\widetilde{\mathcal{K}}_H + \frac{1}{\rho}(\widetilde{\mathcal{K}}_G + \widetilde{\mathcal{K}}_F)(1 + \mathcal{K}_h)\right]O < O.$$

Thus $\psi_1(w) + \psi_2(w_1) \in B_O$. Next, we prove that ψ_1 is a contraction. Due to the continuity of H, G, h, for $w, \overline{w} \in B_O$, using (3.3) and (M1)(ii)(iii), we estimate

$$\begin{aligned}\|(\psi_1 w)(\delta) - (\psi_1 \overline{w})(\delta)\| &\leq \mathcal{K}_H \|w - \overline{w}\|_Q + \frac{1}{\rho}\mathcal{K}_G(1 + \mathcal{K}_h)\|w - \overline{w}\|_Q\left(1 - e^{\frac{-\rho}{\vartheta}\delta}\right) \\ &\leq \left(\mathcal{K}_H + \frac{1}{\rho}\mathcal{K}_G(1 + \mathcal{K}_h)\right)\|w - \overline{w}\|_Q.\end{aligned}$$

Thus, for all $\delta \in [0, \zeta]$ and $w \in B_O$, we have

$$\|(\psi_1 w) - (\psi_1 \overline{w})\|_Q \leq \left(\mathcal{K}_H + \frac{1}{\rho}\mathcal{K}_G(1 + \mathcal{K}_h)\right)\|w - \overline{w}\|_Q.$$

Hence ψ_1 is a contraction. Since the function F is continuous, it follows that the operator ψ_2 is continuous as well. Additionally, ψ_2 is uniformly bounded on B_O. We have

$$\begin{aligned}\|(\psi_2 w)(\delta)\| &\leq \frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}\|F(\phi, w(\phi), Ew(\phi))\|\, d\phi \\ &\leq \frac{1}{\rho}\left[\widehat{\mathcal{K}}_F + \widetilde{\mathcal{K}}_F\left[(1 + \mathcal{K}_h)O + \overline{\mathcal{K}}_h\right]\right] = A,\end{aligned}$$

and thus $\|\psi_2 w\|_Q \leq A$.

The uniform boundedness of the operator ψ_2 implies that it is compact. It suffices to check the equicontinuity for ψ_2. For all $\tau_1, \tau_2 \in [0, \zeta]$ with $\tau_1 < \tau_2$ and $w \in B_O$, we have

$$\begin{aligned}&\|(\psi_2 w)(\tau_2) - (\psi_2 w)(\tau_1)\| \\ &= \frac{1}{\vartheta}\left\| e^{\frac{-\rho}{\vartheta}\tau_2}\int_0^{\tau_2} e^{\frac{\rho}{\vartheta}\phi}F(\phi, w(\phi), Ew(\phi))\, d\phi - e^{\frac{-\rho}{\vartheta}\tau_1}\int_0^{\tau_1} e^{\frac{\rho}{\vartheta}\phi}F(\phi, w(\phi), Ew(\phi))\, d\phi\right\| \\ &= \frac{1}{\vartheta}\left\|\int_{\tau_1}^{\tau_2} e^{\frac{-\rho}{\vartheta}\tau_2}e^{\frac{\rho}{\vartheta}\phi}F(\phi, w(\phi), Ew(\phi))\, d\phi\right. \\ &\quad \left. - \int_0^{\tau_1}\left[e^{\frac{-\rho}{\vartheta}\tau_1} - e^{\frac{-\rho}{\vartheta}\tau_2}\right]e^{\frac{\rho}{\vartheta}\phi}F(\phi, w(\phi), Ew(\phi))\, d\phi\right\| \\ &\leq \frac{1}{\vartheta}\left\| e^{\frac{-\rho}{\vartheta}\tau_2}\int_{\tau_1}^{\tau_2} e^{\frac{\rho}{\vartheta}\phi}F(\phi, w(\phi), Ew(\phi))\, d\phi\right. \\ &\quad \left. - \left[e^{\frac{-\rho}{\vartheta}\tau_1} - e^{\frac{-\rho}{\vartheta}\tau_2}\right]\int_0^{\tau_1} e^{\frac{\rho}{\vartheta}\phi}F(\phi, w(\phi), Ew(\phi))\, d\phi\right\| \\ &\leq \frac{1}{\rho}\left[\widehat{\mathcal{K}}_F + \widetilde{\mathcal{K}}_F\left[(1 + \mathcal{K}_h)O + \overline{\mathcal{K}}_h\right]\right]\left\|1 - e^{\frac{-\rho}{\vartheta}(\tau_2 - \tau_1)} + \left(e^{\frac{-\rho}{\vartheta}\tau_1} - e^{\frac{-\rho}{\vartheta}\tau_2}\right)\left(e^{\frac{-\rho}{\vartheta}\tau_1}\right)\right\| \\ &\leq \frac{1}{\rho}\left[\widehat{\mathcal{K}}_F + \widetilde{\mathcal{K}}_F\left[(1 + \mathcal{K}_h)O + \overline{\mathcal{K}}_h\right]\right]\left\|2 - 2e^{\frac{-\rho}{\vartheta}(\tau_2 - \tau_1)} - e^{\frac{-\rho}{\vartheta}\tau_1} + e^{\frac{-\rho}{\vartheta}\tau_2}\right\|. \qquad (3.5)\end{aligned}$$

From (3.5) we see that as $\tau_2 \to \tau_1$, the right-hand side of (3.5) goes to zero. So $\|(\psi_2 w)(\tau_2) - (\psi_2 w)(\tau_1)\| \to 0$ as $\tau_1 \to \tau_2$.

Thus ψ_2 is equicontinuous, and so $\psi_2(\mathbb{X}) \subset \mathbb{X}$. Therefore ψ_2 is compact. According to the Arzelà–Ascoli theorem, ψ has at least one fixed point. Hence we can find at least one solution to the fractional deformable perturbed neutral integro-differential system (1.1). □

Now we give another criterion for the existence property.

Theorem 3.3 *Suppose that conditions* (M0)–(M1)(ii)(iii) *and* (M3) *are and that* $[\mathcal{K}_H + \frac{1}{\rho}\mathcal{K}_G(1 + \mathcal{K}_h)] < 1$. *Then the fractional deformable perturbed neutral integro-differential system* (1.1) *has at least one solution on* $[0, \zeta]$.

Proof By (2.5) we define two operators as follows:

$$(\psi_1 w)(\delta) = e^{\frac{-\rho}{\vartheta}\delta}\big[w_0 - H(0, w_0)\big] + H\big(\delta, w(\delta)\big) + \frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}G\big(\phi, w(\phi), Ew(\phi)\big)\,d\phi, \tag{3.6}$$

and

$$(\psi_2 w)(\delta) = \frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}F\big(\phi, w(\phi), Ew(\phi)\big)\,d\phi. \tag{3.7}$$

We now show that $\psi B_O \subset B_O$, where B_O is the ball $B(0, O) = \{w \in \mathcal{C}([0, \zeta], \mathbb{X}) = Q : \|w\|_Q \leq O\}$ with radius $O > \|\Omega_1^*\| + \frac{1}{\vartheta}[\|\widetilde{\mathcal{K}}_G\|_{L^1([0,\zeta])} + \|\widetilde{\mathcal{K}}_F\|_{L^1([0,\zeta])}]$, so that $\|\Omega_1^*\| = \|\Omega\| + \widetilde{\mathcal{K}}_H(\zeta)$ and $\|\Omega\| = [\|w_0\| + \|H(0, w_0)\|]$.

For all $w, w_1 \in B_O$, we may write

$$\begin{aligned}
&\big\|\psi_1 w(\delta) + \psi_2 w_1(\delta)\big\| \\
&\quad\leq \bigg\|e^{\frac{-\rho}{\vartheta}\delta}\big[w_0 - H(0, w_0)\big] + H\big(\delta, w(\delta)\big) + \frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}G\big(\phi, w(\phi), Ew(\phi)\big)\,d\phi \\
&\qquad + \frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}F\big(\phi, w_1(\phi), Ew_1(\phi)\big)\,d\phi\bigg\| \\
&\quad\leq \|\Omega\| + \widetilde{\mathcal{K}}_H(\delta) + \frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}\big\|G\big(\phi, w(\phi), Ew(\phi)\big)\big\|\,d\phi \\
&\qquad + \frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}\big\|F\big(\phi, w(\phi), Ew(\phi)\big)\big\|\,d\phi \\
&\quad\leq \|\Omega\| + \widetilde{\mathcal{K}}_H(\delta) + \frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}e^{\frac{\rho}{\vartheta}\delta}\int_0^\delta \big\|G\big(\phi, w(\phi), Ew(\phi)\big)\big\|\,d\phi \\
&\qquad + \frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}e^{\frac{\rho}{\vartheta}\delta}\int_0^\delta \big\|F\big(\phi, w(\phi), Ew(\phi)\big)\big\|\,d\phi \\
&\quad\leq \|\Omega\| + \widetilde{\mathcal{K}}_H(\delta) + \frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}e^{\frac{\rho}{\vartheta}\delta}\int_0^\delta \widetilde{\mathcal{K}}_G(\phi)\,d\phi \\
&\qquad + \frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}e^{\frac{\rho}{\vartheta}\delta}\int_0^\delta \widetilde{\mathcal{K}}_F(\phi)\,d\phi
\end{aligned}$$

$$\leq \left\|\Omega_1^*\right\| + \frac{1}{\vartheta}\left[\|\widetilde{\mathcal{K}}_G\|_{L^1([0,\zeta])} + \|\widetilde{\mathcal{K}}_F\|_{L^1([0,\zeta])}\right].$$

Thus, for all $\delta \in [0,\zeta]$ and $w \in B_O$, we have

$$\left\|\psi_1(w) + \psi_2(w_1)\right\|_Q \leq \left\|\Omega_1^*\right\| + \frac{1}{\vartheta}\left[\|\widetilde{\mathcal{K}}_G\|_{L^1([0,\zeta])} + \|\widetilde{\mathcal{K}}_F\|_{L^1([0,\zeta])}\right] < O.$$

Thus $\psi_1(w)+\psi_2(w_1) \in B_O$. In the next step, we prove that ψ_1 is a contraction. We know that H, G, h are continuous. Letting $w, \overline{w} \in B_O$, from (3.6) and (M1)(ii)(iii) we get the estimate

$$\begin{aligned}\left\|(\psi_1 w)(\delta) - (\psi_1\overline{w})(\delta)\right\| &\leq \mathcal{K}_H\|p-\overline{w}\|_Q + \frac{1}{\rho}\mathcal{K}_G(1+\mathcal{K}_h)\|w-\overline{w}\|_Q\left(1-e^{\frac{-\rho}{\vartheta}\delta}\right)\\ &\leq \left(\mathcal{K}_H + \frac{1}{\rho}\mathcal{K}_G(1+\mathcal{K}_h)\right)\|w-\overline{w}\|_Q.\end{aligned}$$

Thus, for $\delta \in [0,\zeta]$ and $w \in B_O$, we have

$$\left\|(\psi_1 w) - (\psi_1\overline{w})\right\|_Q \leq \left(\mathcal{K}_H + \frac{1}{\rho}\mathcal{K}_G(1+\mathcal{K}_h)\right)\|w-\overline{w}\|_Q.$$

Hence ψ_1 is a contraction. Since the function F is continuous, it follows that the operator ψ_2 is also continuous. Additionally, ψ_2 is uniformly bounded on B_O, and so

$$\begin{aligned}\left\|(\psi_2 w)(\delta)\right\| &\leq \frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}\left\|F\left(\phi, w(\phi), Ew(\phi)\right)\right\|d\phi\\ &\leq \frac{1}{\rho}\|\widetilde{\mathcal{K}}_F\|_{L^1([0,\zeta])} = A,\end{aligned}$$

which shows that $\|\psi_2 w\|_Q \leq A$.

Thus the uniform boundedness of the operator ψ_2 implies that it is compact. Now we prove that it is equicontinuous. For all $\tau_1, \tau_2 \in [0,\zeta]$ with $\tau_1 < \tau_2$ and $w \in B_O$, by Theorem 3.2 we have

$$\begin{aligned}&\left\|(\psi_2 w)(\tau_2) - (\psi_2 w)(\tau_1)\right\|\\ &\quad\leq \frac{N}{\rho}\left\|1-e^{\frac{-\rho}{\vartheta}(\tau_2-\tau_1)} + \left(e^{\frac{-\rho}{\vartheta}\tau_1} - e^{\frac{-\rho}{\vartheta}\tau_2}\right)\left(e^{\frac{-\rho}{\vartheta}\tau_1}\right)\right\|\\ &\quad\leq \frac{N}{\rho}\left\|2-2e^{\frac{-\rho}{\vartheta}(\tau_2-\tau_1)} - e^{\frac{-\rho}{\vartheta}\tau_1} + e^{\frac{-\rho}{\vartheta}\tau_2}\right\|.\end{aligned} \tag{3.8}$$

From (3.8) we see that if $\tau_1 \to \tau_2$, then the right-hand side of (3.8) goes to zero, so $\|(\psi_2 w)(\tau_2) - (\psi_2 w)(\tau_1)\| \to 0$ as $\tau_1 \to \tau_2$. Thus ψ_2 is equicontinuous, and since $\psi_2(\mathbb{X}) \subset \mathbb{X}$, ψ_2 is compact by the Arzelà–Ascoli theorem. Therefore ψ has at least one fixed point. Finally, there is at least one solution to the associated fractional deformable perturbed neutral integro-differential system (1.1). □

3.1 Nonlocal integro-differential system

This subsection contains a generalization of the results discussed in the previous subsection. We focus on a generalized existence theorem in relation to the nonlocal fractional neutral integro-differential system given by

$$
\begin{aligned}
&{}^{\mathcal{DD}}D^{\vartheta}\big[w(\delta) - H\big(\delta, w(\delta)\big)\big] = F\big(\delta, w(\delta), Ew(\delta)\big) + G\big(\delta, w(\delta), Ew(\delta)\big), \quad \delta \in [0,\zeta], \\
&w(0) + q(w) = w_0, \quad 0 < \vartheta < 1,
\end{aligned} \tag{3.9}
$$

where ${}^{\mathcal{DD}}D^{\vartheta}, F, G, H, E, w_0 \in \mathbb{X}$ are similar to those described in Sect. 3, and $q : \mathcal{C}([0,\zeta], \mathbb{X}) = Q \to \mathbb{X}$ is a continuous function.

During the year 1990, Byszewski was the first person to do groundbreaking research on the nonlocal problem. In [25], Byszewski turns to this purpose that the nonlocal condition $w(\delta_0) + q(\delta_1, \dots, \delta_k, w(\cdot)) = w_0$ can be applied in physics to provide results that are better than those described by the initial condition $w(\delta_0) = w_0$. We recommend the readers to follow [23, 25] for further information regarding nonlocal conditions and related applications.

To study the existence of a unique solution of the nonlocal fractional neutral integro-differential system (3.9), we consider the following assumption:

(M4) The function $q : Q \to \mathbb{X}$ is continuous, and $M = \sup_{w \in Q} \|q(w)\| < \infty$. There is a constant $\mathcal{K}_q > 0$ such that $\|q(u) - q(v)\| \le \mathcal{K}_q \|u - v\|_Q$ for all $u, v \in Q$.

Theorem 3.4 *Let F, G, H, h, and q satisfy hypotheses* (M0)–(M1) *and* (M4). *If*

$$
\Lambda_1 = \left[\mathcal{K}_q + \mathcal{K}_H + \frac{1}{\rho}\big[(\mathcal{K}_F + \mathcal{K}_G)(1 + \mathcal{K}_h)\big]\right] < 1, \tag{3.10}
$$

then the nonlocal integro-differential system (3.9) *has a unique solution on* $[0,\zeta]$.

Proof We first transform system (3.9) into a fixed point problem. Define $\psi : Q \to Q$ by

$$
\begin{aligned}
(\psi w)(\delta) = {}& e^{\frac{-\rho}{\vartheta}\delta}\big[w_0 - q(w) - H(0, w_0)\big] + H\big(\delta, w(\delta)\big) \\
&+ \frac{1}{\vartheta} e^{\frac{-\rho}{\vartheta}\delta} \int_0^{\delta} e^{\frac{\rho}{\vartheta}\phi} F\big(\phi, w(\phi), Ew(\phi)\big)\, d\phi \\
&+ \frac{1}{\vartheta} e^{\frac{-\rho}{\vartheta}\delta} \int_0^{\delta} e^{\frac{\rho}{\vartheta}\phi} G\big(\phi, w(\phi), Ew(\phi)\big)\, d\phi, \quad \delta \in [0,\zeta].
\end{aligned} \tag{3.11}
$$

We now show that $\psi B_O \subset B_O$, where B_O is the ball $B(0,O) = \{w \in \mathcal{C}([0,\zeta], \mathbb{X}) = Q : \|w\|_Q \le O\}$ with radius $O > \frac{\|\Omega_1\|}{1-\widetilde{\mu}}$, so that

$$
\begin{aligned}
\|\Omega_1\| &= \big\|\Omega^*\big\| + \left(\overline{\mathcal{K}}_H + \frac{1}{\rho}\big(\overline{\mathcal{K}}_h(\mathcal{K}_F + \mathcal{K}_G) + \overline{\mathcal{K}}_F + \overline{\mathcal{K}}_G\big)\right), \\
\big\|\Omega^*\big\| &= \big[\|w_0\| + M + \big\|H(0, w_0)\big\|\big],
\end{aligned}
$$

and $\widetilde{\mu} = \mathcal{K}_H + \frac{1}{\rho}[(\mathcal{K}_F + \mathcal{K}_G)(1 + \mathcal{K}_h)]$.

Indeed, let $w \in B_O$. We have

$$
\begin{aligned}
&\big\|(\psi w)(\delta)\big\| \\
&\quad = \left\| e^{\frac{-\rho}{\vartheta}\delta}\big[w_0 - q(w) - H(0, w_0)\big] + H\big(\delta, w(\delta)\big) + \frac{1}{\vartheta} e^{\frac{-\rho}{\vartheta}\delta} \int_0^{\delta} e^{\frac{\rho}{\vartheta}\phi} F\big(\phi, w(\phi), Ew(\phi)\big)\, d\phi \right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}s}G\big(\phi,w(\phi),Ew(\phi)\big)\,d\phi\bigg\| \\
&\le \big\|\Omega^*\big\| + \big\|H\big(\delta,w(\delta)\big)-H(\delta,0)\big\| + \big\|H(\delta,0)\big\| \\
&+\frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}\left(\left\|F\left(\phi,w(\phi),\int_0^\phi h\big(\phi,\tau,w(\tau)\big)\,d\tau\right)-F(\phi,0,0)\right\|\right. \\
&\left.+\big\|F(\phi,0,0)\big\|\right)d\phi \\
&+\frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}\left(\left\|G\left(\phi,w(\phi),\int_0^\phi h\big(\phi,\tau,w(\tau)\big)\,d\tau\right)-G(\phi,0,0)\right\|\right. \\
&\left.+\big\|G(\phi,0,0)\big\|\right)d\phi \\
&\le \big\|\Omega^*\big\| + \mathcal{K}_H O + \overline{\mathcal{K}}_H + \frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}\left[\mathcal{K}_F\left\{\|w\| + \left\|\int_0^\phi h\big(\phi,\tau,w(\tau)\big)\,d\tau\right\|\right\}\right. \\
&\left.+\overline{\mathcal{K}}_F\right]d\phi \\
&+\frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}\left[\mathcal{K}_G\left\{\|w\| + \left\|\int_0^\phi h\big(\phi,\tau,w(\tau)\big)\,d\tau\right\|\right\}+\overline{\mathcal{K}}_G\right]d\phi \\
&\le \big\|\Omega^*\big\| + \mathcal{K}_H O + \overline{\mathcal{K}}_H + \big[\mathcal{K}_F\big[(1+\mathcal{K}_h)O+\overline{\mathcal{K}}_h\big]+\overline{\mathcal{K}}_F\big]\frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}\,d\phi \\
&+\big[\mathcal{K}_G\big[(1+\mathcal{K}_h)O+\overline{\mathcal{K}}_h\big]+\overline{\mathcal{K}}_G\big]\frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}\,d\phi \\
&\le \big\|\Omega^*\big\| + \mathcal{K}_H O + \overline{\mathcal{K}}_H + \big[\mathcal{K}_F\big[(1+\mathcal{K}_h)O+\overline{\mathcal{K}}_h\big]+\overline{\mathcal{K}}_F\big]\frac{1}{\rho}\big(1-e^{\frac{-\rho}{\vartheta}\delta}\big) \\
&+\big[\mathcal{K}_G\big[(1+\mathcal{K}_h)O+\overline{\mathcal{K}}_h\big]+\overline{\mathcal{K}}_G\big]\frac{1}{\rho}\big(1-e^{\frac{-\rho}{\vartheta}\delta}\big) \\
&= \big\|\Omega^*\big\| + \mathcal{K}_H O + \overline{\mathcal{K}}_H + \frac{1}{\rho}\big[(\mathcal{K}_F+\mathcal{K}_G)(1+\mathcal{K}_h)O+\overline{\mathcal{K}}_h(\mathcal{K}_F+\mathcal{K}_G) \\
&+\overline{\mathcal{K}}_F+\overline{\mathcal{K}}_G\big]\big(1-e^{\frac{-\rho}{\vartheta}\delta}\big) \\
&\le \|\Omega_1\| + \left[\mathcal{K}_H + \frac{1}{\rho}(\mathcal{K}_F+\mathcal{K}_G)(1+\mathcal{K}_h)\right]O.
\end{aligned}
$$

For all $\delta \in [0,\zeta]$ and $w \in B_O$, we have

$$
\big\|\psi(w)\big\|_Q \le \|\Omega_1\| + \left[\mathcal{K}_H + \frac{1}{\rho}(\mathcal{K}_F+\mathcal{K}_G)(1+\mathcal{K}_h)\right]O < O.
$$

This proves that the ball B_O is mapped into itself under the operation ψ, that is, $\psi B_O \subset B_O$.

Further, for all $w,\overline{w} \in B_O$, we have the estimate

$$
\begin{aligned}
&\big\|(\psi w)(\delta)-(\psi\overline{w})(\delta)\big\| \\
&\le \big\|q(w)-q(\overline{w})\big\| + \big\|H\big(\delta,w(\delta)\big)-H\big(\delta,\overline{w}(\delta)\big)\big\| \\
&+\frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^\delta e^{\frac{\rho}{\vartheta}\phi}\big\|F\big(\phi,w(\phi),Ew(\phi)\big)-F\big(\phi,\overline{w}(\phi),E\overline{w}(\phi)\big)\big\|\,d\phi
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\vartheta}e^{\frac{-\rho}{\vartheta}\delta}\int_0^{\delta} e^{\frac{\rho}{\vartheta}\phi}\big\|G\big(\phi, w(\phi), Ew(\phi)\big) - G\big(\phi, \overline{w}(\phi), E\overline{w}(\phi)\big)\big\|\,d\phi \\
&\leq \mathcal{K}_q\|w-\overline{w}\|_Q + \mathcal{K}_H\|w-\overline{w}\|_Q + \frac{1}{\rho}\big[(\mathcal{K}_F+\mathcal{K}_G)(1+\mathcal{K}_h)\big]\|w-\overline{w}\|_Q\big(1-e^{\frac{-\rho}{\vartheta}\delta}\big) \\
&\leq \left[\mathcal{K}_q + \mathcal{K}_H + \frac{1}{\rho}\big[(\mathcal{K}_F+\mathcal{K}_G)(1+\mathcal{K}_h)\big]\right]\|w-\overline{w}\|_Q.
\end{aligned}
$$

Thus, for each $\delta \in [0,\zeta]$, we get

$$
\big\|\psi(w) - \psi(\overline{w})\big\|_Q \leq \Lambda_1\|w-\overline{w}\|_Q,
$$

where $\Lambda_1 = \Lambda_1(\mathcal{K}_F, \mathcal{K}_G, \mathcal{K}_H, \mathcal{K}_h, \mathcal{K}_q, \rho)$, defined in (3.10), depends on the parameters of the system. According to (3.10), $\Lambda_1 < 1$, so ψ is a contraction. As a result, the given nonlocal integro-differential system (3.9) has a unique solution on $[0,\zeta]$ by [6, Lemma 2.2] of the Banach contraction principle. This completes the proof. □

4 Applications

Now we provide some examples to validate our results.

Example 4.1 Consider the nonlinear fractional deformable perturbed neutral integro-differential system

$$
\begin{aligned}
&{}^{\mathcal{DD}}D^{\frac{1}{2}}\left[w(\delta) - \frac{e^{-\delta}}{49+e^{\delta}}\cdot\frac{w(\delta)}{1+w(\delta)}\right] \\
&\quad= \frac{1}{(\delta+4)^2}\frac{|w(\delta)|}{1+|w(\delta)|} + \frac{1}{16}\int_0^{\delta}\frac{e^{-\phi}}{9}\frac{|w(\delta)|}{1+|w(\delta)|}\,d\phi \\
&\qquad+ \frac{1}{(\delta+5)^2}\frac{|w(\delta)|}{1+|w(\delta)|} + \frac{1}{25}\int_0^{\delta}\frac{e^{-\phi}}{9}\frac{|w(\delta)|}{1+|w(\delta)|}\,d\phi, \delta \in I, \\
&w(0) = 1 = w_0,
\end{aligned}
\tag{4.1}
$$

where $I = [0,1]$. Set

$$
\begin{aligned}
&F(\delta, w, Ew) = \frac{1}{(\delta+4)^2}\frac{|w(\delta)|}{1+|w(\delta)|} + \frac{1}{16}Ew(\delta), \quad (\delta, w) \in [0,1]\times[0,\infty); \\
&G(\delta, w, Ew) = \frac{1}{(\delta+5)^2}\frac{|w(\delta)|}{1+|w(\delta)|} + \frac{1}{25}Ew(\delta), \quad (\delta, w) \in [0,1]\times[0,\infty); \\
&H(\delta, w) = \frac{e^{-\delta}w}{(49+e^{\delta})(1+w)}, \quad (\delta, w) \in [0,1]\times[0,\infty),
\end{aligned}
$$

where $Ew(\delta) = \int_0^{\delta}\frac{e^{-\phi}}{9}\frac{|w(\delta)|}{1+|w(\delta)|}\,d\phi$.

For arbitrary $y, \overline{y} \in [0,\infty)$ and $\delta \in [0,1]$, we have

$$
\begin{aligned}
\big\|h(\delta,\phi,y) - h(\delta,\phi,\overline{y})\big\| &= \left\|\frac{e^{-\phi}}{9}\frac{y}{1+y} - \frac{e^{-\phi}}{9}\frac{\overline{y}}{1+\overline{y}}\right\| \\
&\leq \frac{1}{9}\|y-\overline{y}\|,
\end{aligned}
$$

$$\begin{aligned}
&\bigl\|F(\delta,y,Hy)-F(\delta,\overline{y},H\overline{y})\bigr\| \\
&\quad\le \frac{1}{(\delta+4)^2}\frac{\|y-\overline{y}\|}{(1+\|y\|)(1+\|\overline{y}\|)}+\frac{1}{16}\bigl\|Hy(\delta)-H\overline{y}(\delta)\bigr\| \\
&\quad\le \frac{1}{(\delta+4)^2}\|y-\overline{y}\|+\frac{1}{16}\|Hy-H\overline{y}\| \\
&\quad\le \frac{1}{16}\bigl[\|y-\overline{y}\|+\|Hy-H\overline{y}\|\bigr], \\
&\bigl\|G(\delta,y,Hy)-G(\delta,\overline{y},H\overline{y})\bigr\| \\
&\quad\le \frac{1}{(\delta+5)^2}\frac{\|y-\overline{y}\|}{(1+\|y\|)(1+\|\overline{y}\|)}+\frac{1}{25}\bigl\|Hy(\delta)-H\overline{y}(\delta)\bigr\| \\
&\quad\le \frac{1}{(\delta+5)^2}\|y-\overline{y}\|+\frac{1}{25}\|Hy-H\overline{y}\| \\
&\quad\le \frac{1}{25}\bigl[\|y-\overline{y}\|+\|Hy-H\overline{y}\|\bigr],
\end{aligned}$$

and

$$\begin{aligned}
\bigl\|H(\delta,y)-H(\delta,\overline{y})\bigr\| &\le \frac{e^{-\delta}}{(49+e^{\delta})}\left\|\frac{y}{1+y}-\frac{\overline{y}}{1+\overline{y}}\right\| \\
&\le \frac{1}{50}\|y-\overline{y}\|.
\end{aligned}$$

Assumptions (M0)–(M1) hold with $\mathcal{K}_h=\frac{1}{9}$, $\mathcal{K}_F=\frac{1}{16}, \mathcal{K}_G=\frac{1}{25}$, and $\mathcal{K}_H=\frac{1}{50}$. Since $\vartheta=\frac{1}{2}$ and $\rho+\vartheta=1$, we get $\rho=\frac{1}{2}$. Then

$$\begin{aligned}
\Lambda &= \left[\mathcal{K}_H+\frac{1}{\rho}\bigl[(\mathcal{K}_F+\mathcal{K}_G)(1+\mathcal{K}_h)\bigr]\right] \\
&= \frac{1}{50}+\frac{1}{0.5}\left[\left(\frac{1}{16}+\frac{1}{25}\right)(1.111)\right] \\
&= 0.2478<1.
\end{aligned}$$

As a result, condition (3.1) is to be held with $\Lambda=0.2478<1$. Consequently, by Theorem 3.1 the given nonlinear fractional deformable perturbed neutral integro-differential system (4.1) has a unique solution on $[0,1]$.

Example 4.2 Consider the nonlinear fractional deformable perturbed neutral integro-differential system given by (4.1). In view of Example 4.1, we have

$$\bigl\|F(\delta,y,Hy)\bigr\|\le\frac{5}{72};\qquad \bigl\|G(\delta,y,Hy)\bigr\|\le\frac{2}{45};\qquad \bigl\|H(\delta,y)\bigr\|\le\frac{1}{50}.$$

Assumption (M3) holds with $\widetilde{\mathcal{K}}_F(\delta)=\frac{5}{72}, \widetilde{\mathcal{K}}_G(\delta)=\frac{2}{45}$, and $\widetilde{\mathcal{K}}_H(\delta)=\frac{1}{50}$. Moreover,

$$\begin{aligned}
\mathcal{K}_H+\frac{1}{\rho}\mathcal{K}_G(1+\mathcal{K}_h) &= \frac{1}{50}+2\left(\frac{1}{25}\right)\left(1+\frac{1}{9}\right) \\
&= 0.109<1.
\end{aligned}$$

Then all the conditions of Theorem 3.3 are also satisfied. Therefore at least one solution exists on $[0, 1]$ for the given fractional neutral integro-differential system (4.1).

5 Conclusions

In this paper, Theorems 3.1, 3.2, and 3.3 were proved to conclude that we can find unique solutions for the nonlinear fractional deformable perturbed neutral integro-differential system (1.1). In this direction, Krasnoselskii's fixed point theorem along with the Banach contraction principle were used. Furthermore, we have proved Theorem 3.4 utilizing the Banach contraction principle by using two independent cases in conditions for proving the existence of unique solutions of the given fractional neutral integro-differential system (3.9). Two examples showed the applicability of the main results. In the near future, deformable fractional derivative may be applied in computer artificial intelligence algorithms by using a suitable fixed point iteration method.

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Author contributions
R.S. and S.R.B. and R.U. and SE dealt with the conceptualization, supervision, methodology, investigation, and writing-original draft preparation. R.S. and S.R.B. and R.U. and SE and I.A. and S.R. made the formal analysis, writing-review, editing. All authors read and approved the final manuscript.

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Declarations

Competing interests
The authors declare no competing interests.

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