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Riemann problem for multiply connected domain in Besov spaces

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Abstract

In this paper, we obtain conditions of the solvability of the Riemann boundary value problem for sectionally analytic functions in multiply connected domains in Besov spaces embedded into the class of continuous functions. We indicate a new class of Cauchy-type integrals, which are continuous on a closed domain with continuous (not Hölder) density in terms of Besov spaces, and for which the Sokhotski–Plemelj formulas are valid.

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1 Introduction

Let $L = L_0 + L_1 + \cdots + L_m$ be a collection of $m+1$ disjoint closed contours, with L_0 containing all the others. Let D^+ be the $m+1$ connected domain situated inside the contour L_0 and outside $L_1, L_2, \ldots, L_m$. By D^- we denote the supplement of $\overline{D^+} = D^+ + L$ to the whole plane. For definiteness, we suppose that the origin of the coordinate system is located in D^+. For the positive direction on the contour L, we take that leaving the domain D^+ on the left, i.e., the contour L_0 is to be traversed anticlockwise, and the contours $L_1, \ldots, L_m$, clockwise.

1.1 Formulation of the Riemann boundary value problem

Suppose that on a closed contour L, two functions $G(t)$ and $g(t)$ are given and $G(t)$ does not vanish. We assume that these functions belong to the Besov space $B^r_{p,\theta}(L)$, where the parameters $r > 0, 1 < p < \infty, 1 \le \theta \le \infty$ are determined by one of the following conditions:

(I) $1 < p < 2, \theta = 1, r = 1/p$,

(II) $1 < p < 2, \theta \ge 1, r > 1/p$,

(III) $p \ge 2, \theta \ge 1, r > 1 - 1/p$.

Moreover, we assume that $r + 1/p - 1 < \nu \le 1$ and $L \in C^1_\nu$, where C^1_ν is the class of functions with continuous derivative concerning Hölder with index ν.

It is required to find a sectionally analytic function $\Phi(z)$, consisting of $\Phi^+(z)$, analytical in the domain D^+, and $\Phi^-(z)$, analytical and domain D^-, including $z = \infty$, that satisfies on

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the contour L either the linear relation

$$\Phi^+(t) = G(t)\Phi^-(t) \quad \text{(homogeneous problem)} \tag{1.1}$$

or

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t) \quad \text{(nonhomogeneous problem).} \tag{1.2}$$

The boundary values of the functions $\Phi^+(z)$ and $\Phi^-(z)$ on a closed contour L are denoted $\Phi^+(t)$ and $\Phi^-(t)$, respectively.

The function $G(t)$ is called the coefficient of the Riemann boundary value problem, and the function $g(t)$ is its free term.

1.2 Determination of a sectionally analytic function in accordance with given jump

Let us consider the Riemann boundary value problem of a certain type, which will be used later.

Problem 1.1 Let $\varphi(t) \in B^r_{p,\theta}(L)$. It is required to find a sectionally analytic function $\Phi(z)$ ($\Phi(z) = \Phi^+(z)$ for $z \in D^+$, $\Phi(z) = \Phi^-(z)$ for $z \in D^-$) taking a finite value $A \geq 0$ at infinity and undergoing in passing through the contour L a jump $\varphi(t)$, i.e., satisfying on L the condition

$$\Phi^+(t) - \Phi^-(t) = \varphi(t). \tag{1.3}$$

Problem 1.1 is solvable and has a unique solution (for given A), expressed through a Cauchy-type integral

$$\Phi(z) = \frac{1}{2\pi \mathrm{i}} \int_L \frac{\varphi(\tau)}{\tau - z}\, \mathrm{d}\tau + A \tag{1.4}$$

as in the case of a simply connected domain. It follows from the Sokhotski–Plemelj formulas, which are the same for multiply connected as for simply connected domains, as shown in [5, Ch. 1, Sect. 8.7],. Under our conditions, the Cauchy-type integral in (1.4) belongs to the space $B^{1+\alpha}_{p,\theta}(D^+)$, $\alpha = r + 1/p - 1$ ([4, Ch. 1, Lemma 1.2]), embedded in the space $C(\overline{D^+})$ of continuous functions in the closed domain $\overline{D^+}$ [3], and

$$\|\Phi\|_{B^{1+\alpha}_{p,\theta}(D^+)} \leq M\|\varphi\|_{B^r_{p,\theta}(L)}, \tag{1.5}$$

where the constant $M > 0$ does not depend on φ. For the boundary values $\Phi^+(t)$ and $\Phi^-(t)$ of the Cauchy-type integral ($A = 0$), the Sokhotski–Plemelj formula holds ([4, Ch. 1, Lemma 1.2]; see the proof of Corollary 2).

By definition, $f(x) \in B^r_{p,\theta}(E^n), 1 \leq p < \infty, 1 \leq \theta \leq \infty$, $r = \bar{r} + \alpha$, $\bar{r}$-integer, $0 < \alpha \leq 1$, and E^n is the n-dimensional Euclidean space with finite norm

$$\|f\|_{B^r_{p,\theta}(E^n)} = \|f\|_{L_p(E^n)} + \left(\int_{E^n} |u|^{-n-\alpha\theta} \left\| \Delta^2_u f^{(\bar{r})} \right\|^{\theta}_{L_p(E^n)} \mathrm{d}h \right)^{\frac{1}{\theta}},$$

where $f_u^{(\overline{r})}$ is the generalized (according to Sobolev) derivative of order $\overline{r}$ in some direction $u \in E^n$.

We denote $\Delta_u f(x) = f(x+u) - f(x)$ and $\Delta_u^m f(x) = \Delta_u[\Delta_u^{m-1} f(x)]$, a finite difference of order m. If $\alpha < 1$, then in the seminorm the second difference Δ_u^2 can be replaced by the first Δ_u.

The Besov space can be defined by preserving the corresponding embedding theorems. For example, the theorem for the embedding of different metrics holds for $B_{p,\theta}^r(D^+)$, where $D^+ \subset E^n$ is a domain with Lipschitz boundary [3]. For other equivalent standards and details, see [3].

Referring to the above, we note the following interesting result. In condition (I), the space $B_{p,1}^{1/p}(L)$, $1 < p < 2$, is embedded in the space $C(L)$ of continuous functions but is not embedded in the space $C_\beta(L)$, $0 < \beta < 1$, of Hölder-continuous functions [3].

In this way, note that the Besov class $B_{p,1}^{1/p}(L)$, $1 < p < 2$, contains continuous functions $\varphi(t)$ that are not Hölder continuous on L, the Cauchy-type integral of which is a continuous function in a closed domain $\overline{D^+}$, and the Sokhotski–Plemelj formulas are valid. This result seems interesting because in general it is well known that a Cauchy-type integral with arbitrary continuous density is not necessarily a continuous function in a closed domain. Functions from $B_{p,1}^{1/p}(L)$, $1 < p < 2$, satisfy Hölder's inequality in the L_p norm.

In conditions (II) and (III), $\varphi(t) \in B_{p,\theta}^r(L) \hookrightarrow C_\beta(L)$ for some $0 < \beta < 1$. Even in these cases, the belonging of the Cauchy-type integral $\Phi(z)$ to $B_{p,\theta}^{1+\alpha}(D^+)$ provides information about the existence of its generalized derivatives belonging to $B_{p,\theta}^{\alpha}(D^+)$.

Further, the problem of finding a sectionally analytic function for a given jump will be encountered several times, so it is convenient to formulate it in the form of the following:

Conclusion 1.2 An arbitrary function $\varphi(t) \in B_{p,\theta}^r(L)$ can be uniquely represented as the difference of functions $\Phi^+(t)$ and $\Phi^-(t)$ that are the boundary values of the analytic functions $\Phi^+(z)$ and $\Phi^-(z)$ with the additional condition $\Phi^-(\infty) = A \geq 0$.

In the famous monographs of Muskhelishvili [9] and Gakhov [5], where the theories of singular integral equations (with the Cauchy kernel) and boundary value problems for analytic functions of a complex variable are summarized, and the main method of research is the apparatus of Cauchy-type integrals in the spaces C_β, $0 < \beta < 1$, and L_p, $p \geq 1$. The works of [1, 2, 6–8, 10] are devoted to the study of various cases of the specified range of problems in the same spaces (sometimes with weights). The results of these works can be generalized or refined by applying the above results to specified Besov spaces with relatively simplified data requirements.

2 Solution of the Riemann boundary value problem

2.1 Solution of the homogeneous problem

We use some commonly accepted terms and concepts from [5, 9]. Let us denote by $\varkappa_k = (1/2\pi)[\arg G(t)]_{L_k}$ the index of the function $G(t)$ on the contour L_k ($k = 1, 2, .., m$), where $[\cdot]_{L_k}$ denotes the increment of the expression enclosed in brackets when going around the L_k contour once in the positive direction. The value of $\varkappa = \sum_{k=0}^{m} \varkappa_k$ is called the index of the problem.

We introduce the function

$$\prod_{k=1}^{m}(t-z_k)^{\varkappa_k},$$

where z_k are some points lying inside the contours L_k $(k=1,2,\dots,m)$. We easily obtain

$$\frac{1}{2\pi}\left[\arg\prod_{k=1}^{m}(t-z_k)^{\varkappa_k}\right]_{L_j}=\frac{1}{2\pi}\left[\arg(t-z_j)^{\varkappa_j}\right]_{L_j}=-\varkappa_j,\quad (j=1,2,\dots,m),$$

$$\begin{aligned}\frac{1}{2\pi}\left[\arg G(t)\prod_{k=1}^{m}(t-z_k)^{\varkappa_k}\right]_{L_0}&=\frac{1}{2\pi}\left[\arg G(t)\right]_{L_0}+\frac{1}{2\pi}\sum_{k=1}^{m}\left[\arg(t-z_k)^{\varkappa_k}\right]_{L_0}\\&=\varkappa_0+\sum_{k=1}^{m}\varkappa_k=\varkappa.\end{aligned}$$

Since the origin is located in the domain D^+, we have

$$\frac{1}{2\pi}\left\{\arg\left[t^{-\varkappa}\prod_{k=1}^{m}(t-z_k)^{\varkappa_k}G(t)\right]\right\}_L=0. \tag{2.1}$$

Now we can formulate the conditions of solvability of the homogeneous Riemann boundary value problem written in the form (1.1).

Theorem 2.1 *Let the parameters r,p,θ be determined by one of the conditions* (I)–(III). *Let $G(t),g(t)\in B^r_{p,\theta}(L)$ and $G(t)\neq 0$, $t\in L$. Then if the index $\varkappa$ of the Riemann boundary value problem is positive, then problem* (1.1) *has $\varkappa+1$ linearly independent solutions*

$$\Phi_k^+(z)=\prod_{k=1}^{m}(z-z_k)^{-\varkappa_k}\mathrm{e}^{\Gamma^+(z)}z^l,\qquad \Phi_k^-(z)=\mathrm{e}^{\Gamma^-(z)}z^{l-\varkappa},\quad l=0,1,\dots,\varkappa, \tag{2.2}$$

where

$$\Gamma(z)=\frac{1}{2\pi\mathrm{i}}\int_L\frac{\log G(\tau)\,\mathrm{d}\tau}{\tau-z}.$$

The general solution contains $\varkappa+1$ arbitrary constants and is given by the formula

$$\Phi^+(z)=\prod_{k=1}^{m}(z-z_k)^{-\varkappa_k}\mathrm{e}^{\Gamma^+(z)}P_\varkappa(z),\qquad \Phi^-(z)=z^{-\varkappa}\mathrm{e}^{\Gamma^-(z)}P_\varkappa(z), \tag{2.3}$$

where $P_\varkappa(z)$ is a polynomial of degree not higher than $\varkappa$ with arbitrary complex coefficients. If $\varkappa=0$, then we come to the case of an auxiliary homogeneous problem, and the problem has a solution of the form

$$\Phi^+(z)=A\mathrm{e}^{\Gamma^+(z)},\qquad \Phi^-(z)=A\mathrm{e}^{\Gamma^-(z)},\qquad \Phi^-(\infty)=A\neq 0.$$

If $\varkappa<0$, then the homogeneous problem has only the trivial solution.

Proof Let us rewrite the homogeneous problem (1.1) in the form

$$\Phi^+(t) = \frac{t^{\varkappa}}{\prod_{k=1}^{m}(t-z_k)^{\varkappa_k}}\left[t^{-\varkappa}\prod_{k=1}^{m}(t-z_k)^{\varkappa_k}G(t)\right]\Phi^-(t). \tag{2.4}$$

The function

$$G_1(t) = t^{-\varkappa}\prod_{k=1}^{m}(t-z_k)^{\varkappa^k}G(t) \tag{2.5}$$

belongs to the space $B^r_{p,\theta}(L)$ ([4, Ch. I, Sect. 6]), and its index on L is zero in (2.1). Consequently, $\log G_1(t)$ is a single-valued function on L, and $\log G_1(t) \in B^r_{p,\theta}(L)$. Note that the Besov space, embedded in the class of continuous functions, is closed under the operation of multiplying elements [4].

A homogeneous problem with condition (2.4) on L can be reduced to the problem of finding a sectionally analytic function with zero jump. This is possible if the function $G_1(t) \in B^r_{p,\theta}(L)$ with zero index is represented as a ratio of the functions $\Phi^+(t)$ and $\Phi^-(t)$, which are the boundary values of functions that are analytic in the domains D^+, D^- and do not have zeros in these domains. For this, it is necessary to solve the following auxiliary homogeneous problem:

Find a sectionally analytic function $\Phi(z)$ that is analytic in the domains D^+, D^- and satisfies the following boundary condition on L:

$$\Phi_1(t) = G_1(t)\Phi_1^-(t), \qquad \Phi_1^-(\infty) = A \neq 0. \tag{2.6}$$

Taking the logarithms in the boundary condition (2.6), we have

$$\log\Phi_1^+(t) - \log\Phi_1^-(t) = \log G_1(t). \tag{2.7}$$

For $\log G_1(t)$, any branch may be taken. It is easy to verify that the final result is independent of the choice of a branch. Then, as shown earlier, $\log G_1(t)$ is a one-valued function, and $\log\Phi_1^+(t)$ and $\log\Phi_1^-(t)$ are analytic in the domains D^+ and D^-, respectively (see also Conclusion 1.2).

Thus we have arrived at the problem of determining a sectionally analytic function with the given jump on L (see Problem 1.1). Its solution, under the additional condition $\Phi_1^-(\infty) = A \neq 0$, is given by the formula

$$\log\Phi_1(z) = \frac{1}{2\pi\mathrm{i}}\int_L \frac{\log G_1(\tau)\,\mathrm{d}\tau}{\tau - z} + \log A.$$

Let us denote for brevity $\log\Phi_1(z) = \Gamma(z)$. It follows directly from the Sokhotski–Plemelj formulas that the solution of the boundary value problem (2.6) is given by the functions

$$\Phi_1^+(z) = A\mathrm{e}^{\Gamma^+(z)}, \qquad \Phi_1^-(z) = A\mathrm{e}^{\Gamma^-(z)},$$

which do not vanish ($A \neq 0$ by condition).

In this case, as we can see from (2.7), the function $G_1(t)$ can be represented in the form

$$G_1(t) = \frac{e^{\Gamma^+(t)}}{e^{\Gamma^-(t)}}, \tag{2.8}$$

where

$$\Gamma(z) = \frac{1}{2\pi i}\int_L \frac{\log G_1(\tau)\,d\tau}{\tau - z} + \log A \in B^{1+\alpha}_{p,\theta}(D^+) \hookrightarrow C(\overline{D^+}).$$

The function $G_1(t) \in B^r_{p,\theta}(L)$ with zero index is represented as the ratio of the functions $\Phi_1^+(t)$ and $\Phi_1^-(t)$, which are boundary values on L of the solution to the auxiliary homogeneous problem, which does not vanish anywhere (and on L).

Let us arrive at a homogeneous problem with condition (2.4). Introducing representation (2.8) of the function $G_1(t)$, we rewrite it in the form

$$\prod_{k=1}^{m}(z - z_k)^{\varkappa_k}\frac{\Phi^+(z)}{e^{\Gamma^+(z)}} = z^{\varkappa}\frac{\Phi^-(z)}{e^{\Gamma^-(z)}},$$

from which it is clear that the function $\prod_{k=1}^{m}(z - z_k)^{\varkappa_k}\frac{\Phi^+(z)}{e^{\Gamma^+(z)}}$ is analytic in the domain D^+, and the function $z^{\varkappa}\frac{\Phi^-(z)}{e^{\Gamma^-(z)}}$ is analytic in the domain D^-, except at infinity, where it can have a pole of order not higher than $\varkappa$, constitute the analytic continuation of each other through the contour L. Consequently, they are branches of a unique analytic function that can have, in the entire plane, only one singularity, a pole of order not higher than $\varkappa$ at infinity. According to the generalized Liouville theorem, this function is a polynomial of degree not higher than $\varkappa$ with arbitrary complex coefficients. Hence we obtain the general solution of the problem

$$\Phi^+(z) = \prod_{k=1}^{m}(z - z_k)^{-\varkappa_k} e^{\Gamma^+(z)} P_{\varkappa}(z), \qquad \Phi^-(z) = z^{-\varkappa} e^{\Gamma^-(z)} P_{\varkappa}(z), \tag{2.9}$$

where $P_{\varkappa}(z)$ is a polynomial of degree not higher than $\varkappa$ with arbitrary complex coefficients.

Thus, if the index $\varkappa$ of the Riemann boundary value problem is positive, then the problem has $\varkappa + 1$ linearly independent solutions:

$$\Phi_k^+(z) = \prod_{k=1}^{m}(z - z_k)^{-\varkappa_k} e^{\Gamma^+(z)} z^l, \qquad \Phi_k^-(z) = e^{\Gamma^-(z)} z^{l-\varkappa}, \quad l = 0, 1, \dots, \varkappa. \tag{2.10}$$

The general solution contains $\varkappa + 1$ arbitrary constants and is given by formula (2.9). If $\varkappa = 0$, then we come to the case of an auxiliary homogeneous problem, and the problem has a solution of the form

$$\Phi^+(z) = Ae^{\Gamma^+(z)}, \qquad \Phi^-(z) = Ae^{\Gamma^-(z)},$$

where

$$\Gamma(z) = \frac{1}{2\pi i}\int_L \frac{\log G(\tau)\,d\tau}{\tau - z}, \quad \Phi^-(\infty) = A \neq 0.$$

If $\varkappa < 0$, then the homogeneous problem only has the trivial solution.

It follows from formula (2.9) that $\Phi^-(\infty)$ is equal to the coefficient of $z^\varkappa$ in the polynomial $P_\varkappa(z)$. Therefore, under the condition $\Phi^-(\infty) = 0$, the general solution of the problem can be represented in the form (2.9), where the polynomial $P_\varkappa(z)$ must be replaced by the polynomial $P_{\varkappa-1}(z)$ of degree $\varkappa - 1$. In this case, the problem has $\varkappa$ linearly independent solutions. This completes the proof. □

2.2 Nonhomogeneous problem

To solve the nonhomogeneous problem, it is convenient to use the representation of the coefficient $G(t)$ of the problem in the form $G(t) = X^+(t)/X^-(t)$, where $X(z)$ is some particular solution of a homogeneous problem that does not vanish in the entire finite part of the plane; this implies that the boundary values of this solution also do not vanish anywhere on L. This type of solution is usually called canonical. It is assumed that this solution at infinity may have a pole. The existence and form of the required functions $X^+(t), X^-(t)$ can be obtained from equations (2.5) and (2.8):

$$X^+(t) = \prod_{k=1}^{m}(t - z_k)^{-\varkappa_k} \mathrm{e}^{\Gamma^+(t)}, \qquad X^-(t) = t^{-\varkappa} \mathrm{e}^{\Gamma^-(t)}. \tag{2.11}$$

For $\varkappa \geq 0$, the function $X(z)$ is a solution to the homogeneous problem and $X^+(t) \in B^r_{p,\theta}(L)$ ([4, Ch. I, Sect. 6]). If $\varkappa < 0$, then it has a pole of order $\varkappa$ at infinity.

The next result shows the conditions of the solvability of the nonhomogeneous problem (1.2).

Theorem 2.2 *Let the parameters r, p, θ be determined by one of conditions* (I)–(III). *Let $G(t), g(t) \in B^r_{p,\theta}(L)$ and $G(t) \neq 0$, $t \in L$. Then if $\varkappa \geq 0$, then the general solution to the nonhomogeneous problem, vanishing at infinity, is given by*

$$\Phi(z) = \frac{X(z)}{2\pi \mathrm{i}} \int_L \frac{g(\tau)\,\mathrm{d}\tau}{X^+(\tau)(\tau - z)} + X(z) P_{\varkappa-1}(z), \tag{2.12}$$

where $P_{\varkappa-1}$ is an arbitrary polynomial of degree not higher than $\varkappa - 1$, and for $\varkappa = 0$, we should consider $P_{\varkappa-1} \equiv 0$. If $\varkappa < 0$, then for a solution to exist, it is necessary and sufficient that the following conditions are satisfied:

$$\int_L \frac{\tau^k g(\tau)}{X^+(\tau)}\,\mathrm{d}\tau = 0, \quad k = 1, 2, \ldots, -\varkappa - 1. \tag{2.13}$$

In this case, the solution is given by formula (2.12), *where $P_\varkappa(z) \equiv 0$.*

Proof Let $X(z)$ be the canonical solution indicated above, and represent the coefficient of the problem in the form

$$G(t) = \frac{X^+(t)}{X^-(t)}.$$

Then we can reduce the boundary condition (1.2) to the form

$$\frac{\Phi^+(t)}{X^+(t)} - \frac{\Phi^-(t)}{X^-(t)} = \frac{g(t)}{X^+(t)}, \quad t \in L,$$

where $\frac{g(t)}{X^+(t)} \in B^r_{p,\theta}(L)$, since $X^+(t) \in B^r_{p,\theta}(L)$ and does not vanish on L ([4, Ch. I, Sect. 6]).

We again come to the problem of finding a sectionally analytic function with a given jump, considered in Problem 1.1. The behavior of the canonical solution of the homogeneous problem $X(z)$ at infinity is described above by (2.11). It follows that the function $\frac{\Phi^-(z)}{X^-(z)}$ at infinity has an order of growth not higher than $\varkappa > 0$. Therefore, reasoning as in Problem 1.1, we obtain

$$\frac{\Phi(z)}{X(z)} = \frac{1}{2\pi \mathrm{i}} \int_L \frac{g(\tau)\,\mathrm{d}\tau}{X^+(\tau)(\tau - z)} + P_\varkappa(z),$$

where $P_\varkappa(z)$ is an arbitrary polynomial of degree $\varkappa$. Then we obtain the general solution of the nonhomogeneous problem (1.2):

$$\Phi(z) = \frac{X(z)}{2\pi \mathrm{i}} \int_L \frac{g(\tau)\,\mathrm{d}\tau}{X^+(\tau)(\tau - z)} + X(z)P_\varkappa(z). \tag{2.14}$$

Thus, for $\varkappa \geq 0$, the nonhomogeneous problem (1.2) is solvable for any free term, and its solution is given by formula (2.14), and for $\varkappa = 0$, we should assume that $P_\varkappa \equiv 0$. If $\varkappa < 0$, then the function $X(z)$ has a pole of order $-\varkappa$ at infinity, and therefore in formula (2.14), we should set $P_\varkappa \equiv 0$. The Cauchy-type integral in the general case has a first-order zero at infinity. Consequently, $\Phi^-(z)$ has a pole of order at infinity not higher than $-\varkappa - 1$. Therefore, if $\varkappa < -1$, then the nonhomogeneous problem (1.2) is generally unsolvable. It will be solvable only if the free term satisfies some additional conditions. To obtain these conditions, we expand in a series the indicated Cauchy-type integral from (2.14) in the vicinity of infinity:

$$\sum_{k=1}^{\infty} c_k z^{-k}, \qquad c_k = -\frac{1}{2\pi \mathrm{i}} \int_L \frac{g(\tau)}{X^+(\tau)} \tau^{k-1}\,\mathrm{d}\tau.$$

From here it is easy to see that for the nonhomogeneous problem to be solvable in the case of $\varkappa < -1$, it is necessary and sufficient that the following conditions are satisfied:

$$\int_L \frac{g(\tau)}{X^+(\tau)} \tau^{k-1}\,\mathrm{d}\tau = 0, \quad k = 1, 2, \dots, -\varkappa - 1. \tag{2.15}$$

If $\varkappa = -1$, then the nonhomogeneous problem (1.2) is also solvable and has a unique solution.

If we require that the desired solution vanishes at infinity, then in formulas (2.14) the polynomial $P_\varkappa(z)$ should be replaced by the polynomial $P_{\varkappa-1}(z)$ of degree $\varkappa - 1$.

Thus, if $\varkappa \geq 0$, then the general solution to the nonhomogeneous problem, vanishing at infinity, is given by

$$\Phi(z) = \frac{X(z)}{2\pi \mathrm{i}} \int_L \frac{g(\tau)\,\mathrm{d}\tau}{X^+(\tau)(\tau - z)} + X(z)P_{\varkappa-1}(z), \tag{2.16}$$

where $P_{\varkappa-1}$ is an arbitrary polynomial of degree not higher than $\varkappa - 1$, and for $\varkappa = 0$, we should consider $P_{\varkappa-1} \equiv 0$.

If $\varkappa < 0$, then for a solution to exist, it is necessary and sufficient that the following conditions are satisfied:

$$\int_L \frac{\tau^k g(\tau)}{X^+(\tau)}\,\mathrm{d}\tau = 0, \quad k = 1, 2, \dots, -\varkappa - 1. \tag{2.17}$$

In this case, the solution is given by the formula

$$\Phi(z) = \frac{X(z)}{2\pi \mathrm{i}} \int_L \frac{g(\tau)\,\mathrm{d}\tau}{X^+(\tau)(\tau - z)}. \tag{2.18}$$

Note that for $\varkappa = 0$, there is always a unique solution that vanishes at infinity; for $\varkappa < 0$, a solution that vanishes at infinity, if it exists, is also unique; for $\varkappa < 0$, the solution depends on $\varkappa$ arbitrary constants. This completes the proof. □

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Author contributions
This work was carried out in collaboration between both authors. NB designed the study and guided the research. NY performed the analysis and wrote the first draft of the manuscript. NB and NY managed the analysis of the study. Both authors read and approved the final manuscript.

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

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