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Weighted fractional inequalities for new conditions on h-convex functions

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Abstract

We use a new function class called B-function to establish a novel version of Hermite–Hadamard inequality for weighted ψ-Hilfer operators. Additionally, we prove two new identities involving weighted ψ-Hilfer operators for differentiable functions. Moreover, by employing these equalities and the properties of the B-function, we derive several trapezoid- and midpoint-type inequalities for h-convex functions. Furthermore, the obtained results are reduced to several well-known and some new inequalities by making specific choices of the function h.

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1 Introduction & preliminaries

In recent decades, various publications have focused on generalizing the Hermite–Hadamard inequality and developing trapezoid- and midpoint-type inequalities that provide bounds for the right- and left-hand sides of the aforementioned inequality. The authors [11] demonstrated various similar trapezoid-type inequalities and developed the Hermite–Hadamard inequality for Riemann–Liouville fractional integrals. Kara et al. [8] identified the following Hermite–Hadamard inequalities:

Let $\psi : [a,b] \to \mathbb{R}$ be a monotone increasing function such that the derivative $\psi' > 0$ is continuous on (a,b). If g is a convex function on $[a,b]$, then

$$g\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\beta+1)}{2A_{(\psi,\beta)}(1)}\left[{}^{\beta}\mathcal{J}^{\psi}_{b^-}G\left(\frac{a+b}{2}\right) + {}^{\beta}\mathcal{J}^{\psi}_{a^+}G\left(\frac{a+b}{2}\right)\right] \le \frac{g(a)+g(b)}{2}, \tag{1.1}$$

where the ψ-Hilfer operators are defined as follows:

$$\begin{aligned}
{}^{\beta}\mathcal{J}^{\psi}_{a^+}g(x) &= \frac{1}{\Gamma(\beta)}\int_a^x \psi'(t)\big(\psi(x)-\psi(t)\big)^{\beta-1}g(t)\,dt,\\
{}^{\beta}\mathcal{J}^{\psi}_{b^-}g(x) &= \frac{1}{\Gamma(\beta)}\int_x^b \psi'(t)\big(\psi(t)-\psi(x)\big)^{\beta-1}g(t)\,dt,
\end{aligned}$$

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and

$$G(s) = g(s) + g(a+b-s),$$
$$A_{(\psi,\beta)}(1) = \left(\psi(b) - \psi\left(\frac{a+b}{2}\right)\right)^{\beta} + \left(\psi\left(\frac{a+b}{2}\right) - \psi(a)\right)^{\beta}.$$

See [3, 7, 9, 12] for further information on comparable results.

In [13], the author introduces a novel class of functions, called h-convex functions.

Definition 1 Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$, where $(0,1) \subseteq J$, be a nonnegative function, $h \neq 0$. We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is an h-convex function if f is nonnegative and for all $x, y \in I$, $\alpha \in (0,1)$ we have

$$f\big(\alpha x + (1-\alpha)y\big) \le h(\alpha)f(x) + h(1-\alpha)f(y). \tag{1.2}$$

If the inequality in (1.2) is reversed, then f is said to be h-concave.

By setting

- $h(\lambda) = \lambda$, Definition 1 reduces to that of the classical convex function.
- $h(\lambda) = 1$, Definition 1 reduces to that of P-functions [4, 10].
- $h(\lambda) = \lambda^s$, Definition 1 reduces to that of s-convex functions [2].
- $h(\lambda) = \frac{1}{n}\sum_{k=1}^{n} \lambda^{\frac{1}{k}}$, Definition 1 reduces to that of polynomial n-fractional convex functions [5].

Recently, the authors of [1] presented a new class of function, called B-function.

Definition 2 Let $a < b$ and $g : (a,b) \subset \mathbb{R} \to \mathbb{R}$ be a nonnegative function. The function g is a B-function, or g belongs to the class $B(a,b)$, if for all $x \in (a,b)$, we have

$$g(x-a) + g(b-x) \le 2g\left(\frac{a+b}{2}\right). \tag{1.3}$$

If the inequality (1.3) is reversed, g is called an A-function, or we say that g belongs to the class $A(a,b)$.

If we have the equality in (1.3), then g is called an AB-function, or we say that g belongs to the class $AB(a,b)$.

Corollary 1 *Let $h : (0,1) \to \mathbb{R}$ be a nonnegative function. The function h is a B-function if and only if for all $\lambda \in (0,1)$, we have*

$$h(\lambda) + h(1-\lambda) \le 2h\left(\frac{1}{2}\right). \tag{1.4}$$

- *The functions $h(\lambda) = \lambda$ and $h(\lambda) = 1$ are AB-functions, B-functions, and A-functions.*
- *The function $h(\lambda) = \lambda^s$, $s \in (0,1]$ is a B-function.*
- *The function $h(\lambda) = \frac{1}{n}\sum_{k=1}^{n} \lambda^{\frac{1}{k}}, n, k \in \mathbb{N}$ is a B-function.*

The weighted fractional integrals are defined as follows:

Definition 3 ([6]) Let $[a,b] \subseteq [0,+\infty)$. Let $\beta > 0$ and ψ be a positive, increasing differentiable function such that $\psi'(s) \neq 0$ for all $s \in [a,b]$. The left- and right-sided weighted fractional integrals of a function f with respect to the function ψ on $[a,b]$ are respectively defined as follows:

$$\mathrm{J}^{\beta,\psi}_{w,a^+}f(x) = \frac{1}{w(x)\Gamma(\beta)}\int_a^x \psi'(t)\big(\psi(x)-\psi(t)\big)^{\beta-1}w(t)f(t)\,dt, \quad a < x \leq b; \tag{1.5}$$

$$\mathrm{J}^{\beta,\psi}_{w,b^-}f(x) = \frac{1}{w(x)\Gamma_k(\beta)}\int_x^b \psi'(t)\big(\psi(t)-\psi(x)\big)^{\beta-1}w(t)f(t)\,dt, \quad a \leq x < b, \tag{1.6}$$

where w is a weighted function and the gamma function defined by

$$\Gamma(\beta) = \int_0^\infty t^{\beta-1}e^{-t}\,dt \quad \text{and} \quad \beta\Gamma(\beta) = \Gamma(\beta+1).$$

For these operators, consider the following space:

$$X[a,b] = \left\{f : \| f \|_X = \left(\int_a^b \big|w(t)f(t)\big|\psi'(t)\,dt\right) < \infty\right\}.$$

For special choices of ψ, w, and β, we get already known results.

(1) Taking $w(t) = 1$, the operators reduce to the ψ-Hilfer integral operators of order $\beta > 0$.
(2) For $\psi(t) = t$, we get the weighted Riemann–Liouville operators.
(3) For $\psi(t) = t$ and $w(t) = 1$, the operators are simplified to Riemann–Liouville integral operators.
(4) Taking $\psi(t) = t$, $w(t) = 1$, and $\beta = 1$, the operators reduce to classical Riemann integrals.
(5) Setting $\psi(t) = \ln(t)$ and $a > 1$, we get the weighted Hadamard operators of order $\beta > 0$.
(6) Setting $\psi(t) = \ln(t)$, $w(t) = 1$, and $a > 1$, the operators are simplified to Hadamard operators of order $\beta > 0$.

The purpose of this study is to generalize the Hermite–Hadamard inequality given in [8] for the h-convex function and weighted ψ-Hilfer operator with conditions. For this aim, we assume h is a B-function.

2 Hermite–Hadamard inequality

This section establishes Hermite–Hadamard-type inequalities for h-convex functions using ψ-Hilfer operators. Throughout this paper, we consider that $0 \leq a < b < \infty$, $\beta > 0$, and ψ is a positive differentiable increasing function on (a,b).

Theorem 2.1 *Let h be a B-function and w a nondecreasing function. If $f \in X[a,b]$ is an h-convex function, then the following inequalities hold*:

$$\begin{aligned}\frac{w(a)}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right) &\leq \frac{\Gamma(\beta+1)w(\frac{a+b}{2})}{2\Omega(\psi,\beta)}\left[\mathrm{J}^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right) + \mathrm{J}^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right] \\ &\leq 2h\left(\frac{1}{2}\right)w(b)\left(\frac{f(b)+f(a)}{2}\right),\end{aligned} \tag{2.1}$$

where

$$F(\tau) = f(\tau) + f(a+b-\tau) \tag{2.2}$$

and

$$\Omega(\psi,\beta) = \left(\psi(b) - \psi\left(\frac{a+b}{2}\right)\right)^{\beta} + \left(\psi\left(\frac{a+b}{2}\right) - \psi(a)\right)^{\beta}. \tag{2.3}$$

Proof Since w is a positive nondecreasing function on $[a,b]$,

(1) for all $\tau \in [a, \frac{a+b}{2}]$, we have $0 < w(a) \leq w(\tau) \leq w(\frac{a+b}{2}) \leq w(b)$, and then

$$\begin{aligned} \frac{w(a)}{\beta}\left(\psi\left(\frac{a+b}{2}\right) - \psi(a)\right)^{\beta} &\leq \int_a^{\frac{a+b}{2}} \left(\psi\left(\frac{a+b}{2}\right) - \psi(\tau)\right)^{\beta-1} w(\tau)\psi'(\tau)\,d\tau \\ &\leq \frac{w(b)}{\beta}\left(\psi\left(\frac{a+b}{2}\right) - \psi(a)\right)^{\beta}; \end{aligned} \tag{2.4}$$

(2) for all $\tau \in [\frac{a+b}{2}, b]$, we have $0 < w(a) \leq w(\frac{a+b}{2}) \leq w(\tau) \leq w(b)$, and then

$$\begin{aligned} \frac{w(a)}{\beta}\left(\psi(b) - \psi\left(\frac{a+b}{2}\right)\right)^{\beta} &\leq \int_{\frac{a+b}{2}}^{b} \left(\psi(\tau) - \psi\left(\frac{a+b}{2}\right)\right)^{\beta-1} w(\tau)\psi'(\tau)\,d\tau \\ &\leq \frac{w(b)}{\beta}\left(\psi(b) - \psi\left(\frac{a+b}{2}\right)\right)^{\beta}. \end{aligned} \tag{2.5}$$

Letting f be an h-convex function, we have for any $\tau \in [a,b]$,

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{1}{2}(a+b-\tau) + \frac{1}{2}\tau\right) \\ &\leq h\left(\frac{1}{2}\right)f(a+b-\tau) + h\left(\frac{1}{2}\right)f(\tau), \end{aligned}$$

and then

$$f\left(\frac{a+b}{2}\right) \leq h\left(\frac{1}{2}\right)F(\tau). \tag{2.6}$$

Multiplying (2.6) by $(\psi(\frac{a+b}{2}) - \psi(\tau))^{\beta-1}\psi'(\tau)w(\tau)$ and integrating over $\tau \in [a, \frac{a+b}{2}]$, we obtain

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)\int_a^{\frac{a+b}{2}} \psi'(\tau)\left(\psi\left(\frac{a+b}{2}\right) - \psi(\tau)\right)^{\beta-1} w(\tau)\,d\tau \\ &\quad\leq h\left(\frac{1}{2}\right)\int_a^{\frac{a+b}{2}} \psi'(\tau)\left(\psi\left(\frac{a+b}{2}\right) - \psi(\tau)\right)^{\beta-1} w(\tau)F(\tau)\,d\tau. \end{aligned}$$

By using the left-hand side of (2.4), we deduce

$$f\left(\frac{a+b}{2}\right)\left(\psi\left(\frac{a+b}{2}\right) - \psi(a)\right)^{\beta} \leq \frac{h(\frac{1}{2})\Gamma(\beta+1)w(\frac{a+b}{2})}{w(a)} J^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right). \tag{2.7}$$

Now, multiplying (2.6) by $(\psi(\tau) - \psi(\frac{a+b}{2}))^{\beta-1}\psi'(\tau)w(\tau)$ and integrating over $\tau \in [\frac{a+b}{2}, b]$, we get

$$\begin{aligned} &f\left(\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^{b}\left(\psi(\tau)-\psi\left(\frac{a+b}{2}\right)\right)^{\beta-1}\psi'(\tau)w(\tau)\,d\tau \\ &\quad\leq h\left(\frac{1}{2}\right)\int_{\frac{a+b}{2}}^{b}\left(\psi(\tau)-\psi\left(\frac{a+b}{2}\right)\right)^{\beta-1}\psi'(\tau)w(\tau)F(\tau)\,d\tau. \end{aligned}$$

By using the left-hand side of (2.5), we deduce

$$f\left(\frac{a+b}{2}\right)\left(\psi(b)-\psi\left(\frac{a+b}{2}\right)\right)^{\beta} \leq \frac{h(\frac{1}{2})\Gamma(\beta+1)w(\frac{a+b}{2})}{w(a)} J^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right). \tag{2.8}$$

Adding the inequalities (2.7) and (2.8), we obtain

$$\frac{w(a)}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\beta+1)w(\frac{a+b}{2})}{2\Omega(\psi,\beta)}\left[J^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+J^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]. \tag{2.9}$$

Let us prove the second inequality in (2.1). Since any $\tau \in [a,b]$ can be written as $\tau = (1-t)a + tb$ for $t \in [0,1]$, we have

$$F(\tau) = f\big((1-t)a+tb\big) + f\big(ta+(1-t)b\big).$$

Applying the h-convexity of the function f, we get

$$\begin{aligned} F(\tau) &= f\big((1-t)b+ta\big) + f\big((1-t)a+tb\big) \\ &\leq h(1-t)\big[f(b)+f(a)\big] + h(t)\big[f(b)+f(a)\big] \\ &= \big(h(t)+h(1-t)\big)\big[f(b)+f(a)\big]. \end{aligned}$$

Applying (1.4), we deduce

$$F(\tau) \leq 2h\left(\frac{1}{2}\right)\big[f(b)+f(a)\big]. \tag{2.10}$$

Multiplying (2.10) by $(\psi(\frac{a+b}{2}) - \psi(\tau))^{\beta-1}\psi'(\tau)w(\tau)$ and integrating over $\tau \in [a, \frac{a+b}{2}]$, we obtain

$$\begin{aligned} &\int_{a}^{\frac{a+b}{2}}\psi'(\tau)\left(\psi\left(\frac{a+b}{2}\right)-\psi(\tau)\right)^{\beta-1}w(\tau)F(\tau)\,d\tau \\ &\quad\leq 2h\left(\frac{1}{2}\right)\big[f(b)+f(a)\big]\int_{a}^{\frac{a+b}{2}}\psi'(\tau)\left(\psi\left(\frac{a+b}{2}\right)-\psi(\tau)\right)^{\beta-1}w(\tau)\,d\tau. \end{aligned}$$

By using the right-hand side of (2.4), we deduce

$$\Gamma(\beta+1)J^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right) \leq \frac{2h(\frac{1}{2})w(b)}{w(\frac{a+b}{2})}\big[f(b)+f(a)\big]\left(\psi\left(\frac{a+b}{2}\right)-\psi(a)\right)^{\beta}. \tag{2.11}$$

Now, multiplying (2.10) by $(\psi(\tau)-\psi(\frac{a+b}{2}))^{\beta-1}\psi'(\tau)w(\tau)$ and integrating over $\tau\in[\frac{a+b}{2},b]$, we get

$$\begin{aligned}&\int_{\frac{a+b}{2}}^{b}\left(\psi(\tau)-\psi\left(\frac{a+b}{2}\right)\right)^{\beta-1}\psi'(\tau)w(\tau)F(\tau)\,d\tau\\&\quad\le 2h\left(\frac{1}{2}\right)\big[f(b)+f(a)\big]\int_{\frac{a+b}{2}}^{b}\left(\psi(\tau)-\psi\left(\frac{a+b}{2}\right)\right)^{\beta-1}\psi'(\tau)w(\tau)\,d\tau.\end{aligned}$$

By using the right-hand side of (2.5), we deduce

$$\Gamma(\beta+1)\mathrm{J}^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)\le\frac{2h(\frac{1}{2})w(b)}{w(\frac{a+b}{2})}\big[f(b)+f(a)\big]\left(\psi(b)-\psi\left(\frac{a+b}{2}\right)\right)^{\beta}. \tag{2.12}$$

Adding inequalities (2.11) and (2.12), we obtain

$$\begin{aligned}&\frac{\Gamma(\beta+1)w(\frac{a+b}{2})}{2\Omega(\psi,\beta)}\left[\mathrm{J}^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+\mathrm{J}^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]\\&\quad\le 2h\left(\frac{1}{2}\right)w(b)\left(\frac{f(b)+f(a)}{2}\right).\end{aligned}\tag{2.13}$$

This finishes the proof. □

The following results are dependent on the function h presented in Theorem 2.1. First, assuming $h(\alpha)=\alpha$, we get the following result using the weighted ψ-Hilfer operators for convex functions.

Corollary 2 *Let $f\in X[a,b]$ be a convex function. Then the following inequalities hold*:

$$\begin{aligned}w(a)f\left(\frac{a+b}{2}\right)&\le\frac{\Gamma(\beta+1)w(\frac{a+b}{2})}{2\Omega(\psi,\beta)}\left[\mathrm{J}^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+\mathrm{J}^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]\\&\le w(b)\left(\frac{f(b)+f(a)}{2}\right),\end{aligned}\tag{2.14}$$

where $F(t)$ and $\Omega(\psi,\beta)$ are defined by (2.2) and (2.3), respectively.

By setting $h(\alpha)=1$, we get the following result using the weighted ψ-Hilfer operators with an f being a P-function.

Corollary 3 *Let $\beta>0$ and $f\in X[a,b]$ be a P-function. Then the following inequalities hold*:

$$\begin{aligned}w(a)f\left(\frac{a+b}{2}\right)&\le\frac{\Gamma(\beta+1)w(\frac{a+b}{2})}{\Omega(\psi,\beta)}\left[\mathrm{J}^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+\mathrm{J}^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]\\&\le 2w(b)\big(f(b)+f(a)\big),\end{aligned}\tag{2.15}$$

where $F(t)$ and $\Omega(\psi,\beta)$ are defined by (2.2) and (2.3), respectively.

Using $h(\alpha)=\alpha^s$, we obtain the following result through the weighted ψ-Hilfer operators and s-convex functions.

Corollary 4 *Let $\beta > 0, s \in (0,1]$, and $f \in X[a,b]$ be an s-convex function. Then the following inequalities hold*:

$$\begin{aligned} \frac{w(a)}{2^{1-s}} f\left(\frac{a+b}{2}\right) &\le \frac{\Gamma(\beta+1)w(\frac{a+b}{2})}{2\Omega(\psi,\beta)}\left[\mathrm{J}^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+\mathrm{J}^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right] \\ &\le 2^{1-s}w(b)\left(\frac{f(b)+f(a)}{2}\right). \end{aligned} \tag{2.16}$$

where $F(t)$ and $\Omega(\psi,\beta)$ are defined by (2.2) *and* (2.3), *respectively.*

Taking $h(\alpha) = \frac{1}{n}\sum_{k=1}^{n}\alpha^{\frac{1}{k}}$, we deduce the following result through the weighted ψ-Hilfer operators and n-fractional polynomial convex functions.

Corollary 5 *Let $\beta > 0$ and $f \in X[a,b]$ be an n-fractional polynomial convex function. Then the following inequalities hold*:

$$\begin{aligned} \frac{w(a)}{C_n} f\left(\frac{a+b}{2}\right) &\le \frac{\Gamma(\beta+1)w(\frac{a+b}{2})}{2\Omega(\psi,\beta)}\left[\mathrm{J}^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+\mathrm{J}^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right] \\ &\le C_n w(b)\left(\frac{f(b)+f(a)}{2}\right). \end{aligned} \tag{2.17}$$

where $F(t)$, $\Omega(\psi,\beta)$ are defined by (2.2), (2.3), *respectively, and $C_n = \frac{2}{n}\sum_{k=1}^{n}(\frac{1}{2})^{\frac{1}{k}}$.*

Remark 1 If we choose $\psi(\tau) = \tau$ and $\psi(\tau) = \ln\tau$ in Corollaries 3, 4, and 5, we obtain Hermite–Hadamard inequality for P-functions, s-convex functions, and n-fractional polynomial convex functions involving the weighted Riemann–Liouville fractional operator and the weighted Hadamard fractional operator, respectively.

3 Weighted trapezoid-type inequalities

This section presents weighted trapezoid inequalities and their particular results utilizing weighted ψ-Hilfer operators with w being symmetric with respect to $\frac{a+b}{2}$ (i.e., $w(t) = w(b + a - t)$). To accomplish this, we must first establish an equality in the following lemma.

Lemma 3.1 *Assume w is a differentiable and symmetric with respect to $\frac{a+b}{2}$ function, and suppose h is a B-function. Let $f : [a,b] \to \mathbb{R}$ be a function where (wf) is a differentiable mapping on (a,b). Then the following identity holds*:

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\beta+1)w(\frac{a+b}{2})}{2\Phi(\psi,\beta,w)}\left[\mathrm{J}^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+\mathrm{J}^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right] \\ &= \frac{b-a}{4\Phi(\psi,\beta,w)}\int_0^1 A_{\psi,\beta}(\tau) \\ &\quad\times\left[(wf)'\left(\frac{1-\tau}{2}a+\frac{1+\tau}{2}b\right)-(wf)'\left(\frac{1+\tau}{2}a+\frac{1-\tau}{2}b\right)\right]d\tau, \end{aligned} \tag{3.1}$$

where

$$\Phi(\psi,\beta,w) = \left(\psi(b)-\psi\left(\frac{a+b}{2}\right)\right)^{\beta}w(b)+\left(\psi\left(\frac{a+b}{2}\right)-\psi(a)\right)^{\beta}w(a). \tag{3.2}$$

$$A_{\psi,\beta}(\tau)=\left(\psi\left(\frac{a+b}{2}\right)-\psi\left(\frac{1+\tau}{2}a+\frac{1-\tau}{2}b\right)\right)^{\beta}+\left(\psi\left(\frac{1-\tau}{2}a+\frac{1+\tau}{2}b\right)-\psi\left(\frac{a+b}{2}\right)\right)^{\beta}. \tag{3.3}$$

Proof Let

$$J_1=\frac{2}{b-a}\int_a^{\frac{a+b}{2}}\left(\psi\left(\frac{a+b}{2}\right)-\psi(\tau)\right)^{\beta}(wF)'(\tau)\,d\tau. \tag{3.4}$$

Integrating by parts (3.4) and using (2.2), we get

$$\frac{b-a}{2}J_1=\left(\psi\left(\frac{a+b}{2}\right)-\psi(\tau)\right)^{\beta}w(\tau)F(\tau)\Big|_a^{\frac{a+b}{2}}+\beta\int_a^{\frac{a+b}{2}}\left(\psi\left(\frac{a+b}{2}\right)-\psi(\tau)\right)^{\beta-1}\psi'(\tau)w(\tau)F(\tau)\,d\tau.$$

Therefore

$$\frac{b-a}{2}J_1=-\left(\psi\left(\frac{a+b}{2}\right)-\psi(a)\right)^{\beta}w(a)F(a)+\Gamma(\beta+1)w\left(\frac{a+b}{2}\right)\mathrm{J}^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right). \tag{3.5}$$

Similarly, let

$$J_2=\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}\left(\psi(\tau)-\psi\left(\frac{a+b}{2}\right)\right)^{\beta}(wF)'(\tau)\,d\tau. \tag{3.6}$$

Integrating by parts (3.6), we obtain

$$\frac{b-a}{2}J_2=\left(\psi(b)-\psi\left(\frac{a+b}{2}\right)\right)^{\beta}w(b)F(b)-\Gamma(\beta+1)w\left(\frac{a+b}{2}\right)\mathrm{J}^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right). \tag{3.7}$$

Since $F(a)=F(b)=f(a)+f(b)$, we conclude from (3.5) and (3.7) that

$$\frac{b-a}{2}(J_2-J_1)=\Phi(\psi,\beta,w)\big(f(a)+f(b)\big)-\Gamma(\beta+1)w\left(\frac{a+b}{2}\right)\left[\mathrm{J}^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+\mathrm{J}^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right],$$

thus

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)w(\frac{a+b}{2})}{2\Phi(\psi,\beta,w)}\left[\mathrm{J}^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+\mathrm{J}^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]=\frac{b-a}{4\Phi(\psi,\beta,w)}(J_2-J_1). \tag{3.8}$$

On the other hand, since $F'(\tau) = f'(\tau) - f'(a+b-\tau)$ and $w(\tau) = w(a+b-\tau)$, we get

$$\begin{aligned}(wF)'(\tau) &= w'(\tau)\big(f(\tau) + f(a+b-\tau)\big) + w(\tau)\big(f'(\tau) - f'(a+b-\tau)\big)\\ &= w'(\tau)f(\tau) + w(\tau)f'(\tau) - w'(a+b-\tau)f(a+b-\tau)\\ &\quad - w(a+b-\tau)f'(a+b-\tau))\\ &= (wf)'(\tau) - (wf)'(a+b-\tau).\end{aligned}$$

From (3.4), we get

$$J_1 = \frac{2}{b-a}\int_a^{\frac{a+b}{2}} \left(\psi\left(\frac{a+b}{2}\right) - \psi(\tau)\right)^{\beta} \big((wf)'(\tau) - (wf)'(a+b-\tau)\big)\,d\tau.$$

By changing the variable $\tau = \frac{1+s}{2}a + \frac{1-s}{2}b$, we obtain

$$\begin{aligned}J_1 = \int_0^1 &\left(\psi\left(\frac{a+b}{2}\right) - \psi\left(\frac{1+s}{2}a + \frac{1-s}{2}b\right)\right)^{\beta}\\ &\times \left[(wf)'\left(\frac{1+s}{2}a + \frac{1-s}{2}b\right) - (wf)'\left(\frac{1-s}{2}a + \frac{1+s}{2}b\right)\right] ds.\end{aligned}$$

Similarly, from (3.6) we deduce

$$\begin{aligned}J_2 = \int_0^1 &\left(\psi\left(\frac{1-s}{2}a + \frac{1+s}{2}b\right) - \psi\left(\frac{a+b}{2}\right)\right)^{\beta}\\ &\times \left[(wf)'\left(\frac{1-s}{2}a + \frac{1+s}{2}b\right) - (wf)'\left(\frac{1+s}{2}a + \frac{1-s}{2}b\right)\right] ds.\end{aligned}$$

Consequently,

$$J_2 - J_1 = \int_0^1 A_{\psi,\beta}(s)\left[(wf)'\left(\frac{1-s}{2}a + \frac{1+s}{2}b\right) - (wf)'\left(\frac{1+s}{2}a + \frac{1-s}{2}b\right)\right] ds. \tag{3.9}$$

Finally, we acquire the needed equality (3.1) by substituting (3.9) into (3.8). □

Remark 2 Putting $w = 1$ in Lemma 3.1, we get [8, Lemma 3.1].

Theorem 3.1 *Under the hypotheses of Lemma 3.1, if $|(wf)'|$ is an h-convex mapping on $[a,b]$ and h is a B-function, then the trapezoid-type inequality holds, namely*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\beta+1)w(\frac{a+b}{2})}{2\Phi(\psi,\beta,w)}\left[J^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right) + J^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]\right|\\ &\quad\le \frac{(b-a)h(\frac{1}{2})}{2\Phi(\psi,\beta,w)}\big[|(wf)'(a)| + |(wf)'(b)|\big]\int_0^1 |A_{\psi,\beta}(s)|\,ds.\end{aligned} \tag{3.10}$$

Proof Taking the absolute value of the identity (3.1) and using the h-convexity of the function $|(wf)'|$, we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)w(\frac{a+b}{2})}{2\Phi(\psi,\beta,w)}\left[\mathrm{J}^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+\mathrm{J}^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]\right|\\
&\leq\frac{b-a}{4\Phi(\psi,\beta,w)}\int_0^1\left|A_{\psi,\beta}(s)\right|\left[\left|(wf)'\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)\right|\right.\\
&\quad\left.+\left|(wf)'\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)\right|\right]ds\\
&\leq\frac{b-a}{4\Phi(\psi,\beta,w)}\int_0^1\left|A_{\psi,\beta}(s)\right|\left[h\left(\frac{1-s}{2}\right)+h\left(\frac{1+s}{2}\right)\right]\left[\left|(wf)'(a)\right|+\left|(wf)'(b)\right|\right]ds,
\end{aligned}$$

Given that h is a B-function, setting $\alpha=\frac{1-s}{2}$ and $1-\alpha=\frac{1+s}{2}$ yields

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)w(\frac{a+b}{2})}{2\Phi(\psi,\beta,w)}\left[\mathrm{J}^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+\mathrm{J}^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]\right|\\
&\leq\frac{(b-a)h(\frac{1}{2})}{2\Phi(\psi,\beta,w)}\int_0^1\left|A_{\psi,\beta}(s)\right|\left[\left|(wf)'(a)\right|+\left|(wf)'(b)\right|\right]ds.
\end{aligned}$$

□

The following results are obtained via the weighted ψ-Hilfer operators and depend on the function h given in Theorem 3.1.

Corollary 6

(1) *If* $|(wf)'|$ *is a convex mapping on* $[a,b]$, *then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)w(\frac{a+b}{2})}{2\Phi(\psi,\beta,w)}\left[\mathrm{J}^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+\mathrm{J}^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]\right|\\
&\leq\frac{b-a}{4\Phi(\psi,\beta,w)}\left[\left|(wf)'(a)\right|+\left|(wf)'(b)\right|\right]\int_0^1\left|A_{\psi,\beta}(s)\right|ds.
\end{aligned}$$

Particularly, putting $w=1$, *we get* [8, *Corollary* 3.4].

(2) *If* $|(wf)'|$ *is a P-function on* $[a,b]$, *then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)w(\frac{a+b}{2})}{2\Phi(\psi,\beta,w)}\left[\mathrm{J}^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+\mathrm{J}^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]\right|\\
&\leq\frac{b-a}{2\Phi(\psi,\beta,w)}\left[\left|(wf)'(a)\right|+\left|(wf)'(b)\right|\right]\int_0^1\left|A_{\psi,\beta}(s)\right|ds.
\end{aligned}$$

(3) *If* $|(wf)'|$ *is an s-convex mapping on* $[a,b]$, *then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)w(\frac{a+b}{2})}{2\Phi(\psi,\beta,w)}\left[\mathrm{J}^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+\mathrm{J}^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]\right|\\
&\leq\frac{b-a}{2^{s+1}\Phi(\psi,\beta,w)}\left[\left|(wf)'(a)\right|+\left|(wf)'(b)\right|\right]\int_0^1\left|A_{\psi,\beta}(s)\right|ds.
\end{aligned}$$

(4) *If* $|(wf)'|$ *is an n-fractional polynomial convex mapping on* $[a,b]$, *then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)w(\frac{a+b}{2})}{2\Phi(\psi,\beta,w)}\left[J^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+J^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]\right| \leq \frac{(b-a)C_n}{4\Phi(\psi,\beta,w)}\left[\left|(wf)'(a)\right|+\left|(wf)'(b)\right|\right]\int_0^1\left|A_{\psi,\beta}(s)\right|ds,$$

where $\Phi(\psi,\beta,w)$, $A_{\psi,\beta}(s)$ *are defined by* (3.2), (3.3), *respectively, and* $C_n=\frac{2}{n}\sum_{k=1}^{n}(\frac{1}{2})^{\frac{1}{k}}$.

Theorem 3.2 *Let* $p>1$ *and* $\frac{1}{p'}+\frac{1}{p}=1$. *If* $|(wf)'|^p$ *is an h-convex mapping on* $[a,b]$, *then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)w(\frac{a+b}{2})}{2\Phi(\psi,\beta,w)}\left[J^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+J^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]\right|\\&\quad\leq\frac{(b-a)(2h(\frac{1}{2}))^{\frac{1}{p}}}{4\Phi(\psi,\beta,w)}\left(2\int_0^1\left|A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\left|(wf)'(a)\right|^p+\left|(wf)'(b)\right|^p\right)^{\frac{1}{p}}\\&\quad\leq\frac{(b-a)(2h(\frac{1}{2}))^{\frac{1}{p}}}{4\Phi(\psi,\beta,w)}\left(2\int_0^1\left|A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\left|(wf)'(a)\right|+\left|(wf)'(b)\right|\right).\end{aligned}\tag{3.11}$$

Proof Taking absolute value of (3.1) and using the well-known Hölder's inequality, we obtain

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)w(\frac{a+b}{2})}{2\Phi(\psi,\beta,w)}\left[J^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+J^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]\right|\\&\quad\leq\frac{b-a}{4\Phi(\psi,\beta,w)}\int_0^1\left|A_{\psi,\beta}(s)\right|\left|(wf)'\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)\right|ds\\&\qquad+\frac{b-a}{4\Phi(\psi,\beta,w)}\int_0^1\left|A_{\psi,\beta}(s)\right|\left|(wf)'\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)\right|ds\\&\quad\leq\frac{b-a}{4\Phi(\psi,\beta,w)}\left(\int_0^1\left|A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\times\left(\int_0^1\left|(wf)'\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)\right|^p ds\right)^{\frac{1}{p}}\\&\qquad+\frac{b-a}{4\Phi(\psi,\beta,w)}\left(\int_0^1\left|A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\times\left(\int_0^1\left|(wf)'\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)\right|^p ds\right)^{\frac{1}{p}}.\end{aligned}$$

Notice that for $p>1$, $A,B\geq 0$, $A^{\frac{1}{p}}+B^{\frac{1}{p}}\leq 2^{1-\frac{1}{p}}(A+B)^{\frac{1}{p}}$, and $|(wf)'|^p$ an h-convex function, we get

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)w(\frac{a+b}{2})}{2\Phi(\psi,\beta,w)}\left[J^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+J^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]\right|\\&\quad\leq\frac{b-a}{4\Phi(\psi,\beta,w)}\left(\int_0^1\left|A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}2^{1-\frac{1}{p}}\\&\qquad\times\left[\int_0^1\left|(wf)'\left(\frac{1-s}{2}a+\frac{1+s}{2}\right)\right|^p ds+\int_0^1\left|(wf)'\left(\frac{1+s}{2}a+\frac{1-s}{2}\right)\right|^p ds\right]^{\frac{1}{p}}\\&\quad\leq\frac{b-a}{4\Phi(\psi,\beta,w)}\left(2\int_0^1\left|A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\end{aligned}$$

$$\times\left(\int_0^1\left[h\left(\frac{1-s}{2}\right)+h\left(\frac{1+s}{2}\right)\right]\left[\left|(wf)'(a)\right|^p+\left|(wf)'(b)\right|^p\right]ds\right)^{\frac{1}{p}}.$$

Since h is a B-function, we get

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)w(\frac{a+b}{2})}{2\Phi(\psi,\beta,w)}\left[\mathrm{J}^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+\mathrm{J}^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]\right|\\&\quad\le\frac{(b-a)(2h(\frac{1}{2}))^{\frac{1}{p}}}{4\Phi(\psi,\beta,w)}\left(2\int_0^1\left|A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\left|(wf)'(a)\right|^p+\left|(wf)'(b)\right|^p\right)^{\frac{1}{p}}.\end{aligned}$$

This proves the first inequality in (3.11).

Notice that the inequality $A^p+B^p\le(A+B)^p$ yields the second inequality in (3.11). □

Setting $w=1$ and $h(s)=s$ in Theorem 3.2, we get the following corollary.

Corollary 7 *Let $p>1$ and $\frac{1}{p'}+\frac{1}{p}=1$. If $|f'|^p$ is a convex mapping on $[a,b]$, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)}{2\Omega(\psi,\beta)}\left[{}^{\beta}\mathcal{J}^{\psi}_{b^-}F\left(\frac{a+b}{2}\right)+{}^{\beta}\mathcal{J}^{\psi}_{a^+}F\left(\frac{a+b}{2}\right)\right]\right|\\&\quad\le\frac{b-a}{4\Omega(\psi,\beta)}\left(2\int_0^1\left|A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\left|f'(a)\right|^p+\left|f'(b)\right|^p\right)^{\frac{1}{p}}\\&\quad\le\frac{b-a}{4\Omega(\psi,\beta)}\left(2\int_0^1\left|A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\left|f'(a)\right|+\left|f'(b)\right|\right),\end{aligned}\tag{3.12}$$

which is a better estimate compared with [8, *Theorem* 3.5].

4 Weighted midpoint-type inequalities

This section establishes some weighted midpoint inequalities for weighted ψ-Hilfer operators using the identity in the following lemma.

Lemma 4.1 *Under the hypothesis of Lemma* 3.1, *the following identity holds*:

$$\begin{aligned}&\frac{\Gamma(\beta+1)}{2\Omega(\psi,\beta)}\left[\mathrm{J}^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+\mathrm{J}^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\\&\quad=\frac{b-a}{4\Omega(\psi,\beta)w(\frac{a+b}{2})}\\&\qquad\times\int_0^1\big(\Omega(\psi,\beta)\\&\qquad-A_{\psi,\beta}(s)\big)\left[(wf)'\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)-(wf)'\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)\right]ds,\end{aligned}\tag{4.1}$$

where $\Omega(\psi,\beta)$ and $A_{\psi,\beta}(\tau)$ are defined in (2.3) *and* (3.3), *respectively.*

Proof Let

$$R_1=\frac{2}{b-a}\int_a^{\frac{a+b}{2}}\left[\left(\psi\left(\frac{a+b}{2}\right)-\psi(a)\right)^{\beta}-\left(\psi\left(\frac{a+b}{2}\right)-\psi(\tau)\right)^{\beta}\right](wF)'(\tau)\,d\tau.\tag{4.2}$$

By using (3.4), we get

$$\begin{aligned}\frac{b-a}{2}R_1 &= \int_a^{\frac{a+b}{2}} \left(\psi\left(\frac{a+b}{2}\right)-\psi(a)\right)^{\beta}(wF)'(\tau)\,d\tau \\ &\quad - \int_a^{\frac{a+b}{2}} \left(\psi\left(\frac{a+b}{2}\right)-\psi(\tau)\right)^{\beta}(wF)'(\tau)\,d\tau \\ &= \left(\psi\left(\frac{a+b}{2}\right)-\psi(a)\right)^{\beta}(wF)(\tau)\Big|_a^{\frac{a+b}{2}} - \frac{2}{b-a}J_1 \\ &= \left(\psi\left(\frac{a+b}{2}\right)-\psi(a)\right)^{\beta}2(wf)\left(\frac{a+b}{2}\right) \\ &\quad - \left(\psi\left(\frac{a+b}{2}\right)-\psi(a)\right)^{\beta}(wF)(a) - \frac{2}{b-a}J_1.\end{aligned}$$

Applying (3.5), we obtain

$$\begin{aligned}\frac{b-a}{2}R_1 &= 2\left(\psi\left(\frac{a+b}{2}\right)-\psi(a)\right)^{\beta}(wf)\left(\frac{a+b}{2}\right) \\ &\quad - \Gamma(\beta+1)w\left(\frac{a+b}{2}\right)\mathrm{J}^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right).\end{aligned} \tag{4.3}$$

Similarly, let

$$R_2 = \frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}\left[\left(\psi(b)-\psi\left(\frac{a+b}{2}\right)\right)^{\beta} - \left(\psi(\tau)-\psi\left(\frac{a+b}{2}\right)\right)^{\beta}\right](wF)'(\tau)\,d\tau. \tag{4.4}$$

Using (3.6), then we have

$$\begin{aligned}\frac{b-a}{2}R_2 &= \int_{\frac{a+b}{2}}^{b}\left(\psi(b)-\psi\left(\frac{a+b}{2}\right)\right)^{\beta}(wF)'(\tau)\,d\tau \\ &\quad - \int_{\frac{a+b}{2}}^{b}\left(\psi(\tau)-\psi\left(\frac{a+b}{2}\right)\right)^{\beta}(wF)'(\tau)\,d\tau \\ &= \left(\psi(b)-\psi\left(\frac{a+b}{2}\right)\right)^{\beta}(wF)(\tau)\Big|_{\frac{a+b}{2}}^{b} - \frac{2}{b-a}J_2 \\ &= \left(\psi(b)-\psi\left(\frac{a+b}{2}\right)\right)^{\beta}(wF)(b) \\ &\quad - 2\left(\psi(b)-\psi\left(\frac{a+b}{2}\right)\right)^{\beta}(wf)\left(\frac{a+b}{2}\right) - \frac{2}{b-a}J_2,\end{aligned}$$

and applying (3.7), we get

$$\begin{aligned}\frac{b-a}{2}R_2 &= \Gamma(\beta+1)w\left(\frac{a+b}{2}\right)\mathrm{J}^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right) \\ &\quad - 2\left(\psi(b)-\psi\left(\frac{a+b}{2}\right)\right)^{\beta}(wf)\left(\frac{a+b}{2}\right).\end{aligned} \tag{4.5}$$

From (4.3) and (4.5), we have

$$\begin{aligned} &\frac{b-a}{4\Omega(\psi,\beta)w(\frac{a+b}{2})}(R_2-R_1) \\ &= \frac{\Gamma(\beta+1)}{2\Omega(\psi,\beta)}\left[\mathrm{J}^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+\mathrm{J}^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right). \end{aligned} \tag{4.6}$$

In addition, according to (4.2),

$$\begin{aligned} R_1 &= \frac{2}{b-a}\int_a^{\frac{a+b}{2}}\left[\left(\psi\left(\frac{a+b}{2}\right)-\psi(a)\right)^\beta-\left(\psi\left(\frac{a+b}{2}\right)-\psi(\tau)\right)^\beta\right] \\ &\quad\times\big((wf)'(\tau)-(wf)'(a+b-\tau)\big)\,d\tau \\ &= \int_0^1\left[\left(\psi\left(\frac{a+b}{2}\right)-\psi(a)\right)^\beta-\left(\psi\left(\frac{a+b}{2}\right)-\psi\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)\right)^\beta\right] \\ &\quad\times\left[(wf)'\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)-(wf)'\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)\right]ds. \end{aligned}$$

Similarly, from (4.4) we get

$$\begin{aligned} R_2 &= \frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}\left[\left(\psi(b)-\psi\left(\frac{a+b}{2}\right)\right)^\beta-\left(\psi(\tau)-\psi\left(\frac{a+b}{2}\right)\right)^\beta\right] \\ &\quad\times\big((wf)'(\tau)-(wf)'(a+b-\tau)\big)\,d\tau \\ &= \int_0^1\left[\left(\psi(b)-\psi\left(\frac{a+b}{2}\right)\right)^\beta-\left(\psi\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)-\psi\left(\frac{a+b}{2}\right)\right)^\beta\right] \\ &\quad\times\left[(wf)'\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)-(wf)'\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)\right]ds. \end{aligned}$$

As a result,

$$\begin{aligned} R_2-R_1 &= \int_0^1\big(\Omega(\psi,\beta)-A_{\psi,\beta}(s)\big) \\ &\quad\times\left[(wf)'\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)-(wf)'\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)\right]ds. \end{aligned} \tag{4.7}$$

To obtain the desired equality (4.1), substitute (4.7) into (4.6). □

Remark 3 Put $w=1$ in Lemma 4.1, we get [8, Lemma 4.1].

Theorem 4.1 *If* $|(wf)'|$ *is an h-convex mapping on* $[a,b]$ *and h is a B-function, then*

$$\begin{aligned} &\left|\frac{\Gamma(\beta+1)}{2\Omega(\psi,\beta)}\left[\mathrm{J}^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+\mathrm{J}^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right| \\ &\leq \frac{(b-a)h(\frac{1}{2})}{2\Omega(\psi,\beta)w(\frac{a+b}{2})}\big[\left|(wf)'(a)\right|+\left|(wf)'(b)\right|\big]\int_0^1\left|\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|ds. \end{aligned} \tag{4.8}$$

Proof Taking the absolute value of the identity (4.1) and using the h-convexity of $|(wf)'|$ and inequality (1.4), we deduce

$$\begin{aligned}
&\left|\frac{\Gamma(\beta+1)}{2\Omega(\psi,\beta)}\left[J^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+J^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right| \\
&\quad\le \frac{b-a}{4\Omega(\psi,\beta)w(\frac{a+b}{2})}\int_0^1\left|\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right| \\
&\qquad\times\left[\left|(wf)'\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)\right|+\left|(wf)'\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)\right|\right]ds \\
&\quad\le \frac{b-a}{4\Omega(\psi,\beta)w(\frac{a+b}{2})}\int_0^1\left|\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right| \\
&\qquad\times\left[\left(h\left(\frac{1-s}{2}\right)+h\left(\frac{1+s}{2}\right)\right)\left(\left|(wf)'(a)\right|+\left|(wf)'(b)\right|\right)\right]ds \\
&\quad= \frac{(b-a)h(\frac{1}{2})}{2\Omega(\psi,\beta)w(\frac{a+b}{2})}\left[\left|(wf)'(a)\right|+\left|(wf)'(b)\right|\right]\int_0^1\left|\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|ds.
\end{aligned}$$

This ends the proof. □

The following results are obtained using the weighted ψ-Hilfer operators and depend on the function h given in Theorem 4.1.

Corollary 8

(1) *If $|(wf)'|$ is a convex mapping on $[a,b]$, then*

$$\begin{aligned}
&\left|\frac{\Gamma(\beta+1)}{2\Omega(\psi,\beta)}\left[J^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+J^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right| \\
&\quad\le \frac{b-a}{4\Omega(\psi,\beta)w(\frac{a+b}{2})}\left[\left|(wf)'(a)\right|+\left|(wf)'(b)\right|\right]\int_0^1\left|\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|ds.
\end{aligned}$$

Particularly, putting $w=1$, we get [8, Theorem 4.2].

(2) *If $|(wf)'|$ is a P-function on $[a,b]$, then*

$$\begin{aligned}
&\left|\frac{\Gamma(\beta+1)}{2\Omega(\psi,\beta)}\left[J^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+J^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right| \\
&\quad\le \frac{b-a}{2\Omega(\psi,\beta)w(\frac{a+b}{2})}\left[\left|(wf)'(a)\right|+\left|(wf)'(b)\right|\right]\int_0^1\left|\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|ds.
\end{aligned}$$

(3) *If $|(wf)'|$ is an s-convex mapping on $[a,b]$, then*

$$\begin{aligned}
&\left|\frac{\Gamma(\beta+1)}{2\Omega(\psi,\beta)}\left[J^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+J^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right| \\
&\quad\le \frac{b-a}{2^{s+1}\Omega(\psi,\beta)w(\frac{a+b}{2})}\left[\left|(wf)'(a)\right|+\left|(wf)'(b)\right|\right]\int_0^1\left|\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|ds.
\end{aligned}$$

(4) *If* $|(wf)'|$ *is an n-fractional polynomial convex mapping on* $[a,b]$, *then*

$$\begin{aligned}&\left|\frac{\Gamma(\beta+1)}{2\Omega(\psi,\beta)}\left[J^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+J^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|\\&\quad\le\frac{(b-a)C_n}{4\Omega(\psi,\beta)w(\frac{a+b}{2})}\left[\left|(wf)'(a)\right|+\left|(wf)'(b)\right|\right]\int_0^1\left|\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|ds,\end{aligned}$$

where $\Omega(\psi,\beta)$, $A_{\psi,\beta}(s)$ *are defined by* (2.3), (3.3), *respectively, and* $C_n=\frac{2}{n}\sum_{k=1}^{n}(\frac{1}{2})^{\frac{1}{k}}$.

Theorem 4.2 *Let* $p>1$ *and* $\frac{1}{p'}+\frac{1}{p}=1$. *If* $|(wf)'|^p$ *is an h-convex mapping on* $[a,b]$, *then*

$$\begin{aligned}&\left|\frac{\Gamma(\beta+1)}{2\Omega(\psi,\beta)}\left[J^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+J^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|\\&\quad\le\frac{(b-a)(2h(\frac{1}{2}))^{\frac{1}{p}}}{4\Omega(\psi,\beta)w(\frac{a+b}{2})}\left(2\int_0^1\left|\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\\&\quad\times\left(\left|(wf)'(a)\right|^p+\left|(wf)'(b)\right|^p\right)^{\frac{1}{p}}\\&\quad\le\frac{(b-a)(2h(\frac{1}{2}))^{\frac{1}{p}}}{4\Omega(\psi,\beta)w(\frac{a+b}{2})}\left(2\int_0^1\left|\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\left|(wf)'(a)\right|+\left|(wf)'(b)\right|\right).\end{aligned}\tag{4.9}$$

Proof Taking the absolute value of (4.1) and using the well-known Hölder's inequality, we obtain

$$\begin{aligned}&\left|\frac{\Gamma(\beta+1)}{2\Omega(\psi,\beta)}\left[J^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+J^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|\\&\quad\le\frac{b-a}{4\Omega(\psi,\beta)w(\frac{a+b}{2})}\int_0^1\left|\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|\left|(wf)'\left(\frac{1-s}{2}a+\frac{1+s}{2}b\right)\right|ds\\&\quad+\frac{b-a}{4\Omega(\psi,\beta)w(\frac{a+b}{2})}\int_0^1\left|\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|\left|(wf)'\left(\frac{1+s}{2}a+\frac{1-s}{2}b\right)\right|ds\\&\quad\le\frac{b-a}{4\Omega(\psi,\beta)w(\frac{a+b}{2})}\left(\int_0^1\left|\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\\&\quad\times\left(\int_0^1\left|(wf)'\left(\frac{1-s}{2}a+\frac{1+s}{2}\right)\right|^{p}ds\right)^{\frac{1}{p}}\\&\quad+\frac{b-a}{4\Omega(\psi,\beta)w(\frac{a+b}{2})}\left(\int_0^1\left|\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\\&\quad\times\left(\int_0^1\left|(wf)'\left(\frac{1+s}{2}a+\frac{1-s}{2}\right)\right|^{p}ds\right)^{\frac{1}{p}}.\end{aligned}$$

Noticing that $A^{\frac{1}{p}} + B^{\frac{1}{p}} \leq 2^{1-\frac{1}{p}}(A+B)^{\frac{1}{p}}$ and $|(wf)'|^p$ is an h-convex function, we conclude

$$\begin{aligned}
&\left|\frac{\Gamma(\beta+1)}{2\Omega(\psi,\beta)}\left[\mathrm{J}^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+\mathrm{J}^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{b-a}{4\Omega(\psi,\beta)w(\frac{a+b}{2})}\left(\int_0^1\left|\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}2^{1-\frac{1}{p}} \\
&\quad\times\left[\int_0^1\left|(wf)'\left(\frac{1-s}{2}a+\frac{1+s}{2}\right)\right|^p ds+\int_0^1\left|(wf)'\left(\frac{1+s}{2}a+\frac{1-s}{2}\right)\right|^p ds\right]^{\frac{1}{p}} \\
&\leq \frac{b-a}{4\Omega(\psi,\beta)w(\frac{a+b}{2})}\left(2\int_0^1\left|\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}} \\
&\quad\times\left(\int_0^1\left[h\left(\frac{1-s}{2}\right)+h\left(\frac{1+s}{2}\right)\right]\left[\left|(wf)'(a)\right|^p+\left|(wf)'(b)\right|^p\right]ds\right)^{\frac{1}{p}}.
\end{aligned}$$

Putting $\alpha = \frac{1-s}{2}$ and $1-\alpha = \frac{1+s}{2}$ yields

$$\begin{aligned}
&\left|\frac{\Gamma(\beta+1)}{2\Omega(\psi,\beta)}\left[\mathrm{J}^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+\mathrm{J}^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{(b-a)(2h(\frac{1}{2}))^{\frac{1}{p}}}{4\Omega(\psi,\beta)w(\frac{a+b}{2})}\left(2\int_0^1\left|\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\left|(wf)'(a)\right|^p+\left|(wf)'(b)\right|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

This proves the first inequality in (4.9).

The second inequality in (4.9) is clear from the inequality $A^p + B^p \leq (A+B)^p$. □

Setting $w = 1$ and $h(s) = s$ in Theorem 4.2, we get the following corollary.

Corollary 9 *Let* $p > 1$ *and* $\frac{1}{p'} + \frac{1}{p} = 1$. *If* $|f'|^p$ *is a convex mapping on* $[a,b]$, *then*

$$\begin{aligned}
&\left|\frac{\Gamma(\beta+1)}{2\Omega(\psi,\beta)}\left[\mathrm{J}^{\beta,\psi}_{w,b^-}F\left(\frac{a+b}{2}\right)+\mathrm{J}^{\beta,\psi}_{w,a^+}F\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{b-a}{4\Omega(\psi,\beta)}\left(2\int_0^1\left|\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\left|f'(a)\right|^p+\left|f'(b)\right|^p\right)^{\frac{1}{p}} \qquad (4.10)\\
&\leq \frac{b-a}{4\Omega(\psi,\beta)}\left(2\int_0^1\left|\Omega(\psi,\beta)-A_{\psi,\beta}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\left|f'(a)\right|+\left|f'(b)\right|\right),
\end{aligned}$$

which is a better estimate compared with [8, *Theorem* 4.5].

5 Conclusions

In this study, we recalled a new function class, namely that of B-functions, and utilized it to derive a novel version of the Hermite–Hadamard inequality for weighted ψ-Hilfer operators. We also established two new identities involving weighted ψ-Hilfer operators for differentiable functions. By combining these identities and the properties of the B-function, we obtained several trapezoid- and midpoint-type inequalities for h-convex functions. Our results not only extend the existing literature on inequalities involving fractional operators but also provide new insights into the behavior of h-convex functions under these operators. Additionally, our methods can be applied to other fractional integral operators by using B-functions.

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B. B. and N. A. wrote the main results. H. B. revised the paper.

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No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

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