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Sign-changing solutions for coupled Schrödinger system

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Abstract

In this paper we study the following nonlinear Schrödinger system:

$$\begin{cases} -\Delta u + \alpha u = |u|^{p-1}u + \frac{2}{q+1}\lambda |u|^{\frac{p-3}{2}} u|v|^{\frac{q+1}{2}}, & x \in \mathbb{R}^3, \\ -\Delta v + \beta v = |v|^{q-1}v + \frac{2}{p+1}\lambda |u|^{\frac{p+1}{2}} |v|^{\frac{q-3}{2}} v, & x \in \mathbb{R}^3, \\ u(x) \to 0, \quad v(x) \to 0, & \text{as } |x| \to \infty, \end{cases}$$

where $3 \le p, q < 5$, α, β are positive parameters. We show that there exists $\lambda_k > 0$ such that the equation has at least k radially symmetric sign-changing solutions and at least k seminodal solutions for each $k \in \mathbb{N}$ and $\lambda \in (0, \lambda_k)$. Moreover, we show the existence of a least energy radially symmetric sign-changing solution for each $\lambda \in (0, \lambda_0)$ where $\lambda_0 \in (0, \lambda_1]$.

1 Background and main results

Consider the following nonlinear coupled Schrödinger system:

$$\begin{cases} -\Delta u + \alpha u = |u|^{p-1}u + \frac{2}{q+1}\lambda |u|^{\frac{p-3}{2}} u|v|^{\frac{q+1}{2}}, & x \in \Omega, \\ -\Delta v + \beta v = |v|^{q-1}v + \frac{2}{p+1}\lambda |u|^{\frac{p+1}{2}} |v|^{\frac{q-3}{2}} v, & x \in \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $\Omega = \mathbb{R}^N$ or Ω is a smooth bounded domain in $\mathbb{R}^N$, α, β are positive parameters and $\lambda \neq 0$ is a coupling constant.

In the case $p = q = 3$, system (1.1) becomes the cubic system:

$$\begin{cases} -\Delta u + \alpha u = u^3 + \lambda u v^2, & x \in \Omega, \\ -\Delta v + \beta v = v^3 + \lambda u^2 v, & x \in \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

which arises in the study of many physical phenomena like nonlinear optics and Bose–Einstein condensation (cf. [15, 17]). Therefore, in the last decades, system (1.2) has received great interest from mathematicians. When Ω is the entire space $\mathbb{R}^N$, the existence of least energy and other finite energy solutions of (1.2) was studied in [2, 11, 12, 18, 21, 22, 27] and the references therein. In particular, when $\lambda > 0$ is sufficiently large, infinitely many radially symmetric sign-changing solutions of (1.2) were obtained in [23]. Liu and Wang [20] studied a general m-coupled system ($m \geq 2$) and proved that system (1.2) has infinitely many nontrivial solutions, but whether solutions obtained in [20] are positive or sign-changing cannot be determined there (see also [21]). When $\Omega \subset \mathbb{R}^N$ ($N = 2, 3$) is a smooth bounded domain, there are also many papers studying (1.2). Lin and Wei [18] proved that a least energy solution of (1.2) exists within an appropriate range of λ. Dancer, Wei, and Weth [14] and Noris and Ramos [24] proved the existence of infinitely many positive solutions of (1.2). When Ω is a ball, a multiplicity result on positive radially symmetric solutions was given in [29]. Later, by using a global bifurcation approach, the result of [29] was reproved by [4] without requiring the symmetric condition. Under some more general assumptions, Sato and Wang [26] proved that system (1.2) has infinitely many semipositive solutions (i.e., at least one component is positive). In [14], the authors proved the existence of unbounded sequence solutions for $N \leq 3$ and $\lambda \leq -1$. As pointed out above, for $\lambda \leq -1$, Wei and Weth [29] proved that (1.2) has a radially symmetric solution, which turns out to be a positive solution.

We remark that the existence of infinitely many sign-changing solutions or seminodal solutions to (1.2) was solved by Chen, Lin, and Zou [10] and Liu, Liu, and Wang [19] independently, where $N \leq 3$ and $\lambda < 0$.

To the best of our knowledge, the existence of sign-changing solutions to (1.1) has not ever been studied in the literature when $\Omega = \mathbb{R}^3$ and $3 \leq p, q < 5$. The main goal of this paper is to study the existence of sign-changing solutions, seminodal solutions, and least energy sign-changing solutions to problem (1.1) when $\lambda > 0$ is small. This will complement the study made in [14, 19, 21, 22, 29].

Definition 1.1 A solution (u, v) is called nontrivial if $u \not\equiv 0$ and $v \not\equiv 0$, a solution (u, v) is semitrivial if (u, v) is type of $(u, 0)$ or $(0, v)$. We call a solution (u, v) positive if $u > 0$ and $v > 0$ in $\mathbb{R}^N$, a solution (u, v) sign-changing if both u and v change sign, a solution (u, v) seminodal if one changes sign and the other one is positive.

The first main result of the current paper is as follows.

Theorem 1.1 *Assume $\alpha, \beta > 0$. Then for any $k \in \mathbb{N}$ there exists $\lambda_k > 0$ such that system* (1.1) *possesses at least k radially symmetric sign-changing solutions for each fixed $\lambda \in (0, \lambda_k)$.*

We can also study some further properties of the sign-changing solutions obtained in Theorem 1.1. It is well known that a nontrivial solution $(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ is called a least energy solution if its energy is minimal among the energy of all nontrivial solutions. A sign-changing solution is called a least energy sign-changing solution if it has the least energy among all sign-changing solutions. Precisely, we have the following theorem.

Theorem 1.2 *Assume $\alpha, \beta > 0$. Then there exists $\lambda_0 \in (0, \lambda_1]$ such that system* (1.1) *possesses a least energy radially symmetric sign-changing solution for each fixed $\lambda \in (0, \lambda_0)$.*

Theorem 1.3 *Assume $\alpha, \beta > 0$. Then for any $k \in \mathbb{N}$ there exists $\lambda_k > 0$ such that system* (1.1) *possesses at least k seminodal solutions for each fixed $\lambda \in (0, \lambda_k)$.*

Remark 1.1 We can prove that system (1.1) possesses at least k seminodal solutions with the first component positive and the second component radially symmetric sign-changing or the first component radially symmetric sign-changing and the second component positive.

The structure of this paper is as follows. In Sect. 2 we prove the existence of at least k radially symmetric sign-changing solutions. The main tool will be the use of a new notion of vector genus by [28] and a new constrained problem by [10], which will be used to construct minimax values. Remark that the ideas in [10, 28] cannot be used directly, and here we will give some new ideas. The crucial idea in this paper is turning to study a new problem with two constraints to obtain sign-changing solutions of (1.1). This idea has never been used for (1.1) in the literature up to our knowledge. We will give all the necessary details of the proof. Section 3 is then dedicated to the proof of Theorem 1.2 by using a minimizing argument. Finally in Sect. 4 we will present the proof of Theorem 1.3 applying the arguments in Sect. 2 and Sect. 3.

We give some notations here. Throughout this paper, we denote the norm of $L^p(\mathbb{R}^N)$ by $|u|_p = (\int_{\mathbb{R}^N} |u|^p\,dx)^{\frac{1}{p}}$, the norm of $H^1(\mathbb{R}^N)$ by $\|u\|^2 = \int_{\mathbb{R}^N}(|\nabla u|^2 + |u|^2)\,dx$, and positive constants (possibly different in different places) by C. Define $H_r := H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$ as a subspace of $H := H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ with norm $\|(u,v)\|_{H_r}^2 := \|u\|_\alpha^2 + \|v\|_\beta^2$ where

$$H_r^1(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) : u \text{ is radially symmetric}\},$$
$$\|u\|_\alpha^2 := \int_{\mathbb{R}^N} (|\nabla u|^2 + \alpha |u|^2)\,dx.$$

2 Proof of Theorem 1.1

In this section, we assume that $N = 3$, $3 \le p, q < 2^* - 1 = 5$ and $\alpha, \beta > 0$. Without loss of generality, we assume $p \le q$. Let $\lambda \in (0,1)$. For any $k \in \mathbb{N}$, let $X_{k+1} \subset H_r^1(\mathbb{R}^3)$, $\dim X_{k+1} = k+1$, and there exists $u_0 \in X_{k+1}$ and $u_0 > 0$. Then there exists $m > 0$ such that for any $(u,v) \in X_{k+1} \times X_{k+1}$ satisfying $|u|_{p+1}^{p+1}, |v|_{q+1}^{q+1} < 2$, we have

$$\|u\|_\alpha^2 < m, \qquad \|v\|_\beta^2 < m. \tag{2.1}$$

Without loss of generality, we can assume $m > 1$. Obviously, the sign-changing solutions of system (1.1) are the critical points of the C^2 functional $\Phi_\lambda : H_r \to \mathbb{R}$ given by

$$\begin{aligned}\Phi_\lambda(u,v) := {} & \frac{1}{2}(\|u\|_\alpha^2 + \|v\|_\beta^2) - \frac{1}{p+1}|u|_{p+1}^{p+1} - \frac{1}{q+1}|v|_{q+1}^{q+1} \\ & - \frac{4\lambda}{(p+1)(q+1)} \int_{\mathbb{R}^3} |u|^{\frac{p+1}{2}} |v|^{\frac{q+1}{2}}\,dx.\end{aligned} \tag{2.2}$$

We will look for solutions of Eq. (1.1) as critical points of the functional Φ_λ restricted to the sphere

$$\mathcal{A} := \{(u,v) \in H_r : |u|_{p+1} = 1, |v|_{q+1} = 1\}.$$

To obtain at least k sign-changing critical points, we need to define several minimax energy levels using a new definition of vector genus introduced by [28]. As in [28], we recall vector genus and take the transformations

$$\sigma_i : \mathcal{A} \to \mathcal{A}, \qquad \sigma_1(u,v) = (-u,v), \qquad \sigma_2(u,v) = (u,-v), \quad i = 1,2.$$

Consider the class of sets

$$\mathcal{F} = \{A \subset \mathcal{A} : A \text{ is a closed set and } \sigma_i(u,v) \in A, \forall (u,v) \in A, i = 1,2\}$$

and for each $A \in \mathcal{F}$ and $k_1, k_2 \in \mathbb{N}$, the class of functions

$$F_{(k_1,k_2)}(A) = \Big\{ f = (f_1,f_2) : A \to \prod_{i=1}^{2} \mathbb{R}^{k_i - 1} : f_i : A \to \mathbb{R}^{k_i-1} \text{ continuous}, \\ f_i(\sigma_i(u,v)) = -f_i(u,v) \text{ for each } i, f_i(\sigma_j(u,v)) = f_i(u,v) \text{ for } i \neq j \Big\}.$$

where $\mathbb{R}^0 := \{0\}$.

Definition 2.1 (Vector genus, see [28]) For every nonempty and closed set $A \subset H_0^1(\Omega)$ such that $-A = A$, we define

$$\gamma(A) := \inf\{k : \text{there exists } h : A \to \mathbb{R}^k \setminus \{0\} \text{ continuous and odd}\}$$

and $\gamma(A) := \infty$ if no such k exists.

Let $A \in \mathcal{F}$ and take any $k_1, k_2 \in \mathbb{N}$. We say that $\gamma(A) \geq (k_1,k_2)$ if for every $f \in F_{(k_1,k_2)}(A)$ there exists $(u,v) \in A$ such that $f(u,v) = (f_1(u,v), f_2(u,v)) = (0,0)$. We denote

$$\Gamma^{(k_1,k_2)} := \{A \in \mathcal{F} : \gamma(A) \geq (k_1,k_2)\}.$$

Remark 2.1 Note that Definition 2.1 does not actually define the quantity $\gamma(A)$ but gives the meaning of $\gamma(A) \geq (k_1,k_2)$ only. A different notation of genus was introduced by Chang, Wang, and Zhang in [8].

Lemma 2.1 (see [28]) *Let $f = (f_1,f_2) : \prod_{i=1}^{2} S^{k_i} \to \prod_{i=1}^{2} \mathbb{R}^{k_i}$ be a continuous function such that $f_i(\sigma_i(u,v)) = -f_i(u,v)$, $f_i(\sigma_j(u,v)) = f_i(u,v)$ for any $i,j = 1,2$, $i \neq j$, then there exists $(u_0,v_0) \in \prod_{i=1}^{2} S^{k_i}$ such that $f(u_0,v_0) = (0,\dots,0)$.*

Lemma 2.2 (see [28]) *The following properties hold.*

(1) *Take $A_1 \times A_2 \subset \mathcal{A}$ and let $\eta_i : S^{k_i-1} \to A_i$ be a homeomorphism such that $\eta_i(-x) = -\eta_i(x)$ for every $x \in S^{k_i-1}$, $i = 1,2$. Then $A_1 \times A_2 \in \Gamma^{(k_1,k_2)}$, where $S^{k_i-1} = \{x \in \mathbb{R}^{k_i} : |x| = 1\}$.*

(2) *We have $\overline{\eta(A)} \in \Gamma^{(k_1,k_2)}$ whenever $A \in \Gamma^{(k_1,k_2)}$ and a continuous map $\eta : A \to \mathcal{A}$ is such that $\eta \circ \sigma_i = \sigma_i \circ \eta$, $\forall i = 1,2$.*

Together with the notation of vector genus, to obtain sign-changing solutions, we will use cones of positive or negative functions based on the works such as [5, 13, 30]. We define the cone

$$\mathcal{P}_1 := \{(u,v) \in H_r : u \geq 0\}, \qquad \mathcal{P}_2 := \{(u,v) \in H_r : v \geq 0\},$$

and take $\mathcal{P} := \bigcup_{i=1}^{2}(\mathcal{P}_i \cup -\mathcal{P}_i)$. Moreover, for any $\delta > 0$, we define

$$\mathcal{P}_\delta := \big\{(u,v) \in H_r : \operatorname{dist}\big((u,v),\mathcal{P}\big) < \delta\big\},$$

where

$$\begin{aligned}\operatorname{dist}\big((u,v),\mathcal{P}\big) &:= \min\big\{\operatorname{dist}_{p+1}(u,\mathcal{P}_1), \operatorname{dist}_{p+1}(u,-\mathcal{P}_1),\\ &\qquad \operatorname{dist}_{q+1}(v,\mathcal{P}_2), \operatorname{dist}_{q+1}(v,-\mathcal{P}_2)\big\},\\ \operatorname{dist}_{p+1}(u,\pm\mathcal{P}_1) &:= \inf_{\omega\in\pm\mathcal{P}_1} |u-\omega|_{p+1} = \big|u^{\mp}\big|_{p+1},\\ \operatorname{dist}_{q+1}(v,\pm\mathcal{P}_2) &:= \inf_{\omega\in\pm\mathcal{P}_2} |v-\omega|_{q+1} = \big|v^{\mp}\big|_{q+1},\end{aligned}$$

where $u^{\pm} := \max\{0, \pm u\}$.

Lemma 2.3 *For any* $0 < \delta < 2^{-\frac{1}{p+1}}$, *there holds* $A \backslash \mathcal{P}_\delta \neq \emptyset$ *whenever* $A \in \Gamma^{(k_1,k_2)}$ *with* $k_1, k_2 \geq 2$.

Proof For any $A \in \Gamma^{(k_1,k_2)}$, define $f = (f_1, f_2)$ by

$$\begin{aligned}f_1(u,v) &= \left(\int_{\mathbb{R}^3} |u|^p u\,dx, 0, \ldots, 0\right),\\ f_2(u,v) &= \left(\int_{\mathbb{R}^3} |v|^q v\,dx, 0, \ldots, 0\right),\end{aligned}$$

then $f \in F_{(k_1,k_2)}(A)$, so by Definition 2.1, there exists $(u_0, v_0) \in A$ such that $f(u_0,v_0) = (0,\ldots,0)$. By $A \in \mathcal{A}$, we deduce that

$$\begin{aligned}\int_{\mathbb{R}^3} \big(u_0^+\big)^{p+1}\,dx &= \int_{\mathbb{R}^3} \big(u_0^-\big)^{p+1}\,dx = \frac{1}{2},\\ \int_{\mathbb{R}^3} \big(v_0^+\big)^{q+1}\,dx &= \int_{\mathbb{R}^3} \big(v_0^-\big)^{q+1}\,dx = \frac{1}{2},\end{aligned}$$

therefore, $\operatorname{dist}((u_0,v_0),\mathcal{P}) = 2^{-\frac{1}{p+1}}$, and so $(u_0,v_0) \in A\backslash\mathcal{P}_\delta$ for any $0 < \delta < 2^{-\frac{1}{p+1}}$. □

For technical reasons, we will work on the neighborhood of $\mathcal{A}$ in $H_r^1(\mathbb{R}^3)$,

$$\mathcal{A}^* := \left\{(u,v) \in H_r : \frac{1}{2} < |u|_{p+1}^{p+1} < 2, \frac{1}{2} < |v|_{q+1}^{q+1} < 2\right\}, \tag{2.3}$$

when $u \in \mathcal{A}^*$, $(u,v) \not\equiv (0,0)$. Define

$$\mathcal{B}_m^* := \big\{(u,v) \in \mathcal{A}^* : \|u\|_\alpha^2 < m, \|v\|_\beta^2 < m\big\}, \tag{2.4}$$

$$\mathcal{B}_m := \big\{(u,v) \in \mathcal{A} : \|u\|_\alpha^2 < m, \|v\|_\beta^2 < m\big\}, \tag{2.5}$$

$$\mathcal{C}_m := \big\{(u,v) \in \mathcal{A} : \|u\|_\alpha^2 = m, \|v\|_\beta^2 = m\big\}. \tag{2.6}$$

Let S_p and S_q be the sharp constants of the Sobolev embedding $H_r^1(\mathbb{R}^3) \hookrightarrow L^{p+1}(\mathbb{R}^3)$ and $H_r^1(\mathbb{R}^3) \hookrightarrow L^{q+1}(\mathbb{R}^3)$, respectively,

$$\|u\|_\alpha^2 \geq S_p|u|_{p+1}^2, \qquad \|v\|_\beta^2 \geq S_q|v|_{q+1}^2, \quad \forall u,v \in H_r^1(\mathbb{R}^3). \tag{2.7}$$

For any $(u,v) \in H_r \backslash \{(0,0)\}$, we have

$$\sup_{t,s\geq 0} \Phi_\lambda(tu,sv) = \Phi_\lambda(t_{u,v,\lambda}u, s_{u,v,\lambda}v) =: \Psi_\lambda(u,v), \tag{2.8}$$

where $t_{u,v,\lambda}, s_{u,v,\lambda} \geq 0$ satisfy

$$\frac{\partial}{\partial t}\Phi_\lambda(tu,sv)|_{(t_{u,v,\lambda},s_{u,v,\lambda})} = \frac{\partial}{\partial s}\Phi_\lambda(tu,sv)|_{(t_{u,v,\lambda},s_{u,v,\lambda})} = 0.$$

Note that for $t,s \geq 0$,

$$\begin{aligned}\Phi_\lambda(tu,sv) :=& \frac{1}{2}\left(t^2\|u\|_\alpha^2 + s^2\|v\|_\beta^2\right) - \frac{t^{p+1}}{p+1}|u|_{p+1}^{p+1} - \frac{s^{q+1}}{q+1}|v|_{q+1}^{q+1} \\ &- \frac{4\lambda}{(p+1)(q+1)}t^{\frac{p+1}{2}}s^{\frac{q+1}{2}}\int_{\mathbb{R}^3}|u|^{\frac{p+1}{2}}|v|^{\frac{q+1}{2}}\,dx.\end{aligned} \tag{2.9}$$

Define

$$\begin{aligned}F(u,v,\lambda;t,s) &:= t\|u\|_\alpha^2 - t^p|u|_{p+1}^{p+1} - \frac{2}{q+1}t^{\frac{p-1}{2}}s^{\frac{q+1}{2}}\lambda\int_{\mathbb{R}^3}|u|^{\frac{p+1}{2}}|v|^{\frac{q+1}{2}}\,dx \\ &:= tF_1(u,v,\lambda;t,s)\end{aligned}$$

and

$$\begin{aligned}G(u,v,\lambda;t,s) &:= s\|v\|_\beta^2 - s^q|v|_{q+1}^{q+1} - \frac{2}{p+1}t^{\frac{p+1}{2}}s^{\frac{q-1}{2}}\lambda\int_{\mathbb{R}^3}|u|^{\frac{p+1}{2}}|v|^{\frac{q+1}{2}}\,dx \\ &:= sG_1(u,v,\lambda;t,s),\end{aligned}$$

which implies

$$F_1(u,v,\lambda;t_{u,v,\lambda},s_{u,v,\lambda}) = G_1(u,v,\lambda;t_{u,v,\lambda},s_{u,v,\lambda}) = 0. \tag{2.10}$$

Since $F_1(u,v,\lambda;t,s)$ and $G_1(u,v,\lambda;t,s)$ are decreasing with respect to $t > 0$ and $s > 0$, respectively, $F_1(u,v,\lambda;0,0) > 0$, $G_1(u,v,\lambda;0,0) > 0$, so $t_{u,v,\lambda}$, $s_{u,v,\lambda}$ are unique. Note that for $t,s \geq 0$, $3 \leq p,q < 5$, by (2.9), we can choose some positive constant T such that $\Phi_\lambda(tu,sv) < 0$ for any $t,s > T$, therefore, $t_{u,v,\lambda}, s_{u,v,\lambda} \in [0,T]$.

Define

$$\widetilde{m} > \left[(q+1)S_p\left(\frac{1}{2}\right)^{\frac{2}{p+1}}\right]^{\frac{2}{p+q-2}} + \frac{4(p+1)(q+1)}{(p-1)(\frac{S_p}{8})^{\frac{2}{p-1}}}m^{\frac{p+1}{p-1}} + m. \tag{2.11}$$

Then $B_m \subset B_{\widetilde{m}}$, $B_m^* \subset B_{\widetilde{m}}^*$.

Lemma 2.4 *For any $k \in \mathbb{N}$, there exist $\widetilde{\lambda} \in (0,1)$ and $T_1 > T_2 > 0$ such that for any $\lambda \in (0,\widetilde{\lambda})$ and $(u,v) \in B^*_{\widetilde{m}}$, we have*

$$T_2 \le t_{u,v,\lambda}, s_{u,v,\lambda} \le T_1. \tag{2.12}$$

Furthermore, there exist $\lambda_k \in (0,\widetilde{\lambda}]$ and $c_k > 0$ such that for any $\lambda \in (0,\lambda_k)$, we have

$$\sup_{(u,v)\in\mathcal{B}_m} \sup_{t,s\ge 0} \Phi_\lambda(tu,sv) < c_k \le \inf_{(u,v)\in\mathcal{C}_{\widetilde{m}}} \sup_{t,s\ge 0} \Phi_\lambda(tu,sv). \tag{2.13}$$

Proof We see from (2.9) and (2.10) that

$$\begin{aligned}\sup_{t,s\ge 0} \Phi_\lambda(tu,sv) &= \Phi_\lambda(t_{u,v,\lambda}u, s_{u,v,\lambda}v)\\ &= \left(\frac{1}{2}-\frac{1}{p+1}\right)t^2_{u,v,\lambda}\|u\|^2_\alpha + \left(\frac{1}{2}-\frac{1}{q+1}\right)s^{q+1}_{u,v,\lambda}|v|^{q+1}_{q+1}\\ &\quad + \frac{(q-1)}{(p+1)(q+1)}t^{\frac{p+1}{2}}_{u,v,\lambda}s^{\frac{q+1}{2}}_{u,v,\lambda}\lambda\int_{\mathbb{R}^3}|u|^{\frac{p+1}{2}}|v|^{\frac{q+1}{2}}\,dx.\end{aligned} \tag{2.14}$$

Firstly, we claim that there exist $\widetilde{\lambda} \in (0,1)$ and $T_1 > T_2 > 0$ such that for any $\lambda \in (0,\widetilde{\lambda})$ and $(u,v) \in B^*_{\widetilde{m}}$, we have

$$T_2 \le t_{u,v,\lambda}, s_{u,v,\lambda} \le T_1.$$

By (2.10),

$$t_{u,v,\lambda} \le \left(\frac{\|u\|^2_\alpha}{|u|^{p+1}_{p+1}}\right)^{\frac{1}{p-1}} < (2\widetilde{m})^{\frac{1}{p-1}} < 2\widetilde{m},$$

$$s_{u,v,\lambda} \le \left(\frac{\|v\|^2_\beta}{|v|^{q+1}_{q+1}}\right)^{\frac{1}{q-1}} < (2\widetilde{m})^{\frac{1}{q-1}} < 2\widetilde{m}.$$

Thus, we obtain that

$$t_{u,v,\lambda}, s_{u,v,\lambda} < 2\widetilde{m} =: T_1.$$

Define

$$\widetilde{\lambda} = \frac{(q+1)S_p(\frac{1}{2})^{\frac{2}{p+1}}}{8(2\widetilde{m})^{\frac{p+q-2}{2}}}.$$

We see from (2.11) that $\widetilde{\lambda} \in (0,1)$. Moreover, by (2.7) and (2.10), for any $\lambda \in (0,\widetilde{\lambda})$, we have

$$\begin{aligned}t^{p-1}_{u,v,\lambda}|u|^{p+1}_{p+1} &= \|u\|^2_\alpha - \frac{2}{q+1}t^{\frac{p-3}{2}}_{u,v,\lambda}s^{\frac{q+1}{2}}_{u,v,\lambda}\lambda\int_{\mathbb{R}^3}|u|^{\frac{p+1}{2}}|v|^{\frac{q+1}{2}}\,dx\\ &> S_p\left(\frac{1}{2}\right)^{\frac{2}{p+1}} - \frac{2}{q+1}(2\widetilde{m})^{\frac{p+q-2}{2}}\lambda|u|^{\frac{p+1}{2}}_{p+1}|v|^{\frac{q+1}{2}}_{q+1}\end{aligned}$$

$$
\begin{aligned}
&> S_p\left(\frac{1}{2}\right)^{\frac{2}{p+1}} - \frac{4}{q+1}(2\widetilde{m})^{\frac{p+q-2}{2}}\lambda \\
&> \frac{1}{2}S_p\left(\frac{1}{2}\right)^{\frac{2}{p+1}} > \frac{S_p}{4}.
\end{aligned}
$$

Then we get $t_{u,v,\lambda} > (\frac{S_p}{8})^{\frac{1}{p-1}}$. Similarly, we have $s_{u,v,\lambda} > (\frac{S_q}{8})^{\frac{1}{q-1}}$. Thus, we get

$$
t_{u,v,\lambda}, s_{u,v,\lambda} > \min\left\{\left(\frac{S_p}{8}\right)^{\frac{1}{p-1}}, \left(\frac{S_q}{8}\right)^{\frac{1}{q-1}}\right\} =: T_2.
$$

This completes $T_2 \le t_{u,v,\lambda} \le T_1$.

Now we prove the existence of λ_k and c_k. For any $(u,v) \in \overline{B}_{\widetilde{m}}$ and $\lambda \in (0,\widetilde{\lambda}]$, by (2.14), there holds

$$
\begin{aligned}
&\left|\sup_{t,s\ge 0}\Phi_\lambda(tu,sv) - \left(\frac{1}{2}-\frac{1}{p+1}\right)t_{u,v,\lambda}^2\|u\|_\alpha^2 - \left(\frac{1}{2}-\frac{1}{q+1}\right)s_{u,v,\lambda}^{q+1}|v|_{q+1}^{q+1}\right| \\
&\quad = \left|\frac{(q-1)}{(p+1)(q+1)}t_{u,v,\lambda}^{\frac{p+1}{2}}s_{u,v,\lambda}^{\frac{q+1}{2}}\lambda\int_{\mathbb{R}^3}|u|^{\frac{p+1}{2}}|v|^{\frac{q+1}{2}}\,dx\right| \le C\lambda.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\sup_{(u,v)\in B_m}\sup_{t,s\ge 0}\Phi_\lambda(tu,sv) \\
&\quad \le \sup_{(u,v)\in B_m}\left[\left(\frac{1}{2}-\frac{1}{p+1}\right)t_{u,v,\lambda}^2\|u\|_\alpha^2 + \left(\frac{1}{2}-\frac{1}{q+1}\right)s_{u,v,\lambda}^{q+1}|v|_{q+1}^{q+1}\right] + C\lambda \\
&\quad \le \sup_{(u,v)\in B_m}\left[\left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\frac{\|u\|_\alpha^2}{|u|_{p+1}^{p+1}}\right)^{\frac{2}{p-1}}\|u\|_\alpha^2 + \left(\frac{1}{2}-\frac{1}{q+1}\right)\left(\frac{\|v\|_\beta^2}{|v|_{q+1}^{q+1}}\right)^{\frac{q+1}{q-1}}\right] + C\lambda \\
&\quad \le \left(\frac{1}{2}-\frac{1}{p+1}\right)m^{\frac{p+1}{p-1}} + \left(\frac{1}{2}-\frac{1}{q+1}\right)m^{\frac{q+1}{q-1}} + C\lambda \\
&\quad \le 2\left(\frac{1}{2}-\frac{1}{q+1}\right)m^{\frac{p+1}{p-1}} + C\lambda < (q+1)m^{\frac{p+1}{p-1}} + C\lambda,
\end{aligned}
$$

and

$$
\begin{aligned}
&\inf_{(u,v)\in\mathcal{C}_{\widetilde{m}}}\sup_{t,s\ge 0}\Phi_\lambda(tu,sv) \\
&\quad \ge \inf_{(u,v)\in\mathcal{C}_{\widetilde{m}}}\left[\left(\frac{1}{2}-\frac{1}{p+1}\right)t_{u,v,\lambda}^2\|u\|_\alpha^2 + \left(\frac{1}{2}-\frac{1}{q+1}\right)s_{u,v,\lambda}^{q+1}|v|_{q+1}^{q+1}\right] - C\lambda \\
&\quad > \inf_{(u,v)\in\mathcal{C}_{\widetilde{m}}}\left(\frac{1}{2}-\frac{1}{p+1}\right)t_{u,v,\lambda}^2\|u\|_\alpha^2 - C\lambda \\
&\quad \ge \left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\frac{S_p}{8}\right)^{\frac{2}{p-1}}\widetilde{m} - C\lambda,
\end{aligned}
$$

then by (2.11), we can choose

$$
\lambda_k = \min\left\{\frac{q+1}{2C}m^{\frac{p+1}{p-1}}, \widetilde{\lambda}\right\},
$$

$$c_k = \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{S_p}{8}\right)^{\frac{2}{p-1}} \widetilde{m} - C\lambda_k$$

such that $c_k > 0$ for any $0 < \lambda < \lambda_k$ the conclusion holds. □

For any $(u,v) \in B^*_{\widetilde{m}}$, the following linear problem

$$\begin{cases} -\Delta\varphi + \alpha\varphi - \frac{2}{q+1} t_{u,v,\lambda}^{\frac{p-3}{2}} s_{u,v,\lambda}^{\frac{q+1}{2}} \lambda |u|^{\frac{p-3}{2}} \varphi |v|^{\frac{q+1}{2}} = t_{u,v,\lambda}^{p-1} |u|^{p-1} u, \\ -\Delta\psi + \beta\psi - \frac{2}{p+1} t_{u,v,\lambda}^{\frac{p+1}{2}} s_{u,v,\lambda}^{\frac{q-3}{2}} \lambda |u|^{\frac{p+1}{2}} |v|^{\frac{q-3}{2}} \psi = s_{u,v,\lambda}^{q-1} |v|^{q-1} v, \\ \varphi(x) \to 0, \qquad \psi(x) \to 0, \quad \text{as } |x| \to \infty, \end{cases} \tag{2.15}$$

has a unique solution $(\varphi,\psi) \in H_r \setminus \{(0,0)\}$. Then we can choose λ_k small enough such that for any $\varphi, \psi \in H_r^1(\mathbb{R}^3)$,

$$\begin{aligned} \int_{\mathbb{R}^3} |u|^{p-1} u\varphi \, dx &= \frac{\|\varphi\|_\alpha^2 - \frac{2}{q+1} t_{u,v,\lambda}^{\frac{p-3}{2}} s_{u,v,\lambda}^{\frac{q+1}{2}} \lambda \int_{\mathbb{R}^3} |u|^{\frac{p-3}{2}} \varphi^2 |v|^{\frac{q+1}{2}} \, dx}{t_{u,v,\lambda}^{p-1}} \\ &\geq \frac{\frac{1}{2}\|\varphi\|_\alpha^2}{t_{u,v,\lambda}^{p-1}} > 0 \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^3} |v|^{q-1} v\psi \, dx &= \frac{\|\psi\|_\beta^2 - \frac{2}{p+1} t_{u,v,\lambda}^{\frac{p+1}{2}} s_{u,v,\lambda}^{\frac{q-3}{2}} \lambda \int_{\mathbb{R}^3} |u|^{\frac{p+1}{2}} |v|^{\frac{q-3}{2}} \psi^2 \, dx}{s_{u,v,\lambda}^{q-1}} \\ &\geq \frac{\frac{1}{2}\|\psi\|_\beta^2}{s_{u,v,\lambda}^{q-1}} > 0. \end{aligned}$$

Define

$$\mu := \frac{1}{\int_{\mathbb{R}^3} |u|^{p-1} u\varphi \, dx}, \qquad \nu := \frac{1}{\int_{\mathbb{R}^3} |v|^{q-1} v\psi \, dx},$$

then $\mu > 0$, $\nu > 0$ and $(\widetilde{\varphi}, \widetilde{\psi}) := (\mu\varphi, \nu\psi)$ is the unique solution of

$$\begin{cases} -\Delta\widetilde{\varphi} + \alpha\widetilde{\varphi} - \frac{2}{q+1} t_{u,v,\lambda}^{\frac{p-3}{2}} s_{u,v,\lambda}^{\frac{q+1}{2}} \lambda |u|^{\frac{p-3}{2}} \widetilde{\varphi} |v|^{\frac{q+1}{2}} = \mu t_{u,v,\lambda}^{p-1} |u|^{p-1} u, \\ -\Delta\widetilde{\psi} + \beta\widetilde{\psi} - \frac{2}{p+1} t_{u,v,\lambda}^{\frac{p+1}{2}} s_{u,v,\lambda}^{\frac{q-3}{2}} \lambda |u|^{\frac{p+1}{2}} |v|^{\frac{q-3}{2}} \widetilde{\psi} = \nu s_{u,v,\lambda}^{q-1} |v|^{q-1} v, \\ \int_{\mathbb{R}^3} |u|^{p-1} u\widetilde{\varphi} \, dx = \int_{\mathbb{R}^3} |v|^{q-1} v\widetilde{\psi} \, dx = 1, \\ \widetilde{\varphi}(x) \to 0, \qquad \widetilde{\psi}(x) \to 0, \quad \text{as } |x| \to \infty. \end{cases} \tag{2.16}$$

Fixed any $k \in \mathbb{N}$, we define

$$A_1 := \{u \in X_{k+1} : |u|_{p+1} = 1\}, \qquad A_2 := \{v \in X_{k+1} : |v|_{q+1} = 1\}.$$

There is an odd homeomorphism from S^k to A_1 and A_2. By Lemma 2.2(1), $A := A_1 \times A_2 \in \Gamma^{(k+1,k+1)}$. Observe that from (2.1) we deduce that $A \subset B_m$, and so by (2.13),

$$\sup_{(u,v)\in A} \sup_{t,s\geq 0} \Phi_\lambda(tu, sv) < c_k.$$

Define

$$\Gamma_\lambda^{(k_1,k_2)} := \Big\{A \in \Gamma^{(k_1,k_2)} : A \subset B_{\widetilde{m}}, \sup_{(u,v)\in A} \sup_{t,s\geq 0} \Phi_\lambda(tu,sv) < c_k\Big\}.$$

Observe that $\Gamma_\lambda^{(k_1,k_2)} \neq \emptyset$, $\Gamma_\lambda^{(k_1,k_2)} \subset \Gamma_\lambda^{(k_1',k_2')}$ when $k_1 \geq k_1'$ and $k_2 \geq k_2'$. We are now ready to define a sequence of minimax energy levels which will turn out to be critical levels for Φ_λ over $\mathcal{A}$. For every $k_1, k_2 \in [2, k+1]$ and $0 < \delta < 2^{-\frac{1}{p+1}}$, define

$$d_{\lambda,\delta}^{k_1,k_2} := \inf_{A\in\Gamma_\lambda^{(k_1,k_2)}} \sup_{A\setminus\mathcal{P}_\delta} \sup_{t,s\geq 0} \Phi_\lambda(tu,sv). \tag{2.17}$$

It is easy to see that

$$d_{\lambda,\delta}^{k_1,k_2} < c_k \quad \text{for any } 0 < \delta < 2^{-\frac{1}{p+1}}, 2 \leq k_1, k_2 \leq k+1. \tag{2.18}$$

As a step towards to the proof of Theorem 1.1, we will prove that $d_{\lambda,\delta}^{k_1,k_2}$ is indeed a critical level of Φ_λ for δ sufficiently small. To prove Theorem 1.1, it is necessary to find a pseudogradient for Φ_λ over $\mathcal{A}$ for which $\mathcal{P}_\delta$ is positively invariant for the associated flow. We can now define the operator

$$K : B_{\widetilde{m}}^* \to H_r; \quad (u,v) \mapsto (\widetilde{\varphi}, \widetilde{\psi}),$$

that is, for any $(u,v) \in B_{\widetilde{m}}^*$, $K(u,v) = (\widetilde{\varphi}, \widetilde{\psi})$ is the unique solution of (2.16). It is easy to prove that $K(\sigma_i(u,v)) = \sigma_i(K(u,v))$, $i = 1,2$.

Now, we give some property of the operator K. We can now prove that K is a compact C^1 operator.

Lemma 2.5 *The operator K is of class C^1.*

Proof Define C^1 maps $J_i : B_{\widetilde{m}}^* \times H_r^1(\mathbb{R}^3) \times \mathbb{R} \to H_r^1(\mathbb{R}^3) \times \mathbb{R}$, $i = 1,2$, by

$$\begin{aligned} &J_1\big((u,v),\omega,\gamma\big) \\ &= \bigg(\omega - (-\Delta + \alpha)^{-1}\bigg(\frac{2}{q+1} t_{u,v,\lambda}^{\frac{p-3}{2}} s_{u,v,\lambda}^{\frac{q+1}{2}} \lambda |u|^{\frac{p-3}{2}} \omega |v|^{\frac{q+1}{2}} + \gamma t_{u,v,\lambda}^{p-1} |u|^{p-1} u\bigg), \\ &\quad \int_{\mathbb{R}^3} |u|^{p-1} u\omega \, dx - 1\bigg) \end{aligned}$$

and

$$\begin{aligned} &J_2\big((u,v),\omega,\gamma\big) \\ &\quad = \bigg(\omega - (-\Delta + \beta)^{-1}\bigg(\frac{2}{p+1} t_{u,v,\lambda}^{\frac{p+1}{2}} s_{u,v,\lambda}^{\frac{q-3}{2}} \lambda |u|^{\frac{p+1}{2}} |v|^{\frac{q-3}{2}} \omega + \gamma s_{u,v,\lambda}^{q-1} |v|^{q-1} v\bigg), \\ &\quad \int_{\mathbb{R}^3} |v|^{q-1} v\omega \, dx - 1\bigg) \end{aligned}$$

then by (2.16), $J_1((u,v),\widetilde{\varphi},\mu) = J_2((u,v),\widetilde{\psi},\nu) = 0$. Moreover, the derivatives of J_1 and J_2 with respect to (ω,γ) at the point $((u,v),\widetilde{\varphi},\mu)$ and $((u,v),\widetilde{\psi},\nu)$ in the direction (ω_0,γ_0), respectively, are

$$
\begin{aligned}
&D_{\omega,\gamma}J_1\big((u,v),\widetilde{\varphi},\mu\big)(\omega_0,\gamma_0)\\
&\quad=\left(\omega_0-(-\Delta+\alpha)^{-1}\left(\frac{2}{q+1}t_{u,v,\lambda}^{\frac{p-3}{2}}s_{u,v,\lambda}^{\frac{q+1}{2}}\lambda|u|^{\frac{p-3}{2}}\omega_0|v|^{\frac{q+1}{2}}+\gamma_0t_{u,v,\lambda}^{p-1}|u|^{p-1}u\right),\right.\\
&\quad\left.\int_{\mathbb{R}^3}|u|^{p-1}u\omega_0\,dx\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&D_{\omega,\gamma}J_2\big((u,v),\widetilde{\psi},\nu\big)(\omega_0,\gamma_0)\\
&\quad=\left(\omega_0-(-\Delta+\beta)^{-1}\left(\frac{2}{p+1}t_{u,v,\lambda}^{\frac{p+1}{2}}s_{u,v,\lambda}^{\frac{q-3}{2}}\lambda|u|^{\frac{p+1}{2}}|v|^{\frac{q-3}{2}}\omega_0+\gamma_0s_{u,v,\lambda}^{q-1}|v|^{q-1}v\right),\right.\\
&\quad\left.\int_{\mathbb{R}^3}|v|^{q-1}v\omega_0\,dx\right).
\end{aligned}
$$

We claim that $D_{\omega,\gamma}J_1((u,v),\widetilde{\varphi},\mu)$ and $D_{\omega,\gamma}J_2((u,v),\widetilde{\psi},\nu)$ are bijective maps. In fact, for any $(\omega,\gamma)\in H_r^1(\mathbb{R}^3)\times\mathbb{R}$, the following linear problems

$$
\begin{aligned}
&-\Delta\omega_1+\alpha\omega_1-\frac{2}{q+1}t_{u,v,\lambda}^{\frac{p-3}{2}}s_{u,v,\lambda}^{\frac{q+1}{2}}\lambda|u|^{\frac{p-3}{2}}\omega_1|v|^{\frac{q+1}{2}}=-\Delta\omega+\alpha\omega,\\
&-\Delta\omega_2+\alpha\omega_2-\frac{2}{q+1}t_{u,v,\lambda}^{\frac{p-3}{2}}s_{u,v,\lambda}^{\frac{q+1}{2}}\lambda|u|^{\frac{p-3}{2}}\omega_2|v|^{\frac{q+1}{2}}=t_{u,v,\lambda}^{p-1}|u|^{p-1}u,
\end{aligned}
$$

have unique solutions $\omega_1,\omega_2\in H_r^1(\mathbb{R}^3)$, $\omega_2\not\equiv 0$ by $u\in B_{\widetilde{m}}^*$ and (2.12), then we define

$$
\gamma_0=\frac{\gamma-\int_{\mathbb{R}^3}|u|^{p-1}u\omega_1\,dx}{\int_{\mathbb{R}^3}|u|^{p-1}u\omega_2\,dx},
$$

we have

$$
D_{\omega,\gamma}J_1\big((u,v),\widetilde{\varphi},\mu\big)(\omega_1+\gamma_0\omega_2,\gamma_0)=(\omega,\gamma),
$$

that is, $D_{\omega,\gamma}J_1((u,v),\widetilde{\varphi},\mu)$ is surjective. Similarly, $D_{\omega,\gamma}J_2((u,v),\widetilde{\psi},\nu)$ is surjective.

If $D_{\omega,\gamma}J_1((u,v),\widetilde{\varphi},\mu)(\omega_0,\gamma_0)=(0,0)$, then

$$
\begin{cases}
-\Delta\omega_0+\alpha\omega_0=\frac{2}{q+1}t_{u,v,\lambda}^{\frac{p-3}{2}}s_{u,v,\lambda}^{\frac{q+1}{2}}\lambda|u|^{\frac{p-3}{2}}\omega_0|v|^{\frac{q+1}{2}}+\gamma_0t_{u,v,\lambda}^{p-1}|u|^{p-1}u,\\
\int_{\mathbb{R}^3}|u|^{p-1}u\omega_0\,dx=0,
\end{cases}
$$

so $\omega_0\equiv 0$, $\gamma_0t_{u,v,\lambda}^{p-1}|u|^{p-1}u\equiv 0$, by $t_{u,v,\lambda}>0$, $u\in B_{\widetilde{m}}^*$, we have $\gamma_0=0$, this implies $D_{\omega,\gamma}J_1((u,v),\widetilde{\varphi},\mu)$ is injective. Therefore, $D_{\omega,\gamma}J_1((u,v),\widetilde{\varphi},\mu)$ is bijective. Similarly, $D_{\omega,\gamma}J_2((u,v),\widetilde{\psi},\nu)$ is a bijective map. Then we can apply the implicit theorem to the C^1 maps $D_{\omega,\gamma}J_1((u,v),\widetilde{\varphi},\mu)$ and $D_{\omega,\gamma}J_2((u,v),\widetilde{\psi},\nu)$, we have the conclusions. □

Lemma 2.6 *Let $\{(u_n,v_n)\}_{n\geq 1}\subset B_{\widetilde{m}}$. For any $0<\lambda<\lambda_k$, there exists $(\widetilde{\varphi}_0,\widetilde{\psi}_0)\in H_r$ such that, up to a subsequence,*

$$
K(u_n,v_n)\to(\widetilde{\varphi}_0,\widetilde{\psi}_0),\quad\text{strongly in } H_r.
$$

Proof Since $\{(u_n, v_n)\}_{n\geq 1} \subset B_{\widetilde{m}}$, we have

$$\begin{aligned}
&(u_n, v_n) \rightharpoonup (u_0, v_0) \quad \text{weakly in } H_r, \\
&u_n \to u_0, \quad \text{strongly in } L^{p+1}(\mathbb{R}^3), \\
&v_n \to v_0, \quad \text{strongly in } L^{q+1}(\mathbb{R}^3),
\end{aligned}$$

and $|u_0|_{p+1} = |v_0|_{q+1} = 1$. By (2.12), we also have

$$t_{u_n,v_n,\lambda} \to t_{u_0,v_0,\lambda} > 0, \qquad s_{u_n,v_n,\lambda} \to s_{u_0,v_0,\lambda} > 0.$$

Then by (2.3), (2.7), (2.12), and (2.15),

$$\begin{aligned}
\frac{1}{2}\|\varphi_n\|_\alpha^2 &\leq \|\varphi_n\|_\alpha^2 - \frac{2}{q+1} t_{u_n,v_n,\lambda}^{\frac{p-3}{2}} s_{u_n,v_n,\lambda}^{\frac{q+1}{2}} \lambda \int_{\mathbb{R}^3} |u_n|^{\frac{p-3}{2}} \varphi_n^2 |v_n|^{\frac{q+1}{2}}\, dx \\
&= t_{u_n,v_n,\lambda}^{p-1} \int_{\mathbb{R}^3} |u_n|^{p-1} u_n \varphi_n \, dx \\
&\leq C \int_{\mathbb{R}^3} |u_n|^p |\varphi_n| \, dx \\
&\leq C |u_n|_{p+1}^p |\varphi_n|_{p+1} \leq C \|\varphi_n\|_\alpha.
\end{aligned}$$

Similar estimates hold for ψ_n, we get $\|\psi_n\|_\beta^2 \leq C\|\psi_n\|_\beta$, so $\{(\varphi_n, \psi_n)\}_{n\geq 1} \subset H_r$ are bounded. Thus

$$\begin{aligned}
&(\varphi_n, \psi_n) \rightharpoonup (\varphi_0, \psi_0) \quad \text{weakly in } H_r, \\
&\varphi_n \to \varphi_0, \quad \text{strongly in } L^{p+1}(\mathbb{R}^3), \\
&\psi_n \to \psi_0, \quad \text{strongly in } L^{q+1}(\mathbb{R}^3).
\end{aligned}$$

Then by (2.15) and Hölder's inequality,

$$\begin{aligned}
&\int_{\mathbb{R}^3} \big(\nabla \varphi_n \nabla(\varphi_n - \varphi_0) + \alpha \varphi_n (\varphi_n - \varphi_0)\big) \, dx \\
&\quad = \frac{2}{q+1} t_{u_n,v_n,\lambda}^{\frac{p-3}{2}} s_{u_n,v_n,\lambda}^{\frac{q+1}{2}} \lambda \int_{\mathbb{R}^3} |u_n|^{\frac{p-3}{2}} \varphi_n (\varphi_n - \varphi_0) |v_n|^{\frac{q+1}{2}} \, dx \\
&\qquad + t_{u_n,v_n,\lambda}^{p-1} \int_{\mathbb{R}^3} |u_n|^{p-1} u_n (\varphi_n - \varphi_0) \, dx \\
&\quad \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

Hence,

$$\|\varphi_n\|_\alpha^2 = \int_{\mathbb{R}^3} (\nabla \varphi_n \nabla \varphi_0 + \alpha \varphi_n \varphi_0) \, dx + o(1) = \|\varphi_0\|_\alpha^2 + o(1).$$

Similarly, we have $\|\psi_n\|_\beta^2 = \|\psi_0\|_\beta^2 + o(1)$. Therefore, we have $(\varphi_n, \psi_n) \to (\varphi_0, \psi_0)$ strongly in H_r and (φ_0, ψ_0) satisfies

$$\begin{cases} -\Delta\varphi_0 + \alpha\varphi_0 - \frac{2}{q+1} t_{u_0,v_0,\lambda}^{\frac{p-3}{2}} s_{u_0,v_0,\lambda}^{\frac{q+1}{2}} \lambda |u_0|^{\frac{p-3}{2}} \varphi_0 |v_0|^{\frac{q+1}{2}} = t_{u_0,v_0,\lambda}^{p-1} |u_0|^{p-1} u_0, \\ -\Delta\psi_0 + \beta\psi_0 - \frac{2}{p+1} t_{u_0,v_0,\lambda}^{\frac{p+1}{2}} s_{u_0,v_0,\lambda}^{\frac{q-3}{2}} \lambda |u_0|^{\frac{p+1}{2}} |v_0|^{\frac{q-3}{2}} \psi_0 = s_{u_0,v_0,\lambda}^{q-1} |v_0|^{q-1} v_0, \\ \varphi_0(x) \to 0, \qquad \psi_0(x) \to 0, \quad \text{as } |x| \to \infty, \end{cases}$$

since $|u_0|_{p+1} = |v_0|_{q+1} = 1$, so $\varphi_0 \neq 0$, $\psi_0 \neq 0$ and

$$\mu_n := \frac{1}{\int_{\mathbb{R}^3} |u_n|^{p-1} u_n \varphi_n \, dx} \to \frac{1}{\int_{\mathbb{R}^3} |u_0|^{p-1} u_0 \varphi_0 \, dx} =: \mu_0,$$
$$\nu_n := \frac{1}{\int_{\mathbb{R}^3} |v_n|^{q-1} v_n \psi_n \, dx} \to \frac{1}{\int_{\mathbb{R}^3} |v_0|^{q-1} v_0 \psi_0 \, dx} =: \nu_0.$$

We see that

$$(\widetilde{\varphi}_n, \widetilde{\psi}_n) = (\mu_n \varphi_n, \nu_n \psi_n) \to (\mu_0 \varphi_0, \nu_0 \psi_0) =: (\widetilde{\varphi}_0, \widetilde{\psi}_0), \quad \text{strongly in } H_r.$$

This completes the proof. □

Define

$$B_{\widetilde{m},\lambda} := \Big\{ (u,v) \in B_{\widetilde{m}} : \sup_{t,s \geq 0} \Phi_\lambda(tu, sv) < c_k \Big\},$$

then by (2.13) we obtain $B_m \subset B_{\widetilde{m},\lambda}$.

Lemma 2.7 *For any* $0 < \delta < 2^{-\frac{1}{p+1}}$ *sufficiently small, we have that*

$$\operatorname{dist}\big(K(u,v), \mathcal{P}\big) < \frac{\delta}{2}, \quad \forall (u,v) \in B_{\widetilde{m},\lambda}, \qquad \operatorname{dist}\big((u,v), \mathcal{P}\big) < \delta.$$

Proof Suppose by contradiction that there exist $\delta_n \to 0$ and $(u_n, v_n) \in B_{\widetilde{m},\lambda}$ satisfying $\operatorname{dist}((u_n,v_n),\mathcal{P}) < \delta_n$ and $\operatorname{dist}(K(u_n,v_n),\mathcal{P}) \geq \frac{\delta_n}{2}$. We suppose that $\operatorname{dist}((u_n,v_n),\mathcal{P}) = |u_n^-|_{p+1}$ without loss of generality. Let $(\widetilde{\varphi}_n, \widetilde{\psi}_n) = K(u_n, v_n)$ and $\widetilde{\varphi}_n = \mu_n \varphi_n$, $\widetilde{\psi}_n = \nu_n \psi_n$. By a similar proof as in Lemma 2.6, we have that μ_n and ν_n are uniformly bounded. By (2.12), we can take λ_k smaller if necessary such that for any $\lambda \in (0, \lambda_k)$ and $(u,v) \in B_{\widetilde{m}}^*$, we get

$$\frac{1}{2} \|\widetilde{\varphi}_n^-\|_\alpha^2 \leq \|\widetilde{\varphi}_n^-\|_\alpha^2 - \frac{2}{q+1} t_{u_n,v_n,\lambda}^{\frac{p-3}{2}} s_{u_n,v_n,\lambda}^{\frac{q+1}{2}} \lambda \int_{\mathbb{R}^3} |u_n|^{\frac{p-3}{2}} \big(\widetilde{\varphi}_n^-\big)^2 |v_n|^{\frac{q+1}{2}} \, dx.$$

This together with (2.7) and (2.16) allows us to get

$$\begin{aligned} |\widetilde{\varphi}_n^-|_{p+1}^2 &\leq \frac{1}{S_p} \|\widetilde{\varphi}_n^-\|_\alpha^2 \\ &\leq C \bigg(\|\widetilde{\varphi}_n^-\|_\alpha^2 - \frac{2}{q+1} t_{u_n,v_n,\lambda}^{\frac{p-3}{2}} s_{u_n,v_n,\lambda}^{\frac{q+1}{2}} \lambda \int_{\mathbb{R}^3} |u_n|^{\frac{p-3}{2}} \big(\widetilde{\varphi}_n^-\big)^2 |v_n|^{\frac{q+1}{2}} \, dx \bigg) \end{aligned}$$

$$= -C\mu_n t_{u_n,v_n,\lambda}^{p-1}\int_{\mathbb{R}^3}|u_n|^{p-1}u_n\widetilde{\varphi}_n^-\,dx$$
$$\le C\int_{\mathbb{R}^3}(u_n^-)^p\widetilde{\varphi}_n^-\,dx \le C|u_n^-|_{p+1}^p|\widetilde{\varphi}_n^-|_{p+1} \le C\delta_n^p|\widetilde{\varphi}_n^-|_{p+1},$$

and hence $\mathrm{dist}(K(u_n,v_n),\mathcal{P}) \le |\widetilde{\varphi}_n^-|_{p+1} \le C\delta_n^p < \frac{\delta_n}{2}$ for n sufficiently large, which is a contradiction. This completes the proof. □

Now define a map

$$V : B_{\widetilde{m}}^* \to H_r; \qquad (u,v)\mapsto (u,v) - K(u,v).$$

It is easy to prove that $V(\sigma_i(u,v)) = \sigma_i(V(u,v))$, $i = 1,2$. We will prove that if $(u,v) \in B_{\widetilde{m}}\backslash\mathcal{P}$, $V(u,v) = 0$, then $(t_{u,v,\lambda}u, s_{u,v,\lambda}v)$ is a sign-changing solution of Eq. (1.1). Firstly, we prove that V satisfies the Palais–Smale type condition and V is a pseudogradient for $\sup_{t,s\ge 0}\Phi_\lambda(tu,sv)$ over $B_{\widetilde{m}}$. Denote $\Psi_\lambda(u,v) := \sup_{t,s\ge 0}\Phi_\lambda(tu,sv)$.

Lemma 2.8 (*Palais–Smale type condition*) *Let* $(u_n,v_n) \in B_{\widetilde{m}}$ *be such that*

$$\Psi_\lambda(u_n,v_n) \to c < c_k \quad \textit{and} \quad V(u_n,v_n) \to 0 \quad \textit{strongly in } H_r.$$

Then there exists $(u_0,v_0) \in B_{\widetilde{m}}$ *such that* $(u_n,v_n) \to (u_0,v_0)$ *strongly in* H_r*, up to a subsequence, and* $V(u_0,v_0) = 0$*. We also have*

$$\textit{For any } (u,v) \in B_{\widetilde{m}},\quad \langle\nabla\Psi_\lambda(u,v), V(u,v)\rangle_{H_r} \ge \frac{T_2^2}{2}\|V(u,v)\|_{H_r}^2.$$

Proof Similar as Lemma 2.6, we have, up to a subsequence,

$$(u_n,v_n) \rightharpoonup (u_0,v_0) \quad \text{weakly in } H_r,$$
$$K(u_n,v_n) \to (\widetilde{\varphi}_0,\widetilde{\psi}_0) \quad \text{strongly in } H_r.$$

Then we have, as $n\to\infty$,

$$o(1) = \langle V(u_n,v_n),(u_n-u_0,v_n-v_0)\rangle_{H_r}$$
$$= \langle u_n-\widetilde{\varphi}_n, u_n-u_0\rangle_{H_r} + \langle v_n-\widetilde{\psi}_n, v_n-v_0\rangle_{H_r}$$
$$= \langle u_n, u_n-u_0\rangle_{H_r} - \langle\widetilde{\varphi}_n, u_n-u_0\rangle_{H_r} + \langle v_n, v_n-v_0\rangle_{H_r} - \langle\widetilde{\psi}_n, v_n-v_0\rangle_{H_r}$$

whence

$$\langle u_n, u_n-u_0\rangle_{H_r} + \langle v_n, v_n-v_0\rangle_{H_r} = o(1).$$

Then $(u_n,v_n) \to (u_0,v_0)$ strongly in H_r and $(u_0,v_0) \in \overline{B}_{\widetilde{m}}$,

$$\Phi_\lambda(t_{u_0,v_0,\lambda}u_0, s_{u_0,v_0,\lambda}v_0) = \lim_{n\to\infty}\Phi_\lambda(t_{u_n,v_n,\lambda}u_n, s_{u_n,v_n,\lambda}v_n) = c < c_k,$$

then by (2.13), we have $(u_0,v_0) \in B_{\widetilde{m}}$, $V(u_0,v_0) = \lim_{n\to\infty}V(u_n,v_n) = 0$.

Finally, we prove that V is a pseudogradient for $\Psi_\lambda(u,v)$ over $B_{\widetilde{m}}$. By (2.9) and (2.10) we can prove that

$$\begin{aligned}\langle\nabla\Psi_\lambda(u,v),(\omega,0)\rangle_{H_r} &= t_{u,v,\lambda}^2\int_{\mathbb{R}^3}(\nabla u\nabla\omega+\alpha u\omega)\,dx\\ &\quad-\frac{2\lambda}{q+1}t_{u,v,\lambda}^{\frac{p+1}{2}}s_{u,v,\lambda}^{\frac{q+1}{2}}\int_{\mathbb{R}^3}|u|^{\frac{p-3}{2}}u\omega|v|^{\frac{q+1}{2}}\,dx,\end{aligned}\tag{2.19}$$

$$\begin{aligned}\langle\nabla\Psi_\lambda(u,v),(0,\omega)\rangle_{H_r} &= s_{u,v,\lambda}^2\int_{\mathbb{R}^3}(\nabla v\nabla\omega+\beta v\omega)\,dx\\ &\quad-\frac{2\lambda}{p+1}t_{u,v,\lambda}^{\frac{p+1}{2}}s_{u,v,\lambda}^{\frac{q+1}{2}}\int_{\mathbb{R}^3}|u|^{\frac{p+1}{2}}|v|^{\frac{q-3}{2}}v\omega\,dx\end{aligned}\tag{2.20}$$

hold for any $(u,v)\in\mathcal{B}_{\widetilde{m}}$ and $\omega\in H_r^1(\mathbb{R}^3)$. We can take λ_k smaller if necessary such that for any $\lambda\in(0,\lambda_k)$ by (2.19), (2.20), (2.12), and (2.16)

$$\begin{aligned}&\langle\nabla\Psi_\lambda(u,v),V(u,v)\rangle_{H_r}\\ &\quad= t_{u,v,\lambda}^2\int_{\mathbb{R}^3}\big(\nabla u\nabla(u-\widetilde{\varphi})+\alpha u(u-\widetilde{\varphi})\big)\,dx\\ &\qquad+s_{u,v,\lambda}^2\int_{\mathbb{R}^3}\big(\nabla v\nabla(v-\widetilde{\psi})+\beta v(v-\widetilde{\psi})\big)\,dx\\ &\qquad-\frac{2}{q+1}t_{u,v,\lambda}^{\frac{p+1}{2}}s_{u,v,\lambda}^{\frac{q+1}{2}}\lambda\int_{\mathbb{R}^3}|u|^{\frac{p-3}{2}}u(u-\widetilde{\varphi})|v|^{\frac{q+1}{2}}\,dx\\ &\qquad-\frac{2}{p+1}t_{u,v,\lambda}^{\frac{p+1}{2}}s_{u,v,\lambda}^{\frac{q+1}{2}}\lambda\int_{\mathbb{R}^3}|u|^{\frac{p+1}{2}}|v|^{\frac{q-3}{2}}v(v-\widetilde{\psi})\,dx\\ &\quad= t_{u,v,\lambda}^2\|u-\widetilde{\varphi}\|_\alpha^2+s_{u,v,\lambda}^2\|v-\widetilde{\psi}\|_\beta^2\\ &\qquad+t_{u,v,\lambda}^2\int_{\mathbb{R}^3}\big(\nabla\widetilde{\varphi}\nabla(u-\widetilde{\varphi})+\alpha\widetilde{\varphi}(u-\widetilde{\varphi})\big)\,dx\\ &\qquad+s_{u,v,\lambda}^2\int_{\mathbb{R}^3}\big(\nabla\widetilde{\psi}\nabla(u-\widetilde{\psi})+\alpha\widetilde{\psi}(v-\widetilde{\psi})\big)\,dx\\ &\qquad-\frac{2}{q+1}t_{u,v,\lambda}^{\frac{p+1}{2}}s_{u,v,\lambda}^{\frac{q+1}{2}}\lambda\int_{\mathbb{R}^3}|u|^{\frac{p-3}{2}}u(u-\widetilde{\varphi})|v|^{\frac{q+1}{2}}\,dx\\ &\qquad-\frac{2}{p+1}t_{u,v,\lambda}^{\frac{p+1}{2}}s_{u,v,\lambda}^{\frac{q+1}{2}}\lambda\int_{\mathbb{R}^3}|u|^{\frac{p+1}{2}}|v|^{\frac{q-3}{2}}v(v-\widetilde{\psi})\,dx\\ &\quad= t_{u,v,\lambda}^2\|u-\widetilde{\varphi}\|_\alpha^2+s_{u,v,\lambda}^2\|v-\widetilde{\psi}\|_\beta^2\\ &\qquad-\frac{2}{q+1}t_{u,v,\lambda}^{\frac{p+1}{2}}s_{u,v,\lambda}^{\frac{q+1}{2}}\lambda\int_{\mathbb{R}^3}|u|^{\frac{p-3}{2}}(u-\widetilde{\varphi})^2|v|^{\frac{q+1}{2}}\,dx\\ &\qquad-\frac{2}{p+1}t_{u,v,\lambda}^{\frac{p+1}{2}}s_{u,v,\lambda}^{\frac{q+1}{2}}\lambda\int_{\mathbb{R}^3}|u|^{\frac{p+1}{2}}|v|^{\frac{q-3}{2}}(v-\widetilde{\psi})^2\,dx\\ &\quad\geq\frac{t_{u,v,\lambda}^2}{2}\|u-\widetilde{\varphi}\|_\alpha^2+\frac{s_{u,v,\lambda}^2}{2}\|v-\widetilde{\psi}\|_\beta^2\\ &\quad\geq\frac{T_2^2}{2}\big(\|u-\widetilde{\varphi}\|_\alpha^2+\|v-\widetilde{\psi}\|_\beta^2\big)=\frac{T_2^2}{2}\big\|V(u,v)\big\|_{H_r}^2.\end{aligned}$$

This completes the proof. □

Lemma 2.9 *There exists a unique global solution* $\eta = (\eta_1, \eta_2) : \mathbb{R}^+ \times B_{\widetilde{m},\lambda} \to H_r$ *for the initial value problem*

$$\begin{cases} \frac{d}{dt}\eta(t,(u,v)) = -V(\eta(t,(u,v))), \\ \eta(0,(u,v)) = (u,v) \in B_{\widetilde{m},\lambda}. \end{cases} \tag{2.21}$$

Moreover,

(1) *For any* $t > 0$ *and* $(u,v) \in B_{\widetilde{m},\lambda}$, *there holds* $\eta(t,(u,v)) \in B_{\widetilde{m},\lambda}$;

(2) *For any* $t > 0$, $(u,v) \in B_{\widetilde{m},\lambda}$, *there holds* $\eta(t,\sigma_i(u,v)) = \sigma_i(\eta(t,(u,v)))$, $i = 1,2$;

(3) *For any* $(u,v) \in B_{\widetilde{m},\lambda}$, $\Psi_\lambda(\eta(t,(u,v)))$ *is nonincreasing in* t;

(4) *There exists* $\delta_0 \in (0, 2^{-\frac{1}{p+1}})$ *such that, for any* $0 < \delta < \delta_0$, $(u,v) \in B_{\widetilde{m},\lambda} \cap \mathcal{P}_\delta$ *and* $t > 0$, *there holds* $\eta(t,(u,v)) \in \mathcal{P}_\delta$.

Proof It follows from Lemma 2.5 that $V \in C^1(B^*_{\widetilde{m}}, H_r)$. As $B_{\widetilde{m},\lambda} \subset B_{\widetilde{m}} \subset B^*_{\widetilde{m}}$, we get that $V \in C^1(B_{\widetilde{m},\lambda}, H_r)$. Then there exists a solution $\eta : [0, T_{\max}) \times B_{\widetilde{m},\lambda} \to H_r$, where $T_{\max}$ is the maximal time such that (2.21) has a solution $\eta \in B^*_{\widetilde{m}}$.

For any $(u,v) \in B_{\widetilde{m},\lambda}$ and $t \in (0, T_{\max})$, there holds

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^3} \left|\eta_1\big(t,(u,v)\big)\right|^{p+1} dx \\ &\quad = -(p+1)\int_{\mathbb{R}^3} \left|\eta_1\big(t,(u,v)\big)\right|^{p-1} \eta_1\big(t,(u,v)\big) V_1\big(\eta\big(t,(u,v)\big)\big)\, dx \\ &\quad = -(p+1)\int_{\mathbb{R}^3} \left|\eta_1\big(t,(u,v)\big)\right|^{p-1} \eta_1\big(t,(u,v)\big)\big[\eta_1\big(t,(u,v)\big) - K_1\big(\eta\big(t,(u,v)\big)\big)\big]\, dx \\ &\quad = (p+1) - (p+1)\int_{\mathbb{R}^3} \left|\eta_1\big(t,(u,v)\big)\right|^{p+1} dx, \end{aligned}$$

so we have

$$\frac{d}{dt}\left[e^{(p+1)t}\left(\int_{\mathbb{R}^3} \left|\eta_1\big(t,(u,v)\big)\right|^{p+1} dx - 1\right)\right] = 0.$$

Then

$$\begin{aligned} e^{(p+1)t}\left(\int_{\mathbb{R}^3} \left|\eta_1\big(t,(u,v)\big)\right|^{p+1} dx - 1\right) &= \int_{\mathbb{R}^3} \left|\eta_1\big(0,(u,v)\big)\right|^{p+1} dx - 1 \\ &= \int_{\mathbb{R}^3} |u|^{p+1} dx - 1 \equiv 0. \end{aligned}$$

Similarly, there holds

$$\begin{aligned} e^{(q+1)t}\left(\int_{\mathbb{R}^3} \left|\eta_2\big(t,(u,v)\big)\right|^{q+1} dx - 1\right) &= \int_{\mathbb{R}^3} \left|\eta_2\big(0,(u,v)\big)\right|^{q+1} dx - 1 \\ &= \int_{\mathbb{R}^3} |v|^{q+1} dx - 1 \equiv 0, \end{aligned}$$

we deduce that for any $(u,v) \in B_{\widetilde{m},\lambda}$ and $t \in [0, T_{\max})$,

$$\int_{\mathbb{R}^3} \left|\eta_1\big(t,(u,v)\big)\right|^{p+1} dx \equiv \int_{\mathbb{R}^3} \left|\eta_2\big(t,(u,v)\big)\right|^{q+1} dx \equiv 1.$$

Thus, for any $t \in [0, T_{\max})$, $(u,v) \in B_{\widetilde{m}}$, we have $\eta(t,(u,v)) \in B^*_{\widetilde{m}} \cap \mathcal{A} = B_{\widetilde{m}}$. If $T_{\max} < +\infty$, then $\eta(T_{\max},(u,v)) \in \mathcal{C}_{\widetilde{m}}$. There holds $\Psi_\lambda(\eta(T_{\max},(u,v))) \geq c_k$ by (2.13). Moreover,

$$\begin{aligned}\frac{d}{dt}\Psi_\lambda\big(\eta\big(t,(u,v)\big)\big) &= \Big\langle \nabla\Psi_\lambda\big(\eta\big(t,(u,v)\big)\big), \frac{d}{dt}\eta\big(t,(u,v)\big)\Big\rangle_{H_r} \\ &= -\big\langle \nabla\Psi_\lambda\big(\eta\big(t,(u,v)\big)\big), V\big(\eta\big(t,(u,v)\big)\big)\big\rangle_{H_r} \\ &\leq -\frac{T_2^2}{2}\big\| V\big(\eta\big(t,(u,v)\big)\big)\big\|^2_{H_r} \leq 0.\end{aligned} \tag{2.22}$$

On the other hand, we see from $(u,v) \in B_{\widetilde{m},\lambda}$ and (2.22),

$$\Psi_\lambda\big(\eta\big(T_{\max},(u,v)\big)\big) \leq \Psi_\lambda\big(\eta\big(0,(u,v)\big)\big) = \Psi_\lambda(u,v) < c_k,$$

it yields a contradiction, so $T_{\max} = +\infty$, $\eta(t,(u,v)) \in B_{\widetilde{m},\lambda}$ and (1)(3) hold.

Since $V(\sigma_i(u,v)) = \sigma_i(V(u,v))$, $i = 1,2$, then (2) holds.

Take $\delta_0 > 0$ as in Lemma 2.7, note that as $t \to 0$,

$$\begin{aligned}\eta\big(t,(u,v)\big) &= (u,v) + t\frac{d}{dt}\eta\big(t,(u,v)\big)|_{t=0} + o(t) \\ &= (u,v) - tV(u,v) + o(t) = (1-t)(u,v) + tK(u,v) + o(t),\end{aligned}$$

hence for any $0 < \delta < \delta_0$, $(u,v) \in B_{\widetilde{m},\lambda} \cap \mathcal{P}_\delta$, we have

$$\begin{aligned}\operatorname{dist}\big(\eta\big(t,(u,v)\big),\mathcal{P}\big) &= \operatorname{dist}\big((1-t)(u,v) + tK(u,v) + o(t),\mathcal{P}\big) \\ &\leq (1-t)\operatorname{dist}\big((u,v),\mathcal{P}\big) + t\operatorname{dist}\big(K(u,v),\mathcal{P}\big) + o(t) \\ &< (1-t)\delta + \frac{t\delta}{2} + o(t) < \delta,\end{aligned}$$

for sufficiently small $t > 0$, and (4) holds. This completes the proof. □

To prove Theorem 1.1, we will give that $d^{k_1,k_2}_{\lambda,\delta}$ is indeed critical energy level for $\delta > 0$ sufficiently small.

Lemma 2.10 *For any $k \in \mathbb{N}$, $k_1,k_2 \in [2,k+1]$, $0 < \delta < \delta_0$, and $0 < \lambda < \lambda_k$, there exists $(\widetilde{u}_0,\widetilde{v}_0) \in H_r$ such that $(\widetilde{u}_0,\widetilde{v}_0)$ is a sign-changing solution of Eq.* (1.1) *and $\Phi_\lambda(\widetilde{u}_0,\widetilde{v}_0) = d^{k_1,k_2}_{\lambda,\delta}$.*

Proof By (2.18) we see that $d^{k_1,k_2}_{\lambda,\delta} < c_k$. Assume that there is small $0 < \varepsilon < 1$ such that for any $(u,v) \in B_{\widetilde{m},\lambda}$, $|\Psi_\lambda(u,v) - d^{k_1,k_2}_{\lambda,\delta}| \leq 2\varepsilon$, $\operatorname{dist}((u,v),\mathcal{P}) \geq \delta$, there holds $\|V(u,v)\|^2_{H_r} \geq \varepsilon$. By (2.17), there exists $A \in \Gamma^{(k_1,k_2)}_\lambda$ such that

$$\sup_{A\setminus\mathcal{P}_\delta} \Psi_\lambda(u,v) < d^{k_1,k_2}_{\lambda,\delta} + \varepsilon, \tag{2.23}$$

then $\sup_A \Psi_\lambda(u,v) < c_k$, $A \subset B_{\widetilde{m},\lambda}$. Thus we consider the set $A_0 = \eta(\frac{4}{T_2^2}, A)$, $A_0 \in B_{\widetilde{m},\lambda}$ by Lemma 2.9(1). From Lemma 2.2(2), Lemma 2.3, and Lemma 2.9(3), we get

$$\sup_{A_0} \Psi_\lambda(u,v) \leq \sup_{A} \Psi_\lambda(u,v) < c_k,$$

so $A_0 \in \Gamma_\lambda^{(k_1,k_2)}$ and $A_0 \backslash \mathcal{P}_\delta \neq \emptyset$. Then, by (2.15), (2.19), and Lemma 2.9(3), for the $\varepsilon > 0$, $t \in [0, \frac{4}{T_2^2}]$, there exists $(u,v) \in A$ such that $\eta(\frac{4}{T_2^2}, (u,v)) \in A_0 \backslash \mathcal{P}_\delta$ satisfying

$$\begin{aligned} d_{\lambda,\delta}^{k_1,k_2} &\le \sup_{A_0 \backslash \mathcal{P}_\delta} \Psi_\lambda(u,v) < \Psi_\lambda\left(\eta\left(\frac{4}{T_2^2}, (u,v)\right)\right) + \varepsilon \\ &\le \Psi_\lambda\big(\eta\big(t,(u,v)\big)\big) + \varepsilon \le \Psi_\lambda(u,v) + \varepsilon < d_{\lambda,\delta}^{k_1,k_2} + 2\varepsilon. \end{aligned} \tag{2.24}$$

We conclude that $\|V(\eta(t,(u,v)))\|_{H_r}^2 \ge \varepsilon$ for any $t \in [0, \frac{4}{T_2^2}]$ and

$$\begin{aligned} \frac{d}{dt}\Psi_\lambda\big(\eta\big(t,(u,v)\big)\big) &= -\big\langle \nabla\Psi_\lambda\big(\eta\big(t,(u,v)\big)\big), V\big(\eta\big(t,(u,v)\big)\big)\big\rangle_{H_r} \\ &\le -\frac{T_2^2}{2}\big\|V\big(\eta\big(t,(u,v)\big)\big)\big\|_{H_r}^2 \le -\frac{T_2^2}{2}\varepsilon. \end{aligned}$$

Therefore, by integrating over 0 to $\frac{4}{T_2^2}$ and (2.24), we have

$$\begin{aligned} \big(d_{\lambda,\delta}^{k_1,k_2} - \varepsilon\big) - \big(d_{\lambda,\delta}^{k_1,k_2} + \varepsilon\big) &< \Psi_\lambda\left(\eta\left(\frac{4}{T_2^2}, (u,v)\right)\right) - \Psi_\lambda(u,v) \\ &\le -\frac{T_2^2}{2}\varepsilon \int_0^{\frac{4}{T_2^2}} dt = -2\varepsilon, \end{aligned}$$

it yields a contradiction, and therefore, for any $\varepsilon = \frac{1}{n} > 0$, there exists $(u_n, v_n) \in B_{\widetilde{m},\lambda}$ such that

$$\big|\Psi_\lambda(u_n,v_n) - d_{\lambda,\delta}^{k_1,k_2}\big| \le 2\varepsilon, \qquad \big\|V(u_n,v_n)\big\|_{H_r}^2 \le \varepsilon \quad \text{and} \quad \operatorname{dist}\big((u_n,v_n),\mathcal{P}\big) \ge \delta.$$

By Lemma 2.8, there exists $(u_0,v_0) \in B_{\widetilde{m},\lambda}$ such that $(u_n,v_n) \to (u_0,v_0)$ strongly in H_r, up to a subsequence. Hence, we have

$$\Psi_\lambda(u_0,v_0) = d_{\lambda,\delta}^{k_1,k_2}, \qquad V(u_0,v_0) = 0 \quad \text{and} \quad \operatorname{dist}\big((u_0,v_0),\mathcal{P}\big) \ge \delta.$$

We conclude that (u_0,v_0) is sign-changing and $(u_0,v_0) = K(u_0,v_0) = (\widetilde{\varphi}_0, \widetilde{\psi}_0)$. It follows from (2.16) that (u_0,v_0) satisfies

$$\begin{cases} -\Delta u_0 + \alpha u_0 = \mu t_{u_0,v_0,\lambda}^{p-1}|u_0|^{p-1}u_0 + \frac{2}{q+1} t_{u_0,v_0,\lambda}^{\frac{p-3}{2}} s_{u_0,v_0,\lambda}^{\frac{q+1}{2}} \lambda |u_0|^{\frac{p-3}{2}} u_0 |v_0|^{\frac{q+1}{2}}, \\ -\Delta v_0 + \beta v_0 = \nu s_{u_0,v_0,\lambda}^{q-1}|v_0|^{q-1}v_0 + \frac{2}{p+1} t_{u_0,v_0,\lambda}^{\frac{p+1}{2}} s_{u_0,v_0,\lambda}^{\frac{q-3}{2}} \lambda |u_0|^{\frac{p+1}{2}} |v_0|^{\frac{q-3}{2}} v_0, \\ u_0(x) \to 0, \qquad v_0(x) \to 0, \quad \text{as } |x| \to \infty. \end{cases} \tag{2.25}$$

On the other hand, $t_{u_0,v_0,\lambda}$ and $s_{u_0,v_0,\lambda}$ satisfy

$$\begin{aligned} \|u_0\|_\alpha^2 &= t_{u_0,v_0,\lambda}^{p-1}|u_0|_{p+1}^{p+1} + \frac{2}{q+1} t_{u_0,v_0,\lambda}^{\frac{p-3}{2}} s_{u_0,v_0,\lambda}^{\frac{q+1}{2}} \lambda \int_{\mathbb{R}^3} |u_0|^{\frac{p+1}{2}} |v_0|^{\frac{q+1}{2}}\,dx, \\ \|v_0\|_\beta^2 &= s_{u_0,v_0,\lambda}^{q-1}|v_0|_{q+1}^{q+1} + \frac{2}{p+1} t_{u_0,v_0,\lambda}^{\frac{p+1}{2}} s_{u_0,v_0,\lambda}^{\frac{q-3}{2}} \lambda \int_{\mathbb{R}^3} |u_0|^{\frac{p+1}{2}} |v_0|^{\frac{q+1}{2}}\,dx, \end{aligned}$$

then we have $\mu = \nu = 1$. Hence, we have that $(t_{u_0,v_0,\lambda}u_0, s_{u_0,v_0,\lambda}v_0)$ is a sign-changing solution of Eq. (1.1) by problem (2.25) and

$$\Phi_\lambda(\widetilde{u}_0,\widetilde{v}_0) := \Phi_\lambda(t_{u_0,v_0,\lambda}u_0, s_{u_0,v_0,\lambda}v_0) = \Psi_\lambda(u_0,v_0) = d_{\lambda,\delta}^{k_1,k_2}.$$

This completes the proof. □

Proof of Theorem 1.1 Observe that from Lemma 2.10 we know that for any $k \in \mathbb{N}$, $k_1,k_2 \in [2,k+1]$, $0 < \delta < \delta_0$, and $0 < \lambda < \lambda_k$, there exists a sign-changing solution $(\widetilde{u}_0,\widetilde{v}_0)$ with $\Phi_\lambda(\widetilde{u}_0,\widetilde{v}_0) = d_{\lambda,\delta}^{k_1,k_2}$. For any fixed $k_1 \in [2,k+1]$, we have

$$d_{\lambda,\delta}^{k_1,2} \le d_{\lambda,\delta}^{k_1,3} \le \cdots \le d_{\lambda,\delta}^{k_1,k} \le d_{\lambda,\delta}^{k_1,k+1} < c_k.$$

Suppose that problem (1.1) has at most $k-1$ sign-changing solutions by contradiction, then there exists $k_2 \in [2,k]$ satisfying

$$d := d_{\lambda,\delta}^{k_1,k_2} = d_{\lambda,\delta}^{k_1,k_2+1} < c_k.$$

Now define

$$\mathcal{M} := \big\{(u,v) \in B_{\widetilde{m}} : (u,v) \text{ sign-changing}, \Psi_\lambda(u,v) = d, V(u,v) = 0\big\},$$

then $\mathcal{M} \subset \mathcal{F}$ is finite. So there exist $N \in [1,k-1]$ and $\{(u_n,v_n)\}_{1\le n\le N} \subset \mathcal{M}$ such that

$$\mathcal{M} = \big\{\big\{(u_n,v_n)\big\} \cup \big\{(-u_n,v_n)\big\} \cup \big\{(u_n,-v_n)\big\} \cup \big\{(-u_n,-v_n)\big\}\big\}_{1\le n\le N}.$$

For any $1 \le n \le N$, there exist open neighborhoods Ω_n^1, Ω_n^2, Ω_n^3, Ω_n^4 of $\{(u_n,v_n)\}$, $\{(-u_n,v_n)\}$, $\{(u_n,-v_n)\}$, $\{(-u_n,-v_n)\}$, respectively, such that

$$\Omega_n^1 \cap \Omega_n^2 \cap \Omega_n^3 \cap \Omega_n^4 = \emptyset,$$
$$\mathcal{M} \subset \bigcup_{n=1}^{3}\big(\Omega_n^1 \cup \Omega_n^2 \cup \Omega_n^3 \cup \Omega_n^4\big) =: \Omega.$$

Define

$$\mathcal{M}_\rho := \big\{(u,v) \in B_{\widetilde{m}} : \operatorname{dist}_{H_r}\big((u,v),\mathcal{M}\big) < \rho\big\},$$

we can choose $\rho > 0$ small enough such that $\mathcal{M}_{2\rho} \subset \Omega$. Since $\mathcal{M}$ is finite, then there is $\varepsilon_0 \in (0, \frac{c_k-d}{2})$ such that for any $(u,v) \in B_{\widetilde{m}} \backslash (\mathcal{P}_\delta \cup \mathcal{M}_\rho)$, $|\Psi_\lambda(u,v) - d| \le 2\varepsilon_0$, we have

$$\big\|V(u,v)\big\|_{H_r}^2 \ge \varepsilon_0. \tag{2.26}$$

In fact, if for any $\varepsilon = \frac{1}{n} > 0$ there exists $(u_n,v_n) \in B_{\widetilde{m}} \backslash (\mathcal{P}_\delta \cup \mathcal{M}_\rho)$ satisfying $|\Psi_\lambda(u_n,v_n) - d| \le 2\varepsilon$, then there holds $\|V(u_n,v_n)\|_{H_r}^2 \le \varepsilon$. Then, by Lemma 2.8, there exists $(u_0,v_0) \in B_{\widetilde{m}} \backslash (\mathcal{P}_\delta \cup \mathcal{M}_\rho)$ such that $(u_n,v_n) \to (u_0,v_0)$ strongly in H_r, up to a subsequence, $\Psi_\lambda(u_0,v_0) = d$ and $V(u_0,v_0) = 0$. Therefore, $(u_0,v_0) \in \mathcal{M}_\rho$. It yields a contradiction.

Moreover, for $(u,v) \in \mathcal{M}$, $V(u,v) = 0$, then for $\rho > 0$ small enough, there exists $T_0 > 0$ such that for any $(u,v) \in \overline{\mathcal{M}}_{2\rho}$,

$$\|V(u,v)\|_{H_r} \le T_0. \tag{2.27}$$

Let

$$T := \frac{1}{2}\min\left\{1, \frac{\rho T_2^2}{4T_0}\right\}. \tag{2.28}$$

By (2.17), for $\varepsilon_0 > 0$, there exists $A \in \Gamma_\lambda^{(k_1,k_2+1)}$ such that

$$\sup_{A\backslash\mathcal{P}_\delta} \Psi_\lambda(u,v) < d_{\lambda,\delta}^{k_1,k_2+1} + \frac{T\varepsilon_0}{2} = d + \frac{T\varepsilon_0}{2}. \tag{2.29}$$

Let $B := A\backslash\mathcal{M}_{2\rho}$, then $B \subset \mathcal{F}$.

We claim that $\gamma(B) \ge (k_1,k_2)$. In view of a contradiction, suppose that $\gamma(B) < (k_1,k_2)$. From Definition 2.1, we know that there exists $f \in F_{(k_1,k_2)}(B)$ such that $f(u,v) = (f_1(u,v), f_2(u,v)) \ne (0,0)$ for any $(u,v) \in B$. Take $\widetilde{f} = (\widetilde{f_1},\widetilde{f_2}) \in C(H_r,\mathbb{R}^{k_1-1}\times\mathbb{R}^{k_2-1})$ such that $\widetilde{f}|_B = f$ by Tietze's extension theorem. Define

$$\begin{aligned} F_1(u,v) &:= \widetilde{f_1}(u,v) + \widetilde{f_1}\big(\sigma_2(u,v)\big) - \widetilde{f_1}\big(\sigma_1(u,v)\big) - \widetilde{f_1}(-u,-v), \\ F_2(u,v) &:= \widetilde{f_2}(u,v) + \widetilde{f_2}\big(\sigma_1(u,v)\big) - \widetilde{f_2}\big(\sigma_2(u,v)\big) - \widetilde{f_2}(-u,-v), \end{aligned}$$

then $F := (F_1,F_2) \in C(H_r,\mathbb{R}^{k_1-1}\times\mathbb{R}^{k_2-1})$, $F|_B = 4\widetilde{f}$, $F_i(\sigma_i(u,v)) = -4\widetilde{f_i}(u,v) = -F_i(u,v)$ and $F_i(\sigma_j(u,v)) = 4\widetilde{f_i}(u,v) = F_i(u,v)$, $i \ne j$, $i,j = 1,2$.

Define the continuous function

$$g(u,v) := \begin{cases} 1, & (u,v) \in \bigcup_{n=1}^{3}(\overline{\Omega_n^1} \cup \overline{\Omega_n^2}), \\ -1, & (u,v) \in \bigcup_{n=1}^{3}(\overline{\Omega_n^3} \cup \overline{\Omega_n^4}) \end{cases}$$

and $g(\sigma_1(u,v)) = g(u,v)$, $g(\sigma_2(u,v)) = -g(u,v)$. Take $\widetilde{g} \in C(H_r,\mathbb{R})$ such that $\widetilde{g}|_\Omega = g$ by Tietze's extension theorem. Define

$$G(u,v) := \widetilde{g}(u,v) + \widetilde{g}\big(\sigma_1(u,v)\big) - \widetilde{g}\big(\sigma_2(u,v)\big) - \widetilde{g}(-u,-v),$$

then $G \in C(H_r,\mathbb{R})$, $G|_\Omega = 4\widetilde{g}$, $G(\sigma_1(u,v)) = G(u,v)$, and $G(\sigma_2(u,v)) = -G(u,v)$. Therefore, we can define

$$\begin{aligned} H_1(u,v) &:= F_1(u,v) \in \mathbb{R}^{k_1-1}, \\ H_2(u,v) &:= \big(F_2(u,v), G(u,v)\big) \in \mathbb{R}^{k_2}, \end{aligned}$$

then $H := (H_1,H_2) \in C(A,\mathbb{R}^{k_1-1}\times\mathbb{R}^{k_2})$ and $H \in F_{(k_1,k_2+1)}(A)$. Since $A \in \Gamma_\lambda^{(k_1,k_2+1)}$, $\gamma(A) \ge (k_1,k_2+1)$, so there exists $(u,v) \in A$ such that $H(u,v) = (0,0)$. If $(u,v) \in B = A\backslash\mathcal{M}_{2\rho}$, then

$$F(u,v) = 4\widetilde{f}(u,v) = 4f(u,v) \ne (0,0),$$

a contradiction. Thus $(u,v) \in \mathcal{M}_{2\rho}$, then

$$G(u,v) = 4\widetilde{g}(u,v) = 4g(u,v) \neq (0,0),$$

a contradiction. Therefore, $\gamma(B) \geq (k_1,k_2)$.

Since $B \subset A \subset B_{\widetilde{m}}$, $\sup_B \Psi_\lambda(u,v) \leq \sup_A \Psi_\lambda(u,v) < c_k$, then we have $B \subset B_{\widetilde{m},\lambda}$ and $B \in \Gamma_\lambda^{(k_1,k_2)}$. Define $B_0 := \eta(\frac{\rho}{2T_0}, B)$, then $B_0 \subset B_{\widetilde{m},\lambda}$, $B_0 \in \Gamma^{(k_1,k_2)}$, $B_0 \backslash P_\delta \neq \emptyset$, and $\sup_{B_0} \Psi_\lambda(u,v) \leq \sup_B \Psi_\lambda(u,v) < c_k$ by Lemma 2.2(2) and Lemma 2.3, so $B_0 \in \Gamma_\lambda^{(k_1,k_2)}$. Thus $\sup_{B_0\backslash \mathcal{P}_\delta} \Psi_\lambda(u,v) \geq d_{\lambda,\delta}^{k_1,k_2}$ by (2.17).

We claim that $\eta(t,(u,v)) \notin \mathcal{M}_\rho$ for any $t \in (0,\frac{\rho}{2T_0})$, $(u,v) \in B$. In view of a contradiction, if there exists $t_0 \in (0,\frac{\rho}{2T_0})$ such that $\eta(t_0,(u,v)) \in \mathcal{M}_\rho$, for $(u,v) \in B = A\backslash \mathcal{M}_{2\rho}$, by the continuity of η, there exists $0 \leq t_1 < t_2 \leq t_0$ satisfying $\eta(t_1,(u,v)) \in \partial\mathcal{M}_{2\rho}$, $\eta(t_2,(u,v)) \in \partial\mathcal{M}_\rho$, and $\eta(t,(u,v)) \in \mathcal{M}_{2\rho}\backslash\mathcal{M}_\rho$ for any $t \in (t_1,t_2)$. Then by (2.27) we have

$$\rho \leq \left\|\eta\big(t_1,(u,v)\big) - \eta\big(t_2,(u,v)\big)\right\|_{H_r} = \left\|\int_{t_1}^{t_2} V\big(\eta\big(t,(u,v)\big)\big)\right\|_{H_r} \leq 2T_0(t_2-t_1),$$

so $\frac{\rho}{2T_0} \leq t_2 - t_1 \leq t_0 - 0 < \frac{\rho}{2T_0}$, this yields a contradiction.

For $\varepsilon_0 > 0$, there exists $(u,v) \in B$ such that $\eta(\frac{\rho}{2T_0},(u,v)) \in B_0\backslash\mathcal{P}_\delta$ satisfies

$$d_{\lambda,\delta}^{k_1,k_2} \leq \sup_{B_0\backslash\mathcal{P}_\delta} \Psi_\lambda(u,v) < \Psi_\lambda\left(\eta\left(\frac{\rho}{2T_0},(u,v)\right)\right) + \frac{T\varepsilon_0}{2}.$$

Moreover, $\eta(t,(u,v)) \in B_{\widetilde{m},\lambda}$ for any $t \geq 0$, then by Lemma 2.9(4), $\eta(t,(u,v)) \notin \mathcal{P}_\delta$ for any $t \in [0,\frac{\rho}{2T_0}]$. Therefore,

$$\eta\big(t,(u,v)\big) \in B_{\widetilde{m}}\backslash(\mathcal{P}_\delta \cup \mathcal{M}_\rho). \tag{2.30}$$

In particular, $(u,v) \notin P_\delta$. Moreover, by (2.29) and Lemma 2.9 (3), we get

$$\begin{aligned} d_{\lambda,\delta}^{k_1,k_2} &\leq \sup_{B_0\backslash\mathcal{P}_\delta} \Psi_\lambda(u,v) < \Psi_\lambda\left(\eta\left(\frac{\rho}{2T_0},(u,v)\right)\right) + \frac{T\varepsilon_0}{2} \\ &\leq \Psi_\lambda\big(\eta\big(t,(u,v)\big)\big) + \frac{T\varepsilon_0}{2} \\ &\leq \Psi_\lambda(u,v) + \frac{T\varepsilon_0}{2} < d_{\lambda,\delta}^{k_1,k_2+1} + \frac{T\varepsilon_0}{2} + \frac{T\varepsilon_0}{2}, \end{aligned} \tag{2.31}$$

that is,

$$\big|\Psi_\lambda(u,v) - d\big| \leq \frac{T\varepsilon_0}{2} < 2\varepsilon_0.$$

So we see from (2.26) and Lemma 2.8 that

$$\begin{aligned} \frac{d}{dt}\Psi_\lambda\big(\eta\big(t,(u,v)\big)\big) &= -\big\langle\nabla\Psi_\lambda\big(\eta\big(t,(u,v)\big)\big), V\big(\eta\big(t,(u,v)\big)\big)\big\rangle_{H_r} \\ &\leq -\frac{T_2^2}{2}\left\|V\big(\eta\big(t,(u,v)\big)\big)\right\|_{H_r}^2 \leq -\frac{T_2^2}{2}\varepsilon_0. \end{aligned} \tag{2.32}$$

Finally, we deduce from (2.28), (2.31), and (2.32) that

$$
\begin{aligned}
d_{\lambda,\delta}^{k_1,k_2} &< \Psi_\lambda\left(\eta\left(\frac{\rho}{2T_0},(u,v)\right)\right) + \frac{T\varepsilon_0}{2} \\
&\le \Psi_\lambda(u,v) + \frac{T\varepsilon_0}{2} - \int_0^{\frac{\rho}{2T_0}} \frac{T_2^2}{2}\varepsilon_0\,dt \\
&< d_{\lambda,\delta}^{k_1,k_2} + \frac{T\varepsilon_0}{2} + \frac{T\varepsilon_0}{2} - \frac{T_2^2}{2}\varepsilon_0\frac{\rho}{2T_0} \\
&= d_{\lambda,\delta}^{k_1,k_2} + \frac{\varepsilon_0}{2}\left(2T - \frac{T_2^2\rho}{2T_0}\right) \le d_{\lambda,\delta}^{k_1,k_2},
\end{aligned}
$$

this yields a contradiction. This completes the proof. □

3 Proof of Theorem 1.2

Using Theorem 1.1, for $k = 1$, there exists $\lambda_1 > 0$ such that system (1.1) has a radially symmetric sign-changing solution (u_1,v_1) for any $\lambda \in (0,\lambda_1)$ and for $k_1 = k_2 = 2$,

$$
\Phi_\lambda(u_1,v_1) = d_{\lambda,\delta}^{2,2} < c_1.
$$

Let

$$
U_\lambda := \left\{(u,v) \in H_r : (u,v) \text{ is a sign-changing solution of } (1.1)\right\},
$$

then $U_\lambda \ne \emptyset$ by Theorem 1.1, we can define

$$
d_\lambda := \inf_{(u,v)\in U_\lambda} \Phi_\lambda(u,v)
$$

and $d_\lambda < c_1$. Let $(u_n,v_n) \in U_\lambda$ be a minimizing sequence of d_λ with $\Phi_\lambda(u_n,v_n) \to d_\lambda$, $\Phi_\lambda(u_n,v_n) < c_1$ and $\Phi_\lambda'(u_n,v_n) = 0$. Then

$$
\begin{aligned}
&\left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\|u_n\|_\alpha^2 + \|v_n\|_\beta^2\right) \\
&\quad \le \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\|u_n\|_\alpha^2 + \|v_n\|_\beta^2\right) + \left(\frac{1}{p+1} - \frac{1}{q+1}\right)|v_n|_{q+1}^{q+1} \\
&\qquad + \frac{2}{p+1}\left(\frac{1}{p+1} - \frac{1}{q+1}\right)\lambda\int_{\mathbb{R}^3} |u_n|^{\frac{p+1}{2}}|v_n|^{\frac{q+1}{2}}\,dx \\
&\quad = \Phi_\lambda(u_n,v_n) - \frac{1}{p+1}\Phi_\lambda'(u_n,v_n)(u_n,v_n) < c_1.
\end{aligned}
\tag{3.1}
$$

Observe that $\{(u_n,v_n)\}_{n\ge1}$ is bounded in H_r, we may assume that, up to a subsequence,

$$
\begin{aligned}
&(u_n,v_n) \rightharpoonup (u_0,v_0) \quad \text{weakly in } H_r, \\
&u_n \to u_0, \quad \text{strongly in } L^{p+1}(\mathbb{R}^3), \\
&v_n \to v_0, \quad \text{strongly in } L^{q+1}(\mathbb{R}^3).
\end{aligned}
$$

Since $\Phi'_\lambda(u_n, v_n) = 0$, it is standard to prove that

$$(u_n, v_n) \to (u_0, v_0) \quad \text{strongly in } H_r,$$

and $\Phi'_\lambda(u_0, v_0) = 0$, $\Phi_\lambda(u_0, v_0) = d_\lambda$.

Moreover, $\Phi'_\lambda(u_n, v_n)(u_n^\pm, 0) = 0$ and $\Phi'_\lambda(u_n, v_n)(0, v_n^\pm) = 0$, we deduce from (2.7) and (3.1) that

$$\begin{aligned} S_p\left|u_n^\pm\right|_{p+1}^2 \le \left\|u_n^\pm\right\|_\alpha^2 &= \left|u_n^\pm\right|_{p+1}^{p+1} + \frac{2}{q+1}\lambda \int_{\mathbb{R}^3} \left|u_n^\pm\right|^{\frac{p+1}{2}} |v_n|^{\frac{q+1}{2}}\,dx \\ &\le \left|u_n^\pm\right|_{p+1}^{p+1} + \frac{2}{q+1}\lambda \left|u_n^\pm\right|_{p+1}^{\frac{p+1}{2}} |v_n|_{q+1}^{\frac{q+1}{2}} \\ &< \left|u_n^\pm\right|_{p+1}^{p+1} + \frac{2}{q+1}\left[\frac{c_1}{(\frac{1}{2} - \frac{1}{p+1})S_q}\right]^{\frac{q+1}{4}} \lambda \left|u_n^\pm\right|_{p+1}^{\frac{p+1}{2}}. \end{aligned}$$

We can choose $0 < \lambda_0 < \lambda_1$ small enough such that for any $\lambda \in (0, \lambda_0)$ we have

$$S_p\left|u_n^\pm\right|_{p+1}^2 < 2\left|u_n^\pm\right|_{p+1}^{p+1},$$

which implies $|u_n^\pm|_{p+1} \ge \xi_1 > 0$ for any $n \ge 1$. Similarly, $|v_n^\pm|_{q+1} \ge \xi_2 > 0$ for any $n \ge 1$. Therefore, $|u_0^\pm|_{p+1} \ge \xi_1 > 0$, $|v_0^\pm|_{q+1} \ge \xi_2 > 0$, and so Eq. (1.1) has a least energy sign-changing solution (u_0, v_0). This completes the proof. □

4 The proof of Theorem 1.3

In this section, we obtain seminodal solutions (u, v) such that u is positive, v is sign-changing and use the same notations as in Sect. 2 for convenience. Define the C^1 functional

$$\begin{aligned} \Phi_\lambda(u, v) := \frac{1}{2}\left(\|u\|_\alpha^2 + \|v\|_\beta^2\right) - \frac{1}{p+1}\left|u^+\right|_{p+1}^{p+1} - \frac{1}{q+1}|v|_{q+1}^{q+1} \\ - \frac{4\lambda}{(p+1)(q+1)} \int_{\mathbb{R}^3} |u|^{\frac{p+1}{2}} |v|^{\frac{q+1}{2}}\,dx, \end{aligned}$$

where $(u, v) \in \widetilde{H}_r := \{(u, v) \in H_r : u^+ \ne 0, v \ne 0\}$,

$$\mathcal{A} := \left\{(u, v) \in H_r : \left|u^+\right|_{p+1} = 1, |v|_{q+1} = 1\right\},$$

$$\mathcal{A}^* := \left\{(u, v) \in H_r : \frac{1}{2} < \left|u^+\right|_{p+1}^{p+1} < 2, \frac{1}{2} < |v|_{q+1}^{q+1} < 2\right\},$$

$$\mathcal{B}_m^* := \left\{(u, v) \in \mathcal{A}^* : \|u\|_\alpha^2 < m, \|v\|_\beta^2 < m\right\}, \qquad \mathcal{B}_m := \mathcal{B}_m^* \cap \mathcal{A}.$$

As in Sect. 2, for any $(u, v) \in \mathcal{A}$, we define

$$\sup_{t,s\ge 0} \Phi_\lambda(tu, sv) = \Phi_\lambda(t_{u,v,\lambda}u, s_{u,v,\lambda}v) =: \Psi_\lambda(u, v). \tag{4.1}$$

It is easy to prove that Lemma 2.4 also holds in this section by trivial modifications. Then define

$$B_{\widetilde{m},\lambda} := \left\{(u, v) \in B_{\widetilde{m}} : \sup_{t,s\ge 0} \Phi_\lambda(tu, sv) < c_k\right\}.$$

For any $(u,v) \in \mathcal{B}^*_{\widetilde{m}}$, $\lambda \in (0,\lambda_k)$, we consider the following linear problem:

$$\begin{cases} -\Delta\varphi + \alpha\varphi - \frac{2}{q+1} t_{u,v,\lambda}^{\frac{p-3}{2}} s_{u,v,\lambda}^{\frac{q+1}{2}} \lambda |u|^{\frac{p-3}{2}} \varphi |v|^{\frac{q+1}{2}} = t_{u,v,\lambda}^{p-1}(u^+)^p, \\ -\Delta\psi + \beta\psi - \frac{2}{p+1} t_{u,v,\lambda}^{\frac{p+1}{2}} s_{u,v,\lambda}^{\frac{q-3}{2}} \lambda |u|^{\frac{p+1}{2}} |v|^{\frac{q-3}{2}} \psi = s_{u,v,\lambda}^{q-1} |v|^{q-1} v, \\ \varphi(x) \to 0, \qquad \psi(x) \to 0, \quad \text{as } |x| \to \infty, \end{cases} \tag{4.2}$$

then (4.2) has a unique solution $(\varphi,\psi) \in H_r \backslash \{(0,0)\}$. Define

$$\mu := \frac{1}{\int_{\mathbb{R}^3} (u^+)^p \varphi \, dx} > 0, \qquad \nu := \frac{1}{\int_{\mathbb{R}^3} |v|^{q-1} v\psi \, dx} > 0.$$

Then $(\widetilde{\varphi},\widetilde{\psi}) := (\mu\varphi, \nu\psi)$ is the unique solution of the following problem:

$$\begin{cases} -\Delta\widetilde{\varphi} + \alpha\widetilde{\varphi} - \frac{2}{q+1} t_{u,v,\lambda}^{\frac{p-3}{2}} s_{u,v,\lambda}^{\frac{q+1}{2}} \lambda |u|^{\frac{p-3}{2}} \widetilde{\varphi} |v|^{\frac{q+1}{2}} = \mu t_{u,v,\lambda}^{p-1}(u^+)^p, \\ -\Delta\widetilde{\psi} + \beta\widetilde{\psi} - \frac{2}{p+1} t_{u,v,\lambda}^{\frac{p+1}{2}} s_{u,v,\lambda}^{\frac{q-3}{2}} \lambda |u|^{\frac{p+1}{2}} |v|^{\frac{q-3}{2}} \widetilde{\psi} = \nu s_{u,v,\lambda}^{q-1} |v|^{q-1} v, \\ \int_{\mathbb{R}^3} (u^+)^p \widetilde{\varphi} \, dx = \int_{\mathbb{R}^3} |v|^{q-1} v \widetilde{\psi} \, dx = 1, \\ \widetilde{\varphi}(x) \to 0, \qquad \widetilde{\psi}(x) \to 0, \quad \text{as } |x| \to \infty. \end{cases} \tag{4.3}$$

We can now also define the operator

$$\begin{aligned} &K : B^*_{\widetilde{m}} \to H_r; \qquad (u,v) \mapsto (\widetilde{\varphi},\widetilde{\psi}), \\ &K\bigl(\sigma_2(u,v)\bigr) = \sigma_2\bigl(K(u,v)\bigr). \end{aligned} \tag{4.4}$$

Then, by similar proofs as in Lemma 2.5 and Lemma 2.6, we have that $K \in C^1(B^*_{\widetilde{m}}, H_r)$ and K satisfies the Palais–Smale type condition. Define the map

$$V : B^*_{\widetilde{m}} \to H_r; \qquad (u,v) \mapsto (u,v) - K(u,v).$$

Consider the class of sets

$$\mathcal{F} = \bigl\{A \in \mathcal{A} : A \text{ is a closed set and } \sigma_2(u,v) \in A, \forall (u,v) \in A\bigr\} \tag{4.5}$$

for each $A \in \mathcal{F}$ and $k_2 \geq 2$, the class of functions

$$F_{(1,k_2)}(A) = \bigl\{f : A \to \mathbb{R}^{k_2-1} : f \text{ continuous and } f\bigl(\sigma_2(u,v)\bigr) = -f(u,v)\bigr\}. \tag{4.6}$$

To obtain seminodal solutions, we should also define a cone of positive functions, that is,

$$\begin{aligned} &\mathcal{P}_2 := \bigl\{(u,v) \in H_r : v \geq 0\bigr\}, \qquad \mathcal{P} = \mathcal{P}_2 \cup -\mathcal{P}_2, \\ &\operatorname{dist}_{q+1}\bigl((u,v),\mathcal{P}\bigr) := \min\bigl\{\operatorname{dist}_{q+1}(v,\mathcal{P}_2), \operatorname{dist}_{q+1}(v,-\mathcal{P}_2)\bigr\}, \end{aligned} \tag{4.7}$$

thus, v is sign-changing if $\operatorname{dist}_{q+1}((u,v),\mathcal{P}) > 0$.

Under the new definitions (4.4)–(4.6), we define vector genus, slightly different from Definition 2.1.

Definition 4.1 Let $A \in \mathcal{F}$ and take any $k_2 \in \mathbb{N}$ with $k_2 \geq 2$. We say that $\gamma(A) \geq (1, k_2)$ if for every $f \in F_{(1,k_2)}(A)$ there exists $(u, v) \in A$ such that $f(u, v) = 0$. We denote

$$\Gamma^{(1,k_2)} := \{A \in \mathcal{F} : \gamma(A) \geq (1, k_2)\}.$$

Lemma 4.1

(1) *Take $A := A_1 \times A_2 \subset \mathcal{A}$ and let $\eta : S^{k_2-1} \to A_2$ be a homeomorphism such that $\eta(-x) = -\eta(x)$ for every $x \in S^{k_2-1}$. Then $A \in \Gamma^{(1,k_2)}$;*

(2) *We have $\overline{\eta(A)} \in \Gamma^{(1,k_2)}$ whenever $A \in \Gamma^{(1,k_2)}$ and a continuous map $\eta : A \to \mathcal{A}$ is such that $\eta \circ \sigma_2 = \sigma_2 \circ \eta$.*

Proof (1) For every $f \in F_{(1,k_2)}(A)$ and $u \in A_1$, we define a map

$$h : S^{k_2-1} \to \mathbb{R}^{k_2-1}; \qquad h(x) := f\bigl(u, \eta(x)\bigr),$$

then by (4.6) it is easy to see that h is continuous and

$$h(-x) = f\bigl(u, \eta(-x)\bigr) = f\bigl(u, -\eta(x)\bigr) = -f\bigl(u, \eta(x)\bigr) = -h(x).$$

Then Borsuk–Ulam theorem yields $x_0 \in S^{k_2-1}$ such that $h(x_0) = f(u, \eta(x_0)) = 0$. By Definition 4.1, we have $A \in \Gamma^{(1,k_2)}$.

(2) Fix any $f \in F_{(1,k_2)}(\overline{\eta(A)})$, then by (4.6) we have $f \circ \eta \in F_{(1,k_2)}(A)$. Since $A \in \Gamma^{(1,k_2)}$, there exists $(u_0, v_0) \in A$ such that $f \circ \eta(u_0, v_0) = 0$. Then by $\eta(u_0, v_0) \in \overline{\eta(A)}$ we have $\gamma(\overline{\eta(A)}) \geq (1, k_2)$, that is, $\overline{\eta(A)} \in \Gamma^{(1,k_2)}$. This completes the proof. □

Lemma 4.2 *Assume $k_2 \geq 2$. Then, for any $0 < \delta < 2^{-\frac{1}{q+1}}$ and $A \in \Gamma^{(1,k_2)}$, we have $A \backslash \mathcal{P}_\delta \neq \emptyset$.*

Proof For any $A \in \Gamma^{(1,k_2)}$, define f by

$$f(u, v) = \left(\int_{\mathbb{R}^3} |v|^q v \, dx, 0, \ldots, 0\right),$$

then $f \in F_{(1,k_2)}(A)$, so by Definition 4.1, there exists $(u_0, v_0) \in A$ such that $f(u_0, v_0) = 0$. We deduce from $A \in \mathcal{A}$ that

$$\int_{\mathbb{R}^3} \bigl(v_0^+\bigr)^{q+1} \, dx = \int_{\mathbb{R}^3} \bigl(v_0^-\bigr)^{q+1} \, dx = \frac{1}{2}.$$

Therefore, $\operatorname{dist}_{q+1}((u_0, v_0), \mathcal{P}) = 2^{-\frac{1}{q+1}}$, and so $(u_0, v_0) \in A \backslash \mathcal{P}_\delta$ for any $0 < \delta < 2^{-\frac{1}{q+1}}$. This completes the proof. □

Fixed any $k \in \mathbb{N}$, we define

$$A_1 := \left\{cu_0 : c = \frac{1}{|u_0|_{p+1}}, u_0 > 0\right\}, \qquad A_2 := \bigl\{v \in X_{k+1} : |v|_{q+1} = 1\bigr\}.$$

By Lemma 4.1(1), $A := A_1 \times A_2 \in \Gamma^{(1,k+1)}$, $A \subset B_{\widetilde{m}}$, and $\sup_A \Psi_\lambda(u, v) < c_k$. Then we can define

$$\Gamma_\lambda^{(1,k_2)} := \Bigl\{A \in \Gamma^{(1,k_2)} : A \subset B_{\widetilde{m}}, \sup_A \Psi_\lambda(u, v) < c_k\Bigr\}.$$

For any $k_2 \in [2, k+1]$ and $0 < \delta < 2^{-\frac{1}{q+1}}$, we define a sequence of minimax energy level:

$$d_{\lambda,\delta}^{1,k_2} := \inf_{A \in \Gamma_\lambda^{(1,k_2)}} \sup_{A \setminus \mathcal{P}_\delta} \sup_{t,s \geq 0} \Phi_\lambda(tu, sv).$$

It is easy to see that

$$d_{\lambda,\delta}^{1,k_2} < c_k \quad \text{for any } 0 < \delta < 2^{-\frac{1}{q+1}} \text{ and } 2 \leq k_2 \leq k+1.$$

Lemma 2.7 and Lemma 2.8 also hold in Sect. 4.

Lemma 4.3 *There exists a unique global solution $\eta : \mathbb{R}^+ \times B_{\widetilde{m},\lambda} \to H_r$ for the initial value problem*

$$\begin{cases} \frac{d}{dt}\eta(t,(u,v)) = -V(\eta(t,(u,v))), \\ \eta(0,(u,v)) = (u,v) \in B_{\widetilde{m},\lambda}. \end{cases} \tag{4.8}$$

Moreover, (1), (3), (4) *of Lemma* 2.9 *hold and*

(2) *For any $t > 0$, $(u,v) \in B_{\widetilde{m},\lambda}$, $\eta(t, \sigma_2(u,v)) = \sigma_2(\eta(t,(u,v)))$.*

Proof From the above discussion, we see that $V \in C^1(B_{\widetilde{m}}^*, H_r)$. As $B_{\widetilde{m},\lambda} \subset B_{\widetilde{m}} \subset B_{\widetilde{m}}^*$, we get that $V \in C^1(B_{\widetilde{m},\lambda}, H_r)$, then there exists a solution $\eta : [0, T_{\max}) \times B_{\widetilde{m},\lambda} \to H_r$, where $T_{\max}$ is the maximal time such that (4.8) has s solution $\eta \in B_{\widetilde{m}}^*$.

For any $(u,v) \in B_{\widetilde{m},\lambda}$ and $t \in (0, T_{\max})$, there holds

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^3} \bigl(\eta_1^+(t,(u,v))\bigr)^{p+1}\,dx \\ &\quad = -(p+1)\int_{\mathbb{R}^3} \bigl(\eta_1^+(t,(u,v))\bigr)^p V\bigl(\eta_1^+(t,(u,v))\bigr)\,dx \\ &\quad = -(p+1)\int_{\mathbb{R}^3} \bigl(\eta_1^+(t,(u,v))\bigr)^p \bigl[\eta_1^+(t,(u,v)) - K_1\bigl(\eta^+(t,(u,v))\bigr)\bigr]\,dx \\ &\quad = (p+1) - (p+1)\int_{\mathbb{R}^3} \bigl(\eta_1^+(t,(u,v))\bigr)^{p+1}\,dx, \end{aligned}$$

so we have

$$\frac{d}{dt}\left[e^{(p+1)t}\left(\int_{\mathbb{R}^3} \bigl(\eta_1^+(t,(u,v))\bigr)^{p+1}\,dx - 1\right)\right] = 0.$$

Since $\int_{\mathbb{R}^3}(\eta_1^+(0,(u,v)))^{p+1}\,dx = \int_{\mathbb{R}^3}(u^+)^{p+1}\,dx = 1$, then for any $t \in [0, T_{\max})$,

$$\int_{\mathbb{R}^3} \bigl(\eta_1^+(t,(u,v))\bigr)^{p+1}\,dx \equiv 1.$$

The rest of the proof is the same as Lemma 2.9. This completes the proof. □

Proof of Theorem 1.2 Observe that from Lemma 2.10, for any $k_2 \in [2, k+1]$, $0 < \delta < \delta_0$ small, there exists $(u_0, v_0) \in B_{\widetilde{m}}$ such that

$$\Psi_\lambda(u_0,v_0) = d_{\lambda,\delta}^{1,k_2}, \qquad V(u_0,v_0) = 0 \quad \text{and} \quad \operatorname{dist}_{q+1}\bigl((u_0,v_0), \mathcal{P}\bigr) \geq \delta.$$

We conclude that v_0 is sign-changing and $(u_0, v_0) = K(u_0, v_0) = (\widetilde{\varphi}_0, \widetilde{\psi}_0)$. It follows from (4.3) that (u_0, v_0) satisfies

$$\begin{cases} -\Delta u_0 + \alpha u_0 = \mu t_{u,v,\lambda}^{p-1}(u_0^+)^p + \frac{2}{q+1} t_{u_0,v_0,\lambda}^{\frac{p-3}{2}} s_{u_0,v_0,\lambda}^{\frac{q+1}{2}} \lambda |u_0|^{\frac{p-3}{2}} u_0 |v_0|^{\frac{q+1}{2}}, \\ -\Delta v_0 + \beta v_0 = \nu s_{u_0,v_0,\lambda}^{q-1} |v_0|^{q-1} v_0 + \frac{2}{p+1} t_{u_0,v_0,\lambda}^{\frac{p+1}{2}} s_{u_0,v_0,\lambda}^{\frac{q-3}{2}} \lambda |u_0|^{\frac{p+1}{2}} |v_0|^{\frac{q-3}{2}} v_0, \\ u_0(x) \to 0, \qquad v_0(x) \to 0, \quad \text{as } |x| \to \infty, \end{cases} \tag{4.9}$$

and $|u_0^+|_{p+1} = |v_0|_{q+1} = 1$, then by (4.1) we have $\mu = \nu = 1$. Moreover, (4.9) yields

$$\|u_0^-\|_\alpha^2 = \frac{2}{q+1} t_{u_0,v_0,\lambda}^{\frac{p-3}{2}} s_{u_0,v_0,\lambda}^{\frac{q+1}{2}} \lambda \int_{\mathbb{R}^3} |u_0|^{\frac{p-3}{2}} (u_0^-)^2 |v_0|^{\frac{q+1}{2}}.$$

We can take λ_k small enough if necessary such that for any $\lambda \in (0, \lambda_k)$ and $(u_0, v_0) \in \mathcal{B}_{\widetilde{m}}^*$,

$$\|u_0^-\|_\alpha^2 - \frac{2}{q+1} t_{u_0,v_0,\lambda}^{\frac{p-3}{2}} s_{u_0,v_0,\lambda}^{\frac{q+1}{2}} \lambda \int_{\mathbb{R}^3} |u_0|^{\frac{p-3}{2}} (u_0^-)^2 |v_0|^{\frac{q+1}{2}} \geq \frac{1}{2} \|u_0^-\|_\alpha^2,$$

then $\|u_0^-\|_\alpha^2 = 0$, so $u_0 \geq 0$. By the strong maximum principle, $u_0 > 0$. Hence we have that $(t_{u_0,v_0,\lambda} u_0, s_{u_0,v_0,\lambda} v_0)$ is a seminodal solution of (1.1) with $t_{u_0,v_0,\lambda} u_0$ positive and $s_{u_0,v_0,\lambda} v_0$ sign-changing,

$$\Phi_\lambda(t_{u_0,v_0,\lambda} u_0, s_{u_0,v_0,\lambda} v_0) = \Psi_\lambda(u_0, v_0) = d_{\lambda,\delta}^{1,k_2}.$$

By similar proof as Theorem 1.1, we complete the proof. □

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Data Availability
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