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Impulsive coupled systems with regular and singular ϕ-Laplacians and generalized jump conditions

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Abstract

This work contains sufficient conditions for the solvability of a third-order coupled system with two differential equations involving different Laplacians, fully discontinuous nonlinearities, two-point boundary conditions, and two sets of impulsive effects. The first existing result is obtained from Schauder's fixed point theorem, and the second one provides also the localization of a solution via the lower and upper solutions technique.

We point out that it is the first time that impulsive coupled systems with strongly nonlinear fully differential equations and generalized impulse effects are considered simultaneously. Moreover, the singular case is applied to a special relativity model in classical electrodynamics.

Mathematics Subject Classification: 34A34; 34B37; 34B16; 34B15

Keywords: Coupled systems; Regular and singular Laplacian equations; Generalized impulsive conditions; Special relativity

1 Introduction

In this article we consider the third-order impulsive coupled system

$$\begin{cases} (\phi(u''(x)))' + f(x,u(x),u'(x),u''(x),v(x),v'(x),v''(x)) = 0, & x \in M, \\ (\psi(v''(x)))' + g(x,u(x),u'(x),u''(x),v(x),v'(x),v''(x)) = 0, & x \in N, \end{cases} \tag{1.1}$$

where $M = [a,b]\backslash\{x_1,\dots,x_m\}$ and $N = [a,b]\backslash\{\tau_1,\dots,\tau_n\}$, ϕ, $\psi : \mathbb{R} \to \mathbb{R}$ are increasing homeomorphisms such that $\phi(0) = \psi(0) = 0$ and $\phi(\mathbb{R}) = \psi(\mathbb{R}) = \mathbb{R}$, $f,g : [a,b] \times \mathbb{R}^6 \mapsto \mathbb{R}$ are L^1-Carathéodory functions, together with the boundary conditions

$$\begin{cases} u(a) = A_0, & u'(a) = A_1, & u''(b) = A_2, \\ v(a) = B_0, & v'(a) = B_1, & v''(b) = B_2, \end{cases} \tag{1.2}$$

with $A_k, B_k \in \mathbb{R}$, $k = 0,1,2$.

The impulsive conditions are given by

$$\begin{aligned}
&\Delta u(x_i) = I_{0i}\big(x_i, u(x_i), u'(x_i), v(x_i), v'(x_i)\big),\\
&\Delta u'(x_i) = I_{1i}\big(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i)\big),\\
&\Delta \phi\big(u''(x_i)\big) = I_{2i}\big(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i), v''(x_i)\big),\\
&\Delta v(\tau_j) = J_{0j}\big(\tau_j, u(\tau_j), u'(\tau_j), v(\tau_j), v'(\tau_j)\big),\\
&\Delta v'(\tau_j) = J_{1j}\big(\tau_j, u(\tau_j), u'(\tau_j), v(\tau_j), v'(\tau_j), v''(\tau_j)\big),\\
&\Delta \psi\big(v''(\tau_j)\big) = J_{2j}\big(\tau_j, u(\tau_j), u'(\tau_j), u''(\tau_j), v(\tau_j), v'(\tau_j), v''(\tau_j)\big),
\end{aligned} \tag{1.3}$$

where $\Delta u(x_i) = u(x_i^+) - u(x_i^-)$, $i = 1, 2, \dots, m$, $\Delta v(\tau_j) = v(\tau_j^+) - v(\tau_j^-)$, $j = 1, 2, \dots, n$, $I_{0i}, J_{0j} \in C([a,b] \times \mathbb{R}^4, \mathbb{R})$, $I_{1i}, J_{1j} \in C([a,b] \times \mathbb{R}^5, \mathbb{R})$, $I_{2i}, J_{2j} \in C([a,b] \times \mathbb{R}^6, \mathbb{R})$, and x_k are fixed points such that $a = x_0 < x_1 < x_2 < \cdots < x_m < x_{m+1} = b$, $a = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_n < \tau_{n+1} = b$.

For a particular case, without jumps on the ϕ, ψ-Laplacians, that is, for

$$\begin{aligned}
&\Delta u(x_i) = I_{0i}\big(x_i, u(x_i), u'(x_i), v(x_i), v'(x_i)\big),\\
&\Delta u'(x_i) = I_{1i}\big(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i)\big),\\
&\Delta v(\tau_j) = J_{0j}\big(\tau_j, u(\tau_j), u'(\tau_j), v(\tau_j), v'(\tau_j)\big),\\
&\Delta v'(\tau_j) = J_{1j}\big(\tau_j, u(\tau_j), u'(\tau_j), v(\tau_j), v'(\tau_j), v''(\tau_j)\big),
\end{aligned} \tag{1.4}$$

an existence and localization theorem is proved, where we present the sufficient assumptions to localize a solution in a strip bounded by lower and upper solutions.

Usually ϕ and ψ are known as ϕ, ψ-Laplacian as they generalize the one-dimensional Laplacian and the p-Laplacian, and they were used by many authors in a broad range of problems. Some examples: [34] to obtain a positive periodic solution for a φ-Laplacian Liénard equation with a singularity; [21] proving the multiplicity of solutions of p-Laplacian Dirichlet boundary value problem with discontinuous nonlinearities; [35] giving sufficient conditions for the existence of at least three positive solutions of one-dimensional p-Laplacian boundary value problem; [7, 31] to obtain positive solutions for some p-Laplacian problem in superlinear cases; and [29] based on nonnegative nonlinearities under a version of the Krasnosel'skii expansion and compression cone theory.

Beyond the classic regular Laplacians, the singular cases, that is, homeomorphisms $\phi : (-a, a) \to \mathbb{R}$ with $0 < a < +\infty$, have been recently studied by several authors, such as, for example: [15, 16] for p-Laplacian; [2, 3, 26] with existence and multiplicity results; [5] obtaining heteroclinic solutions; and [12] for equations on the half-line with functional boundary conditions.

Nonlinear coupled systems, where the unknown functions and their derivatives can interact, have been considered in several works in recent years, such as, among others: [30] via Schauder's fixed point theorem; [11] for fractional differential equations at resonance applying coincidence degree theory; [24] including the study of different types of differential and integral equations; [36] via lower and upper solutions technique; and [18] applied to reaction–diffusion Robin problems.

Impulsive differential equations model many real phenomena in which the nonlinearities have sudden discontinuous jumps in their values. These types of events can occur in population dynamics, control, and optimization theory, ecology, biology and biotechnology, economics, pharmacokinetics, and other physics and mechanics problems. For some examples of the approach to impulsive differential equations, we refer to: [22] for a general theory; [19] via fixed point index; [23, 27] applied to functional impulsive problems; and [38] with a monotone iterative technique for approximating the solution. The study of ϕ-Laplacian impulsive problems can be seen, for instance, in: [17] in periodic problems applying a continuation theorem; [9, 10] for bounded and unbounded intervals; [33] for fractional equations with p-Laplacian; and [1, 8, 28] for Brownian motion.

Combining all these areas and results, we consider, to the best of our knowledge, for the first time the methods and techniques suggested in, for example, [6, 14] to an impulsive coupled system with fully differential equations including different regular and singular Laplacians and generalized impulsive conditions, whose jumps depend on both variables and some of its derivatives.

The paper is organized as follows: Sect. 2 contains the functional framework and some preliminary results, namely the explicit solution for the associated impulsive linear problem, Nagumo-type growth conditions, and *a priori* bounds for the second derivatives. In Sect. 3, we present an existence theorem for the general case. Section 4 contains an existence and localization result applied to a particular case of the initial impulsive conditions and a concrete example to show the applicability of the localization tool. Section 5 applies our method to the singular case and to special relativity theory.

2 Definitions and preliminary results

This section introduces some preliminary results and the functional framework.

Define

$$y\left(x_\kappa^{\pm}\right) := \lim_{x \to x_\kappa^{\pm}} y(x),$$

and consider the sets of piecewise continuous functions:

$$PC_1\left([a,b]\right) := \left\{ \begin{matrix} u : u \in C\left([a,b],\mathbb{R}\right) \text{ continuous for } x \neq x_i, u(x_i) = u\left(x_i^-\right), \\ u\left(x_i^+\right) \text{ is finite for } i = 1,2,3,\dots,m \end{matrix} \right\},$$

$$PC_2\left([a,b]\right) := \left\{ \begin{matrix} v : v \in C\left([a,b],\mathbb{R}\right) \text{ continuous for } x \neq \tau_j, v(\tau_j) = v\left(\tau_j^-\right), \\ v\left(\tau_j^+\right) \text{ is finite for } j = 1,2,3,\dots,n \end{matrix} \right\},$$

and

$$PC_k^l[a,b] = \left\{ y : y^{(l)} \in PC_k[a,b], l,k = 1,2 \right\}.$$

Let $X_k := PC_k^2[a,b]$, $k = 1,2$, be the usual Banach space equipped with the norm $\|\cdot\|_\infty$, defined by

$$\|y\|_{X_k} := \max\left\{ \|y\|_\infty, \|y'\|_\infty, \|y''\|_\infty \right\},$$

where

$$\|y\|_\infty := \sup_{a \leq x \leq b} |y(x)|$$

and $X^2 := X_1 \times X_2$ with the norm

$$\|(u,v)\|_{X^2} = \max\{\|u\|_{X_1}, \|v\|_{X_2}\}.$$

Definition 2.1 A function $h : [a,b] \times \mathbb{R}^6 \to \mathbb{R}$ is L^1-Carathéodory if

i. For each $(y_0,y_1,y_2,z_0,z_1,z_2) \in \mathbb{R}^6$, $x \mapsto h(x,y_0,y_1,y_2,z_0,z_1,z_2)$ is measurable on $[a,b]$;
ii. For almost every $x \in [a,b]$, $(y_0,y_1,y_2,z_0,z_1,z_2) \mapsto h(x,y_0,y_1,y_2,z_0,z_1,z_2)$ is continuous on $\mathbb{R}^6$;
iii. For each $L > 0$, there is a positive function $\rho_L \in L^1[a,b]$ such that, for a.e. $x \in [a,b]$ and $(y_0,y_1,y_2,z_0,z_1,z_2) \in \mathbb{R}^6$ with

$$\max\{|y_0|,|y_1|,|y_2|,|z_0|,|z_1|,|z_2|\} < L,$$

we have

$$\left|h(x,y_0,y_1,y_2,z_0,z_1,z_2)\right| \le \rho_L(x).$$

For (u,v) solution of problem (1.1)–(1.3), one must consider $(u(x),v(x)) \in X^2$ satisfying (1.1), boundary conditions (1.2), and impulsive effects (1.3).

The next lemma gives the unique solution for the homogeneous problem related to (1.1)–(1.3).

Lemma 2.2 *Let ϕ, $\psi : \mathbb{R} \to \mathbb{R}$ be increasing homeomorphisms and $p,q \in L^1[a,b]$.*

The problem composed by the differential system

$$\begin{cases} (\phi(u''(x)))' + p(x) = 0 \\ (\psi(v''(x)))' + q(x) = 0 \end{cases} \tag{2.1}$$

and conditions (1.2) *and* (1.3) *has a unique solution given by*

$$\begin{aligned} u(x) &= A_0 + A_1(x-a) + \sum_{i:x_i<x} I_{0i}\big(x_i,u(x_i),u'(x_i),v(x_i),v'(x_i)\big) \\ &\quad + (x-a)\sum_{i:x_i<x} I_{1i}\big(x_i,u(x_i),u'(x_i),u''(x_i),v(x_i),v'(x_i)\big) \\ &\quad + \int_a^x \int_a^r \phi^{-1}\Bigg(\phi(A_2) - \sum_{i:x_i>s} I_{2i}\big(x_i,u(x_i),u'(x_i),u''(x_i),v(x_i),v'(x_i),v''(x_i)\big) \\ &\quad + \int_s^b p(\xi)\,d\xi\Bigg)\,ds\,dr \end{aligned}$$

and

$$\begin{aligned} v(x) &= B_0 + B_1(x-a) + \sum_{j:\tau_j<x} J_{0j}\big(\tau_j,u(\tau_j),u'(\tau_j),v(\tau_j),v'(\tau_j)\big) \\ &\quad + (x-a)\sum_{j:\tau_j<x}\big(\tau_j,u(\tau_j),u'(\tau_j),v(\tau_j),v'(\tau_j),v''(\tau_j)\big) \end{aligned}$$

$$+\int_a^x \int_a^r \psi^{-1}\Big(\psi(B_2) - \sum_{j:\tau_j>s} J_{2j}\big(\tau_j, u(\tau_j), u'(\tau_j), u''(\tau_j), v(\tau_j), v'(\tau_j), v''(\tau_j)\big)$$
$$+\int_s^b q(\xi)\,d\xi\Big)\,ds\,dr.$$

Proof Integrating the first equation of (2.1), for $x \in (x_n, b]$, we have, by (1.2),

$$\phi\big(u''(x)\big) = \phi(A_2) + \int_x^b p(\xi)\,d\xi. \tag{2.2}$$

For $x \in (x_{n-1}, x_n]$, integrating (2.1), by (1.3) and (2.2), we obtain

$$\begin{aligned}\phi\big(u''(x)\big) &= \phi\big(u''(x_n^-)\big) + \int_x^{x_n} p(\xi)\,d\xi \\ &= \phi\big(u''(x_n^+)\big) - I_{2n}\big(x_n, u(x_n), u'(x_n), u''(x_n), v(x_n), v'(x_n), v''(x_n)\big) + \int_x^{x_n} p(\xi)\,d\xi \\ &= \phi(A_2) + \int_{x_n}^b p(\xi)\,d\xi - I_{2n}\big(x_n, u(x_n), u'(x_n), u''(x_n), v(x_n), v'(x_n), v''(x_n)\big) \\ &\quad + \int_x^{x_n} p(\xi)\,d\xi \\ &= \phi(A_2) - I_{2n}\big(x_n, u(x_n), u'(x_n), u''(x_n), v(x_n), v'(x_n), v''(x_n)\big) + \int_x^b p(\xi)\,d\xi.\end{aligned}$$

So, by mathematical induction, for $x \in [a, b]$,

$$\phi\big(u''(x)\big) = \phi(A_2) - \sum_{i:x_i>x} I_{2i}\big(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i), v''(x_i)\big) + \int_x^b p(\xi)\,d\xi,$$

and therefore

$$\begin{aligned}u''(x) = \phi^{-1}\Big(&\phi(A_2) - \sum_{i:x_i>x} I_{2i}\big(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i), v''(x_i)\big) \\ &+ \int_s^b p(\xi)\,d\xi\Big).\end{aligned} \tag{2.3}$$

By a new integration of (2.3) from a to x, when $x \in [a, x_1]$,

$$\begin{aligned}u'(x) = A_1 + \int_a^x \phi^{-1}\Big(&\phi(A_2) - \sum_{i:x_i>s} I_{2i}\big(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i), v''(x_i)\big) \\ &+ \int_s^b p(\xi)\,d\xi\Big)\,ds.\end{aligned} \tag{2.4}$$

According to (1.3), when $x \to x_1^+$, we have

$$u'\big(x_1^+\big) = u'\big(x_1^-\big) + I_{11}\big(x_1, u(x_1), u'(x_i), u''(x_1), v(x_1), v'(x_1)\big),$$

by (2.4),

$$
\begin{aligned}
u'(x_1^+) = A_1 + \int_a^{x_1} \phi^{-1}\Bigg(&\phi(A_2) - \sum_{i:x_i>s} I_{2i}\big(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i), v''(x_i)\big) \\
&+ \int_s^b p(\xi)\,d\xi\Bigg)\,ds \\
&+ I_{11}\big(x_1, u(x_1), u'(x_i), u''(x_1), v(x_1), v'(x_1)\big),
\end{aligned}
$$

and for $x \in [a,b]$,

$$
\begin{aligned}
u'(x) = [A_1 &+ \sum_{i:x_i<x} I_{1i}\big(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i)\big) \\
&+ \int_a^{x} \phi^{-1}\Bigg(\phi(A_2) - \sum_{i:x_i>s} I_{2i}\big(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i), v''(x_i)\big) \\
&+ \int_s^b p(\xi)\,d\xi\Bigg)\,ds.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
u(x_1^+) = A_0 + A_1(x_1 - a) &+ I_{01}\big(x_1, u(x_1), u'(x_1), v(x_1), v'(x_1)\big) \\
&+ (x_1 - a)\sum_{i:x_i<x} I_{1i}\big(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i)\big) \\
&+ \int_a^{x_1}\int_a^{r} \phi^{-1}\Bigg(\phi(A_2) - \sum_{i:x_i>s} I_{2i}\big(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i), v''(x_i)\big) \\
&+ \int_s^b p(\xi)\,d\xi\Bigg)\,ds\,dr,
\end{aligned}
$$

and by induction it can be proved that the solution of the first equation of problem (2.1), (1.2), (1.3), for $x \in [a,b]$, is given by

$$
\begin{aligned}
u(x) = A_0 + A_1(x - a) &+ \sum_{i:x_i<x} I_{0i}\big(x_i, u(x_i), u'(x_i), v(x_i), v'(x_i)\big) \\
&+ (x - a)\sum_{i:x_i<x} I_{1i}\big(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i)\big) \\
&+ \int_a^{x}\int_a^{r} \phi^{-1}\Bigg(\phi(A_2) - \sum_{i:x_i>s} I_{2i}\big(x_i, u(x_i), u'(x_i), u''(x_i), v(x_i), v'(x_i), v''(x_i)\big) \\
&+ \int_s^b p(\xi)\,d\xi\Bigg)\,ds\,dr.
\end{aligned}
$$

Likewise, for the second equation, we have

$$
\begin{aligned}
v(x) = B_0 + B_1(x - a) &+ \sum_{j:\tau_j<x} J_{0j}\big(\tau_j, u(\tau_j), u'(\tau_j), v(\tau_j), v'(\tau_j)\big) \\
&+ (x - a)\sum_{j:\tau_j<x} J_{0j}\big(\tau_j, u(\tau_j), u'(\tau_j), v(\tau_j), v'(\tau_j), v''(\tau_j)\big)
\end{aligned}
$$

$$+\int_a^x \int_a^r \psi^{-1}\left(\psi(B_2) - \sum_{j:\tau_j>s} J_{2j}\big(\tau_j, u(\tau_j), u'(\tau_j), u''(\tau_j), v(\tau_j), v'(\tau_j), v''(\tau_j)\big) + \int_s^b q(\xi)\,d\xi\right) ds\,dr. \qquad \square$$

The Nagumo condition, introduced in [25], is an important tool for controlling the second derivatives. We consider here a Nagumo-type condition given by the following definition.

Definition 2.3 Let $\gamma_k^{(l)}(x), \Gamma_k^{(l)}(x)$, $k = 1,2$, $l = 0,1$, be piecewise continuous functions such that

$$\gamma_1^{(l)}(x) \le \Gamma_1^{(l)}(x), \qquad \gamma_2^{(l)}(x) \le \Gamma_2^{(l)}(x), \quad \text{a.e. } x \in [a,b],$$

and consider the set

$$S = \left\{ \begin{array}{c} (x, y_0, y_1, y_2, z_0, z_1, z_2) \in [a,b] \times \mathbb{R}^6 : \\ \gamma_1^{(l)}(x) \le y_l \le \Gamma_1^{(l)}(x), \gamma_2^{(l)}(x) \le z_l \le \Gamma_2^{(l)}(x), l = 0,1 \end{array} \right\}. \tag{S}$$

The L^1-Carathéodory functions $f, g : [a,b] \times \mathbb{R}^6 \to \mathbb{R}$ satisfy a Nagumo-type condition if there are $\mu_k > 0$, $k = 1,2$ with

$$\begin{aligned} \mu_1 &:= \max_{i=0,1,2,\ldots,m} \left\{ \frac{|\Gamma_1'(x_{i+1}) - \gamma_1'(x_i)|}{x_{i+1} - x_i}, \frac{|\gamma_1'(x_{i+1}) - \Gamma_1'(x_i)|}{x_{i+1} - x_i} \right\}, \\ \mu_2 &:= \max_{j=0,1,2,\ldots,n} \left\{ \frac{|\Gamma_2'(\tau_{j+1}) - \gamma_2'(\tau_j)|}{\tau_{j+1} - \tau_j}, \frac{|\gamma_2'(\tau_{j+1}) - \Gamma_2'(\tau_j)|}{\tau_{j+1} - \tau_j} \right\}, \end{aligned} \tag{2.5}$$

and continuous positive functions $\varphi_k : [0,+\infty) \to (0,+\infty)$, $k = 1,2$, verifying

$$\begin{aligned} \big|f(x, y_0, y_1, y_2, z_0, z_1, z_2)\big| &\le \varphi_1\big(|y_2|\big), \quad \forall (x, y_0, y_1, y_2, z_0, z_1, z_2) \in S, \\ \big|g(x, y_0, y_1, y_2, z_0, z_1, z_2)\big| &\le \varphi_2\big(|z_2|\big), \quad \forall (x, y_0, y_1, y_2, z_0, z_1, z_2) \in S, \end{aligned} \tag{2.6}$$

with

$$\int_{\phi(\mu_1)}^{\phi(+\infty)} \frac{|\phi^{-1}(s)|}{\varphi_1(|\phi^{-1}(s)|)}\,ds = +\infty, \qquad \int_{\psi(\mu_2)}^{\psi(+\infty)} \frac{|\psi^{-1}(s)|}{\varphi_2(|\psi^{-1}(s)|)}\,ds = +\infty. \tag{2.7}$$

This growth condition allows *a priori* estimations on the second derivatives.

Lemma 2.4 *Consider* $\gamma_k', \Gamma_k' \in PC^1[a,b]$, $k = 1,2$, *such that*

$$\gamma_k'(x) \le \Gamma_k'(x), \quad \text{a.e. } x \in [a,b],$$

and let $f, g : [a,b] \times \mathbb{R}^6 \to \mathbb{R}$ *be* L^1*-Carathéodory functions satisfying a Nagumo-type condition according to Definition 2.3. Then there exist* $N_k > 0$, $k = 1,2$, *such that every solution* (u,v) *of* (1.1) *on the set* (S) *satisfies*

$$\|u''\|_\infty \le N_1 \quad \text{and} \quad \|v''\|_\infty \le N_2.$$

Remark 2.5 Note that N_1 depends only on γ_1', Γ_1', and φ_1 and N_2 on γ_2', Γ_2', and φ_2.

Proof Let $(u(x), v(x))$ be a solution of (1.1) on the set (S).

By the mean value theorem, there are $\bar{x} \in (x_i, x_{i+1})$ and $\tilde{x} \in (\tau_j, \tau_{j+1})$ such that

$$u''(\bar{x}) = \frac{u'(x_{i+1}) - u'(x_i^+)}{x_{i+1} - x_i} \quad \text{and} \quad v''(\tilde{x}) = \frac{v'(\tau_{j+1}) - v'(\tau_j^+)}{\tau_{j+1} - \tau_j}. \tag{2.8}$$

If $|u''(x)| \le \mu_1$, $\forall x \in [a, b]$, then it is enough to define $N_1 := \mu_1$, and the proof is complete.

The case $|u''(t)| > \mu_1$, $\forall x \in [a, b]$, with μ_1 defined in (2.5), is not possible.

In fact, if $u''(x) > \mu_1$, $\forall x \in (x_i, x_{i+1})$, we obtain by (2.8), (S), and (2.5) the contradiction

$$u''(\bar{x}) = \frac{u'(x_{i+1}) - u'(x_i^+)}{x_{i+1} - x_i} \le \frac{\Gamma_1'(x_{i+1}) - \gamma_1'(x_i)}{x_{i+1} - x_i} \le \mu_1. \tag{2.9}$$

If $u''(x) < -\mu_1$, $\forall x \in [a, b]$, the contradiction is similar.

Assume now that there are $\check{x}, x^* \in (x_i, x_{i+1})$ with $\check{x} < x^*$ such that

$$u''(\check{x}) \le \mu_1 \quad \text{and} \quad u''\bigl(x^*\bigr) > \mu_1.$$

By the continuity of $u''(x)$, there exists $\hat{x} \in [\check{x}, x^*]$ such that $u''(\hat{x}) = \mu_1$ and $u''(x) > 0$, $\forall x \in [\check{x}, x^*]$.

Consider $N_k > \mu_k$, $k = 1, 2$, such that

$$\int_{\phi(\mu_1)}^{\phi(N_1)} \frac{|\phi^{-1}(s)|}{\varphi_1(|\phi^{-1}(s)|)}\, ds > \mu_1(b - a) \quad \text{and} \quad \int_{\psi(\mu_2)}^{\psi(N_2)} \frac{|\psi^{-1}(s)|}{\varphi_2(|\psi^{-1}(s)|)}\, ds > \mu_2(b - a). \tag{2.10}$$

Making the change of variable $\phi(u''(x)) = s$ and using (2.6) and (2.9),

$$\begin{aligned}
\int_{\phi(u''(\hat{x}))}^{\phi(u''(x^*))} \frac{|\phi^{-1}(s)|}{\varphi_1(|\phi^{-1}(s)|)}\, ds &= \int_{\hat{x}}^{x^*} \frac{|\phi^{-1}(\phi(u''(x)))|}{\varphi_1(|\phi^{-1}(\phi(u''(x)))|)} \bigl(\phi\bigl(u''(x)\bigr)\bigr)'\, dx \\
&\le \int_{\hat{x}}^{x^*} \frac{|u''(x)|}{\varphi_1(|u''(x)|)} \bigl(\phi\bigl(u''(x)\bigr)\bigr)'\, dx \\
&\le \int_{\hat{x}}^{x^*} \frac{|u''(x)| \cdot |-f(x, u(x), u'(x), u''(x), v(x), v'(x), v''(x))|}{\varphi_1(|u''(x)|)}\, dx, \\
&\le \int_{\hat{x}}^{x^*} \frac{|u''(x)| \cdot |\varphi_1(|u''(x)|)|}{\varphi(|u''(t)|)}\, dx, \\
&\le \int_{\hat{x}}^{x^*} u''(x)\, dx = u'\bigl(x^*\bigr) - u'(\hat{x}) \le \mu_1(b - a),
\end{aligned}$$

and by (2.10)

$$\int_{\phi(u''(\hat{x}))}^{\phi(u''(x^*))} \frac{|\phi^{-1}(s)|}{\varphi_1(|\phi^{-1}(s)|)}\, ds \le \mu_1(b - a) < \int_{\phi(\mu_1)}^{\phi(N_1)} \frac{|\phi^{-1}(s)|}{\varphi_1(|\phi^{-1}(s)|)}\, ds.$$

Therefore $u''(x^*) < N_1$, and as x^* is taken arbitrarily, then $u''(x) < N_1$ for the values of x whenever $u''(x) > \mu_1$.

The case for $\breve{x} > x^*$ follows similar arguments.

The other possible case where

$$u''(\breve{x}) > -\mu_1 \quad \text{and} \quad u''\left(x^*\right) < -\mu_1$$

can be proved by the previous techniques. Therefore $\|u''\|_\infty \leq N_1$.

By a similar method as above, it can be shown that

$$\int_{\psi(v''(\hat{x}))}^{\psi(v''(x^*))} \frac{|\psi^{-1}(s)|}{\varphi_2(|\psi^{-1}(s)|)}\,ds \leq \mu_2(b-a) < \int_{\psi(\mu_2)}^{\psi(N_2)} \frac{|\psi^{-1}(s)|}{\varphi_2(|\psi^{-1}(s)|)}\,ds,$$

and, with the same type of arguments, obtaining that $\|v''\|_\infty \leq N_2$. □

The arguments forward will require the following lemma of [32].

Lemma 2.6 *For $v, w \in C(I)$ such that $v(x) \leq w(x)$, for every $x \in I$, define*

$$q(x,u) = \max\left\{v, \min\{u,w\}\right\}.$$

Then, for each $u \in C^1(I)$, the next two properties hold:

(a) $\frac{d}{dx}q(x,u(x))$ *exists for a.e. $x \in I$.*

(b) *If $u, u_m \in C^1(I)$ and $u_m \to u$ in $C^1(I)$, then*

$$\frac{d}{dx}q\left(x,u_m(x)\right) \to \frac{d}{dx}q\left(x,u(x)\right) \quad \text{for a.e. } x \in I.$$

Schauder's fixed point theorem will be the key existence tool.

Theorem 2.7 [37] *Let Y be a nonempty, closed, bounded, and convex subset of a Banach space X, and suppose that $P : Y \to Y$ is a compact operator. Then P has at least one fixed point in Y.*

3 Existence result

The next theorem will guarantee the existence of a solution of (1.1)–(1.3) through the existence of fixed points of a convenient operator.

(H1) ϕ, $\psi : \mathbb{R} \to \mathbb{R}$ are increasing homeomorphisms such that $\phi(0) = \psi(0) = 0$ and $\phi(\mathbb{R}) = \psi(\mathbb{R}) = \mathbb{R}$, and

$$\left|\phi^{-1}(w)\right| \leq \phi^{-1}\left(|w|\right) \quad \text{and} \quad \left|\psi^{-1}(w)\right| \leq \psi^{-1}\left(|w|\right).$$

Theorem 3.1 *Consider $A_k, B_k \in \mathbb{R}$, $k = 0,1,2$, and the homeomorphisms ϕ and ψ verifying (H1). Let $f, g : [a,b] \times \mathbb{R}^6 \to \mathbb{R}$ be L^1-Carathéodory functions satisfying a Nagumo-type condition as in Definition 2.3, and I_{ki}, J_{kj}, $k = 0,1,2$, $i = 1,\dots,m$, $j = 1,\dots,n$, be continuous functions. Then there is at least one pair of functions $(u,v) \in X^2$ solution to problem (1.1)–(1.3).*

Proof Define the operators $T_1 : X^2 \to X_1$, $T_2 : X^2 \to X_2$, and $T : X^2 \to X^2$ given by

$$T(u,v) = \left(T_1(u,v), T_2(u,v)\right), \tag{3.1}$$

with

$$\begin{aligned}\big(T_1(u,v)\big)(x) &= A_0 + A_1(x-a) + \sum_{i:x_i<x} I_{0i}\big(x_i,u(x_i),u'(x_i),v(x_i),v'(x_i)\big)\\ &\quad + (x-a)\sum_{i:x_i<x} I_{1i}\big(x_i,u(x_i),u'(x_i),u''(x_i),v(x_i),v'(x_i)\big)\\ &\quad + \int_a^x\int_a^r \phi^{-1}\Bigg(\phi(A_2) - \sum_{i:x_i>s} I_{2i}\big(x_i,u(x_i),u'(x_i),u''(x_i),v(x_i),v'(x_i),v''(x_i)\big)\\ &\quad + \int_s^b f\big(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi)\big)\,d\xi\Bigg)\,ds\,dr\end{aligned}$$

and

$$\begin{aligned}\big(T_2(u,v)\big)(x) &= B_0 + B_1(x-a) + \sum_{j:\tau_j<x} J_{0j}\big(\tau_j,u(\tau_j),u'(\tau_j),v(\tau_j),v'(\tau_j)\big)\\ &\quad + (x-a)\sum_{j:\tau_j<x} J_{1j}\big(\tau_j,u(\tau_j),u'(\tau_j),v(\tau_j),v'(\tau_j),v''(\tau_j)\big)\\ &\quad + \int_a^x\int_a^r \psi^{-1}\Bigg(\psi(B_2) - \sum_{j:\tau_j>s} J_{2j}\big(\tau_j,u(\tau_j),u'(\tau_j),u''(\tau_j),v(\tau_j),v'(\tau_j),v''(\tau_j)\big)\\ &\quad + \int_s^b g\big(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi)\big)\,d\xi\Bigg)\,ds\,dr.\end{aligned}$$

Define $L>0$ and $M>0$ such that

$$L > \|(u,v)\|_{X^2} \tag{3.2}$$

and

$$M > \max_{k=0,1,2}\left\{\sum_{i=1}^{m}|I_{ki}|,\sum_{j=1}^{n}|J_{kj}|\right\}. \tag{3.3}$$

Since f and g are L^1-Carathéodory functions and a nonnegative function $\rho_{\kappa L}(x)\in L^1([a,b])$, $\kappa=1,2$, such that

$$\begin{aligned}&\big|f\big(x,u(x),u'(x),u''(x),v(x),v'(x),v''(x)\big)\big| \le \rho_{1L}(x),\\ &\big|g\big(x,u(x),u'(x),u''(x),v(x),v'(x),v''(x)\big)\big| \le \rho_{2L}(x), \quad a.e.\,x\in[a,b].\end{aligned} \tag{3.4}$$

The proof will follow several steps that, for clarity, are detailed for the $T_1(u,v)$ operator. The technique for the $T_2(u,v)$ operator is similar.

Step 1: *T is well defined, continuous, and uniformly bounded.*

By the Lebesgue dominated convergence theorem, (3.4), (1.3), (H1), and (3.3), then

$$\begin{aligned}\big|\big(T_1(u,v)\big)(x)\big| &\le |A_0| + \big|A_1(x-a)\big| + \sum_{i:x_i<x}|I_{0i}| + \big|(x-a)\big|\sum_{i:x_i<x}|I_{1i}|\\ &\quad + \left|\int_a^x\int_a^r \phi^{-1}\Bigg(\phi(A_2) - \sum_{i:x_i>s} I_{2i}\right.\end{aligned}$$

$$
\begin{aligned}
&+\int_s^b f\big(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi)\big)\,d\xi\bigg)\,ds\,dr\bigg| \\
&\leq |A_0|+\big|A_1(x-a)\big|+M+M\big|(x-a)\big| \\
&+\int_a^x\int_a^r\phi^{-1}\bigg(\big|\phi(A_2)\big|+M \\
&+\int_s^b\big|f\big(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi)\big)\big|\,d\xi\bigg)\,ds\,dr \\
&\leq |A_0|+\big|A_1(b-a)\big|+M+M\big|(b-a)\big| \\
&+\int_a^b\int_a^r\phi^{-1}\bigg(\big|\phi(A_2)\big|+M+\int_s^b\rho_{1L}(\xi)\,d\xi\bigg)\,ds\,dr<+\infty,
\end{aligned}
$$

$$
\begin{aligned}
\big|\big(T_1(u,v)\big)'(x)\big| \leq{}& |A_1|+\sum_{i:x_i<x}|I_{1i}| \\
&+\bigg|\int_a^x\phi^{-1}\bigg(\phi(A_2)-\sum_{i:x_i>s}I_{2i} \\
&+\int_s^b\big|f\big(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi)\big)\big|\,d\xi\bigg)\,ds\bigg| \\
\leq{}& |A_1|+M+\int_a^b\phi^{-1}\bigg(\big|\phi(A_2)\big|+M+\int_s^b\rho_{1L}(\xi)\,d\xi\bigg)\,ds<+\infty
\end{aligned}
$$

and

$$
\begin{aligned}
\big|\big(T_1(u,v)\big)''(x)\big| \leq{}& \bigg|\phi^{-1}\bigg(\phi(A_2)-\sum_{i:x_i>s}I_{2i} \\
&+\int_s^b\big|f\big(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi)\big)\big|\,d\xi\bigg)\bigg| \\
\leq{}& \phi^{-1}\bigg(\big|\phi(A_2)\big|+M+\int_a^b\rho_{1L}(\xi)\,d\xi\bigg)<+\infty.
\end{aligned}
$$

Therefore $(T_1(u,v))(x)\in X_1$. The proof that $(T_2(u,v))(x)\in X_2$ is similar, and so T is well defined in X^2.

Moreover, defining $\mathcal{B}\subseteq X^2$ as

$$
\mathcal{B}.=\big\{(u,v)\in X^2:\big\|(u,v)\big\|_{X^2}\leq L\big\}, \tag{3.5}
$$

from the above, it is clear that $T\mathcal{B}$ is uniformly bounded.

Step 2: *T is equicontinuous, that is, $T_1\mathcal{B}$ is equicontinuous on each interval $]x_i,x_{i+1}]$ for $i=0,1,\ldots,m$ with $x_0=a$ and $x_{m+1}=b$, and $T_2\mathcal{B}$ is equicontinuous on each interval $]\tau_j,\tau_{j+1}]$ for $j=0,1,\ldots,n$ with $\tau_0=a$ and $\tau_{n+1}=b$.*

Consider $\mathcal{I}\subseteq]x_i,x_{i+1}]$ and $\tilde{x},x^*\in\mathcal{I}$ such that, without loss of generality, $\tilde{x}\leq x^*$. For $(u,v)\in\mathcal{B}$, we have

$$
\begin{aligned}
&\lim_{\tilde{x}\to x^*}\big|\big(T_1(u,v)\big)\big(x^*\big)-\big(T_1(u,v)\big)(\tilde{x})\big| \\
&\quad\leq\lim_{\tilde{x}\to x^*}\big|A_1\big(x^*-a\big)-A_1(\tilde{x}-a)\big|
\end{aligned}
$$

$$
\begin{aligned}
&+\lim_{\tilde{x}\to x^*}\left|\sum_{a<x_i<x^*} I_{0i}-\sum_{a<x_i<\tilde{x}} I_{0i}+\left(x^*-a\right)\sum_{a<x_i<x^*} I_{1i}-(\tilde{x}-a)\sum_{a<x_i<\tilde{x}} I_{1i}\right| \\
&+\lim_{\tilde{x}\to x^*}\left|\int_{x_i}^{x^*}\int_a^r \phi^{-1}\left(\phi(A_2)-\sum_{i:x_i>s} I_{2i}\right.\right. \\
&\left.+\int_s^b f\left(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi)\right)d\xi\right)ds\,dr \\
&-\int_{x_i}^{\tilde{x}}\int_a^r \phi^{-1}\left(\phi(A_2)-\sum_{i:x_i>s} I_{2i}\right. \\
&\left.\left.+\int_s^b f\left(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi)\right)d\xi\right)ds\,dr\right| \\
&\leq \lim_{\tilde{x}\to x^*}\left|\int_{\tilde{x}}^{x^*}\int_a^r \phi^{-1}\left(\phi(A_2)-\sum_{i:x_i>s} I_{2i}\right.\right. \\
&\left.\left.+\int_s^b f\left(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi)\right)d\xi\right)ds\,dr\right| = 0, \\
&\lim_{\tilde{x}\to x^*}\left|\left(T_1(u,v)\right)'\left(x^*\right)-\left(T_1(u,v)\right)'(\tilde{x})\right| \\
&\leq \lim_{\tilde{x}\to x^*}\left|\sum_{a<x_i<x^*} I_{1i}-\sum_{a<x_i<\tilde{x}} I_{1i}\right| \\
&+\lim_{\tilde{x}\to x^*}\left|\int_{x_i}^{x^*}\phi^{-1}\left(\phi(A_2)-\sum_{i:x_i>s} I_{2i}\right.\right. \\
&\left.+\int_s^b f\left(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi)\right)d\xi\right)ds \\
&-\int_{x_i}^{\tilde{x}}\phi^{-1}\left(\phi(A_2)-\sum_{i:x_i>s} I_{2i}\right. \\
&\left.\left.+\int_s^b f\left(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi)\right)d\xi\right)ds\right| \\
&\leq \lim_{\tilde{x}\to x^*}\left|\int_{\tilde{x}}^{x^*}\phi^{-1}\left(\phi(A_2)-\sum_{i:x_i>s} I_{2i}\right.\right. \\
&\left.\left.+\int_s^b f\left(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi)\right)d\xi\right)ds\right| = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{\tilde{x}\to x^*}\left|\left(T_1(u,v)\right)''\left(x^*\right)-\left(T_1(u,v)\right)''(\tilde{x})\right| \\
&\leq \lim_{\tilde{x}\to x^*}\left|\phi^{-1}\left(\phi(A_2)-\sum_{i:x_i>x^*} I_{2i}+\int_{x^*}^b f\left(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi)\right)d\xi\right)\right. \\
&\left.-\phi^{-1}\left(\phi(A_2)-\sum_{i:x_i>\tilde{x}} I_{2i}+\int_{\tilde{x}}^b f\left(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi)\right)d\xi\right)\right| = 0.
\end{aligned}
$$

Therefore, $T_1\mathcal{B}$ is equicontinuous on X_1. Similarly, we can show that $T_2\mathcal{B}$ is equicontinuous on X_2, too. Thus, $T\mathcal{B}$ is equicontinuous on X^2.

Step 3: *$T\mathcal{B}: X^2 \to X^2$ is equiconvergent at $x = x_i$* and $x = \tau_j$.

First, let us prove the equiconvergence at $x = x_i^+$ for $i = 1, 2, \dots m$. The proof of equiconvergence at $x = \tau_j^+$ for $j = 1, 2, \dots n$ is analogous.

So, it follows for $i = 1, 2, \dots, m$ that

$$\begin{aligned}
&\left|\big(T_1(u,v)\big)(x_i) - \lim_{x\to x_i^+}\big(T_1(u,v)\big)(x)\right| \\
&\quad\le \left|A_1(x_i-a) - \lim_{x\to x_i^+} A_1(x-a)\right| \\
&\qquad + \left|\sum_{a<x_\lambda<x_i} I_{0\lambda} - \lim_{x\to x_i^+}\sum_{a<x_\lambda<x} I_{0\lambda} + (x_i-a)\sum_{a<x_\lambda<x_i} I_{1\lambda} - \lim_{x\to x_i^+}(x-a)\sum_{a<x_\lambda<x} I_{1\lambda}\right| \\
&\qquad + \left|\int_a^{x_i}\int_a^r \phi^{-1}\left(\phi(A_2) - \sum_{i:x_i>s} I_{2i}\right.\right. \\
&\qquad + \left.\int_s^b f\big(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi)\big)\,d\xi\right) ds\,dr \\
&\qquad - \lim_{x\to x_i^+}\int_a^x\int_a^r \phi^{-1}\left(\phi(A_2) - \sum_{i:x_i>s} I_{2i}\right. \\
&\qquad + \left.\left.\int_s^b f\big(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi)\big)\,d\xi\right) ds\,dr\right| \\
&\quad\le \lim_{x\to x_i^+}\left|\int_x^{x_i}\int_a^r \phi^{-1}\left(\phi(A_2) - \sum_{i:x_i>s} I_{2i}\right.\right. \\
&\qquad + \left.\left.\int_s^b f\big(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi)\big)\,d\xi\right) ds\,dr\right| = 0,
\end{aligned}$$

$$\begin{aligned}
&\left|\big(T_1(u,v)\big)'(x_i) - \lim_{x\to x_i^+}\big(T_1(u,v)\big)'(x)\right| \\
&\quad\le \left|\sum_{a<x_\lambda<x_i} I_{1\lambda} - \lim_{x\to x_i^+}\sum_{a<x_\lambda<x} I_{1\lambda}\right| \\
&\qquad + \left|\int_a^{x_i} \phi^{-1}\left(\phi(A_2) - \sum_{i:x_i>s} I_{2i}\right.\right. \\
&\qquad + \left.\int_s^b f\big(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi)\big)\,d\xi\right) ds \\
&\qquad - \lim_{x\to x_i^+}\int_a^x \phi^{-1}\left(\phi(A_2) - \sum_{i:x_i>s} I_{2i}\right. \\
&\qquad + \left.\left.\int_s^b f\big(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi)\big)\,d\xi\right) ds\right| \\
&\quad\le \lim_{x\to x_i^+}\left|\int_{x_i}^{x} \phi^{-1}\left(\phi(A_2) - \sum_{i:x_i>s} I_{2i}\right.\right. \\
&\qquad + \left.\left.\int_s^b f\big(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi)\big)\,d\xi\right) ds\right| = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\left(T_1(u,v)\right)''(x_i) - \lim_{x\to x_i^+}\left(T_1(u,v)\right)''(x)\right| \\
&\quad\leq \left|\phi^{-1}\left(\phi(A_2) - \sum_{a<x_\lambda<x_i} I_{2\lambda} + \int_{x_i}^b f\left(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi)\right)d\xi\right)\right. \\
&\qquad - \lim_{x\to x_i^+}\phi^{-1}\left(\phi(A_2) - \sum_{a<x_\lambda<x} I_{2\lambda}\right. \\
&\qquad \left.\left. + \int_x^b f\left(\xi,u(\xi),u'(\xi),u''(\xi),v(\xi),v'(\xi),v''(\xi)\right)d\xi\right)\right| = 0.
\end{aligned}$$

Therefore, $T_1\mathcal{B}$ is equiconvergent at each point $x = x_i^+$ for $i = 1,2,\ldots,m$.

Analogously, it can be proved that $T_2\mathcal{B}$ is equiconvergent at each point $x = \tau_j^+$ for $j = 1,2,\ldots,n$.

So, $T\mathcal{B}$ is equiconvergent at each impulsive point.

Step 4: *$T : X^2 \to X^2$ has a fixed point.*

Consider

$$\Omega := \left\{(u,v)\in X^2 : \|(u,v)\|_{X^2} \leq K\right\}$$

with $K > 0$ such that

$$K := \max\left\{\begin{array}{c}
L, |A_0| + \left|A_1(b-a)\right| + M + M\left|(b-a)\right| \\
+ \int_a^b\int_a^r \phi^{-1}\left(\left|\phi(A_2)\right| + M + \int_s^b \rho_{1L}(\xi)\,d\xi\right)ds\,dr, \\
|A_1| + M + \int_a^b \phi^{-1}\left(\left|\phi(A_2)\right| + M + \int_s^b \rho_{1L}(\xi)\,d\xi\right)ds, \\
\phi^{-1}\left(\left|\phi(A_2)\right| + M + \int_a^b \rho_{1L}(\xi)\,d\xi\right), \\
|B_0| + \left|B_1(b-a)\right| + M + M\left|(b-a)\right| \\
+ \int_a^b\int_a^r \psi^{-1}\left(\left|\psi(B_2)\right| + M + \int_s^b \rho_{2L}(\xi)\,d\xi\right)ds\,dr, \\
|B_1| + M + \int_a^b \psi^{-1}\left(\left|\psi(B_2)\right| + M + \int_s^b \rho_{2L}(\xi)\,d\xi\right)ds, \\
\psi^{-1}\left(\left|\psi(B_2)\right| + M + \int_a^b \rho_{2L}(\xi)\,d\xi\right)
\end{array}\right\},$$

with $L > 0$ given by (3.2) and (3.5), and $M > 0$ defined in (3.3) such that $\mathcal{B}\subset\Omega$.

According to Step 1, we have

$$\begin{aligned}
\left\|T(u,v)\right\|_{X^2} &= \left\|\left(T_1(u,v),T_2(u,v)\right)\right\|_{X^2} \\
&= \max\left\{\left\|T_1(u,v)\right\|_{X_1}, \left\|T_2(u,v)\right\|_{X_2}\right\} \\
&= \max\left\{\begin{array}{c}\|T_1(u,v)\|_\infty, \|T_1'(u,v)\|_\infty, \|T_1''(u,v)\|_\infty, \\ \|T_2(u,v)\|_\infty, \|T_2'(u,v)\|_\infty, \|T_2''(u,v)\|_\infty\end{array}\right\} \\
&\leq K.
\end{aligned}$$

So, $T\Omega \subset \Omega$, and by Theorem 2.7, the operator $T(u,v) = (T_1(u,v), T_2(u,v))$ has a fixed point (u^*, v^*).

By standard techniques and Lemma 2.2, it can be shown that this fixed point is a solution of problem (1.1)–(1.3). □

Example 3.2 Consider the following system of coupled differential equations:

$$\begin{cases} \frac{u'''(x)}{\sqrt{1+(u''(x))^2}} - (u(x))^3 v(x) + u'(x)\arctan(v'(x)) + (u''(x))^2 - \sqrt[3]{v''(x)} = 0, \\ \quad x \in [0,1] \setminus \{(x_i)\}, \\ v'''(x) - u(x)v'(x) + \cos(u''(x))v(x) + (u'(x))^2 - v''(x) = 0, \\ \quad x \in [0,1] \setminus \{(\tau_j)\} \end{cases} \tag{3.6}$$

with the boundary conditions

$$\begin{cases} u(0) = 0, \qquad u'(0) = 1, \qquad u''(1) = 0 \\ v(0) = 0, \qquad v'(0) = -1, \qquad v''(1) = 0, \end{cases} \tag{3.7}$$

and the impulsive effects given by

$$\begin{cases} \Delta u(x_i) = 2u(x_i) + u'(x_i) + v(x_i), \\ \Delta u'(x_i) = (u'(x_i))^2 - u''(x_i) + v(x_i), \\ \Delta \phi(u''(x_i)) = \frac{\sin(x_i)}{x_i}, \\ \Delta v(\tau_i) = u(\tau_i) + 3v(\tau_i) + v'(\tau_i), \\ \Delta v'(\tau_i) = u(\tau_i) + v(\tau_i) + 4\sin(v'(\tau_i)) - (v''(\tau_i))^2, \\ \Delta \psi(v''(\tau_j)) = e^{-\tau_j} \end{cases} \tag{3.8}$$

with $x_i = \frac{i}{5}$ for $i = 1,2,3,4$ and $\tau_j = \frac{j^2}{10}$ for $j = 1,2,3$.

This problem is a particular case of (1.1)–(1.3) with $[a,b] = [0,1]$,

$$\begin{aligned} &\phi(y_2) = \operatorname{arcsinh} y_2, \qquad \psi(z_2) = z_2, \\ &f(x,y_0,y_1,y_2,z_0,z_1,z_2) = -y_0^3 z_0 + y_1 \arctan z_1 + y_2^2 - \sqrt[3]{z_2}, \\ &g(x,y_0,y_1,y_2,z_0,z_1,z_2) = -y_0 z_1 + \cos(y_2) z_0 + y_1^2 - z_2, \\ &A_0 = A_2 = B_0 = B_2 = 0, \qquad A_1 = 1, \qquad B_1 = -1, \\ &I_0(\cdot,y_0,y_1,z_0,z_1) = 2y_0 + y_1 + z_0, \\ &I_1(\cdot,y_0,y_1,y_2,z_0,z_1) = y_1^2 - y_2 + z_0, \\ &I_2(\cdot,y_0,y_1,y_2,z_0,z_1) = \frac{\sin(x_i)}{x_i}, \\ &J_0(\cdot,y_0,y_1,z_0,z_1) = y_0 + 3z_0 + z_1, \\ &J_1(\cdot,y_0,y_1,z_0,z_1,z_2) = y_0 + z_0 + 4\sin z_1 - z_2^2, \\ &J_1(\cdot,y_0,y_1,z_0,z_1,z_2) = e^{-\tau_j} \end{aligned} \tag{3.9}$$

and $m = 4$, $n = 3$.

It is clear that the functions in (3.9) verify assumption (H1) and f and g satisfy a Nagumo-type condition in sets such as, for some piecewise continuous functions $\gamma_k^{(l)}(x)$, $\Gamma_k^{(l)}(x)$, $k = 1, 2$, $l = 0, 1$,

$$\left\{ \begin{array}{c} (x, y_0, y_1, y_2, z_0, z_1, z_2) \in [0,1] \times \mathbb{R}^6 : \gamma_1(x) \leq y_0 \leq \Gamma_1(x), \\ \gamma_1'(x) \leq y_1 \leq \Gamma_1'(x), \gamma_2(x) \leq z_0 \leq \Gamma_2(x), \gamma_2'(x) \leq z_1 \leq \Gamma_2'(x) \end{array} \right\},$$

with

$$\varphi_1(|y_2|) := K_0 + y_2^2, \quad \text{and} \quad \varphi_2(|z_2|) := K_1 + |z_2|,$$

where K_0, K_1 are some real positive numbers.

Therefore, by Theorem 3.1, problem (3.6)–(3.8) has at least a solution.

4 Existence and localization results

In addition to the existence of a solution, it is possible to obtain an existence and localization theorem, that is, not only it guarantees the existence of at least a solution, but provides also a strip where this solution is localized.

However, the localization part is obtained for a particular case of impulsive conditions (1.4), applying lower and upper functions, defined as follows.

Definition 4.1 The pair of functions $(\alpha_1(x)\alpha_2(x)) \in X^2$ such that $(\phi(\alpha_1'(x)), \psi(\alpha_2'(x))) \in (AC[a,b])^2$ is a lower solution of problem (1.1), (1.2), (1.4) if

$$\begin{cases} (\phi(\alpha_1''(x)))' + f(x, \alpha_1(x), \alpha_1'(x), \alpha_1''(x), \alpha_2(x), \alpha_2'(x), z) \geq 0, & \text{for } z \in \mathbb{R} \\ (\psi(\alpha_2''(x)))' + g(x, \alpha_1(x), \alpha_1'(x), y, \alpha_2(x), \alpha_2'(x), \alpha_2''(x)) \geq 0, & \text{for } y \in \mathbb{R}, \\ \alpha_1^{(l)}(a) \leq A_l, \qquad \alpha_2^{(l)}(a) \leq B_l, & l = 0, 1, \\ \alpha_1''(b) \leq A_2, \qquad \alpha_2''(b) \leq B_2, & \\ \Delta\alpha_1(x_i) \leq I_{0i}(x_i, \alpha_1(x_i), \alpha_1'(x_i), \alpha_2(x_i), \alpha_2'(x_i)), & \\ \Delta\alpha_1'(x_i) \leq I_{1i}(x_i, \alpha_1(x_i), \alpha_1'(x_i), \alpha_1''(x_i), \alpha_2(x_i), \alpha_2'(x_i)), & \\ \Delta\alpha_2(\tau_j) \leq J_{0j}(\tau_j, \alpha_1(\tau_j), \alpha_1'(\tau_j), \alpha_2(\tau_j), \alpha_2'(\tau_j)), & \\ \Delta\alpha_2'(\tau_j) \leq J_{1j}(\tau_j, \alpha_1(\tau_j), \alpha_1'(\tau_j), \alpha_2(\tau_j), \alpha_2'(\tau_j), \alpha_2''(\tau_j)). & \end{cases} \tag{4.1}$$

A pair of functions $(\beta_1(x), \beta_2(x)) \in X^2$ such that $(\phi(\beta_1'(x)), \psi(\beta_2'(x))) \in (AC[a,b])^2$ is an upper solution of problem (1.1), (1.2), (1.4) if the opposite inequalities hold.

To obtain this goal, we consider local monotone assumptions:

(H2) $f, g : [a,b] \times \mathbb{R}^4 \mapsto \mathbb{R}$ are L^1-Carathéodory such that

$$\begin{aligned} f\big(x, \alpha_1(x), y_1, \alpha_1''(x), \alpha_2(x), \alpha_2'(x), z_2\big) &\leq f(x, y_0, y_1, y_2, z_0, z_1, z_2) \\ &\leq f\big(x, \beta_1(x), y_1, \beta_1''(x), \beta_2(x), \beta_2'(x), z_2\big), \end{aligned}$$

for $\alpha_1(x) \le y_0 \le \beta_1(x)$, $\alpha_1''(x) \le y_2 \le \beta_1''(x)$, $\alpha_2(x) \le z_0 \le \beta_2(x)$, $\alpha_2'(x) \le z_1 \le \beta_2'(x)$, and $(x, y_1, z_2) \in [a,b] \times \mathbb{R}^2$, and

$$\begin{aligned} g\big(x, \alpha_1(x), \alpha_1'(x), y_2, \alpha_2(x), z_1, \alpha_2''(x)\big) &\le g(x, y_0, y_1, y_2, z_0, z_1, z_2) \\ &\le g\big(x, \beta_1(x), \beta_1'(x), y_2, \beta_2(x), z_1, \beta_2''(x)\big), \end{aligned}$$

for $\alpha_1(x) \le y_0 \le \beta_1(x)$, $\alpha_1'(x) \le y_1 \le \beta_1'(x)$, $\alpha_2(x) \le z_0 \le \beta_2(x)$, $\alpha_2''(x) \le z_2 \le \beta_2''(x)$, and $(x, y_2, z_1) \in [a,b] \times \mathbb{R}^2$.

(H3) $I_{0i}, J_{0j} \in C([a,b] \times \mathbb{R}^4, \mathbb{R})$ verify

$$\begin{aligned} I_{0i}\big(x_i, \alpha_1(x_i), \alpha_1'(x_i), \alpha_2(x_i), \alpha_2'(x_i)\big) &\le I_{0i}(x_i, y_0, y_1, z_0, z_1) \\ &\le I_{0i}\big(x_i, \beta_1(x_i), \beta_1'(x_i), \beta_2(x_i), \beta_2'(x_i)\big) \end{aligned}$$

for $i = 1, 2, \dots, m$, $l = 0, 1$, $\alpha_1^{(l)}(x) \le y_l \le \beta_1^{(l)}(x)$, $\alpha_2^{(l)}(x) \le z_l \le \beta_2^{(l)}(x)$ and

$$\begin{aligned} J_{0j}\big(\tau_j, \alpha_1(\tau_j), \alpha_1'(\tau_j), \alpha_2(\tau_j), \alpha_2'(\tau_j)\big) &\le J_{0j}(\tau_j, y_0, y_1, z_0, z_1) \\ &\le J_{0j}\big(\tau_j, \beta_1(\tau_j), \beta_1'(\tau_j), \beta_2(\tau_j), \beta_2'(\tau_j)\big) \end{aligned}$$

for $j = 1, 2, \dots, n$, $l = 0, 1$, $\alpha_1^{(l)}(x) \le y_l \le \beta_1^{(l)}(x)$, $\alpha_2^{(l)}(x) \le z_l \le \beta_2^{(l)}(x)$,
and $I_{1i}, J_{1j} \in C([a,b] \times \mathbb{R}^5, \mathbb{R})$ satisfy

$$\begin{aligned} I_{1i}\big(x_i, \alpha_1(x_i), y_1, y_2, \alpha_2(x_i), \alpha_2'(x_i)\big) &\le I_{1i}(x_i, y_0, y_1, y_2, z_0, z_1) \\ &\le I_i\big(x_i, \beta_1(x_i), y_1, y_2, \beta_2(x_i), \beta_2'(x_i)\big) \end{aligned}$$

for $i = 1, 2, \dots, m$, $l = 0, 1$, $\alpha_1(x) \le y_0 \le \beta_1(x)$, $\alpha_2^{(l)}(x) \le z_l \le \beta_2^{(l)}(x)$, $\forall (y_1, y_2) \in \mathbb{R}^2$ and

$$\begin{aligned} J_{1j}\big(\tau_j, \alpha_1(\tau_j), \alpha_1'(\tau_j), \alpha_2(\tau_j), z_1, z_2\big) &\le J_{1j}(\tau_j, y_0, y_1, z_0, z_1, z_2) \\ &\le J_{1j}\big(\tau_j, \beta_1(\tau_j), \beta_1'(\tau_j), \beta_2(\tau_j), z_1, z_2\big) \end{aligned}$$

for $j = 1, 2, \dots, n$, $l = 0, 1$, $\alpha_1^{(l)}(x) \le y_l \le \beta_1^{(l)}(x)$, $\alpha_2(x) \le z \le \beta_2(x)$, $\forall (z_1, z_2) \in \mathbb{R}^2$.

The existence and localization theorem is given as follows.

Theorem 4.2 *Let $A_k, B_k \in \mathbb{R}$, $k = 0, 1, 2$, and the homeomorphisms ϕ and ψ verify* (*H*1). *Assume that there are lower and upper solutions of* (1.1), (1.2), (1.4), $(\alpha_1^{(l)}, \alpha_2^{(l)})$ *and* $(\beta_1^{(l)}, \beta_2^{(l)})$, *respectively, such that*

$$\alpha_\kappa^{(l)}(x) \le \beta_\kappa^{(l)}(x), \quad \kappa = 1, 2, l = 0, 1, \forall x \in [a,b],$$

the L^1-Carathéodory functions $f, g : [a,b] \times \mathbb{R}^4 \to \mathbb{R}$ satisfy Nagumo conditions as in Definition 2.3 *in the set*

$$S^* = \left\{ \begin{array}{c} (x, y_0, y_1, y_2, z_0, z_1, z_2) \in [a,b] \times \mathbb{R}^6 : \\ \alpha_1^{(l)}(x) \le y_l \le \beta_1^{(l)}(x), \alpha_2^{(l)}(x) \le z_l \le \beta_2^{(l)}(x),\ l = 0, 1 \end{array} \right\}.$$

If assumptions $(H2)$ *and* $(H3)$ *hold, then there is at least a pair* $(u(x),v(x))\in X^2$ *solution of* (1.1), (1.2), (1.4) *and, moreover,*

$$\alpha_1^{(l)}(x)\le u^{(l)}(x)\le \beta_1^{(l)}(x),\qquad \alpha_2^{(l)}(x)\le v^{(l)}(x)\le \beta_2^{(l)}(x),\quad l=0,1,\forall x\in[a,b],$$

and

$$\|u''\|\le N_1\quad \text{and}\quad \|v''\|\le N_2,$$

with N_1 *and* N_2 *given by Lemma* 2.4.

Proof Define the truncate functions $\delta_{im}:[a,b]\times\mathbb{R}\to\ \mathbb{R}$ for $\kappa=1,2$ and $l=0,1$ given by

$$\delta_{\kappa l}(x,w_l)=\begin{cases}\beta_\kappa^{(l)}(x) & \text{if } w_l>\beta_\kappa^{(l)}(x)\\ w_l & \text{if } \alpha_\kappa^{(l)}(x)\le w_l\le \beta_\kappa^{(l)}(x)\\ \alpha_\kappa^{(l)}(x) & \text{if } w_l<\alpha_\kappa^{(l)}(x).\end{cases} \tag{4.2}$$

Consider the following modified coupled system composed by the truncated and perturbed differential equations

$$\begin{cases}(\phi(u''(x)))'+f\begin{pmatrix}x,\delta_{10}(x,u(x)),\delta_{11}(x,u'(x)),u''(x),\\ \delta_{20}(x,v(x)),\delta_{21}(x,v'(x)),v''(x)\end{pmatrix}\\ \quad+\frac{\delta_{11}(x,u'(x))-u'(x)}{1+|\delta_{11}(x,u'(x))-u'(x)|}=0,\\ (\psi(v''(x)))'+g\begin{pmatrix}x,\delta_{10}(x,u(x)),\delta_{11}(x,u'(x)),u''(x),\\ \delta_{20}(x,v(x)),\delta_{21}(x,v'(x)),v''(x)\end{pmatrix}\\ \quad+\frac{\delta_{21}(x,v'(x))-v'(x)}{1+|\delta_{21}(x,v'(x))-v'(x)|}=0,\end{cases} \tag{4.3}$$

with the truncated impulsive conditions

$$\begin{aligned}&\Delta u(x_i)=I_{0i}\big(x_i,\delta_{10}\big(x_i,u(x_i)\big),\delta_{11}\big(x_i,u'(x_i)\big),\delta_{20}\big(x_i,v(x_i)\big),\delta_{21}\big(x_i,v'(x_i)\big)\big),\\ &\Delta u'(x_i)=I_{1i}\begin{pmatrix}x_i,\delta_{10}(x_i,u(x_i)),\delta_{11}(x_i,u'(x_i)),\frac{d}{dx}\delta_{11}(x_i,u'(x_i)),\\ \delta_{20}(x_i,v(x_i)),\delta_{21}(x_i,v'(x_i))\end{pmatrix}\\ &\Delta v(\tau_j)=J_{0j}\big(\tau_j,\delta_{10}\big(\tau_j,u(\tau_j)\big),\delta_{11}\big(\tau_j,u'(\tau_j)\big),\delta_{20}\big(\tau_j,v(\tau_j)\big),\delta_{21}\big(\tau_j,v'(\tau_j)\big)\big),\\ &\Delta v'(\tau_j)=J_{1j}\begin{pmatrix}\tau_j,\delta_{10}(\tau_j,u(\tau_j)),\delta_{11}(\tau_j,u'(\tau_j)),\delta_{20}(\tau_j,v(\tau_j)),\\ \delta_{21}(\tau_j,v'(\tau_j)),\frac{d}{dx}\delta_{21}(x_i,v'(\tau_j))\end{pmatrix},\end{aligned} \tag{4.4}$$

for $i=1,2,\dots,m$, $j=1,2,\dots,n$, and boundary conditions (1.2).

It is clear that the functions F and G, given by

$$\begin{aligned}F(x):=&f\big(x,\delta_{10}\big(x,u(x)\big),\delta_{11}\big(x,u'(x)\big),u''(x),\delta_{20}\big(x,v(x)\big),\delta_{21}\big(x,v'(x)\big),v''(x)\big)\\ &+\frac{\delta_{11}(x,u'(x))-u'(x)}{1+|\delta_{11}(x,u'(x))-u'(x)|}\end{aligned}$$

and

$$G(x) := g\big(x, \delta_{10}\big(x, u(x)\big), \delta_{11}\big(x, u'(x)\big), u''(x), \delta_{20}\big(x, v(x)\big), \delta_{21}\big(x, v'(x)\big), v''(x)\big) + \frac{\delta_{21}(x, v'(x)) - v'(x)}{1 + |\delta_{21}(x, v'(x)) - v'(x)|},$$

satisfy the Nagumo type conditions, as in Definition 2.3, relative to the set S^* with

$$\big|F(x, y_0, y_1, y_2, z_0, z_1, z_2)\big| \le \varphi_1\big(|y_2|\big) + 1$$

and

$$\big|G(x, y_0, y_1, y_2, z_0, z_1, z_2)\big| \le \varphi_2\big(|y_2|\big) + 1.$$

Therefore, applying the same arguments as in Theorem 3.1, it can be proved that problem (4.3), (1.2), (4.4) has at least a solution $(u(x), v(x))$.

To prove that this solution is also a solution to the initial problem (1.1), (1.2), (1.4), it will be enough to show that

$$\alpha_1^{(l)}(x) \le u^{(l)}(x) \le \beta_1^{(l)}(x), \qquad \alpha_2^{(l)}(x) \le v^{(l)}(x) \le \beta_2^{(l)}(x), \quad l = 0, 1, \forall x \in [a, b].$$

For the second inequality, assume, by contradiction, that there is $x \in [a, b]$ such that $u'(x) > \beta_1'(x)$, and define

$$\sup_{a \le x \le b} \big(u'(x) - \beta_1'(x)\big) := u'(\bar{x}) - \beta_1'(\bar{x}) > 0. \tag{4.5}$$

As, by boundary conditions (1.2) and Definition 4.1, $u'(a) - \beta_1'(a) \le 0$, then $\bar{x} \ne a$. In the same way, $u''(b^-) - \beta_1''(b^-) \le 0$, therefore $\bar{x} \ne b$.

Then $\bar{x} \in (a, b)$, two possibilities remain to be studied:

(i) Assume that there is $p \in \{0, 1, 2, \dots, n\}$ such that $\bar{x} \in (x_p, x_{p+1})$. Therefore

$$\max_{x \in (x_p, x_{p+1})} \big(u'(x) - \beta_1'(x)\big) := u'(\bar{x}) - \beta_1'(\bar{x}) > 0$$

and

$$u''(\bar{x}) - \beta_1''(\bar{x}) = 0. \tag{4.6}$$

Choose $\epsilon > 0$ sufficiently small such that

$$u'(x) - \beta_1'(x) > 0 \quad \text{and} \quad u''(x) - \beta_1''(x) \le 0, \quad \forall x \in (\bar{x}, \bar{x} + \epsilon). \tag{4.7}$$

By (H2), for all $x \in (\bar{x}, \bar{x} + \epsilon)$,

$$\big(\phi\big(u''(x)\big)\big)' - \big(\phi\big(\beta_1''(x)\big)\big)' \ge -f\big(x, \delta_{10}\big(x, u(x)\big), \delta_{11}\big(x, u'(x)\big), u''(x), \delta_{20}\big(x, v(x)\big), \delta_{21}\big(x, v'(x)\big), v''(x)\big)$$

$$
\begin{aligned}
&-\frac{\delta_{11}(x,u'(x))-u'(x)}{1+|\delta_{11}(x,u'(x))-u'(x)|}+f\big(x,\beta_1(x),\beta_1'(x),\beta_1''(x),\beta_2(x),\beta_2'(x),\beta_2''(x)\big)\\
&\geq -f\big(x,\delta_{10}\big(x,u(x)\big),\beta_1'(x),\beta_1''(x),\delta_{20}\big(x,v(x)\big),\delta_{21}\big(x,v'(x)\big),v''(x)\big)\\
&-\frac{\beta_1'(x)-u'(x)}{1+|\beta_1'(x)-u'(x)|}+f\big(x,\beta_1(x),\beta_1'(x),\beta_1''(x),\beta_2(x),\beta_2'(x),\beta_2''(x)\big)\\
&\geq \frac{u'(x)-\beta_1'(x)}{1+|u'(x)-\beta_1'(x)|}>0.
\end{aligned}
$$

So $(\phi(u''(x))-\phi(\beta_1''(x)))$ is increasing for $\forall x\in(\bar{x},\bar{x}+\epsilon)$, and, by (4.7), we obtain the contradiction in $(\bar{x},\bar{x}+\epsilon)$ by (4.6) and (4.7):

$$
0=\phi\big(u''(\bar{x})\big)-\phi\big(\beta_1''(\bar{x})\big)<\phi\big(u''(x)\big)-\phi(\beta_1''(x)\leq 0.
$$

Therefore, for $x\in(x_p,x_{p+1})$, $p=0,1,2,\ldots,n$,

$$
u'(x)\leq\beta_1'(x).
$$

(ii) Suppose now that there is $p_*\in\{1,2,\ldots,n\}$ such that $\bar{x}=x_{p_*}$. That is,

$$
\sup_{x\in[a,b]}\big(u'(x)-\beta_1'(x)\big):=u'(x_{p_*})-\beta_1'(x_{p_*})>0. \tag{4.8}
$$

As $u,\beta_1\in X$, by (i), we obtain the contradiction

$$
u'(x_{p_*})=\lim_{x\to x_{p_*}^-}u'(x)\leq\lim_{x\to x_{p_*}^-}\beta_1'(x)=\beta_1'(x_{p_*}).
$$

If $\bar{x}=x_{p_*}^+$, suppose

$$
\sup_{x\in[a,b]}\big(u'(x)-\beta_1'(x)\big):=u'\big(x_{p_*}^+\big)-\beta_1'\big(x_{p_*}^+\big)>0.
$$

By (4.4), (H3), and Definition 4.1, we obtain the contradiction

$$
\begin{aligned}
0&<u'\big(x_{p_*}^+\big)-\beta_1'\big(x_{p_*}^+\big)=u'(x_{p_*})\\
&\quad+I_{1p_*}\begin{pmatrix} x_{p_*},\delta_{10}(x_{p_*},u(x_{p_*})),\delta_{11}(x_{p_*},u'(x_{p_*})),\frac{d}{dx}\delta_{11}(x_{p_*},u'(x_{p_*})),\\ \delta_{20}(x_{p_*},v(x_{p_*})),\delta_{21}(x_{p_*},v'(x_{p_*}))\end{pmatrix}\\
&\quad-\beta_1'(x_{p_*})-I_{p_*}\big(x_{p_*},\beta_1(x_{p_*}),\beta_1'(x_{p_*}),\beta_1''(x_{p_*}),\beta_2(x_{p_*}),\beta_2'(x_{p_*})\big)\\
&\leq I_{1p_*}\big(x_{p_*},\beta_1(x_{p_*}),\beta_1'(x_{p_*}),\beta_1''(x_{p_*}),\delta_{20}\big(x_{p_*},v(x_{p_*})\big),\delta_{21}\big(x_{p_*},v'(x_{p_*})\big)\big)\\
&\quad-I_{1p_*}\big(x_{p_*},\beta_1(x_{p_*}),\beta_1'(x_{p_*}),\beta_1''(x_{p_*}),\beta_2(x_{p_*}),\beta_2'(x_{p_*})\big)\leq 0.
\end{aligned}
$$

Therefore, $u'(x)\leq\beta'(x)$ for $x\in[a,b]$.

By similar arguments, the remaining inequality can be proved, and therefore

$$
\alpha_1'(x)\leq u'(x)\leq\beta_1'(x)\quad\text{for all } x\in[a,b]. \tag{4.9}
$$

The other inequalities follow similar steps.

By integration of (4.9) for $x \in [a, x_1]$,

$$\alpha_1(x) \le u(x) - u(a) + \alpha_1(a) \le u(x),$$

and for $x \in (x_1, x_2]$ we have, by (H3),

$$\begin{aligned}\alpha_1(x) &\le u(x) - u\big(x_1^+\big) + \alpha_1\big(x_1^+\big)\\ &\le u(x) - u(x_1)\\ &\quad - I_{01}\big(x_1, \delta_{10}\big(x_1, u(x_1)\big), \delta_{11}\big(x_1, u'(x_1)\big), \delta_{20}\big(x_1, v(x_1)\big), \delta_{21}\big(x_1, v'(x_1)\big)\big)\\ &\quad + \alpha_1(x_1) + I_{01}\big(x_i, \alpha_1(x_1), \alpha_1'(x_i), \alpha_2(x_i), \alpha_2'(x_i)\big)\\ &\le u(x).\end{aligned}$$

By recurrence, it can be shown, analogously, that

$$\alpha_1(x) \le u(x), \quad \forall x \in (x_i, x_{i+1}] \text{ for } i = 1, \dots, m.$$

Therefore, $\alpha_1(x) \le u(x)$, $\forall x \in [a, b]$.

Analogously, the remaining inequality can be proved, and therefore

$$\alpha_1(x) \le u(x) \le \beta_1(x) \quad \text{for all } x \in [a, b]. \tag{4.10}$$

Analogously, it can be proved that

$$\alpha_2^{(l)}(x) \le v^{(l)}(x) \le \beta_2^{(l)}(x), \quad l = 0, 1, \forall x \in [a, b]. \qquad \square$$

To illustrate the importance of the location arguments, we consider the following example.

Example 4.3 Let the problem be composed by the strongly nonlinear ϕ-Laplacian and p-Laplacian differential equations

$$\begin{cases} \frac{u'''(x)}{1+(u''(x))^2} + u(x) - 4u'(x) + v(x) + \arctan(v''(x)) = 0, \\ \quad x \in [0,1] \setminus \{(x_i)\}, \\ (|v''(x)|^{p-2} v''(x))' + u(x) - 2v'(x)\sin^2(u''(x)) + v(x) - 2\sqrt[3]{v'(x)} = 0, \\ \quad x \in [0,1] \setminus \{(\tau_j)\} \end{cases} \tag{4.11}$$

with $p > 1$, the boundary conditions

$$\begin{cases} u(0) = 0, \quad u'(0) = \frac{1}{2}, \quad u''(1) = 1 \\ v(0) = 0, \quad v'(0) = 0, \quad v''(1) = 1, \end{cases} \tag{4.12}$$

and impulsive conditions are given by

$$\begin{cases} \Delta u(x_i) = \frac{1}{30}u(x_i) + \frac{1}{40}u'(x_i) + \frac{1}{50}v(x_i), \\ \Delta u'(x_i) = \frac{1}{2\pi}\sin(\pi u'(x_i)) + \frac{1}{10}u''(x_i) + \frac{1}{10}v(x_i), \\ \Delta v(\tau_j) = \frac{1}{50}u(\tau_j) + \frac{1}{200e^2}v(\tau_j) + \frac{1}{1000}\sqrt[3]{v'(\tau_j)}, \\ \Delta v'(\tau_j) = \frac{1}{20}u(\tau_j) + \frac{1}{20}v(\tau_j) + \frac{1}{\pi}\sin(\frac{2\pi}{5}v'(\tau_j)), \end{cases} \tag{4.13}$$

with $x_i = \frac{i}{5}$ for $i = 1,2,3,4$ and $\tau_j = \frac{j^2}{10}$ for $j = 1,2,3$.

System (4.11)–(4.13) is a particular case of problem (1.1), (1.2), (1.4) with $[a,b] = [0,1]$,

$$\phi(y_2) = \arctan(y_2), \qquad \psi(z_2) = |z_2|^{p-2}z_2, \tag{4.14}$$

$$f(x,y_0,y_1,y_2,z_0,z_1,z_2) = y_0 - 4y_1 + z_0 + \arctan z_2,$$

$$g(x,y_0,y_1,y_2,z_0,z_1,z_2) = y_0 - 2z_1\sin^2(y_2) + z_0 - 2\sqrt[3]{z_1},$$

$$A_0 = B_0 = B_1 = 0, \qquad A_1 = \frac{1}{2}, \qquad A_2 = B_2 = 1,$$

$$I_0(\cdot,y_0,y_1,z_0,z_1) = \frac{1}{30}y_0 + \frac{1}{40}y_1 + \frac{1}{50}z_0,$$

$$I_1(\cdot,y_0,y_1,y_2,z_0,z_1) = \frac{1}{2\pi}\sin(\pi y_1) + \frac{1}{10}y_2 + \frac{1}{10}z_0,$$

$$J_0(\cdot,y_0,y_1,z_0,z_1) = \frac{1}{50}y_0 + \frac{1}{200e^2}z_0 + \frac{1}{1000}\sqrt[3]{z_1},$$

$$J_1(\cdot,y_0,y_1,z_0,z_1,z_2) = \frac{1}{20}y_0 + \frac{1}{20}z_0 + \frac{1}{\pi}\sin\left(\frac{2\pi}{5}z_1\right)$$

and $m = 4$, $n = 3$.

It is easy to see that the functions $\phi, \psi : \mathbb{R} \to \mathbb{R}$, given in (4.14), verify assumption (H1) and are increasing homeomorphisms such that $\phi(0) = \psi(0) = 0$, $\phi(\mathbb{R}) = \psi(\mathbb{R}) = \mathbb{R}$.

The functions $\alpha_\kappa : [0,1] \to \mathbb{R}$, $\kappa = 1,2$, given by

$$\alpha_1(x) = \begin{cases} -e^{x-1} - \frac{1}{5}x & \text{if } 0 \le x \le \frac{1}{5} \\ -e^{x-1} - \frac{2}{5}x & \text{if } \frac{1}{5} < x \le \frac{2}{5} \\ -e^{x-1} - \frac{3}{5}x & \text{if } \frac{2}{5} < x \le \frac{3}{5} \\ -e^{x-1} - \frac{4}{5}x & \text{if } \frac{3}{5} < x \le \frac{4}{5} \\ -e^{x-1} - x & \text{if } \frac{4}{5} < x \le 1, \end{cases} \qquad \alpha_2(x) = \begin{cases} -x^2 - \frac{1}{10}x & \text{if } 0 \le x \le \frac{1}{10} \\ -x^2 - \frac{2}{5}x & \text{if } \frac{1}{10} < x \le \frac{2}{5} \\ -x^2 - \frac{9}{10}x & \text{if } \frac{2}{5} < x \le \frac{9}{10} \\ -x^2 - x & \text{if } \frac{9}{10} < x \le 1, \end{cases}$$

and $\beta_\kappa : [0,1] \to \mathbb{R}$, $\kappa = 1,2$, given by

$$\beta_1(x) = \begin{cases} e^{x-1} + \frac{1}{5}x & \text{if } 0 \le x \le \frac{1}{5} \\ e^{x-1} + \frac{2}{5}x & \text{if } \frac{1}{5} < x \le \frac{2}{5} \\ e^{x-1} + \frac{3}{5}x & \text{if } \frac{2}{5} < x \le \frac{3}{5} \\ e^{x-1} + \frac{4}{5}x & \text{if } \frac{3}{5} < x \le \frac{4}{5} \\ e^{x-1} + x & \text{if } \frac{4}{5} < x \le 1, \end{cases} \qquad \beta_2(x) = \begin{cases} x^2 + \frac{1}{10}x & \text{if } 0 \le x \le \frac{1}{10} \\ x^2 + \frac{2}{5}x & \text{if } \frac{1}{10} < x \le \frac{2}{5} \\ x^2 + \frac{9}{10}x & \text{if } \frac{2}{5} < x \le \frac{9}{10} \\ x^2 + x & \text{if } \frac{9}{10} < x \le 1, \end{cases}$$

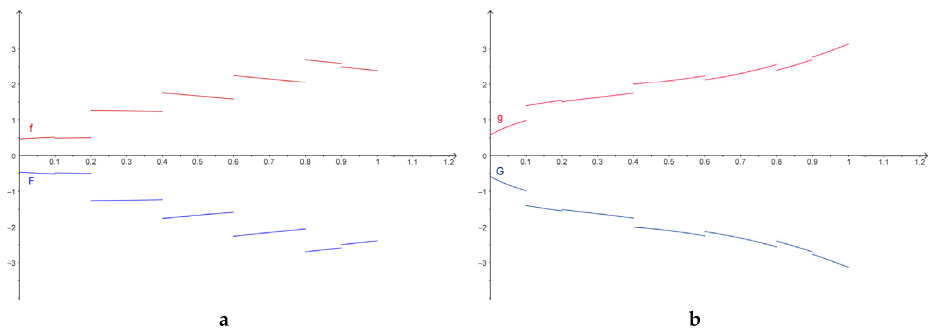

Figure 1 Relationship between nonlinearities depending on the lower and upper solutions, given by the inequalities: a) $(\phi(\alpha_1''(x)))' + f(x,\alpha_1(x),\alpha_1'(x),\alpha_1''(x),\alpha_2(x),\alpha_2'(x),\alpha_2''(x)) \geq 0 \geq (\phi(\beta_1''(x)))' + f(x,\beta_1(x),\beta_1'(x),\beta_1''(x),\beta_2(x),\beta_2'(x),\beta_2''(x))$; b) $(\psi(\alpha_2''(x)))' + g(x,\alpha_1(x),\alpha_1'(x),\alpha_1''(x),\alpha_2(x),\alpha_2'(x),\alpha_2''(x)) \geq 0 \geq (\psi(\beta_2''(x)))' + g(x,\beta_1(x),\beta_1'(x),\beta_1''(x),\beta_2(x),\beta_2'(x),\beta_2''(x))$

Table 1 Impulsive conditions for functions α_1 and β_1

i	x_i	$\Delta\alpha_1(x_i)$	$I_{0i}(\alpha_1)$	$I_{0i}(\beta_1)$	$\Delta\beta_1(x_i)$	$\Delta\alpha_1'(x_i)$	$I_{1i}(\alpha_1)$	$I_{1i}(\beta_1)$	$\Delta\beta_1'(x_i)$
1	0.2	−0.0400	−0.0349	0.0349	0.0400	−0.2000	−0.1989	0.1989	0.2000
2	0.4	−0.0800	−0.0537	0.0537	0.0800	−0.2000	−0.1124	0.1124	0.2000
3	0.6	−0.1200	−0.0841	0.0841	0.1200	−0.2000	−0.0375	0.0375	0.2000
4	0.8	−0.1600	−0.1163	0.1163	0.1600	−0.2000	−0.0697	0.0697	0.2000

Table 2 Impulsive conditions for functions α_2 and β_2

j	τ_j	$\Delta\alpha_2(\tau_j)$	$J_{0j}(\alpha_2)$	$J_{0j}(\beta_2)$	$\Delta\beta_2(\tau_j)$	$\Delta\alpha_2'(\tau_j)$	$J_{1j}(\alpha_2)$	$J_{1j}(\beta_2)$	$\Delta\beta_2'(\tau_j)$
1	0.1	−0.0300	−0.0092	0.0092	0.0300	−0.3000	−0.1395	0.1395	0.2000
2	0.4	−0.2000	−0.0155	0.0155	0.2000	−0.5000	−0.3691	0.3691	0.5000
3	0.9	−0.0900	−0.0386	0.0386	0.0900	−0.1000	−0.0921	0.0921	0.1000

when $x_5 = \tau_4 = 1$, are, respectively, lower and upper solutions of problem (4.11)–(4.13) according to Definition 4.1. The differential inequalities are verified in the interval $[0,1]$, as shown in Fig. 1.

The boundary conditions

$$\alpha_1(0) = -\frac{1}{e} < 0, \qquad \alpha_1'(0) = -\frac{1}{e} - \frac{1}{5} < \frac{1}{2}, \qquad \alpha_1''(1) = -1 < 1,$$
$$\alpha_2(0) = 0, \qquad \alpha_2'(0) = -\frac{1}{10} < 0, \qquad \alpha_2''(1) = -2 < 1,$$
$$\beta_1(0) = \frac{1}{e} > 0, \qquad \beta_1'(0) = \frac{1}{e} + \frac{1}{5} > \frac{1}{2}, \qquad \beta_1''(1) = 1,$$
$$\beta_2(0) = 0, \qquad \beta_2'(0) = \frac{1}{e} > 0, \qquad \beta_2''(1) = 2 > 1,$$

and impulsive conditions verify the inequalities of Definition 4.1, as shown in Table 1 and Table 2.

Let

$$L > \max\left\{\|\alpha_1(x)\|_{X_1}, \|\beta_1(x)\|_{X_1}, \|\alpha_2(x)\|_{X_2}, \|\beta_2(x)\|_{X_2}\right\},$$

then f and g are L^1-Carathéodory functions with

$$\begin{aligned} \left|f(x,y_0,y_1,y_2,z_0,z_1,z_2)\right| &\leq 6L+1 := \rho_{1L}(x), \\ \left|g(x,y_0,y_1,y_2,z_0,z_1,z_2)\right| &\leq 4L+2\sqrt[3]{L} := \rho_{2L}(x), \end{aligned}$$

and the sum of the jumps is bounded.

The functions f and g satisfy the Nagumo condition relative to the sets

$$S_1 = \left\{ \begin{array}{c} (x,y_0,y_1,y_2,z_0,z_1,z_2) \in [0,1]\times\mathbb{R}^6 : \\ \alpha_1^{(l)}(x) \leq y_l \leq \beta_1^{(l)}(x),\ \alpha_2^{(l)}(x) \leq z_l \leq \beta_2^{(l)}(x), l=0,1 \end{array} \right\}.$$

Consider a constant $\mathcal{K}_k > 0$, $k = 1,2$, and μ_k as defined in (2.5), then, in S_1,

$$\begin{aligned} \left|f(x,y_0,y_1,y_2,z_0,z_1,z_2)\right| &= |y_0 - 4y_1 + z_0 + \arctan z_2| \\ &\leq \mathcal{K}_1 := \varphi_1\bigl(|y_2|\bigr) \end{aligned}$$

and

$$\begin{aligned} \left|g(x,y_0,y_1,y_2,z_0,z_1,z_2)\right| &= \left|y_0 - 2z_1\sin^2(y_2) + z_0 - 2\sqrt[3]{z_1}\right| \\ &\leq \mathcal{K}_2 := \varphi_2\bigl(|z_2|\bigr), \end{aligned}$$

it is trivial that

$$\int_{\phi(\mu_1)}^{\phi(+\infty)} \frac{|\phi^{-1}(s)|}{|\phi^{-1}(s)| + \mathcal{K}_1}\,ds = +\infty \quad \text{and} \quad \int_{\psi(\mu_2)}^{\psi(+\infty)} \frac{|\psi^{-1}(s)|}{|\psi^{-1}(s)| + \mathcal{K}_2}\,ds = +\infty.$$

So, by Theorem 4.2, there is at least one pair of functions $(u(x), v(x)) \in X^2$, a solution of problem (4.11)–(4.13); moreover,

$$\begin{aligned} &\alpha_1(x) \leq u(x) \leq \beta_1(x), \qquad \alpha_2(x) \leq v(x) \leq \beta_2(x), \\ &\alpha_1'(x) \leq u'(x) \leq \beta_1'(x), \qquad \alpha_2'(x) \leq v'(x) \leq \beta_2'(x), \quad \forall x \in [0,1], \end{aligned}$$

as shown in Fig. 2 and Fig. 3, and

$$\|u''\| \leq N_1 \quad \text{and} \quad \|v''\| \leq N_2$$

with N_1 and N_2 given by Lemma 2.4.

5 Singular ϕ-Laplacian equations in special relativity

Relativity implies that physical laws do not depend on the chosen reference frame. In special relativity, the speed of light c is recognized as the maximum speed with which information can travel in free space from one frame of reference to another [4]. Let us consider two frames of reference $\mathcal{P}_0$ and $\mathcal{P}$ in uniform relative motion to each other, that is, moving with relative speed v. Taking into account the upper limit c of the speed of information propagation, the space–time coordinates of the frames $\mathcal{P}_0$ and $\mathcal{P}$ must be related

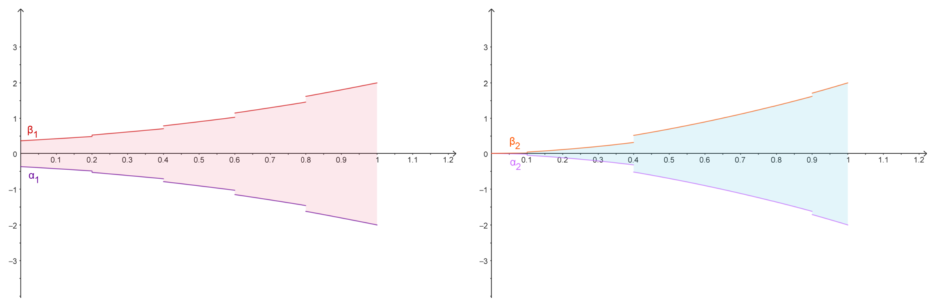

Figure 2 At least one solution $(u(x), v(x))$ of problem (4.11)–(4.13) is located in the colored region, when $x \in [0, 1]$

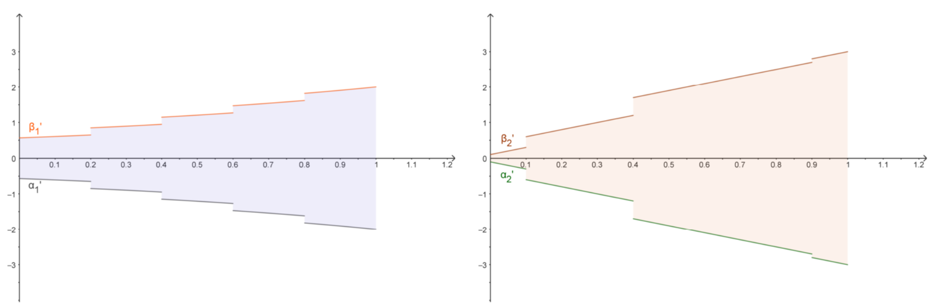

Figure 3 Localization for the first derivative of the solution of problem (4.11)–(4.13), when $x \in [0, 1]$

by *Lorentz transformations* [13]. The Lorentz factor depends nonlinearly on the relative velocity v and is defined by

$$\Gamma \equiv \frac{1}{\sqrt{1-(\frac{v}{c})^2}}.$$

The theory of special relativity is fundamental in the development of the modern theory of classical electrodynamics. The fact that an electric charge q generates an electric field $\mathbf{E}$ and in motion generates a magnetic field $\mathbf{B}$ is intuitively compatible with the statement that the electric and magnetic fields are covariant under a Lorentz transformation from one inertial system to another [20].

The study developed in this article can be adapted and applied to a system of singular ϕ-Laplacian equations, that is, to the system of equations (1.1), with two restrictions:

- In Lemma 2.4, the constants N_1 and N_2 must be chosen such that

$$0 < N_1 < \eta \quad \text{and} \quad 0 < N_2 < \gamma;$$

- Assumption (H1) must be replaced by

(Hs) $\phi : (-\eta, \eta) \to \mathbb{R}$ and $\psi : (-\gamma, \gamma) \to \mathbb{R}$ for some $0 < \eta < +\infty$ and $0 < \gamma < +\infty$ are increasing homeomorphisms with $\phi(0) = \psi(0) = 0$, $\phi(-\eta, \eta) = \mathbb{R}$ and $\psi(-\gamma, \gamma) = \mathbb{R}$ such that

$$\left|\phi^{-1}(w)\right| \le \phi^{-1}\left(|w|\right) \quad \text{and} \quad \left|\psi^{-1}(w)\right| \le \psi^{-1}\left(|w|\right).$$

In this case, a solution to problem (1.1)–(1.3) is a pair of functions $(u(x), v(x)) \in X^2$ such that $(u''(x), v''(x)) \in (-\eta, \eta) \times (-\gamma, \gamma)$ for all $x \in [a, b]$ satisfying (1.1)–(1.3).

Example 5.1 Consider the problem

$$\begin{cases} \left(\frac{u''(x)}{\sqrt{1-(u''(x))^2}}\right)' + \frac{1}{12}u(x) - \frac{5}{2}(u'(x))^2 + \frac{1}{6}v'(x)|v'(x)| = 0, \\ \quad x \in [-1,1] \setminus \{(x_i)\}, \\ \left(\frac{v''(x)}{\sqrt{1-\frac{(v''(x))^2}{9}}}\right)' + \frac{1}{12}u(x) - \frac{5}{2}(v'(x))^2 + \frac{1}{6}u'(x)|u'(x)| = 0, \\ \quad x \in [-1,1] \setminus \{(\tau_j)\} \end{cases} \tag{5.1}$$

with the boundary conditions

$$\begin{cases} u(-1) = 0, & u'(-1) = \frac{1}{3}, & u''(1) = \frac{1}{2} \\ v(-1) = 0, & v'(-1) = -\frac{1}{2}, & v''(1) = 0, \end{cases} \tag{5.2}$$

and impulsive conditions are given by

$$\begin{cases} \Delta u(x_i) = \frac{1}{4\pi} \arctan(u(x_i) + 1), \\ \Delta u'(x_i) = \frac{1}{10}u(x_i) - \frac{1}{10}u'(x_i), \\ \Delta v(\tau_j) = \frac{1}{100}v(\tau_j) + \frac{1}{6}, \\ \Delta v'(\tau_j) = \frac{1}{3\pi} \sin^2(2\pi v'(\tau_j)), \end{cases} \tag{5.3}$$

with $x_1 = 0$ and $\tau_1 = -\frac{1}{2}$, $\tau_2 = 0$, $\tau_3 = \frac{1}{2}$.

System (5.1)–(5.3) is a particular case of problem (1.1), (1.2), (1.4), with $[a, b] = [-1, 1]$,

$$\phi(y_2) = \frac{y_2}{\sqrt{1 - y_2^2}}, \qquad \psi(z_2) = \frac{z_2}{\sqrt{1 - \frac{z_2^2}{9}}}, \tag{5.4}$$

$$f(x, y_0, y_1, y_2, z_0, z_1, z_2) = \frac{1}{12}y_0 - \frac{5}{2}y_1^2 + \frac{1}{6}z_1|z_1|,$$

$$g(x, y_0, y_1, y_2, z_0, z_1, z_2) = \frac{1}{12}y_0 - \frac{5}{2}z_1^2 + \frac{1}{6}y_1|y_1|,$$

$$A_0 = B_0 = B_2 = 0, \qquad A_1 = \frac{1}{3}, \qquad A_2 = \frac{1}{2}, \qquad B_1 = -\frac{1}{2},$$

$$I_0(\cdot, y_0, y_1, z_0, z_1) = \frac{1}{4\pi} \arctan(y_0 + 1),$$

$$I_1(\cdot, y_0, y_1, y_2, z_0, z_1) = \frac{1}{10}y_0 - \frac{1}{10}y_1,$$

$$J_0(\cdot, y_0, y_1, z_0, z_1) = \frac{1}{100}z_0 + \frac{1}{6},$$

$$J_1(\cdot, y_0, y_1, z_0, z_1, z_2) = \frac{1}{3\pi} \sin^2(2\pi z_1),$$

and $m = 1$, $n = 3$.

As the functions $\phi : (-1,1) \to \mathbb{R}$ and $\psi : (-3,3) \to \mathbb{R}$ given in (5.4) are increasing homeomorphisms such that $\phi(0) = \psi(0) = 0$, $\phi(-1,1) = \mathbb{R}$, $\psi(-3,3) = \mathbb{R}$,

$$\phi^{-1}(w) = \psi^{-1}(w) = \frac{w}{\sqrt{1+w^2}},$$

and

$$\left|\phi^{-1}(w)\right| = \frac{|w|}{\sqrt{1+w^2}} = \phi^{-1}\big(|w|\big),$$

then assumption (Hs) holds.

The functions $\alpha_\kappa : [0,1] \to \mathbb{R}$, $\kappa = 1,2$, given by

$$\alpha_1(x) = \begin{cases} \frac{1}{60}x^5 + \frac{1}{18}x^3 + \frac{1}{20} & \text{if } -1 \le x \le 0 \\ \frac{1}{60}x^5 + \frac{1}{36}x^3 + \frac{1}{10} & \text{if } 0 < x \le 1, \end{cases}$$

$$\alpha_2(x) = \begin{cases} \frac{1}{2\pi}\sin(\pi x) & \text{if } -1 \le x \le -\frac{1}{2} \\ -\frac{1}{2\pi}(\sin(\pi x)+1) & \text{if } -\frac{1}{2} < x \le 0 \\ -\frac{1}{2\pi}\sin(\pi x) & \text{if } 0 < x \le \frac{1}{2} \\ \frac{1}{2\pi}(\sin(\pi x)-1) & \text{if } \frac{1}{2} < x \le 1, \end{cases}$$

and $\beta_\kappa : [0,1] \to \mathbb{R}$, $\kappa = 1,2$, given by

$$\beta_1(x) = \begin{cases} \frac{1}{6}\sqrt{(x+3)^3} & \text{if } -1 \le x \le 0 \\ \frac{1}{4}x^2 + \frac{1}{2}x + 1 & \text{if } 0 < x \le 1, \end{cases}$$

$$\beta_2(x) = \begin{cases} -\frac{1}{12}x^3 + \frac{1}{2}x^2 + x + \frac{1}{2} & \text{if } -1 \le x \le -\frac{1}{2} \\ -\frac{1}{6}x^3 - x^2 - \frac{2}{5}x + \frac{9}{25} & \text{if } -\frac{1}{2} < x \le 0 \\ -\frac{1}{6}x^3 + x^2 - \frac{3}{10}x + \frac{3}{5} & \text{if } 0 < x \le \frac{1}{2} \\ -\frac{1}{12}x^3 + \frac{1}{3}x^2 + \frac{1}{3}x + \frac{2}{3} & \text{if } \frac{1}{2} < x \le 1, \end{cases}$$

are, respectively, lower and upper solutions of problem (5.1)–(5.3) according to Definition 4.1. In fact, the differential inequalities are verified in the interval $[-1,1]$, as shown in Fig. 4.

The boundary conditions

$$\alpha_1(-1) = -\frac{1}{45} < 0, \qquad \alpha_1'(-1) = \frac{1}{4} < \frac{1}{3}, \qquad \alpha_1''(1) = \frac{1}{2},$$

$$\alpha_2(-1) = 0, \qquad \alpha_2'(-1) = -\frac{1}{2}, \qquad \alpha_2''(1) = 0,$$

$$\beta_1(-1) = \frac{1}{3}\sqrt{2} > 0, \qquad \beta_1'(-1) = \frac{\sqrt{2}}{4} > \frac{1}{3}, \qquad \beta_1''(1) = \frac{1}{2},$$

$$\beta_2(-1) = \frac{1}{12} > 0, \qquad \beta_2'(-1) = -\frac{1}{4} > -\frac{1}{2}, \qquad \beta_2''(1) = \frac{1}{6} > 0,$$

and impulsive conditions verify the inequalities of Definition 4.1, as shown in Table 3 and Table 4.

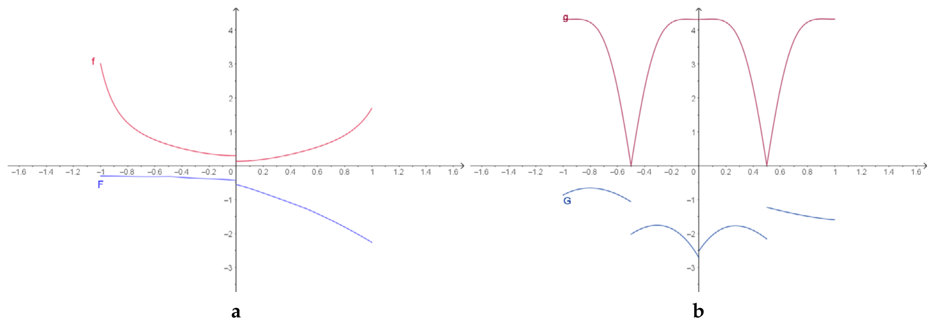

Figure 4 Relationship between nonlinearities depending on the lower and upper solutions, given by the inequalities: a) $(\phi(\alpha_1''(x)))' + f(x, \alpha_1(x), \alpha_1'(x), \alpha_1''(x), \alpha_2(x), \alpha_2'(x), \alpha_2''(x)) \geq 0 \geq (\phi(\beta_1'(x)))' + f(x, \beta_1(x), \beta_1'(x), \beta_1''(x), \beta_2(x), \beta_2'(x), \beta_2''(x))$; b) $(\psi(\alpha_2''(x)))' + g(x, \alpha_1(x), \alpha_1'(x), \alpha_1''(x), \alpha_2(x), \alpha_2'(x), \alpha_2''(x)) \geq 0 \geq (\psi(\beta_2'(x)))' + g(x, \beta_1(x), \beta_1'(x), \beta_1''(x), \beta_2(x), \beta_2'(x), \beta_2''(x))$

Table 3 Impulsive conditions for functions α_1 and β_1

i	x_i	$\Delta\alpha_1(x_i)$	$I_{0i}(\alpha_1)$	$I_{0i}(\beta_1)$	$\Delta\beta_1(x_i)$	$\Delta\alpha_1'(x_i)$	$I_{1i}(\alpha_1)$	$I_{1i}(\beta_1)$	$\Delta\beta_1'(x_i)$
1	0.0	0.0500	0.0644	0.0859	0.1340	0.0000	0.0050	0.0433	0.0670

Table 4 Impulsive conditions for functions α_2 and β_2

j	τ_j	$\Delta\alpha_2(\tau_j)$	$J_{0j}(\alpha_2)$	$J_{0j}(\beta_2)$	$\Delta\beta_2(\tau_j)$	$\Delta\alpha_2'(\tau_j)$	$J_{1j}(\alpha_2)$	$J_{1j}(\beta_2)$	$\Delta\beta_2'(\tau_j)$
1	–0.5	0.1592	0.1651	0.1680	0.1954	0.0000	0.0000	0.0155	0.0375
2	0.0	0.1592	0.1651	0.1703	0.2400	0.0000	0.0000	0.0367	0.1000
3	0.5	0.1592	0.1651	0.1735	0.2271	0.0000	0.0000	0.0219	0.0292

Let

$$L > \max\left\{\|\alpha_1(x)\|_{X_1}, \|\beta_1(x)\|_{X_1}, \|\alpha_2(x)\|_{X_2}, \|\beta_2(x)\|_{X_2}\right\},$$

then f and g are L^1-Carathéodory functions with

$$\begin{aligned} \left|f(x, y_0, y_1, y_2, z_0, z_1, z_2)\right| &\leq \frac{1}{12}L + \frac{8}{3}L^2 := \rho_{1L}(x), \\ \left|g(x, y_0, y_1, y_2, z_0, z_1, z_2)\right| &\leq \frac{1}{12}L + \frac{8}{3}L^2 := \rho_{2L}(x), \end{aligned}$$

and the sum of the jumps is bounded.

The functions f and g satisfy the Nagumo condition relative to the sets

$$S_2 = \left\{ \begin{array}{c} (x, y_0, y_1, y_2, z_0, z_1, z_2) \in [-1, 1] \times \mathbb{R}^6 : \\ \alpha_1^{(l)}(x) \leq y_l \leq \beta_1^{(l)}(x), \alpha_2^{(l)}(x) \leq z_l \leq \beta_2^{(l)}(x), l = 0, 1 \end{array} \right\}.$$

Consider a constant $\mathcal{K}_k > 0$, $k = 1, 2$, and μ_k as defined in (2.5), then, in S_2,

$$\begin{aligned} \left|f(x, y_0, y_1, y_2, z_0, z_1, z_2)\right| &= \left|\frac{1}{12}y_0 - \frac{5}{2}y_1^2 + \frac{1}{6}z_1|z_1|\right| \\ &\leq \mathcal{K}_1 := \varphi_1\left(|y_2|\right) \end{aligned}$$

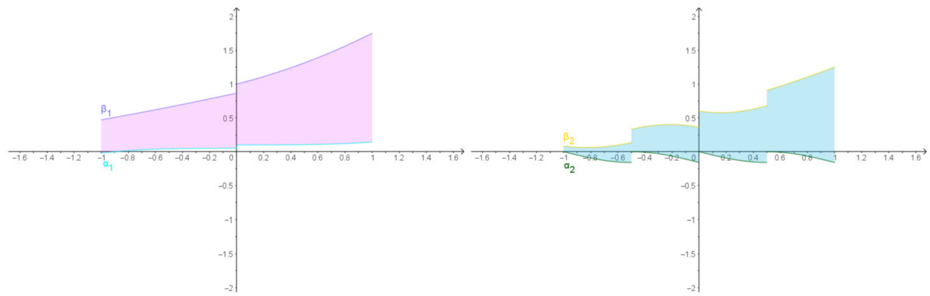

Figure 5 At least one solution $(u(x), v(x))$ of problem (5.1)–(5.3) is located in the colored region, when $x \in [-1, 1]$

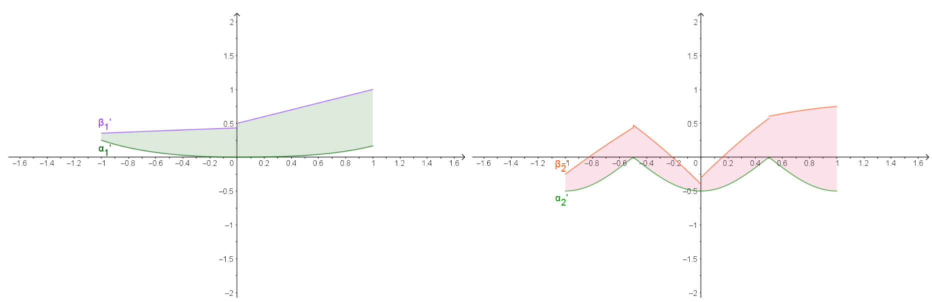

Figure 6 Localization for the first derivative of the solution of problem (5.1)–(5.3), when $x \in [-1, 1]$

and

$$\begin{aligned} \left|g(x, y_0, y_1, y_2, z_0, z_1, z_2)\right| &= \left|\frac{1}{12}y_0 - \frac{5}{2}z_1^2 + \frac{1}{6}y_1|y_1|\right| \\ &\le \mathcal{K}_2 := \varphi_2\left(|z_2|\right), \end{aligned}$$

it is trivial that

$$\int_{\phi(\mu_1)}^{+\infty} \frac{|\phi^{-1}(s)|}{|\phi^{-1}(s)| + \mathcal{K}_1}\, ds = +\infty \quad \text{and} \quad \int_{\psi(\mu_2)}^{+\infty} \frac{|\psi^{-1}(s)|}{|\psi^{-1}(s)| + \mathcal{K}_2}\, ds = +\infty.$$

So, by Theorem 4.2, there is at least one pair of functions $(u(x), v(x)) \in X^2$, a solution of problem (5.1)–(5.3); moreover,

$$\begin{aligned} &\alpha_1(x) \le u(x) \le \beta_1(x), \qquad \alpha_2(x) \le v(x) \le \beta_2(x), \\ &\alpha_1'(x) \le u'(x) \le \beta_1'(x), \qquad \alpha_2'(x) \le v'(x) \le \beta_2'(x), \quad \forall x \in [-1, 1], \end{aligned}$$

as shown in Fig. 5 and Fig. 6, and

$$\|u''\| \le N_1 < 1 \quad \text{and} \quad \|v''\| \le N_2 < 3$$

with N_1 and N_2 given by Lemma 2.4.

6 Conclusion

This work shows, mainly, that local monotonies on the nonlinearities and the impulsive functions are sufficient conditions for the solvability of a third-order impulsive coupled system with two differential equations involving different Laplacians, fully discontinuous nonlinearities, and two-point boundary conditions. The localization information given by the lower and upper solutions had been underused to obtain qualitative data on the solutions, such as growth type, sign, and estimation of the unknown function and its derivatives, as it is illustrated in both examples. To the best of our knowledge, it is the first time where impulsive coupled systems with strongly nonlinear fully differential equations and generalized impulse effects are considered simultaneously. There remain arguments and techniques to be used, to obtain the localization part for coupled systems with jumps on the Laplacians.

Author contributions
The two authors contributed equally to the article.

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

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