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A modified fuzzy Adomian decomposition method for solving time-fuzzy fractional partial differential equations with initial and boundary conditions

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Abstract

This research article introduces a novel approach based on the fuzzy Adomian decomposition method (FADM) to solve specific time fuzzy fractional partial differential equations with initial and boundary conditions (IBCs). The proposed approach addresses the challenge of incorporating both initial and boundary conditions into the FADM framework by employing a modified approach. This approach iteratively generates a new initial solution using the decomposition method. The method presented here offers a significant contribution to solving fuzzy fractional partial differential equations (FFPDEs) with fuzzy IBCs, a topic that has received limited attention in the literature. Furthermore, it satisfies a high convergence rate with minimal computational complexity, establishing a novel aspect of this research. By providing a series solution with a small number of recursive formulas, this method enhances accuracy and emerges as a preferred choice for tackling FFPDEs with mixed initial and boundary conditions. The effectiveness of the proposed technique is further supported by the inclusion of several illustrative examples.

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1 Introduction

Fractional partial differential equations (FPDEs) have recently gained traction as a powerful modeling tool in diverse scientific fields, encompassing biology, physics, chemistry, and engineering [1, 7–10, 15, 35]. In FPDEs, the limitations of crisp quantities in representing inherent imprecision and uncertainty are addressed by employing fuzzy quantities, resulting in FFPDEs. The solution of FFPDEs has attracted significant research interest in the last few years due to the inherent challenges associated with obtaining analytical or numerical solutions. However, researchers have made significant progress in developing novel methodologies to address this challenge. Some notable methods include the ADM

[22, 24, 30], the Laplace transform method [16, 26], the natural transform decomposition approach [37], the Laplace transform decomposition method (LTDM) [29, 34, 36], the Elzaki transform decomposition approach [5, 28], the finite difference technique [38–40], the homotopy perturbation method [6], the differential transform method [13], the Bernstein spectral numerical method [25], the homotopy analysis method [23, 31, 33], the Chebyshev spectral method [17], the optimal He–Laplace algorithm [27] and the variational iteration approach [14, 18].

The phenomenon of diffusion is widespread in nature, characterized by the movement of molecules from areas of high concentration to those of low concentration. In contrast to diffusion, where molecules move randomly, reactions involve the bulk flow of molecules. Many physical processes in nature can be modeled mathematically using the diffusion equation. While several research works have been conducted on solving FFPDEs using numerical and analytical techniques, attempts to solve systems of FFPDEs with IBCs remain limited.

In this work, we will extend the modified technique of ADM introduced by the authors in [2, 19, 21] to solve fractional-order diffusion and advection–diffusion equations with IBCs in a fuzzy concept. The general description of the proposed approach is implemented to solve some examples of the suggested problems. The analytical solutions of FFPDEs with fuzzy IBCs are very difficult to investigate. In this work, the analytical solutions of FFPDEs are obtained in a very simple and straightforward procedure and provide the closed-form solutions. The less computational work and simplicity are the uniqueness of the present modified technique. The obtained results are displayed through graphs. The graphical representations have shown the analytical solutions of the problems at various fractional orders and uncertainty $\varsigma \in [0,1]$. The fractional order solutions provide useful information about the dynamics of the suggested problems within a fuzzy environment.

2 Fundamental concepts of fractional and fuzzy calculus

This section provides essential definitions from fuzzy set theory and fractional calculus [3, 4, 32]. We denote the collection of fuzzy numbers by $\mathcal{F}_R$, whereas normal, fuzzy convex, upper semicontinuous, and compactly supported fuzzy sets can be defined on the real line.

Definition 2.1 A fuzzy number $\mathfrak{q}$ can be expressed in parametric form as $[\underline{\mathfrak{q}}(\varsigma), \overline{\mathfrak{q}}(\varsigma)]$, for $0 \leq \varsigma \leq 1$, if and only if

(i) $\underline{\mathfrak{q}}(\varsigma)$ is a bounded nondecreasing function and left continuous over (0,1],
(ii) $\overline{\mathfrak{q}}(\varsigma)$ is a bounded nonincreasing function and right continuous over (0,1],
(iii) $\underline{\mathfrak{q}}(\varsigma) \leq \overline{\mathfrak{q}}(\varsigma)$.

Definition 2.2 The generalized Hukuhara difference (g$\mathcal{H}$-difference) of two fuzzy number $\mathfrak{b}, \mathfrak{e} \in \mathcal{F}_R$ is defined as the element $\boldsymbol{c} \in \mathcal{F}_R$ such that

$$\mathfrak{b} \ominus_{g\mathcal{H}} \mathfrak{e} = \boldsymbol{c} \quad \Leftrightarrow \quad (i)\ \mathfrak{b} = \mathfrak{e} \oplus \boldsymbol{c} \quad \text{or} \quad (ii)\ \mathfrak{e} = \mathfrak{b} \oplus (-1)\boldsymbol{c}.$$

Note: If case (i) exists, then there is no need to consider case (ii), but if both cases are applicable, it signifies that the two types of the difference are same and equal.

Definition 2.3 Let $\mathcal{W} : \mathsf{J} \longrightarrow \mathcal{F}_R$, $\mathsf{J} \in \mathbb{R}^2$. Then g$\mathcal{H}$-partial derivative of first order at the point $(\mu_0, \tau_0) \in \mathsf{J}$ with respect to variables μ, τ is denoted by $\frac{\partial \mathcal{W}(\mu_0,\tau_0)}{\partial \mu}$, $\frac{\partial \mathcal{W}(\mu_0,\tau_0)}{\partial \tau}$ and given by

$$\frac{\partial \mathcal{W}(\mu_0,\tau_0)}{\partial \mu} = \lim_{h\to 0} \frac{\mathcal{W}(\mu_0+h,\tau_0) \ominus_{g\mathcal{H}} \mathcal{W}(\mu_0,\tau_0)}{h},$$
$$\frac{\partial \mathcal{W}(\mu_0,\tau_0)}{\partial \tau} = \lim_{d\to 0} \frac{\mathcal{W}(\mu_0,\tau_0+d) \ominus_{g\mathcal{H}} \mathcal{W}(\mu_0,\tau_0)}{d},$$

provided that $\frac{\partial \mathcal{W}(\mu_0,\tau_0)}{\partial \mu}$ and $\frac{\partial \mathcal{W}(\mu_0,\tau_0)}{\partial \tau} \in \mathcal{F}_R$.

Definition 2.4 Let $\mathcal{W} : \mathrm{J} \longrightarrow \mathcal{F}_R$ be g$\mathcal{H}$-partial differentiable with respect to μ at $(\mu_0,\tau_0) \in \mathrm{J}$. Then

(1) $\mathcal{W}$ is (i) g$\mathcal{H}$-partial differentiable with respect to μ at $(\mu_0,\tau_0) \in \mathrm{J}$. If

$$\left[\frac{\partial \mathcal{W}(\mu_0,\tau_0,\varsigma)}{\partial \mu}\right] = \left[\frac{\partial \underline{\mathcal{W}}(\mu_0,\tau_0,\varsigma)}{\partial \mu}, \frac{\partial \overline{\mathcal{W}}(\mu_0,\tau_0,\varsigma)}{\partial \mu}\right], \quad \forall \varsigma \in [0,1].$$

(2) $\mathcal{W}$ is (ii) g$\mathcal{H}$-partial differentiable with respect to μ at $(\mu_0,\tau_0) \in \mathrm{J}$. If

$$\left[\frac{\partial \mathcal{W}(\mu_0,\tau_0,\varsigma)}{\partial \mu}\right] = \left[\frac{\partial \overline{\mathcal{W}}(\mu_0,\tau_0,\varsigma)}{\partial \mu}, \frac{\partial \underline{\mathcal{W}}(\mu_0,\tau_0,\varsigma)}{\partial \mu}\right], \quad \forall \varsigma \in [0,1].$$

The g$\mathcal{H}$-partial derivative of $\mathcal{W}$ with respect to τ at $(\mu_0,\tau_0) \in \mathrm{J}$ are defined similarly.

Remark: We assume the existence of (i) g$\mathcal{H}$-partial differentiability throughout this paper.

We represent the space of all continuous fuzzy-valued functions on $\mathcal{I} \in \mathbb{R}$ by $\mathfrak{C}[\mathcal{I}\mathcal{F}_R]$ and we recall the Lebesgue integrable space of fuzzy functions on the bounded interval $\mathcal{I} \to \mathbb{R}$ by $\mathfrak{L}[\mathcal{I},\mathcal{F}_R]$.

Definition 2.5 Let $U(\mu) \in \mathfrak{C}[\mathcal{I},\mathcal{F}_R] \cap \mathfrak{L}[\mathcal{I},\mathcal{F}_R]$, the fuzzy Riemann–Liouville integral of fuzzy-valued function is defined as

$$\mathfrak{I}^{\varphi} U(\mu,\varsigma) = \left[\mathfrak{I}^{\varphi} \underline{U}(\mu,\varsigma), \mathfrak{I}^{\varphi} \overline{U}(\mu,\varsigma)\right], \quad \varsigma \in [0,1],$$

where

$$\mathfrak{I}^{\varphi} \underline{U}(\mu,\varsigma) = \frac{1}{\Gamma(\varphi)} \int_0^{\mu} (\mu - \mathrm{t})^{\varphi-1} \underline{U}(\mathrm{t},\varsigma)\, d\mathrm{t}, \quad \mu > 0,$$
$$\mathfrak{I}^{\varphi} \overline{U}(\mu,\varsigma) = \frac{1}{\Gamma(\varphi)} \int_0^{\mu} (\mu - \mathrm{t})^{\varphi-1} \overline{U}(\mathrm{t},\varsigma)\, d\mathrm{t}, \quad \mu > 0.$$

Definition 2.6 Let $U(\mu) \in \mathfrak{C}[\mathcal{I},\mathcal{F}_R] \cap \mathfrak{L}[\mathcal{I},\mathcal{F}_R]$. Then the fuzzy Caputo's g$\mathcal{H}$-derivative of order $\ell - 1 < \varphi \le \ell$, $\ell \in \mathbb{N}$ under (i) g$\mathcal{H}$-differentiability is given by

$${}^{c}_{g\mathcal{H}}\mathfrak{D}^{\varphi}_{\mu} U(\mu,\varsigma) = \left[{}^{c}\mathfrak{D}^{\varphi}_{\mu} \underline{U}(\mu,\varsigma), {}^{c}\mathfrak{D}^{\varphi}_{\mu} \overline{U}(\mu,\varsigma)\right],$$

where

$${}^{c}\mathfrak{D}^{\varphi}_{\mu} \underline{U}(\mu,\varsigma)] = \frac{1}{\Gamma(\ell-\varphi)} \int_0^{\mu} (\mu - \mathrm{t})^{\ell-\varphi-1}\, \underline{U}^{(\ell)}(\mathrm{t},\varsigma)\, d\mathrm{t},$$

$$^{c}\mathfrak{D}_{\mu}^{\varphi}\overline{U}(\mu,\varsigma) = \frac{1}{\Gamma(\ell-\varphi)}\int_{0}^{\mu}(\mu-\mathfrak{t})^{\ell-\varphi-1}\,\overline{U}^{(\ell)}(\mathfrak{t},\varsigma)\,d\mathfrak{t}.$$

3 Analysis of the fuzzy Adomian decomposition method (FADM)

Consider the equation

$$F\big(U(\mu,\tau,\varsigma)\big) = \mathfrak{g}(\mu,\tau,\varsigma), \tag{1}$$

where F represents a general fuzzy fractional partial differential operator and $\mathfrak{g}$ is known fuzzy-valued function. The linear terms in $F(U(\mu,\tau,\varsigma))$ are decomposed as $\hat{R}U(\mu,\tau,\varsigma) + \hat{L}U(\mu,\tau,\varsigma)$, where $\hat{R}$ represent an invertible operator. This operator corresponds to taking the highest possible derivative and $\hat{L}$ is the linear operator. Thus, Equation (1) can be represented as

$$\hat{R}U(\mu,\tau,\varsigma) + \hat{L}U(\mu,\tau,\varsigma) + \hat{N}U(\mu,\tau,\varsigma) = \mathfrak{g}(\mu,\tau,\varsigma), \tag{2}$$

where $\hat{N}$ represents a nonlinear operator in $F(U(\mu,\tau,\varsigma))$.

Now, applying the operator $\hat{R}^{-1}$ to both sides of Equation (2), we get

$$U(\mu,\tau,\varsigma) = \psi + \hat{R}^{-1}\mathfrak{g}(\mu,\tau,\varsigma) - \hat{R}^{-1}\big[\hat{L}U(\mu,\tau,\varsigma) + \hat{N}U(\mu,\tau,\varsigma)\big], \tag{3}$$

where ψ is the constant of integration and $\hat{R}^{-1}\psi = 0$.

The parametric form of Equation (3) is given by

$$U(\mu,\tau,\varsigma) = \big[\underline{U}(\mu,\tau,\varsigma), \overline{U}(\mu,\tau,\varsigma)\big],$$

where

$$\underline{U}(\mu,\tau,\varsigma) = \underline{\psi} + \hat{R}^{-1}\underline{\mathfrak{g}}(\mu,\tau,\varsigma) - \hat{R}^{-1}\big[\hat{L}\underline{U}(\mu,\tau,\varsigma) + \hat{N}\underline{U}(\mu,\tau,\varsigma)\big], \tag{4}$$

$$\overline{U}(\mu,\tau,\varsigma) = \overline{\psi} + \hat{R}^{-1}\overline{\mathfrak{g}}(\mu,\tau,\varsigma) - \hat{R}^{-1}\big[\hat{L}\overline{U}(\mu,\tau,\varsigma) + \hat{N}\overline{U}(\mu,\tau,\varsigma)\big]. \tag{5}$$

FADM's solution $U(\mu,\tau,\varsigma)$ is given by the following infinite series:

$$\underline{U}(\mu,\tau,\varsigma) = \sum_{\jmath=o}^{\infty}\underline{U}_{\jmath}(\mu,\tau,\varsigma), \tag{6}$$

$$\overline{U}(\mu,\tau,\varsigma) = \sum_{\jmath=o}^{\infty}\overline{U}_{\jmath}(\mu,\tau,\varsigma), \tag{7}$$

the nonlinear term $\hat{N}$ is calculated by

$$\hat{N}\underline{U} = \sum_{\jmath=0}^{\infty}\underline{M}_{\jmath} \quad \text{and} \quad \hat{N}\overline{U} = \sum_{\jmath=0}^{\infty}\overline{M}_{\jmath},$$

where

$$\underline{M}_{\jmath} = \frac{1}{\jmath!}\frac{\partial^{\jmath}}{\partial\mathfrak{q}^{\jmath}}\left[\bar{N}\left(\sum_{i}^{\infty}\mathfrak{q}^{i}\underline{U}_{i}\right)\right]_{\mathfrak{q}=0}$$

and

$$\overline{M}_J = \frac{1}{J!}\frac{\partial^J}{\partial \mathfrak{p}^J}\left[\bar{N}\left(\sum_i^{\infty} \mathfrak{p}^i \overline{U}_i\right)\right]_{\mathfrak{p}=0}$$

are defined as Adomian polynomials.

Moreover, a recurrence relation is constructed as follows:

$$\begin{aligned} &\underline{U}_0(\mu,\tau,\varsigma) = \underline{\psi} + \hat{R}^{-1}\underline{\mathfrak{g}}(\mu,\tau,\varsigma), \\ &\underline{U}_{J+1}(\mu,\tau,\varsigma) = \hat{R}^{-1}\left(\bar{L}\underline{U}_J + \sum_{J=0}^{\infty}\underline{M}_J\right) \end{aligned} \tag{8}$$

and

$$\begin{aligned} &\overline{U}_0(\mu,\tau,\varsigma) = \overline{\psi} + \hat{R}^{-1}\overline{\mathfrak{g}}(\mu,\tau,\varsigma), \\ &\overline{U}_{J+1}(\mu,\tau,\varsigma) = \hat{R}^{-1}\left(\bar{L}\overline{U}_J + \sum_{J=0}^{\infty}\overline{M}_J\right). \end{aligned} \tag{9}$$

4 Modification of FADM (MFADM)

To understand the main idea of the MFADM, we examine the following two types of one-dimensional time FFPDEs.

4.1 Time fuzzy fractional diffusion equations (TFFDEs)

$${}^{c}\mathfrak{D}_{\tau}^{\varphi}U(\mu,\tau,\varsigma) = g(\mu)U\mu\mu(\mu,\tau,\varsigma) + h(\mu,\tau,\varsigma), 0 < \varphi < 1, 0 \leq \mu \leq a, \tau > 0 \tag{10}$$

with the fuzzy IBCs

$$\begin{aligned} &U(\mu,0,\varsigma) = w_1(\mu,\varsigma), \\ &U(0,\tau,\varsigma) = w_2(\tau,\varsigma), \qquad U(a,\tau,\varsigma) = w_3(\tau,\varsigma), \end{aligned} \tag{11}$$

where g, h, w_1, w_2, w_3 are known fuzzy-valued functions and $0 \leq \varsigma \leq 1$.

We generate new successive initial solutions U_n^* at each iteration for Equation (10) using the following novel technique:

$$\begin{aligned} &U_n^*(\mu,\tau,\varsigma) = U_n(\mu,\tau,\varsigma) + (1-\mu)\big[w_2 - U_n(0,\tau,\varsigma)\big] + \mu\big[w_3 - U_n(a,\tau,\varsigma)\big], \\ &\quad n = 0,1,2,\ldots \end{aligned} \tag{12}$$

We write Equation (12) in parameter form as follows:

$$\begin{aligned} &\underline{U_n^*}(\mu,\tau,\varsigma) = \underline{U}_n(\mu,\tau,\varsigma) + (1-\mu)\big[\underline{w_2} - \underline{U}_n(0,\tau,\varsigma)\big] + \mu\big[\underline{w_3} - \underline{U}_n(a,\tau,\varsigma)\big], \\ &\overline{U_n^*}(\mu,\tau,\varsigma) = \overline{U}_n(\mu,\tau,\varsigma) + (1-\mu)\big[\overline{w_2} - \overline{U}_n(0,\tau,\varsigma)\big] + \mu\big[\overline{w_3} - \overline{U}_n(a,\tau,\varsigma)\big]. \end{aligned} \tag{13}$$

Applying FADM procedure, we have $\hat{R} = {}^{c}\mathfrak{D}_{\tau}$, hence $\hat{R}^{-1} = \mathfrak{I}^{\varphi}$.

By operating with $\mathfrak{I}^{\varphi}$ on both sides of Equation (10), we have

$$\begin{aligned} \underline{U}(\mu,\tau,\varsigma) &= \underline{U}(\mu,0,\varsigma) + \mathfrak{I}^{\varphi}\big[\underline{g}(\mu)\underline{U}_{\mu\mu}(\mu,\tau,\varsigma) + \underline{h}(\mu,\tau,\varsigma)\big], \\ \overline{U}(\mu,\tau,\varsigma) &= \overline{U}(\mu,0,\varsigma) + \mathfrak{I}^{\varphi}\big[\overline{g}(\mu)\overline{U}_{\mu\mu}(\mu,\tau,\varsigma) + \overline{h}(\mu,\tau,\varsigma)\big]. \end{aligned} \tag{14}$$

The initial approximation can be given as

$$\begin{aligned} \underline{U_0}(\mu,\tau,\varsigma) &= \underline{U}(\mu,0,\varsigma) + \mathfrak{I}^{\varphi}\big(\underline{h}(\mu,\tau,\varsigma)\big), \\ \overline{U_0}(\mu,\tau,\varsigma) &= \overline{U}(\mu,0,\varsigma) + \mathfrak{I}^{\varphi}\big(\overline{h}(\mu,\tau,\varsigma)\big), \end{aligned} \tag{15}$$

so the iteration formula is

$$\begin{aligned} \underline{U}_{n+1}(\mu,\tau,\varsigma) &= \mathfrak{I}^{\varphi}\big[\underline{g}(\mu)\big(\underline{U}_n^{*}(\mu,\tau,\varsigma)\big)_{\mu\mu}\big], \quad n \geq 0, \\ \overline{U}_{n+1}(\mu,\tau,\varsigma) &= \mathfrak{I}^{\varphi}\big[\overline{g}(\mu)\big(\overline{U}_n^{*}(\mu,\tau,\varsigma)\big)_{\mu\mu}\big], \quad n \geq 0. \end{aligned} \tag{16}$$

4.2 Time fuzzy fractional advection–diffusion equations (TFFADEs)

$$\begin{aligned} &{}^{c}\mathfrak{D}_{\tau}^{\varphi}U(\mu,\tau,\varsigma) + U_{\mu}(\mu,\tau,\varsigma) = U_{\mu\mu}(\mu,\tau,\varsigma) + h(\mu,\tau,\varsigma), \\ &\quad 0 < \varphi < 1, 0 \leq \mu \leq a, \tau > 0 \end{aligned} \tag{17}$$

with the fuzzy IBCs

$$\begin{aligned} &U(\mu,0,\varsigma) = w_1(\mu,\varsigma), \\ &U(0,\tau,\varsigma) = w_2(\tau,\varsigma), \qquad U(a,\tau,\varsigma) = w_3(\tau,\varsigma), \end{aligned} \tag{18}$$

where h, w_1, w_2, w_3 are known fuzzy-valued functions and $0 \leq \varsigma \leq 1$.

We generate new successive initial solutions U_n^{*} at each iteration for Equation (17) using the following novel technique:

$$\begin{aligned} &U_n^{*}(\mu,\tau,\varsigma) = U_n(\mu,\tau,\varsigma) + (1-\mu)\big[w_2 - U_n(0,\tau,\varsigma)\big] + \mu\big[w_3 - U_n(a,\tau,\varsigma)\big], \\ &\quad n = 0,1,2,\ldots \end{aligned} \tag{19}$$

Now, we write Equation (19) in parameter form as follows:

$$\begin{aligned} \underline{U_n^{*}}(\mu,\tau,\varsigma) &= \underline{U}_n(\mu,\tau,\varsigma) + (1-\mu)\big[\underline{w_2} - \underline{U}_n(0,\tau,\varsigma)\big] + \mu\big[\underline{w_3} - \underline{U}_n(a,\tau,\varsigma)\big], \\ \overline{U_n^{*}}(\mu,\tau,\varsigma) &= \overline{U}_n(\mu,\tau,\varsigma) + (1-\mu)\big[\overline{w_2} - \overline{U}_n(0,\tau,\varsigma)\big] + \mu\big[\overline{w_3} - \overline{U}_n(a,\tau,\varsigma)\big]. \end{aligned} \tag{20}$$

Applying FADM procedure, we have $\hat{R} = {}^{c}\mathfrak{D}_{\tau}$, hence $\hat{R}^{-1} = \mathfrak{I}^{\varphi}$.

By operating with $\mathfrak{I}^{\varphi}$ on both sides of Equation (17), we have

$$\begin{aligned} \underline{U}(\mu,\tau,\varsigma) &= \underline{U}(\mu,0,\varsigma) + \mathfrak{I}^{\varphi}\big[\underline{U}_{\mu\mu}(\mu,\tau,\varsigma) - \underline{U}_{\mu}(\mu,\tau,\varsigma) + \underline{h}(\mu,\tau,\varsigma)\big], \\ \overline{U}(\mu,\tau,\varsigma) &= \overline{U}(\mu,0,\varsigma) + \mathfrak{I}^{\varphi}\big[\overline{U}_{\mu\mu}(\mu,\tau,\varsigma) - \overline{U}_{\mu}(\mu,\tau,\varsigma) + \overline{h}(\mu,\tau,\varsigma)\big]. \end{aligned} \tag{21}$$

The initial approximation can be written as

$$\begin{aligned} \underline{U_0}(\mu,\tau,\varsigma) &= \underline{U}(\mu,0,\varsigma) + \mathfrak{I}^{\varphi}\left(\underline{h}(\mu,\tau,\varsigma)\right), \\ \overline{U_0}(\mu,\tau,\varsigma) &= \overline{U}(\mu,0,\varsigma) + \mathfrak{I}^{\varphi}\left(\overline{h}(\mu,\tau,\varsigma)\right), \end{aligned} \tag{22}$$

so the iteration formula is

$$\begin{aligned} \underline{U}_{n+1}(\mu,\tau,\varsigma) &= \mathfrak{I}^{\varphi}\left[\left(\underline{U_n^*}(\mu,\tau,\varsigma)\right)_{\mu\mu} - \left(\underline{U_n^*}(\mu,\tau,\varsigma)\right)_{\mu}\right], \quad n \geq 0, \\ \overline{U}_{n+1}(\mu,\tau,\varsigma) &= \mathfrak{I}^{\varphi}\left[\left(\overline{U}_n^*(\mu,\tau,\varsigma)\right)_{\mu\mu} - \left(\overline{U}_n^*(\mu,\tau,\varsigma)\right)_{\mu}\right], \quad n \geq 0. \end{aligned} \tag{23}$$

Remark: The newly obtained initial solutions, denoted by U_n^*, demonstrably satisfy the IBCs as presented below.

$$\begin{aligned} &\text{If } \tau = 0, \text{ then } \underline{U_n^*}(\mu,0,\varsigma) = \underline{U_n}(\mu,0,\varsigma), \\ &\qquad\qquad\quad\;\; \overline{U_n^*}(\mu,0,\varsigma) = \overline{U_n}(\mu,0,\varsigma), \\ &\text{if } \mu = 0, \text{ then } \underline{U_n^*}(0,\tau,\varsigma) = \underline{w_2}(\tau,\varsigma), \\ &\qquad\qquad\quad\;\; \overline{U_n^*}(0,\tau,\varsigma) = \overline{w_2}(\tau,\varsigma), \\ &\text{if } \mu = a, \text{ then } \underline{U_n^*}(a,\tau,\varsigma) = \underline{w_3}(\tau,\varsigma), \\ &\qquad\qquad\quad\;\; \overline{U_n^*}(a,\tau,\varsigma) = \overline{w_3}(\tau,\varsigma). \end{aligned} \tag{24}$$

5 Applications and results

To demonstrate the effectiveness of the MFADM, this section solves several illustrative examples.

Example 5.1 Consider TFFDE of the following form:

$$\begin{aligned} &{}^{c}\mathfrak{D}_{\tau}^{\varphi} U(\mu,\tau,\varsigma) = U_{\mu\mu}(\mu,\tau,\varsigma) + \frac{\Gamma(4+\varphi)}{6}\mu^4(2-\mu)\tau - 4\mu^2(6-5\mu)\tau^{3+\mu}, \\ &0 \leq \mu \leq 2, \tau > 0, \end{aligned} \tag{25}$$

with the fuzzy IBCs

$$\begin{aligned} &U(\mu,0,\varsigma) = k = [\varsigma - 1, 1 - \varsigma], \quad 0 \leq \varsigma \leq 1, \\ &U(0,\tau,\varsigma) = U(2,\tau,\varsigma) = k = [\varsigma - 1, 1 - \varsigma]. \end{aligned} \tag{26}$$

Applying the MFADM, we have

$$\begin{aligned} \underline{U_n^*}(\mu,\tau,\varsigma) &= \underline{U}_n(\mu,\tau,\varsigma) + (1-\mu)\left[\underline{k} - \underline{U}_n(0,\tau,\varsigma)\right] + \mu\left[\underline{k} - \underline{U}_n(2,\tau,\varsigma)\right], \\ \overline{U_n^*}(\mu,\tau,\varsigma) &= \overline{U}_n(\mu,\tau,\varsigma) + (1-\mu)\left[\overline{k} - \overline{U}_n(0,\tau,\varsigma)\right] + \mu\left[\overline{k} - \overline{U}_n(2,\tau,\varsigma)\right], \end{aligned} \tag{27}$$

where $n = 0, 1, 2, \ldots$. Using FADM solution, we get

$$\begin{aligned} \underline{U}_0(\mu,\tau,\varsigma) &= \underline{k} + \mu^4(2-\mu)^{3+\varphi} + \frac{(20\mu^3 - 24\mu^2)\Gamma(4+\varphi)\tau^{3+2\varphi}}{\Gamma(4+2\varphi)}, \\ \overline{U}_0(\mu,\tau,\varsigma) &= \overline{k} + \mu^4(2-\mu)^{3+\varphi} + \frac{(20\mu^3 - 24\mu^2)\Gamma(4+\varphi)\tau^{3+2\varphi}}{\Gamma(4+2\varphi)} \end{aligned} \tag{28}$$

and

$$\begin{aligned} \underline{U}_{n+1}(\mu,\tau,\varsigma) &= \Im^{\varphi}\bigl[\bigl(\underline{U}_n^*(\mu,\tau,\varsigma)\bigr)_{\mu\mu}\bigr], \quad n \geq 0, \\ \overline{U}_{n+1}(\mu,\tau,\varsigma) &= \Im^{\varphi}\bigl[\bigl(\overline{U}_n^*(\mu,\tau,\varsigma)\bigr)_{\mu\mu}\bigr], \quad n \geq 0. \end{aligned} \tag{29}$$

Now, we use the IBCs in Equation (27) for $n = 0$.

$$\begin{aligned} \underline{U}_0^*(\mu,\tau,\varsigma) &= \underline{U}_0(\mu,\tau,\varsigma) + (1-\mu)\bigl[\underline{k} - \underline{U}_0(0,\tau,\varsigma)\bigr] + \mu\bigl[\underline{k} - \underline{U}_0(2,\tau,\varsigma)\bigr], \\ \overline{U}_0^*(\mu,\tau,\varsigma) &= \overline{U}_0(\mu,\tau,\varsigma) + (1-\mu)\bigl[\overline{k} - \overline{U}_0(0,\tau,\varsigma)\bigr] + \mu\bigl[\overline{k} - \overline{U}_0(2,\tau,\varsigma)\bigr], \end{aligned} \tag{30}$$

which implies

$$\begin{aligned} &\underline{U}_0^*(\mu,\tau,\varsigma) \\ &\quad = \underline{k} + \mu^4(2-\mu)\tau^{3+\varphi} + \frac{(20\mu^3 - 24\mu^2)\Gamma(4+\varphi)\tau^{3+2\varphi}}{\Gamma(4+2\varphi)} - \frac{16\mu\Gamma(4+\varphi)\tau^{3+2\varphi}}{\Gamma(4+2\varphi)}, \\ &\overline{U}_0^*(\mu,\tau,\varsigma) \\ &\quad = \overline{k} + \mu^4(2-\mu)\tau^{3+\varphi} + \frac{(20\mu^3 - 24\mu^2)\Gamma(4+\varphi)\tau^{3+2\varphi}}{\Gamma(4+2\varphi)} - \frac{16\mu\Gamma(4+\varphi)\tau^{3+2\varphi}}{\Gamma(4+2\varphi)}. \end{aligned} \tag{31}$$

From Equation (29), we have

$$\begin{aligned} \underline{U}_1(\mu,\tau,\varsigma) &= \Im^{\varphi}\bigl[\bigl(\underline{U}_0^*(\mu,\tau,\varsigma)\bigr)_{\mu\mu}\bigr], \\ \overline{U}_1(\mu,\tau,\varsigma) &= \Im^{\varphi}\bigl[\bigl(\overline{U}_0^*(\mu,\tau,\varsigma)\bigr)_{\mu\mu}\bigr]. \end{aligned} \tag{32}$$

We get

$$\begin{aligned} \underline{U}_1(\mu,\tau,\varsigma) &= \Im^{\varphi}\left[\left(24\mu^2 - 20\mu^3\right)\tau^{3+\varphi} + \frac{(120\mu - 48)\Gamma(4+\varphi)\tau^{3+2\varphi}}{\Gamma(4+2\varphi)}\right] \\ &= \frac{(24\mu^2 - 20\mu^3)\Gamma(4+\varphi)\tau^{3+2\varphi}}{\Gamma(4+2\varphi)} + \frac{(120\mu - 48)\Gamma(4+\varphi)\tau^{3+3\varphi}}{\Gamma(4+3\varphi)}, \end{aligned} \tag{33}$$

$$\begin{aligned} \overline{U}_1(\mu,\tau,\varsigma) &= \Im^{\varphi}\left[\mu^4(2-\mu)^{3+\varphi} + \frac{(120\mu - 48)\Gamma(4+\varphi)\tau^{3+2\varphi}}{\Gamma(4+2\varphi)}\right] \\ &= \frac{(24\mu^2 - 20\mu^3)\Gamma(4+\varphi)\tau^{3+2\varphi}}{\Gamma(4+2\varphi)} + \frac{(120\mu - 48)\Gamma(4+\varphi)\tau^{3+3\varphi}}{\Gamma(4+3\varphi)}. \end{aligned} \tag{34}$$

Now, for $n = 1$, Equation (27) becomes

$$\begin{aligned}\underline{U_1^*}(\mu,\tau,\varsigma) &= \underline{U}_1(\mu,\tau,\varsigma) + (1-\mu)\big[\underline{k} - \underline{U}_1(0,\tau,\varsigma)\big] + \mu\big[\overline{k} - \overline{U}_1(2,\tau,\varsigma)\big] \\ &= \underline{k} + \frac{(24\mu^2 - 20\mu^3 + 64\mu)\Gamma(4+\varphi)\tau^{3+2\varphi}}{\Gamma(4+2\varphi)} - \frac{120\mu\Gamma(4+\varphi)\tau^{3+3\varphi}}{\Gamma(4+3\varphi)},\end{aligned} \tag{35}$$

$$\begin{aligned}\overline{U_1^*}(\mu,\tau,\varsigma) &= \overline{U}_1(\mu,\tau,\varsigma) + (1-\mu)\big[\overline{k} - \overline{U}_1(0,\tau,\varsigma)\big] + \mu\big[\overline{k} - \overline{U}_1(2,\tau,\varsigma)\big] \\ &= \overline{k} + \frac{(24\mu^2 - 20\mu^3 + 64\mu)\Gamma(4+\varphi)\tau^{3+2\varphi}}{\Gamma(4+2\varphi)} - \frac{120\mu\Gamma(4+\varphi)\tau^{3+3\varphi}}{\Gamma(4+3\varphi)}.\end{aligned} \tag{36}$$

From Equation (29), we obtain

$$\begin{aligned}\underline{U}_2(\mu,\tau,\varsigma) &= \Im^{\varphi}\big[\big(\underline{U_1^*}(\mu,\tau,\varsigma)\big)_{\mu\mu}\big] \\ &= \frac{(48 - 120\mu)\Gamma(4+\varphi)\tau^{3+3\varphi}}{\Gamma(4+3\varphi)},\end{aligned} \tag{37}$$

$$\begin{aligned}\overline{U}_2(\mu,\tau,\varsigma) &= \Im^{\varphi}\big[\big(\overline{U_1^*}(\mu,\tau,\varsigma)\big)_{\mu\mu}\big] \\ &= \frac{(48 - 120\mu)\Gamma(4+\varphi)\tau^{3+3\varphi}}{\Gamma(4+3\varphi)}.\end{aligned} \tag{38}$$

For $n = 2$, Equation (27) becomes

$$\begin{aligned}\underline{U_2^*}(\mu,\tau,\varsigma) &= \underline{U}_2(\mu,\tau,\varsigma) + (1-\mu)\big[\underline{k} - \underline{U}_2(0,\tau,\varsigma)\big] + \mu\big[\underline{k} - \underline{U}_2(2,\tau,\varsigma)\big] \\ &= \underline{k} + \frac{120\mu\Gamma(4+\varphi)\tau^{3+3\varphi}}{\Gamma(4+3\varphi)},\end{aligned} \tag{39}$$

$$\begin{aligned}\overline{U_2^*}(\mu,\tau,\varsigma) &= \overline{U}_2(\mu,\tau,\varsigma) + (1-\mu)\big[\overline{k} - \overline{U}_2(0,\tau,\varsigma)\big] + \mu\big[\overline{k} - \overline{U}_2(2,\tau,\varsigma)\big] \\ &= \overline{k} + \frac{120\mu\Gamma(4+\varphi)\tau^{3+3\varphi}}{\Gamma(4+3\varphi)}.\end{aligned} \tag{40}$$

From Equation (29), we obtain

$$\begin{aligned}\underline{U}_3(\mu,\tau,\varsigma) &= \Im^{\varphi}\big[\big(\underline{U_2^*}(\mu,\tau,\varsigma)\big)_{\mu\mu}\big] \\ &= 0,\end{aligned} \tag{41}$$

$$\begin{aligned}\overline{U}_3(\mu,\tau,\varsigma) &= \Im^{\varphi}\big[\big(\overline{U_2^*}(\mu,\tau,\varsigma)\big)_{\mu\mu}\big] \\ &= 0.\end{aligned} \tag{42}$$

$$\vdots$$

Thus, the MFADM solution is

$$\begin{aligned}\underline{U}(\mu,\tau,\varsigma) &= \underline{U}_0(\mu,\tau,\varsigma) + \underline{U}_1(\mu,\tau,\varsigma) + \underline{U}_2(\mu,\tau,\varsigma) + \cdots \\ &= \underline{k} + \mu^4(2-\mu)\tau^{3+\varphi},\end{aligned} \tag{43}$$

$$\begin{aligned}\overline{U}(\mu,\tau,\varsigma) &= \overline{U}_0(\mu,\tau,\varsigma) + \overline{U}_1(\mu,\tau,\varsigma) + \overline{U}_2(\mu,\tau,\varsigma) + \cdots \\ &= \overline{k} + \mu^4(2-\mu)\tau^{3+\varphi}].\end{aligned} \tag{44}$$

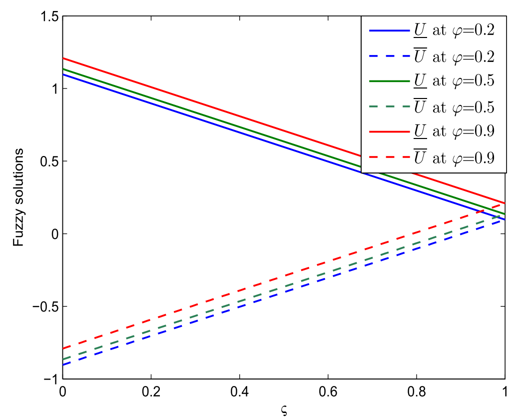

Figure 1 2D graph of the analytical lower and upper solutions of Ex. 5.1 at $\mu = 0.2$ and $\tau = 3$

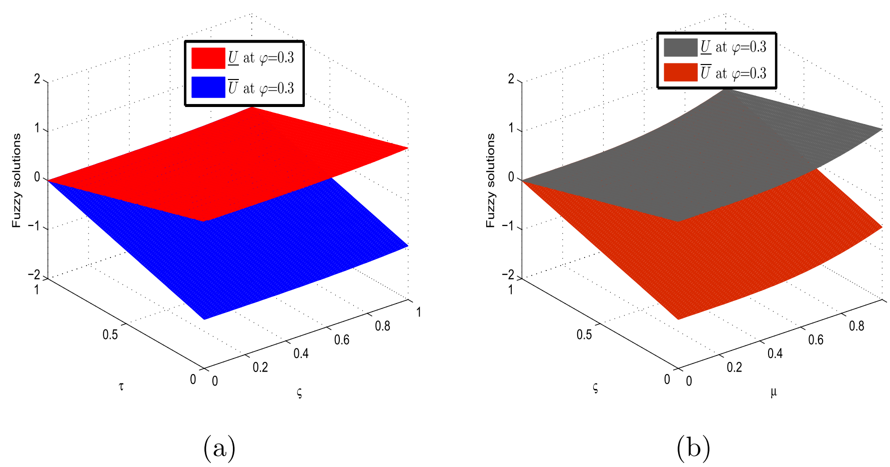

Figure 2 3D graph of the analytical lower and upper solutions of Ex. 5.1 at $\mu = 0.5$ (a) and at $\tau = 0.8$ (b)

In Fig. 1, we plot the analytical fuzzy solutions for Example 5.1 corresponding to different fractional order and uncertainty ς.

Further, we present in Fig. 2(a,b) surface plots of the analytical fuzzy solutions for Example 5.1 corresponding to given fractional order and at different values of μ and τ as well as of uncertainty ς.

Example 5.2 Consider the following TFFDE:

$$^{c}\mathfrak{D}_{\tau}^{\varphi}U(\mu,\tau,\varsigma) = \frac{1}{2}\mu^2 U_{\mu\mu}(\mu,\tau,\varsigma), \quad 0 \le \mu \le 1, \tau > 0 \tag{45}$$

having the fuzzy IBCs as follows:

$$\begin{aligned} &U(\mu,0,\varsigma) = k\mu^2, \\ &U(0,\tau,\varsigma) = 0, \qquad U(1,\tau,\varsigma) = kE_{\varphi}(\tau), \end{aligned} \tag{46}$$

where $E_{\varphi}(\tau) = \sum_{J=0}^{\infty} \frac{\tau^{\varphi}}{\Gamma(\varphi J+1)}$.

Applying the MFADM, we have

$$\begin{aligned}\underline{U_n^*}(\mu,\tau,\varsigma) &= \underline{U}_n(\mu,\tau,\varsigma) + (1-\mu)\left[\underline{k}\mu^2 - \underline{U}_n(0,\tau,\varsigma)\right] + \mu\left[\underline{k}E_\varphi(\tau) - \underline{U}_n(1,\tau,\varsigma)\right],\\ \overline{U_n^*}(\mu,\tau,\varsigma) &= \overline{U}_n(\mu,\tau,\varsigma) + (1-\mu)\left[\overline{k}\mu^2 - \overline{U}_n(0,\tau,\varsigma)\right] + \mu\left[\overline{k}E_\varphi(\tau) - \overline{U}_n(1,\tau,\varsigma)\right],\end{aligned} \tag{47}$$

where $n = 0, 1, 2, \ldots$. Using FADM solution, we have

$$\begin{aligned}\underline{U}_0(\mu,\tau,\varsigma) &= \underline{k}\mu^2,\\ \overline{U}_0(\mu,\tau,\varsigma) &= \overline{k}\mu^2\end{aligned} \tag{48}$$

and

$$\begin{aligned}\underline{U}_{n+1}(\mu,\tau,\varsigma) &= \frac{1}{2}\mu^2\Im^\varphi\left[\left(\underline{U_n^*}(\mu,\tau,\varsigma)\right)_{\mu\mu}\right], \quad n \geq 0,\\ \overline{U}_{n+1}(\mu,\tau,\varsigma) &= \frac{1}{2}\mu^2\Im^\varphi\left[\left(\overline{U_n^*}(\mu,\tau,\varsigma)\right)_{\mu\mu}\right], \quad n \geq 0.\end{aligned} \tag{49}$$

Now, we use the IBCs in Equation (47) for $n = 0$.

$$\begin{aligned}\underline{U_0^*}(\mu,\tau,\varsigma) &= \underline{U}_0(\mu,\tau,\varsigma) + (1-\mu)\left[\underline{k}\mu^2 - \underline{U}_0(0,\tau,\varsigma)\right] + \mu\left[\underline{k}E_\varphi(\tau) - \underline{U}_0(2,\tau,\varsigma)\right]\\ &= \underline{k}\mu^2 + \underline{k}\mu\left[E_\varphi(\tau) - 1\right],\end{aligned} \tag{50}$$

$$\begin{aligned}\overline{U_0^*}(\mu,\tau,\varsigma) &= \overline{U}_0(\mu,\tau,\varsigma) + (1-\mu)\left[\overline{k}\mu^2 - \overline{U}_0(0,\tau,\varsigma)\right] + \mu\left[\overline{k}E_\varphi(\tau) - \overline{U}_0(1,\tau,\varsigma)\right]\\ &= \overline{k}\mu^2 + \overline{k}\mu\left[E_\varphi(\tau) - 1\right].\end{aligned} \tag{51}$$

From Equation (49), we have

$$\underline{U}_1(\mu,\tau,\varsigma) = \frac{1}{2}\mu^2\Im^\varphi\left[\left(\underline{U_0^*}(\mu,\tau,\varsigma)\right)_{\mu\mu}\right] = \frac{\underline{k}\mu^2\tau^\varphi}{\Gamma(\varphi+1)}, \tag{52}$$

$$\overline{U}_1(\mu,\tau,\varsigma) = \frac{1}{2}\mu^2\Im^\varphi\left[\left(\overline{U_0^*}(\mu,\tau,\varsigma)\right)_{\mu\mu}\right] = \frac{\overline{k}\mu^2\tau^\varphi}{\Gamma(\varphi+1)}. \tag{53}$$

Now, for $n = 1$, Equation (47) becomes

$$\begin{aligned}\underline{U_1^*}(\mu,\tau,\varsigma) &= \underline{U}_1(\mu,\tau,\varsigma) + (1-\mu)\left[\underline{k}\mu^2 - \underline{U}_1(0,\tau,\varsigma)\right] + \mu\left[\overline{k}E_\varphi(\tau) - \overline{U}_1(1,\tau,\varsigma)\right]\\ &= \frac{\underline{k}\mu^2\tau^\varphi}{\Gamma(\varphi+1)} + \underline{k}\mu\left[E_\varphi(\tau) - \frac{\tau^\varphi}{\Gamma(\varphi+1)}\right],\end{aligned} \tag{54}$$

$$\begin{aligned}\overline{U_1^*}(\mu,\tau,\varsigma) &= \overline{U}_1(\mu,\tau,\varsigma) + (1-\mu)\left[\overline{k}\mu^2 - \overline{U}_1(0,\tau,\varsigma)\right] + \mu\left[\overline{k}E_\varphi(\tau) - \overline{U}_1(1,\tau,\varsigma)\right]\\ &= \frac{\overline{k}\mu^2\tau^\varphi}{\Gamma(\varphi+1)} + \overline{k}\mu\left[E_\varphi(\tau) - \frac{\tau^\varphi}{\Gamma(\varphi+1)}\right].\end{aligned} \tag{55}$$

From Equation (49), we obtain

$$\underline{U}_2(\mu,\tau,\varsigma) = \frac{1}{2}\mu^2\Im^\varphi\left[\left(\underline{U_1^*}(\mu,\tau,\varsigma)\right)_{\mu\mu}\right] = \frac{\underline{k}\mu^2\tau^{2\varphi}}{\Gamma(2\varphi+1)}. \tag{56}$$

$$\overline{U}_2(\mu,\tau,\varsigma) = \frac{1}{2}\mu^2\Im^\varphi\left[\left(\overline{U_1^*}(\mu,\tau,\varsigma)\right)_{\mu\mu}\right] = \frac{\overline{k}\mu^2\tau^{2\varphi}}{\Gamma(2\varphi+1)}. \tag{57}$$

For $n = 2$, Equation (47) becomes

$$\begin{aligned}\underline{U_2^*}(\mu,\tau,\varsigma) &= \underline{U}_2(\mu,\tau,\varsigma) + (1-\mu)\big[\underline{k}\mu^2 - \underline{U}_2(0,\tau,\varsigma)\big] + \mu\big[\underline{k}E_\varphi(\tau) - \underline{U}_2(1,\tau,\varsigma)\big] \\ &= \frac{\underline{k}\mu^2\tau^\varphi}{\Gamma(\varphi+1)} + \underline{k}\mu\left[E_\varphi(\tau) - \frac{\tau^{2\varphi}}{\Gamma(2\varphi+1)}\right],\end{aligned} \tag{58}$$

$$\begin{aligned}\overline{U_2^*}(\mu,\tau,\varsigma) &= \overline{v}_2(\mu,\tau,\varsigma) + (1-\mu)\big[\overline{k}\mu^2 - \overline{U}_2(0,\tau,\varsigma)\big] + \mu\big[\overline{k}E_\varphi(\tau) - \overline{U}_2(1,\tau,\varsigma)\big] \\ &= \frac{\overline{k}\mu^2\tau^\varphi}{\Gamma(\varphi+1)} + \overline{k}\mu\left[E_\varphi(\tau) - \frac{\tau^{2\varphi}}{\Gamma(2\varphi+1)}\right].\end{aligned} \tag{59}$$

From Equation (49), we obtain

$$\underline{U}_3(\mu,\tau,\varsigma) = \frac{1}{2}\mu^2\mathfrak{I}^\varphi\big[\big(\underline{U_1^*}(\mu,\tau,\varsigma)\big)_{\mu\mu}\big] = \frac{\underline{k}\mu^2\tau^{3\varphi}}{\Gamma(3\varphi+1)}, \tag{60}$$

$$\overline{U}_3(\mu,\tau,\varsigma) = \frac{1}{2}\mu^2\mathfrak{I}^\varphi\big[\big(\overline{U_1^*}(\mu,\tau,\varsigma)\big)_{\mu\mu}\big] = \frac{\overline{k}\mu^2\tau^{3\varphi}}{\Gamma(3\varphi+1)}. \tag{61}$$

$$\vdots$$

The MFADM solution is

$$\begin{aligned}\underline{U}(\mu,\tau,\varsigma) &= \underline{U}_0(\mu,\tau,\varsigma) + \underline{U}_1(\mu,\tau,\varsigma) + \underline{U}_2(\mu,\tau,\varsigma) + \cdots \\ &= \underline{k}\mu^2\left[1 + \frac{\tau^\varphi}{\Gamma(\varphi+1)} + \frac{\tau^{2\varphi}}{\Gamma(2\varphi+1)} + \frac{\tau^{3\varphi}}{\Gamma(3\varphi+1)} + \cdots\right] = \underline{k}\mu^2E_\varphi(\tau),\end{aligned} \tag{62}$$

$$\begin{aligned}\overline{U}(\mu,\tau,\varsigma) &= \overline{U}_0(\mu,\tau,\varsigma) + \overline{U}_1(\mu,\tau,\varsigma) + \overline{U}_2(\mu,\tau,\varsigma) + \cdots \\ &= \overline{k}\mu^2\left[1 + \frac{\tau^\varphi}{\Gamma(\varphi+1)} + \frac{\tau^{2\varphi}}{\Gamma(2\varphi+1)} + \frac{\tau^{3\varphi}}{\Gamma(3\varphi+1)} + \cdots\right] = \overline{k}\mu^2E_\varphi(\tau).\end{aligned} \tag{63}$$

In Fig. 3, we plot the analytical fuzzy solutions for Example 5.2 corresponding to different fractional order and uncertainty ς.

Further, we present in Fig. 4(a,b) surface plots of the analytical fuzzy solutions for Example 5.2 corresponding to given fractional order and at different values of μ and τ as well as of uncertainty ς.

Example 5.3 Consider the following TFFAD equation:

$${}^c\mathfrak{D}_\tau^\varphi U(\mu,\tau,\varsigma) + U_\mu(\mu,\tau,\varsigma) = U_{\mu\mu}(\mu,\tau,\varsigma) + \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\varphi)}e^\mu\tau^{\beta-\varphi} \tag{64}$$

with the fuzzy IBCs

$$\begin{aligned}&U(\mu,0,\varsigma) = k, \\ &U(0,\tau,\varsigma) = \tau^\beta, \qquad U(1,\tau,\varsigma) = e\tau^\beta,\end{aligned} \tag{65}$$

where $0 \le \mu \le 1, \tau > 0$.

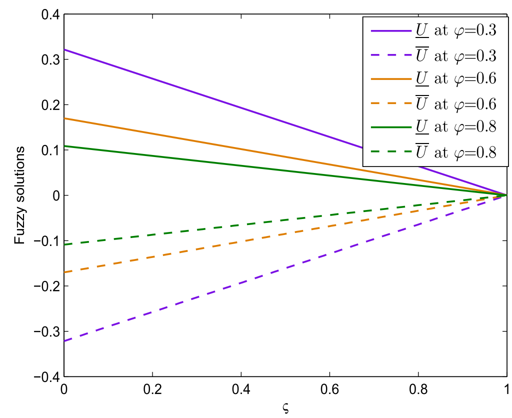

Figure 3 2D graph of the analytical lower and upper solutions of Ex. 5.2 at $\mu = 0.2$ and $\tau = 1$

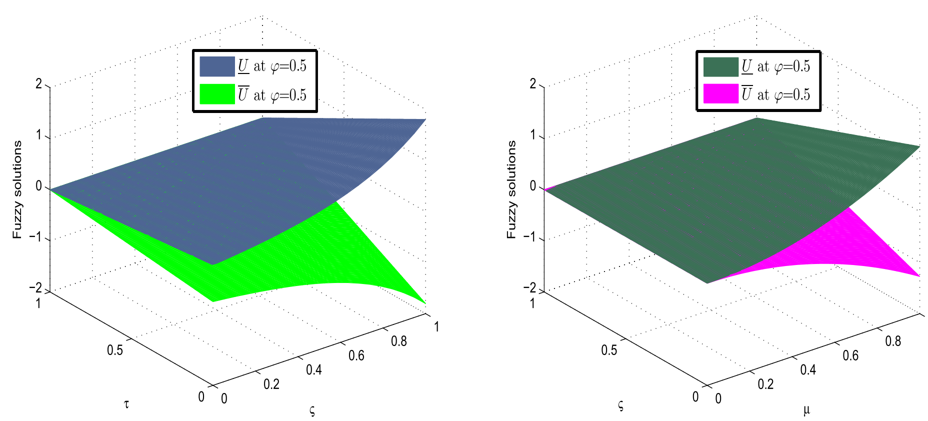

Figure 4 3D graph of the analytical lower and upper solutions of Ex. 5.2 at $\mu = 0.6$ (a) and $\tau = 0.2$ (b)

Applying the MFADM, we have

$$\begin{aligned}\underline{U}_n^*(\mu,\tau,\varsigma) &= \underline{U}_n(\mu,\tau,\varsigma) + (1-\mu)\big[\tau^\beta - \underline{U}_n(0,\tau,\varsigma)\big] + \mu\big[et^\beta - \underline{U}_n(1,\tau,\varsigma)\big],\\ \overline{U}_n^*(\mu,\tau,\varsigma) &= \overline{U}_n(\mu,\tau,\varsigma) + (1-\mu)\big[\tau^\beta - \overline{U}_n(0,\tau,\varsigma)\big] + \mu\big[e\tau^\beta - \overline{U}_n(1,\tau,\varsigma)\big],\end{aligned} \tag{66}$$

where $n = 0, 1, 2, \ldots$.

Using FADM solution, we get

$$\begin{aligned}\underline{U}_0(\mu,\tau,\varsigma) &= \underline{k} + e^\mu \tau^\beta,\\ \overline{U}_0(\mu,\tau,\varsigma) &= \overline{k} + e^\mu \tau^\beta\end{aligned} \tag{67}$$

and

$$\begin{aligned}\underline{U}_{n+1}(\mu,\tau,\varsigma) &= \Im^\varphi\big[\big(\underline{U}_n^*(\mu,\tau,\varsigma)\big)_{\mu\mu} - \big(\underline{U}_n^*(\mu,\tau,\varsigma)\big)_\mu\big], \quad n \ge 0,\\ \overline{U}_{n+1}(\mu,\tau,\varsigma) &= \Im^\varphi\big[\big(\overline{U}_n^*(\mu,\tau,\varsigma)\big)_{\mu\mu} - \big(\overline{U}_n^*(\mu,\tau,\varsigma)\big)_\mu\big], \quad n \ge 0.\end{aligned} \tag{68}$$

Now, we put the IBCs in Equation (66) for $n = 0$.

$$\begin{aligned}\underline{\underline{U}_0^*}(\mu,\tau,\varsigma) &= \underline{U}_0(\mu,\tau,\varsigma) + (1-\mu)\big[\tau^{\beta} - \underline{U}_0(0,\tau,\varsigma)\big] + \mu\big[e\tau^{\beta} - \underline{U}_0(1,\tau,\varsigma)\big] \\ &= 2\underline{k} + e^{\mu}\tau^{\beta},\end{aligned} \tag{69}$$

$$\begin{aligned}\overline{\overline{U}_0^*}(\mu,\tau,\varsigma) &= \overline{U}_0(\mu,\tau,\varsigma) + (1-\mu)\big[\tau^{\beta} - \overline{U}_0(0,\tau,\varsigma)\big] + \mu\big[e\tau^{\beta} - \overline{U}_0(1,\tau,\varsigma)\big] \\ &= 2\overline{k} + e^{\mu}\tau^{\beta}.\end{aligned} \tag{70}$$

From Equation (68), we have

$$\begin{aligned}\underline{U}_1(\mu,\tau,\varsigma) &= \Im^{\varphi}\big[\big(\underline{U}_0^*(\mu,\tau,\varsigma)\big)_{\mu\mu} - \big(\underline{U}_0^*(\mu,\tau,\varsigma)\big)_{\mu}\big] \\ &= \Im^{\varphi}\big[e^{\mu}\tau^{\beta} - e^{\mu}\tau^{\beta}\big] \\ &= 0,\end{aligned} \tag{71}$$

$$\begin{aligned}\overline{U}_1(\mu,\tau,\varsigma) &= \Im^{\varphi}\big[\big(\overline{U}_0^*(\mu,\tau,\varsigma)\big)_{\mu\mu} - \big(\overline{U}_0^*(\mu,\tau,\varsigma)\big)_{\mu}\big] \\ &= \Im^{\varphi}\big[e^{\mu}\tau^{\beta} - e^{\mu}\tau^{\beta}\big] \\ &= 0. \\ &\ \ \vdots\end{aligned} \tag{72}$$

The MFADM solution is

$$\begin{aligned}\underline{U}(\mu,\tau,\varsigma) &= \underline{U}_0(\mu,\tau,\varsigma) + \underline{U}_1(\mu,\tau,\varsigma) + \underline{U}_2(\mu,\tau,\varsigma) + \cdots \\ &= \underline{k} + e^{\mu}\tau^{\beta},\end{aligned} \tag{73}$$

$$\begin{aligned}\overline{U}(\mu,\tau,\varsigma) &= \overline{U}_0(\mu,\tau,\varsigma) + \overline{U}_1(\mu,\tau,\varsigma) + \overline{U}_2(\mu,\tau,\varsigma) + \cdots \\ &= \overline{k} + e^{\mu}\tau^{\beta}.\end{aligned} \tag{74}$$

In Fig. 5, we plot the analytical fuzzy solutions for Example 5.3 to different values of τ and uncertainty ς with $\beta = 5$.

Further, we present in Fig. 6(a,b) surface plots of the analytical fuzzy solutions for Example 5.3 corresponding to different values of μ and τ as well as of uncertainty ς with $\beta = 5$.

Example 5.4 Consider the following TFFADE:

$${}^{c}\mathfrak{D}_{\tau}^{\varphi}U(\mu,\tau,\varsigma) + U_{\mu}(\mu,\tau,\varsigma) = U_{\mu\mu}(\mu,\tau,\varsigma) + h(\mu,\tau,\varsigma), \quad 0 \le \mu \le 1, \tau > 0 \tag{75}$$

having the fuzzy IBCs as follows:

$$\begin{aligned}&U(\mu,0,\varsigma) = k\mu(1-\mu), \\ &U(0,\tau,\varsigma) = U(1,\tau,\varsigma) = 0,\end{aligned} \tag{76}$$

where $h(\mu,\tau,\varsigma) = k\big[\frac{2\mu(1-\mu)}{\Gamma(3-\varphi)}\tau^{2-\varphi} + (3-2\mu)(\tau^2+1)\big]$.

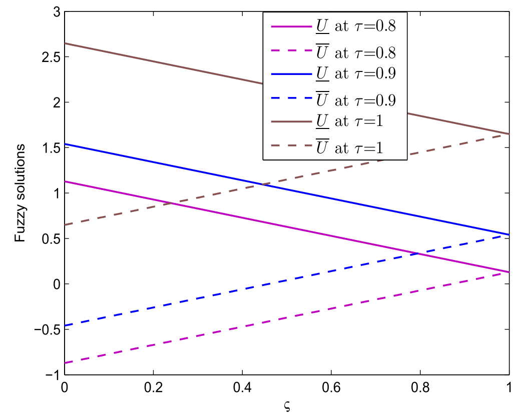

Figure 5 2D graph of the analytical lower and upper solutions of Ex. 5.3 at $\mu = 0.5$

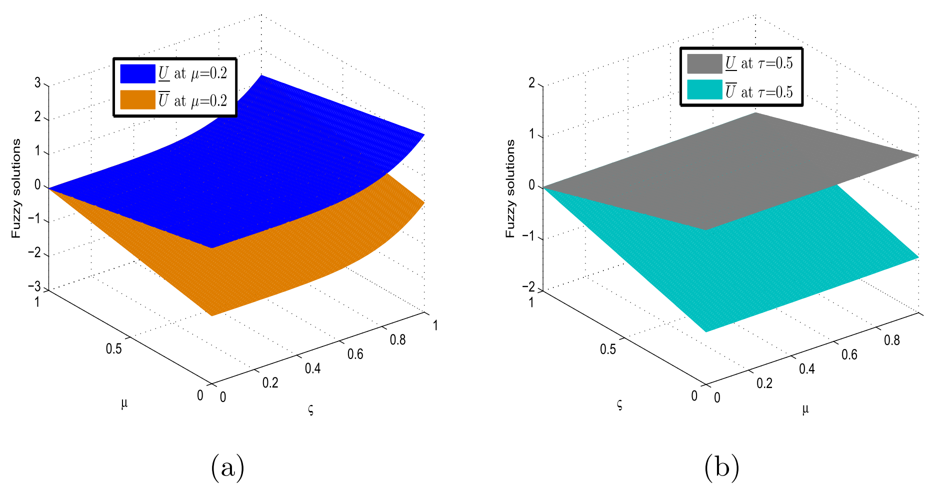

Figure 6 3D graph of the analytical lower and upper solutions of Ex. 5.3 at $\mu = 0.2$ (a) and $\tau = 0.5$ (b)

Applying the MFADM, we have

$$\begin{aligned} \underline{U}_n^*(\mu,\tau,\varsigma) &= \underline{U}_n(\mu,\tau,\varsigma) + (1-\mu)\big[0 - \underline{U}_n(0,\tau,\varsigma)\big] + \mu\big[0 - \underline{U}_n(1,\tau,\varsigma)\big], \\ \overline{U}_n^*(\mu,\tau,\varsigma) &= \overline{U}_n(\mu,\tau,\varsigma) + (1-\mu)\big[0 - \overline{U}_n(0,\tau,\varsigma)\big] + \mu\big[0 - \overline{U}_n(1,\tau,\varsigma)\big], \end{aligned} \tag{77}$$

where $n = 0, 1, 2, \ldots$.

Using the FADM procedure, we have

$$\begin{aligned} \underline{U}_0(\mu,\tau,\varsigma) &= \underline{k}\mu(1-\mu)\big(\tau^2+1\big) + \underline{k}(3-2\mu)\left[\frac{2\tau^{\varphi+2}}{\Gamma(\varphi+3)} + \frac{\tau^{\varphi}}{\Gamma(\varphi+1)}\right], \\ \overline{U}_0(\mu,\tau,\varsigma) &= \overline{k}\mu(1-\mu)\big(\tau^2+1\big) + \overline{k}(3-2\mu)\left[\frac{2\tau^{\varphi+2}}{\Gamma(\varphi+3)} + \frac{\tau^{\varphi}}{\Gamma(\varphi+1)}\right] \end{aligned} \tag{78}$$

and

$$\begin{aligned}\underline{U}_{n+1}(\mu,\tau,\varsigma)&=\Im^{\varphi}\big[\big(\underline{U}_n^*(\mu,\tau,\varsigma)\big)_{\mu\mu}-\big(\underline{U}_n^*(\mu,\tau,\varsigma)\big)_{\mu}\big],\quad n\geq 0,\\ \overline{U}_{n+1}(\mu,\tau,\varsigma)&=\Im^{\varphi}\big[\big(\overline{U}_n^*(\mu,\tau,\varsigma)\big)_{\mu\mu}-\big(\overline{U}_n^*(\mu,\tau,\varsigma)\big)_{\mu}\big],\quad n\geq 0.\end{aligned}\tag{79}$$

Now, we put the IBCs in Equation (77) for $n = 0$.

$$\begin{aligned}\underline{U_0^*}(\mu,\tau,\varsigma)&=\underline{U}_0(\mu,\tau,\varsigma)+(1-\mu)\big[0-\underline{U}_0(0,\tau,\varsigma)\big]+\mu\big[0-\underline{U}_0(1,\tau,\varsigma)\big]\\&=\underline{k}\mu(1-\mu)\big(\tau^2+1\big),\end{aligned}\tag{80}$$

$$\begin{aligned}\overline{U_0^*}(\mu,\tau,\varsigma)&=\overline{U}_0(\mu,\tau,\varsigma)+(1-\mu)\big[0-\overline{U}_0(0,\tau,\varsigma)\big]+\mu\big[0-\overline{U}_0(1,\tau,\varsigma)\big]\\&=\overline{k}\mu(1-\mu)\big(\tau^2+1\big).\end{aligned}\tag{81}$$

From Equation (79), we have

$$\begin{aligned}\underline{U}_1(\mu,\tau,\varsigma)&=\Im^{\varphi}\big[\big(\underline{U}_0^*(\mu,\tau,\varsigma)\big)_{\mu\mu}-\big(\underline{U}_0^*(\mu,\tau,\varsigma)\big)_{\mu}\big]\\&=-\underline{k}(3-2\mu)\bigg[\frac{2\tau^{\varphi+2}}{\Gamma(\varphi+3)}+\frac{\tau^{\varphi}}{\Gamma(\varphi+1)}\bigg],\end{aligned}\tag{82}$$

$$\begin{aligned}\overline{U}_1(\mu,\tau,\varsigma)&=\Im^{\varphi}\big[\big(\overline{U}_0^*(\mu,\tau,\varsigma)\big)_{\mu\mu}-\big(\overline{U}_0^*(\mu,\tau,\varsigma)\big)_{\mu}\big]\\&=-\overline{k}(3-2\mu)\bigg[\frac{2\tau^{\varphi+2}}{\Gamma(\varphi+3)}+\frac{\tau^{\varphi}}{\Gamma(\varphi+1)}\bigg].\end{aligned}\tag{83}$$

Now, for $n = 1$, Equation (77) becomes

$$\begin{aligned}\underline{U_1^*}(\mu,\tau,\varsigma)&=\underline{U}_1(\mu,\tau,\varsigma)+(1-\mu)\big[0-\underline{U}_1(0,\tau,\varsigma)\big]+\mu\big[0-\overline{U}_1(1,\tau,\varsigma)\big]\\&=-\underline{k}(3-2\mu)\bigg[\frac{2\tau^{\varphi+2}}{\Gamma(\varphi+3)}+\frac{\tau^{\varphi}}{\Gamma(\varphi+1)}\bigg]+\underline{k}(3-2\mu)\bigg[\frac{2\tau^{\varphi+2}}{\Gamma(\varphi+3)}+\frac{\tau^{\varphi}}{\Gamma(\varphi+1)}\bigg]\\&=0,\\ \overline{U_1^*}(\mu,\tau,\varsigma)&=\overline{U}_1(\mu,\tau,\varsigma)+(1-\mu)\big[0-\overline{U}_1(0,\tau,\varsigma)\big]+\mu\big[0-\overline{U}_1(1,\tau,\varsigma)\big]\\&=-\overline{k}(3-2\mu)\bigg[\frac{2\tau^{\varphi+2}}{\Gamma(\varphi+3)}+\frac{\tau^{\varphi}}{\Gamma(\varphi+1)}\bigg]+\overline{k}(3-2\mu)\bigg[\frac{2\tau^{\varphi+2}}{\Gamma(\varphi+3)}+\frac{\tau^{\varphi}}{\Gamma(\varphi+1)}\bigg]\\&=0.\end{aligned}$$

From Equation (79), we obtain

$$\begin{aligned}\underline{U}_2(\mu,\tau,\varsigma)&=\Im^{\varphi}\big[\big(\underline{U}_1^*(\mu,\tau,\varsigma)\big)_{\mu\mu}\big]\\&=0.\end{aligned}\tag{84}$$

$$\begin{aligned}\overline{U}_2(\mu,\tau,\varsigma)&=\Im^{\varphi}\big[\big(\overline{U}_1^*(\mu,\tau,\varsigma)\big)_{\mu\mu}\big]\\&=0.\end{aligned}\tag{85}$$

$$\vdots$$

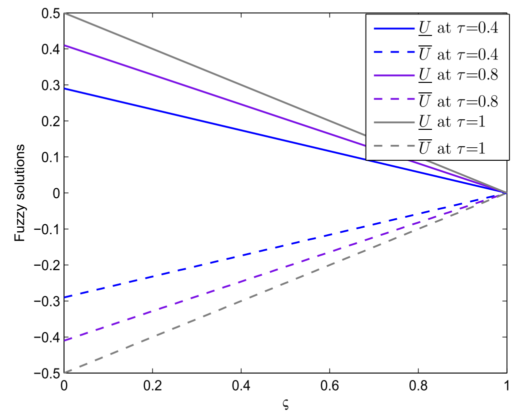

Figure 7 2D graph of the analytical lower and upper solutions of Ex. 5.4 at $\mu = 0.5$

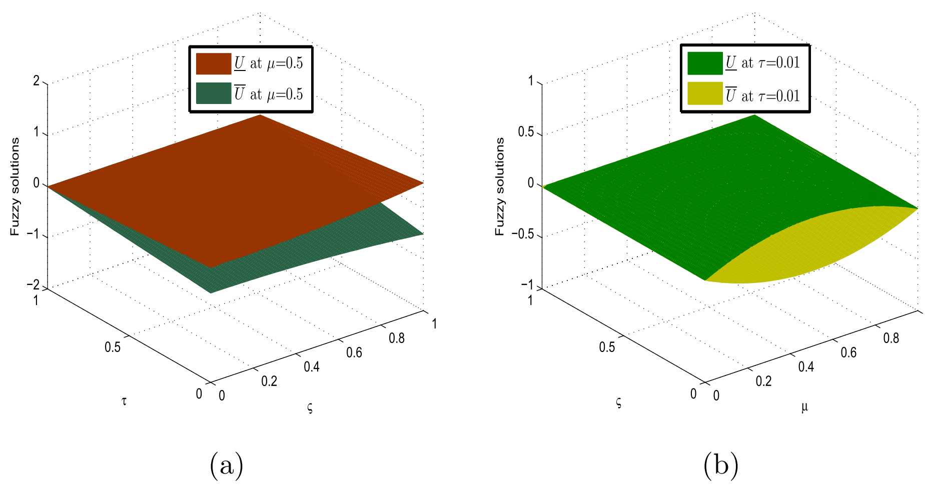

Figure 8 3D graph of the analytical lower and upper solutions of Ex. 5.4 at $\mu = 0.5$ (a) and $\tau = 0.01$ (b)

The MFADM solution is

$$\begin{aligned}\underline{U}(\mu,\tau,\varsigma) &= \underline{U}_0(\mu,\tau,\varsigma) + \underline{U}_1(\mu,\tau,\varsigma) + \underline{U}_2(\mu,\tau,\varsigma) + \cdots \\ &= \underline{k}\mu(1-\mu)\big(\tau^2+1\big),\end{aligned} \tag{86}$$

$$\begin{aligned}\overline{U}(\mu,\tau,\varsigma) &= \overline{U}_0(\mu,\tau,\varsigma) + \overline{U}_1(\mu,\tau,\varsigma) + \overline{U}_2(\mu,\tau,\varsigma) + \cdots \\ &= \overline{k}\mu(1-\mu)\big(\tau^2+1\big).\end{aligned} \tag{87}$$

In Fig. 7, we plot the analytical fuzzy solutions for Example 5.4 corresponding to different values of τ and uncertainty ς.

Further, we present in Fig. 8(a,b) surface plots of the analytical fuzzy solutions for Example 5.4 corresponding to different values of μ and τ as well as of uncertainty ς.

6 Conclusion

In this article, we applied the FADM incorporating novel modifications to solve FFPDEs with IBCs. The MFADM observed to be efficient and simple in handling the solution of fuzzy fractional boundary value problems as compared to other analytical methods [18, 31]. The method we discussed used for solving some special examples of TFFDEs and TFFADEs under (i) g$\mathcal{H}$-partial differentiability. Furthermore, when we substitute $k = 0$ in Example 5.1 and Example 5.3, we recover the analytical solutions of the fractional-order problems as in [19] and [12] respectively. Also, when we substitute $k = 1$ in Example 5.2 and Example 5.4, we recover the analytical solutions of the fractional-order problems as in [20] and [11] respectively. Therefore, the fractional operator with fuzziness provides the global dynamic of the proposed model more than the classical integer and fractional-order model. This suggests that combining fuzzy concepts with fractional calculus leads to a better representation of the dynamics of physical phenomena. Future work will focus on using the proposed method to solve various types of nonlinear FFPDEs.

Author contributions
Author 1 (Nagwa A. Saeed) wrote the whole manuscript and Author 2 (Deepak B. Pachpatte) reviewed the manuscript.

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No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

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