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On generalized (k,ψ)-Hilfer proportional fractional operator and its applications to the higher-order Cauchy problem

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Abstract

In this work, we introduce a novel idea of generalized (k,ψ)-Hilfer proportional fractional operators. The proposed operator combines the (k,ψ)-Riemann–Liouville and (k,ψ)-Caputo proportional fractional operators. Some properties and auxiliary results of the proposed operators are investigated. The ψ-Laplace transform and its properties of the proposed operators are established and utilized to solve Cauchy-type problems. Furthermore, the uniqueness result for a higher-order initial value problem under (k,ψ)-Hilfer proportional fractional operators is proved by using Picard's iterative technique. At the end, examples are provided to present the theoretical results. This new type of proposed operator can help other researchers who are still working on real-world problems.

Mathematics Subject Classification: 26A33; 33B15; 34A08; 34A12; 34D20

Keywords: Fractional calculus; (k,ψ)-Hilfer proportional fractional derivative; Fractional differential equations; Existence and uniqueness; Picard's iterative method

1 Introduction

The field of mathematical analysis known as fractional calculus ($\mathbb{FC}$) is actively expanding. It has much applicability to everyday issues because its nonlocal properties are suitable for describing copious memory phenomena, not only in pure and applied mathematics but also in biology, physics, chemistry, and engineering; we suggest the reader to the famous books [1–3]. The history of $\mathbb{FC}$ is almost as old as classical calculus that deals with integer order, but $\mathbb{FC}$ focuses on differential and integral operators of noninteger order (fractional order). The study of $\mathbb{FC}$ currently holds great interest for many academics and researchers. The abundance of different types of fractional operators derived from various characteristics is one of the fundamental advantages of $\mathbb{FC}$. A variety of these operators have been applied in many works, and the most illustrious ones are Riemann–Liouville ($\mathbb{RL}$) and Caputo types. Afterward, many works continuously tried to create and develop some new fractional operators for over a decade. For example, the Hilfer fractional derivative, a generalized derivative operator between $\mathbb{RL}$ and Caputo, was introduced by Hilfer [4]. Katugampola type is the generalized fractional operator that unifies $\mathbb{RL}$ and Hadamard types into a unique form proposed by Katugampola [5]. Jarad *et al.* [6] generated proportional $\mathbb{RL}$ and Caputo types of fractional derivatives under exponential kernels. Next, Ahmed *et al.* [7] constructed Hilfer type of proportional fractional derivative, which combines the proportional $\mathbb{RL}$ and Caputo types, while Vanterler and Oliveira [8] proposed the ψ-Hilfer fractional derivative and its properties. Mallah *et al.* [9] proposed the ψ-Hilfer proportional fractional derivative operator (ψ-Hilfer-PFDO). However, some researchers developed many other types of new fractional operators, which helped to expand the fractional calculus field, see [10–15] and the references therein.

Over the years, fractional derivatives and integrals have been in the form of the gamma function $\Gamma(\cdot)$. In [16–18], in recent years, Díaz *et al.* proposed the k-gamma function $\Gamma_k(\cdot)$, it is the generalization of $\Gamma(\cdot)$ in which $\Gamma_k(\cdot) \to \Gamma(\cdot)$ when parameter $k \to 1$. Recently, a great deal of research has been done on the idea of fractional derivatives and integrals with k. The fractional operator has been developed utilizing the k-gamma function, and there are a lot of fascinating publications available. For example, Mubeen and Habibullah [19] created the k-$\mathbb{RL}$ fractional integral operator, which is the generalized version of the $\mathbb{RL}$-fractional integral operator. Romero *et al.* [20] constructed the k-$\mathbb{RL}$ fractional derivative, which is the generalized version of the $\mathbb{RL}$ fractional derivative. Moreover, Kucche and Mali [21] introduced the (k,ψ)-Hilfer fractional derivative operator ((k,ψ)-Hilfer-FDO) and some crucial outcomes for obtaining the corresponding to the nonlinear differential equation under the (k,ψ)-Hilfer-FDO. Aljaaidi *et al.* [22] introduced the generalized (k,ψ)-proportional fractional operator. For more relevant results, we suggest the reader to recent application works [23–38] and the references cited therein.

The literature review through the research mentioned above [9, 21, 22, 39] inspired us to fulfill the gap of the study in this area because this has not yet been taken into account by the integrated concepts between the ψ-Hilfer-PFDO and (k,ψ)-Hilfer-FDO. Here, we propose a new concept of the proportional fractional derivative operator, the so-called (k,ψ)-Hilfer proportional fractional derivative operator ((k,ψ)-Hilfer-PFDO), a more generalized version covering a broader range of fractional operators. The (k,ψ)-Hilfer-PFDO has the advantage of allowing you to change the parameters k, ρ, β and the function ψ, which covers the conventional fractional differentiation operators. This allows us to unify and acquire the attributes of the fractional operators discussed previously. Some properties are proved. Using the ψ-Laplace transform of the proportional derivatives in $\mathbb{RL}$ and Caputo types, we can give the following property that links the $\mathbb{RL}$ and Caputo general proportional fractional derivatives. We explain how fractional integral operators affect differential operators and vice versa. Next, we present the relation between the fractional proportional derivatives in sense of $\mathbb{RL}$ and Caputo types applying the ψ-Laplace transform. Furthermore, we study the existence and uniqueness of the solutions for the higher-order initial value problems under (k,ψ)-Hilfer-PFDO using Picard's iterative technique as follows:

$$\begin{cases} {}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi}u(t) = \lambda(k\rho)^{\frac{\alpha}{k}}u(t) + f(t,u(t)), \quad t\in[a,T], \quad n-1<\alpha\le n, \\ \lim\limits_{t\to a^+} {}_{a,k}\mathcal{I}^{(1-\beta)(nk-\alpha)-ik,\rho;\psi}u(t) = \theta_i, \quad \theta_i\in\mathbb{R}, \quad i=0,1,\dots,n-1, \quad \lambda<0, \end{cases} \tag{1}$$

where ${}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi}$ is the (k,ψ)-Hilfer-PFDO of order α and type β, $0\le\beta\le1$, $0<\rho\le1$, $k>0$, ${}_{k}\mathfrak{D}^{i,\rho;\psi}$ is the (k,ψ)-proportional derivative of order i, $i=0,1,\dots,n-1$, $n\in\mathbb{N}$, ${}_{a,k}\mathcal{I}^{nk-\eta,\rho;\psi}$ is the (k,ψ)-proportional fractional integral of order $nk-\eta>0$, where $\eta=\alpha+\beta(nk-\alpha)$, and $f \in \mathcal{C}([a,T] \times \mathbb{R}, \mathbb{R})$. The Hilfer-type fractional model offers an effective mathematical framework for describing and comprehending complicated systems and processes in natural problems. It was widely used to apply numerous theories and cover a wide range of scientific and technical areas such as anomalous diffusion, viscoelasticity, dynamical systems, control theory, signal processing, biomedical engineering, geophysics, environmental science, economics, renewable energy, and so on. Especially, no prior discussion of the existence and uniqueness of solutions to the proposed problem (1) has used this new fractional formulation.

The remaining section of this work is distributed as follows. Sect. 2 provides some essential definitions, lemmas, and theorems that are applied throughout this work. In Sect. 3, we develop new generalized concepts of (k,ψ)-Hilfer-PFDOs and their properties. In Sect. 4, we define the ψ-Laplace transform for the proposed operators and provide examples. In addition, the existence and uniqueness result of the proposed problem is studied. In Sect. 5, two given examples are shown to illustrate the applicability of our results, while the conclusion of the paper is presented in Sect. 6.

2 Preliminaries

In this section, we give some definitions and lemmas for the ψ-Hilfer proportional fractional operator and the (k,ψ)-Hilfer fractional operator and their properties.

Assume that $\mathcal{J} := [a,T]$, $0 \leq a < T < \infty$ is a finite interval, and $\psi : \mathcal{J} \to \mathbb{R}$ is a strictly increasing continuous function with $\psi'(t) \neq 0$. The space of a continuous function x on $\mathcal{J}$ is defined by $\mathcal{C}(\mathcal{J},\mathbb{R})$ equipped with the norm $\|x\| := \sup_{t\in\mathcal{J}} \{|x(t)|\}$. The space $\mathcal{AC}^n(\mathcal{J},\mathbb{R})$ of n-times absolutely continuous differentiable functions x on $\mathcal{J}$ is defined by

$$\mathcal{AC}^n(\mathcal{J},\mathbb{R}) := \left\{x : \mathcal{J} \to \mathbb{R} | x^{(n-1)} \in \mathcal{AC}(\mathcal{J},\mathbb{R})\right\},$$

and the weighted spaces $\mathbb{X} := \mathcal{C}_{\sigma,\psi}(\mathcal{J},\mathbb{R})$ and $\mathbb{X}^n := \mathcal{C}^n_{\alpha,\psi}(\mathcal{J},\mathbb{R})$ of x on $\mathcal{J}$ are provided by

$$\begin{aligned}
&\mathcal{C}_{\alpha,\psi}(\mathcal{J},\mathbb{R}) := \left\{x : (a,b] \to \mathbb{R} | (\psi(t)-\psi(a))^{\alpha} x(t) \in \mathcal{C}(\mathcal{J},\mathbb{R})\right\}, \\
&\mathcal{C}^n_{\alpha,\psi}(\mathcal{J},\mathbb{R}) := \left\{x : \mathcal{J} \to \mathbb{R} | x(t) \in \mathcal{C}^{(n-1)}(\mathcal{J},\mathbb{R}), x^{(n)}(t) \in \mathcal{C}_{\alpha,\psi}(\mathcal{J},\mathbb{R})\right\},
\end{aligned}$$

equipped with the norms $\|x\|_{\mathbb{X}} := \sup_{t\in\mathcal{J}} |(\psi(t)-\psi(a))^{\alpha} x(t)|$ and $\|x\|_{\mathbb{X}^n} := \sum_{i=0}^{n-1} \|x^{(i)}\| + \|x^{(n)}\|_{\mathbb{X}}$. Note that $\mathcal{C}^0_{\alpha,\psi}(\mathcal{J},\mathbb{R}) = \mathcal{C}_{\alpha,\psi}(\mathcal{J},\mathbb{R})$.

2.1 ψ-Hilfer proportional fractional operators (ψ-Hilfer-PFOs) with properties

For convenience and ease of computation in this work, we define a symbol as follows:

$${}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-1}(t,s) := e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(s))}\left(\psi(t)-\psi(s)\right)^{\frac{\alpha}{k}-1}, \quad a \leq s < t \leq T. \tag{2}$$

Definition 1 (ψ-proportional derivative operators (ψ-PDOs) [40, 41]) Assume that $\sigma_i \in \mathcal{C}([0,1] \times \mathbb{R}, [0,\infty))$, $i = 1,2$, $\rho \in [0,1]$, for $t \in \mathbb{R}$ such that

$$\lim_{\rho\to 0^+} \sigma_1(\rho,t) = 1, \quad \lim_{\rho\to 0^+} \sigma_0(\rho,t) = 0, \quad \lim_{\rho\to 1^-} \sigma_1(\rho,t) = 0, \quad \lim_{\rho\to 1^-} \sigma_0(\rho,t) = 1,$$

and $\sigma_1(\rho,t)\neq 0$, $\rho\in[0,1)$, and $\sigma_0(\rho,t)\neq 0$, $\rho\in(0,1]$. Then the ψ-PDO of order ρ is given by

$$\mathfrak{D}^{\rho;\psi}f(t)=\sigma_1(\rho,t)f(t)+\sigma_0(\rho,t)\frac{f'(t)}{\psi'(t)}. \tag{3}$$

Especially, by taking $\sigma_0(\rho,t)=\rho$ and $\sigma_1(\rho,t)=1-\rho$, the operator (3) can be rewritten (called the left-sided ψ-PDO of order ρ) as

$$\mathfrak{D}^{\rho;\psi}f(t)=(1-\rho)f(t)+\rho\frac{f'(t)}{\psi'(t)}, \tag{4}$$

which corresponds to the left-sided ψ-proportional integral operator (ψ-PIO) [40],

$$_a\mathcal{I}^{1,\rho;\psi}f(t)=\frac{1}{\rho}\int_a^t {}_1^\rho\Psi_\psi^0(t,s)\psi'(s)f(s)ds,\quad\text{where}\quad {}_a\mathcal{I}^{0,\rho;\psi}f(t)=f(t). \tag{5}$$

By applying mathematical induction and changing the order of the integral for (5), we obtain the following left-sided ψ-PIO of order n equivalent to the left-sided ψ-PDO, $\mathfrak{D}^{n,\rho;\psi}f(t)$, defined by [40]

$$\begin{aligned}{}_a\mathcal{I}^{n,\rho;\psi}f(t)&=\frac{1}{\rho}\int_a^t {}_1^\rho\Psi_\psi^0(t,s_1)\psi'(s_1)ds_1\frac{1}{\rho}\int_a^{s_1} {}_1^\rho\Psi_\psi^0(s_1,s_2)\psi'(s_2)ds_2\cdots\\&\qquad\times\frac{1}{\rho}\int_a^{s_{n-1}} {}_1^\rho\Psi_\psi^0(s_{n-1},s_n)\psi'(s_n)f(s_n)ds_n\\&=\frac{1}{\rho^n\Gamma(n)}\int_a^t {}_1^\rho\Psi_\psi^{n-1}(t,s)\psi'(s)f(s)ds,\end{aligned} \tag{6}$$

where $\mathfrak{D}^{n,\rho;\psi}=\underbrace{\mathfrak{D}^{\rho;\psi}\mathfrak{D}^{\rho;\psi}\cdots\mathfrak{D}^{\rho;\psi}}_{n-\text{times}}$ and $\Gamma(n)=\int_0^\infty s^{n-1}e^{-s}ds$, $n\in\mathbb{N}$. In addition, the right-sided ψ-PDO of order ρ is defined by

$$\mathfrak{D}_\ominus^{\rho;\psi}f(t)=(1-\rho)f(t)-\rho\frac{f'(t)}{\psi'(t)},\qquad \mathfrak{D}_\ominus^{n,\rho;\psi}=\underbrace{\mathfrak{D}_\ominus^{\rho;\psi}\mathfrak{D}_\ominus^{\rho;\psi}\cdots\mathfrak{D}_\ominus^{\rho;\psi}}_{n-\text{times}}. \tag{7}$$

Now, by applying (6), the definitions of the left-sided and right-sided ψ-proportional fractional integral and derivative operators (ψ-PFIO/ψ-PFDO) are introduced as follows.

Definition 2 (ψ-$\mathbb{RL}$-proportional fractional integral operator (ψ-$\mathbb{RL}$-PFIO) [40, 41]) Assume $f\in L^1(\mathcal{J},\mathbb{R})$, $\alpha\in\mathbb{C}$, $Re(\alpha)>0$, $\rho\in(0,1]$, $\psi(t)\in\mathcal{C}^n(\mathcal{J},\mathbb{R})$ with $\psi'(t)\neq 0$, and $n\in\mathbb{N}$ such that $n=\lfloor\alpha\rfloor+1$. Then the left-sided and right-sided ψ-PFIO of order α of f are given by, respectively,

$$\begin{aligned}{}_a\mathcal{I}^{\alpha,\rho;\psi}f(t)&=\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^t {}_1^\rho\Psi_\psi^{\alpha-1}(t,s)\psi'(s)f(s)ds,\\ \mathcal{I}_b^{\alpha,\rho;\psi}f(t)&=\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_t^b {}_1^\rho\Psi_\psi^{\alpha-1}(s,t)\psi'(s)f(s)ds.\end{aligned}$$

Lemma 1 ([40–42]) *Assume* $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, $Re(\alpha) > 0$, $Re(\beta) > 0$, *and* $\rho \in (0,1]$. *Then, if* f *is a continuous function, we obtain*

$$ {}_a\mathcal{I}^{\,\alpha,\rho;\psi}\left({}_a\mathcal{I}^{\,\beta,\rho;\psi}f(t)\right) = {}_a\mathcal{I}^{\,\beta,\rho;\psi}\left({}_a\mathcal{I}^{\,\alpha,\rho;\psi}f(t)\right) = {}_a\mathcal{I}^{\,\alpha+\beta,\rho;\psi}f(t). $$

Definition 3 (ψ-$\mathbb{RL}$ proportional fractional derivative operator (ψ-$\mathbb{RL}$-PFDO) [40, 41]) Assume $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, $\rho \in (0,1]$, $f \in \mathcal{C}(\mathcal{J},\mathbb{R})$, $\psi(t) \in \mathcal{C}^n(\mathcal{J},\mathbb{R})$ with $\psi'(t) \neq 0$, and $n \in \mathbb{N}$ such that $n = \lfloor\alpha\rfloor + 1$. Then the left-sided and right-sided ψ-$\mathbb{RL}$-PFDO of order α of f are defined by, respectively,

$$ {}_a^{RL}\mathfrak{D}^{\,\alpha,\rho;\psi}f(t) = \mathfrak{D}^{\,n,\rho;\psi}\left({}_a\mathcal{I}^{\,n-\alpha,\rho;\psi}f(t)\right) = \frac{\mathfrak{D}^{n,\rho;\psi}}{\rho^{n-\alpha}\Gamma(n-\alpha)}\int_a^t {}_1^\rho\Psi_\psi^{n-\alpha-1}(t,s)\psi'(t)f(s)ds, $$

$$ {}^{RL}\mathfrak{D}_b^{\alpha,\rho;\psi}f(t) = \mathfrak{D}_\ominus^{n,\rho;\psi}\left(\mathcal{I}_b^{n-\alpha,\rho;\psi}f(t)\right) = \frac{\mathfrak{D}_\ominus^{n,\rho;\psi}}{\rho^{n-\alpha}\Gamma(n-\alpha)}\int_t^b {}_1^\rho\Psi_\psi^{n-\alpha-1}(s,t)\psi'(s)f(s)ds. $$

Definition 4 (ψ-Caputo proportional fractional derivative operator (ψ-Caputo-PFDO) [40, 41]) Assume $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, $\rho \in (0,1]$, $f \in \mathcal{C}^n(\mathcal{J},\mathbb{R})$, $\psi(t) \in \mathcal{C}^n(\mathcal{J},\mathbb{R})$ with $\psi'(t) \neq 0$, and $n \in \mathbb{N}$ so that $n = \lfloor\alpha\rfloor + 1$. Then the left-sided and right-sided ψ-Caputo-PFDO of order α of f are defined by, respectively,

$$ \begin{aligned} {}_a^C\mathfrak{D}^{\alpha,\rho;\psi}f(t) &= {}_a\mathcal{I}^{n-\alpha,\rho;\psi}\left(\mathfrak{D}^{n,\rho;\psi}f(t)\right) \\ &= \frac{1}{\rho^{n-\alpha}\Gamma(n-\alpha)}\int_a^t {}_1^\rho\Psi_\psi^{n-\alpha-1}(t,s)\psi'(s)\left(\mathfrak{D}^{n,\rho;\psi}f(s)\right)ds, \\ {}^C\mathfrak{D}_b^{\alpha,\rho;\psi}f(t) &= \mathcal{I}_b^{n-\alpha,\rho;\psi}\left(\mathfrak{D}_\ominus^{n,\rho;\psi}f(t)\right) \\ &= \frac{1}{\rho^{n-\alpha}\Gamma(n-\alpha)}\int_t^b {}_1^\rho\Psi_\psi^{n-\alpha-1}(s,t)\psi'(s)\left(\mathfrak{D}_\ominus^{n,\rho;\psi}f(s)\right)ds. \end{aligned} $$

Definition 5 (ψ-Hilfer proportional fractional derivative operator (ψ-Hilfer-PFDO) [9]) Assume $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, $\rho \in (0,1]$, $\beta \in [0,1]$, $f \in \mathcal{C}^n(\mathcal{J},\mathbb{R})$, $\psi(t) \in \mathcal{C}^n(\mathcal{J},\mathbb{R})$ with $\psi'(t) \neq 0$ so that $n = \lfloor\alpha\rfloor + 1$, $n = \{1,2,\ldots\}$. Then the left-sided and right-sided ψ-Hilfer-PFDO of order α and types β of f are defined by, respectively,

$$ {}_a^H\mathfrak{D}^{\,\alpha,\beta,\rho;\psi}f(t) = {}_a\mathcal{I}^{\,\beta(n-\alpha),\rho;\psi}\left(\mathfrak{D}^{n,\rho;\psi}\left({}_a\mathcal{I}^{\,(1-\beta)(n-\alpha),\rho;\psi}f(t)\right)\right), \tag{8} $$

$$ {}^H\mathfrak{D}_b^{\alpha,\beta,\rho;\psi}f(t) = \mathcal{I}_b^{\beta(n-\alpha),\rho;\psi}\left(\mathfrak{D}_\ominus^{n,\rho;\psi}\left(\mathcal{I}_b^{(1-\beta)(n-\alpha),\rho;\psi}f(t)\right)\right). \tag{9} $$

Lemma 2 ([9, 40–42]) *If* $\alpha, \mu \in \mathbb{C}$ *with* $Re(\alpha) \geq 0$ *and* $Re(\mu) > 0$, *then for any* $\rho \in (0,1]$, *we obtain the following relations*:

(*i*) ${}_a\mathcal{I}^{\,\alpha,\rho;\psi}\left[{}_1^\rho\Psi_\psi^{\mu-1}(t,a)\right] = \frac{\Gamma(\mu)}{\rho^\alpha\Gamma(\mu+\alpha)}{}_1^\rho\Psi_\psi^{\mu+\alpha-1}(t,a), \mathcal{I}_b^{\alpha,\rho;\psi}\left[{}_1^\rho\Psi_\psi^{\mu-1}(b,t)\right] = \frac{\Gamma(\mu)}{\rho^\alpha\Gamma(\mu+\alpha)} \times {}_1^\rho\Psi_\psi^{\mu+\alpha-1}(b,t)$;

(*ii*) ${}_a^{RL}\mathfrak{D}^{\,\alpha,\rho;\psi}\left[{}_1^\rho\Psi_\psi^{\mu-1}(t,a)\right] = \frac{\Gamma(\mu)}{\rho^\alpha\Gamma(\mu-\alpha)}{}_1^\rho\Psi_\psi^{\mu-\alpha-1}(t,a), {}^{RL}\mathfrak{D}_b^{\alpha,\rho;\psi}\left[{}_1^\rho\Psi_\psi^{\mu-1}(b,t)\right] = \frac{\Gamma(\mu)}{\rho^\alpha\Gamma(\mu-\alpha)} \times {}_1^\rho\Psi_\psi^{\mu-\alpha-1}(b,t)$;

(*iii*) ${}_a^H\mathfrak{D}^{\,\alpha,\beta,\rho;\psi}\left[{}_1^\rho\Psi_\psi^{\mu-1}(t,a)\right] = \frac{\Gamma(\mu)}{\rho^\alpha\Gamma(\mu-\alpha)}{}_1^\rho\Psi_\psi^{\mu-\alpha-1}(t,a), {}^H\mathfrak{D}_b^{\alpha,\beta,\rho;\psi}\left[{}_1^\rho\Psi_\psi^{\mu-1}(b,t)\right] = \frac{\Gamma(\mu)}{\rho^\alpha\Gamma(\mu-\alpha)} \times {}_1^\rho\Psi_\psi^{\mu-\alpha-1}(b,t)$.

2.2 (k,ψ)-Hilfer fractional operators with properties

Definition 6 ((k,ψ)-$\mathbb{RL}$ fractional integral operator ((k,ψ)-$\mathbb{RL}$-FIO) [42]) Assume $f \in L^1(\mathcal{J},\mathbb{R})$, $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, and $k > 0$. Then the left-sided and right-sided (k,ψ)-$\mathbb{RL}$-FIO of order α of f are defined by, respectively,

$$ {}_{a,k}\mathcal{I}^{\alpha;\psi} f(t) = \frac{1}{k\Gamma_k(\alpha)} \int_a^t {}_k^1\Psi_\psi^{\frac{\alpha}{k}-1}(t,s)\psi'(s)f(s)ds, \tag{10} $$

$$ \mathcal{I}_{b,k}^{\alpha,\rho;\psi} f(t) = \frac{1}{k\Gamma_k(\alpha)} \int_t^b {}_k^1\Psi_\psi^{\frac{\alpha}{k}-1}(s,t)\psi'(s)f(s)ds, \tag{11} $$

where

$$ \Gamma_k(z) = \int_0^\infty s^{z-1} e^{-\frac{s^k}{k}} ds, \quad z \in \mathbb{C}, \quad Re(z) > 0. \tag{12} $$

In addition, some well-known important properties of (12) are as follows:

$$ \Gamma_k(z+k) = z\Gamma_k(z), \quad \Gamma_k(k) = 1, \quad \Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right), \quad \Gamma(z) = \lim_{k\to 1}\Gamma_k(z). \tag{13} $$

Furthermore, $\mathbb{B}_k(\cdot,\cdot)$ is the k-beta function [17], which is given by

$$ \mathbb{B}_k(z,w) = \frac{1}{k}\int_0^1 s^{\frac{z}{k}-1}(1-s)^{\frac{w}{k}-1} ds, \quad z,w \in \mathbb{C}, \quad Re(z), Re(w) > 0. \tag{14} $$

The following are some important relations between $\Gamma_k(z)$ and $\mathbb{B}_k(z,w)$:

$$ \mathbb{B}_k(z,w) = \frac{\Gamma_k(z)\Gamma_k(w)}{\Gamma_k(z+w)} \quad \text{and} \quad \mathbb{B}_k(z,w) = \frac{1}{k}\mathbb{B}\left(\frac{z}{k},\frac{w}{k}\right). \tag{15} $$

Definition 7 ((k,ψ)-$\mathbb{RL}$ fractional derivative operator ((k,ψ)-$\mathbb{RL}$-FDO) [21]) Assume $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, $k > 0$, $f \in \mathcal{C}(\mathcal{J},\mathbb{R})$, $\psi(t) \in \mathcal{C}^n(\mathcal{J},\mathbb{R})$ with $\psi'(t) \neq 0$, and $n \in \mathbb{N}$ such that $n = \lfloor \alpha/k \rfloor + 1$. Then the left-sided and right-sided (k,ψ)-$\mathbb{RL}$-FDO of f of order α are defined by, respectively,

$$ {}_{a,k}^{RL}\mathfrak{D}^{\alpha;\psi} f(t) = {}_k\mathfrak{D}^{n;\psi}\left({}_{a,k}\mathcal{I}^{nk-\alpha;\psi}f(t)\right) = \frac{{}_k\mathfrak{D}^{n;\psi}}{k\Gamma_k(nk-\alpha)} \int_a^t {}_k^1\Psi_\psi^{\frac{nk-\alpha}{k}-1}(t,s)\psi'(s)f(s)ds, $$

$$ {}^{RL}\mathfrak{D}_{b,k}^{\alpha;\psi} f(t) = {}_k\mathfrak{D}_\ominus^{n;\psi}\left(\mathcal{I}_{b,k}^{nk-\alpha;\psi}f(t)\right) = \frac{{}_k\mathfrak{D}_\ominus^{n;\psi}}{k\Gamma_k(nk-\alpha)} \int_t^b {}_k^1\Psi_\psi^{\frac{nk-\alpha}{k}-1}(s,t)\psi'(s)f(s)ds, $$

where ${}_k\mathfrak{D}^{n;\psi} = \underbrace{{}_k\mathfrak{D}^\psi {}_k\mathfrak{D}^\psi \cdots {}_k\mathfrak{D}^\psi}_{n \text{ time}} = \left(\frac{k}{\psi(t)}\frac{d}{dt}\right)^n$ and ${}_k\mathfrak{D}_\ominus^{n;\psi} = \underbrace{{}_k\mathfrak{D}_\ominus^\psi {}_k\mathfrak{D}_\ominus^\psi \cdots {}_k\mathfrak{D}_\ominus^\psi}_{n \text{ time}} = \left(-\frac{k}{\psi(t)}\frac{d}{dt}\right)^n$, and the left-sided and right-sided ψ-derivative operator of order α of f are given by, respectively,

$$ {}_k\mathfrak{D}^\psi f(t) = k\frac{f'(t)}{\psi'(t)} \quad \text{and} \quad {}_k\mathfrak{D}_\ominus^\psi f(t) = -k\frac{f'(t)}{\psi'(t)}. \tag{16} $$

Definition 8 ((k,ψ)-Caputo fractional derivative operator ((k,ψ)-Caputo-FDO) [21]) Assume $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, $k > 0$, $f \in \mathcal{C}^n(\mathcal{J},\mathbb{R})$, $\psi(t) \in \mathcal{C}^n(\mathcal{J},\mathbb{R})$ with $\psi'(t) \neq 0$, $n \in \mathbb{N}$ such that $n = \lfloor \alpha/k \rfloor + 1$. Then the left-sided and right-sided (k,ψ)-Caputo-FDO of order α of f are defined by, respectively,

$$ {}^{C}_{a,k}\mathfrak{D}^{\alpha;\psi} f(t) = {}_{a,k}\mathcal{I}^{nk-\alpha;\psi}\left({}_{k}\mathfrak{D}^{n;\psi} f(t)\right) = \frac{1}{k\Gamma_k(nk-\alpha)} \int_a^t {}^{1}_{k}\Psi_{\psi}^{\frac{nk-\alpha}{k}-1}(t,s)\psi'(s)\left({}_{k}\mathfrak{D}^{n;\psi} f(s)\right)ds, $$

$$ {}^{C}\mathfrak{D}^{\alpha;\psi}_{b,k} f(t) = \mathcal{I}^{nk-\alpha;\psi}_{b,k}\left({}_{k}\mathfrak{D}^{n;\psi}_{\ominus} f(t)\right) = \frac{1}{k\Gamma_k(nk-\alpha)} \int_t^b {}^{1}_{k}\Psi_{\psi}^{\frac{nk-\alpha}{k}-1}(s,t)\psi'(s)\left({}_{k}\mathfrak{D}^{n;\psi}_{\ominus} f(s)\right)ds. $$

Definition 9 ((k,ψ)-Hilfer fractional derivative operator ((k,ψ)-Hilfer-FDO) [21]) Assume $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, $k > 0$, $\beta \in [0,1]$, $f \in \mathcal{C}^n(\mathcal{J},\mathbb{R})$, $\psi(t) \in \mathcal{C}^n(\mathcal{J},\mathbb{R})$ with $\psi'(t) \neq 0$ so that $n = \lfloor \alpha/k \rfloor + 1$, $n \in \mathbb{N}$. Then the left-sided and right-sided (k,ψ)-Hilfer-FDO of order α and types β of f are defined by, respectively,

$$ {}^{H}_{a,k}\mathfrak{D}^{\,\alpha,\beta;\psi} f(t) = {}_{a,k}\mathcal{I}^{\,\beta(nk-\alpha);\psi}\left({}_{k}\mathfrak{D}^{n;\psi}\left({}_{a,k}\mathcal{I}^{\,(1-\beta)(nk-\alpha);\psi} f(t)\right)\right), \tag{17} $$

$$ {}^{H}\mathfrak{D}^{\alpha,\beta;\psi}_{b,k} f(t) = \mathcal{I}^{\beta(nk-\alpha);\psi}_{b,k}\left({}_{k}\mathfrak{D}^{n;\psi}_{\ominus}\left(\mathcal{I}^{(1-\beta)(nk-\alpha);\psi}_{b,k} f(t)\right)\right). \tag{18} $$

Lemma 3 ([21, 42]) *If $k > 0$, $\beta \in [0,1]$, $\alpha, \mu \in \mathbb{C}$ so that $Re(\alpha) \geq 0$ and $Re(\mu) > 0$, then we obtain the following relations*:

(*i*) ${}_{a,k}\mathcal{I}^{\,\alpha;\psi}\left[{}^{1}_{k}\Psi_{\psi}^{\frac{\mu}{k}-1}(t,a)\right] = \frac{\Gamma_k(\mu)}{\Gamma_k(\mu+\alpha)}{}^{1}_{k}\Psi_{\psi}^{\frac{\mu+\alpha}{k}-1}(t,a)$, $\mathcal{I}^{\alpha;\psi}_{b,k}\left[{}^{1}_{k}\Psi_{\psi}^{\frac{\mu}{k}-1}(b,t)\right] = \frac{\Gamma_k(\mu)}{\Gamma_k(\mu+\alpha)} \times {}^{1}_{k}\Psi_{\psi}^{\frac{\mu+\alpha}{k}-1}(b,t)$;

(*ii*) ${}^{RL}_{a,k}\mathfrak{D}^{\,\alpha;\psi}\left[{}^{1}_{k}\Psi_{\psi}^{\frac{\mu}{k}-1}(t,a)\right] = \frac{\Gamma_k(\mu)}{\Gamma_k(\mu-\alpha)}{}^{1}_{k}\Psi_{\psi}^{\frac{\mu-\alpha}{k}-1}(t,a)$, ${}^{RL}\mathfrak{D}^{\alpha;\psi}_{b,k}\left[{}^{1}_{k}\Psi_{\psi}^{\frac{\mu}{k}-1}(b,t)\right] = \frac{\Gamma_k(\mu)}{\Gamma_k(\mu-\alpha)} \times {}^{1}_{k}\Psi_{\psi}^{\frac{\mu-\alpha}{k}-1}(b,t)$;

(*iii*) ${}^{H}_{a,k}\mathfrak{D}^{\,\alpha,\beta;\psi}\left[{}^{1}_{k}\Psi_{\psi}^{\frac{\mu}{k}-1}(t,a)\right] = \frac{\Gamma_k(\mu)}{\Gamma_k(\mu-\alpha)}{}^{1}_{k}\Psi_{\psi}^{\frac{\mu-\alpha}{k}-1}(t,a)$, ${}^{H}\mathfrak{D}^{\alpha,\beta;\psi}_{b,k}\left[{}^{1}_{k}\Psi_{\psi}^{\frac{\mu}{k}-1}(b,t)\right] = \frac{\Gamma_k(\mu)}{\Gamma_k(\mu-\alpha)} \times {}^{1}_{k}\Psi_{\psi}^{\frac{\mu-\alpha}{k}-1}(b,t)$.

3 Main results

In this section, we define the generalized (k,ψ)-proportional fractional integral and derivative operators and their properties.

3.1 (k,ψ)-$\mathbb{RL}$ proportional fractional operators and properties

Under all assumptions in Definition 1, we give the definition of (k,ψ)-proportional derivative operators. By taking $\sigma_0(\rho,t) = k\rho$ and $\sigma_1(\rho,t) = 1-\rho$ into (3), the left-sided (k,ψ)-proportional derivative operator ((k,ψ)-PDO) can be defined as

$$ {}_{k}\mathfrak{D}^{\rho;\psi} f(t) = (1-\rho)f(t) + k\rho\frac{f'(t)}{\psi'(t)}, \tag{19} $$

which corresponds to the left-sided (k,ψ)-proportional integral operator ((k,ψ)-PIO),

$$ {}_{a,k}\,\mathcal{I}^{\,1,\rho;\psi} f(t) = \frac{1}{\rho k}\int_a^t {}^{\rho}_{k}\Psi^{0}_{\psi}(t,s)\psi'(s)f(s)ds, \quad \text{where} \quad {}_{a,k}\,\mathcal{I}^{\,0,\rho;\psi} f(t) = f(t). \tag{20} $$

By applying mathematical induction, the left-sided (k,ψ)-$\mathbb{RL}$-PIO of order n of f is defined by

$$ {}_{a,k}\,\mathcal{I}^{\,n,\rho;\psi} f(t) = \frac{1}{\rho k}\int_a^t {}^{\rho}_{k}\Psi^{0}_{\psi}(t,s_1)\psi'(s_1)ds_1 \frac{1}{\rho k}\int_a^{s_1} {}^{\rho}_{k}\Psi^{0}_{\psi}(s_1,s_2)\psi'(s_2)ds_2 \cdots $$

$$\times \frac{1}{\rho k}\int_a^{s_{n-1}} {}_1^{\rho}\Psi_{\psi}^{0}(s_{n-1},s_n)\psi'(s_n)f(s_n)ds_n$$

$$= \frac{1}{\rho^n k\Gamma_k(nk)}\int_a^t {}_k^{\rho}\Psi_{\psi}^{n-1}(t,s)\psi'(s)f(s)ds, \quad n\in\mathbb{N}, \quad k>0, \tag{21}$$

where ${}_k\mathfrak{D}^{n,\rho;\psi} = \underbrace{{}_k\mathfrak{D}^{\rho;\psi}\,{}_k\mathfrak{D}^{\rho;\psi}\cdots{}_k\mathfrak{D}^{\rho;\psi}}_{n-\text{times}}$.

On the other hand, the right-sided (k,ψ)-PDO can be defined as

$${}_k\mathfrak{D}_{\ominus}^{\rho;\psi} f(t) = (1-\rho)f(t) - k\rho\frac{f'(t)}{\psi'(t)}, \tag{22}$$

which corresponds to the right-sided (k,ψ)-PIO

$$\mathcal{I}_{b,k}^{1,\rho;\psi} f(t) = \frac{1}{\rho k}\int_t^b {}_k^{\rho}\Psi_{\psi}^{0}(s,t)\psi'(s)f(s)ds, \quad \text{where} \quad \mathcal{I}_{b,k}^{0,\rho;\psi} f(t) = f(t). \tag{23}$$

By applying mathematical induction, the right-sided (k,ψ)-$\mathbb{RL}$-PIO of order n of f is defined by

$$\mathcal{I}_{b,k}^{n,\rho;\psi} f(t) = \frac{1}{\rho k}\int_t^b {}_k^{\rho}\Psi_{\psi}^{0}(s_1,t)\psi'(s_1)ds_1\frac{1}{\rho k}\int_{s_1}^b {}_k^{\rho}\Psi_{\psi}^{0}(s_2,s_1)\psi'(s_2)ds_2\cdots$$

$$\times\frac{1}{\rho k}\int_{s_{n-1}}^b {}_1^{\rho}\Psi_{\psi}^{0}(s_n,s_{n-1})\psi'(s_n)f(s_n)ds_n$$

$$= \frac{1}{\rho^n k\Gamma_k(nk)}\int_t^b {}_k^{\rho}\Psi_{\psi}^{n-1}(s,t)\psi'(s)f(s)ds, \quad n\in\mathbb{N}, \quad k>0, \tag{24}$$

where ${}_k\mathfrak{D}_{\ominus}^{n,\rho;\psi} = \underbrace{{}_k\mathfrak{D}_{\ominus}^{\rho;\psi}\,{}_k\mathfrak{D}_{\ominus}^{\rho;\psi}\cdots{}_k\mathfrak{D}_{\ominus}^{\rho;\psi}}_{n-\text{times}}$.

Lemma 4 *Let f be an integrable on $\mathcal{J}$, $k>0$, $\rho\in(0,1]$, $n\in\mathbb{N}$. Then we have*

$${}_k\mathfrak{D}^{n,\rho;\psi}\left({}_{a,k}\mathcal{I}^{n,\rho;\psi}f(t)\right) = f(t) \qquad \text{and} \qquad {}_k\mathfrak{D}_{\ominus}^{n,\rho;\psi}\left(\mathcal{I}_{b,k}^{n,\rho;\psi}f(t)\right) = f(t).$$

Proof By using (21) and (19), we have

$$\begin{aligned}
{}_k\mathfrak{D}^{n,\rho;\psi}\left({}_{a,k}\mathcal{I}^{n,\rho;\psi}f(t)\right) &= {}_k\mathfrak{D}^{n,\rho;\psi}\left[\frac{1}{k\rho^n\Gamma_k(nk)}\int_a^t {}_k^{\rho}\Psi_{\psi}^{n-1}(t,s)\psi'(s)f(s)ds\right] \\
&= {}_k\mathfrak{D}^{n-1,\rho;\psi}\left[{}_k\mathfrak{D}^{\rho;\psi}\left(\frac{1}{k\rho^n\Gamma_k(nk)}\int_a^t {}_k^{\rho}\Psi_{\psi}^{n-1}(t,s)\psi'(s)f(s)ds\right)\right] \\
&= \frac{1}{k\rho^n\Gamma_k(nk)}{}_k\mathfrak{D}^{n-1,\rho;\psi}\left[(1-\rho)\int_a^t {}_k^{\rho}\Psi_{\psi}^{n-1}(t,s)\psi'(s)f(s)ds\right. \\
&\qquad \left.+\frac{k\rho}{\psi'(t)}\cdot\frac{d}{dt}\left(\int_a^t {}_k^{\rho}\Psi_{\psi}^{n-1}(t,s)\psi'(s)f(s)ds\right)\right] \\
&= \frac{1}{k\rho^{n-1}\Gamma_k(nk-k)}\,{}_k\mathfrak{D}^{n-1,\rho;\psi}\left[\int_a^t {}_k^{\rho}\Psi_{\psi}^{n-2}(t,s)\psi'(s)f(s)ds\right].
\end{aligned}$$

Repeating the process, it follows that

$$
\begin{aligned}
&{}_{k}\mathfrak{D}^{n,\rho;\psi}\left({}_{a,k}\mathcal{I}^{n,\rho;\psi}f(t)\right)\\
&= \frac{1}{k\rho^{n-(n-1)}\Gamma_k(nk-(n-1)k)}\,{}_{k}\mathfrak{D}^{n-(n-1),\rho;\psi}\left[\int_a^t {}_{k}^{\rho}\Psi_{\psi}^{n-n}(t,s)\psi'(s)f(s)ds\right]\\
&= \frac{1}{\rho k}\left[{}_{k}\mathfrak{D}^{\rho;\psi}\left(\int_a^t e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(s))}\psi'(s)f(s)ds\right)\right]\\
&= \frac{1}{\rho k}\left[(1-\rho)\int_a^t e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(s))}\psi'(s)f(s)ds + \frac{k\rho}{\psi'(t)}\frac{d}{dt}\left(\int_a^t e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(s))}\psi'(s)f(s)ds\right)\right]\\
&= f(t).
\end{aligned}
$$

The proof is done. □

Now, by applying (21), the definitions of the left-sided and right-sided (k,ψ)-proportional fractional integral and derivative operators ((k,ψ)-PFIO/(k,ψ)-PFDO) are introduced as follows.

Definition 10 ((k,ψ)-$\mathbb{RL}$ proportional fractional integral operator ((k,ψ)-$\mathbb{RL}$-PFIO)) Let $f \in L^1(\mathcal{J},\mathbb{R})$, $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, $k > 0$, $\rho \in (0,1]$. Then the left-sided and right-sided (k,ψ)-$\mathbb{RL}$-PFIO of order α of f are defined by, respectively,

$$
{}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}f(t) = \frac{1}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_a^t {}_{k}^{\rho}\Psi_{\psi}^{\frac{\alpha}{k}-1}(t,s)\psi'(s)f(s)ds, \tag{25}
$$

$$
\mathcal{I}_{b,k}^{\alpha,\rho;\psi}f(t) = \frac{1}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_t^b {}_{k}^{\rho}\Psi_{\psi}^{\frac{\alpha}{k}-1}(s,t)\psi'(s)f(s)ds, \tag{26}
$$

where $\Gamma_k(\alpha)$ is given by (12).

Definition 11 ((k,ψ)-$\mathbb{RL}$ proportional fractional derivative operator ((k,ψ)-$\mathbb{RL}$-PFDO)) Let $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, $k > 0$, $\rho \in (0,1]$, $f \in \mathcal{C}(\mathcal{J},\mathbb{R})$, $\psi(t) \in \mathcal{C}^n(\mathcal{J},\mathbb{R})$ with $\psi'(t) \neq 0$, and $n \in \mathbb{N}$ such that $n = \lfloor Re(\alpha)/k \rfloor + 1$. Then the left-sided and right-sided (k,ψ)-$\mathbb{RL}$-PFDO of f of order α are defined by, respectively,

$$
\begin{aligned}
{}_{a,k}^{RL}\mathfrak{D}^{\alpha,\rho;\psi}f(t) &= {}_{k}\mathfrak{D}^{\,n,\rho;\psi}\left({}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\psi}f(t)\right)\\
&= \frac{{}_{k}\mathfrak{D}^{n,\rho;\psi}}{\rho^{\frac{nk-\alpha}{k}}k\Gamma_k(nk-\alpha)}\int_a^t {}_{k}^{\rho}\Psi_{\psi}^{\frac{nk-\alpha}{k}-1}(t,s)\psi'(\tau)f(s)ds,\\
{}^{RL}\mathfrak{D}_{b,k}^{\alpha,\rho;\psi}f(t) &= {}_{k}\mathfrak{D}_{\ominus}^{n,\rho;\psi}\left(\mathcal{I}_{b,k}^{nk-\alpha,\rho;\psi}f(t)\right)\\
&= \frac{{}_{k}\mathfrak{D}_{\ominus}^{n,\rho;\psi}}{\rho^{\frac{nk-\alpha}{k}}k\Gamma_k(nk-\alpha)}\int_t^b {}_{k}^{\rho}\Psi_{\psi}^{\frac{nk-\alpha}{k}-1}(s,t)\psi'(s)f(s)ds.
\end{aligned}
$$

Next, the essential properties of (k,ψ)-$\mathbb{RL}$-PFIO and (k,ψ)-$\mathbb{RL}$-PFDO will be investigated.

Lemma 5 *Let* $\alpha, \omega \in \mathbb{C}$, $Re(\alpha) \geq 0$, $Re(\omega) > 0$, $k > 0$, $\rho \in (0,1]$ *such that* $Re(\omega)/k > -1$. *Then we obtain the following essential relations*:

(i) ${}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(t,a)\right] = \frac{\Gamma_k(\omega)}{\rho^{\frac{\alpha}{k}}\Gamma_k(\omega+\alpha)}{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega+\alpha}{k}-1}(t,a)$, $Re(\alpha) > 0$;

(ii) $\mathcal{I}^{\alpha,\rho;\psi}_{b,k}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(b,t)\right] = \frac{\Gamma_k(\omega)}{\rho^{\frac{\alpha}{k}}\Gamma_k(\omega+\alpha)}{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega+\alpha}{k}-1}(b,t)$, $Re(\alpha) > 0$;

(iii) ${}_{k}\mathfrak{D}^{n,\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(t,a)\right] = \frac{\rho^n\Gamma_k(\omega)}{\Gamma_k(\omega-nk)}{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega-nk}{k}-1}(t,a)$, $Re(\alpha) > 0$;

(iv) ${}_{k}\mathfrak{D}^{n,\rho;\psi}_{\ominus}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(b,t)\right] = \frac{\rho^n\Gamma_k(\omega)}{\Gamma_k(\omega-nk)}{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega-nk}{k}-1}(b,t)$, $Re(\alpha) > 0$;

(v) ${}^{RL}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(t,a)\right] = \frac{\rho^{\frac{\alpha}{k}}\Gamma_k(\omega)}{\Gamma_k(\omega-\alpha)}{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega-\alpha}{k}-1}(t,a)$, $Re(\alpha) \geq 0$;

(vi) ${}^{RL}\mathfrak{D}^{\alpha,\rho;\psi}_{b,k}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(b,t)\right] = \frac{\rho^{\frac{\alpha}{k}}\Gamma_k(\omega)}{\Gamma_k(\omega-\alpha)}{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega-\alpha}{k}-1}(b,t)$, $Re(\alpha) \geq 0$.

In particular, for $m = 0, 1, \ldots, n-1$, *we have*

$${}_{k}\mathfrak{D}^{n,\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{m}(t,a)\right] = 0 \qquad \textit{and} \qquad {}_{k}\mathfrak{D}^{n,\rho;\psi}_{\ominus}\left[{}^{\rho}_{k}\Psi_{\psi}^{m}(b,t)\right] = 0.$$

Proof The proofs for properties (i) and (ii) are fairly similar. We shall prove (i), whereas the proof of (ii) is equivalent. Property (i) will be directly proven. By applying (2) and Definition 10, we have

$$\begin{aligned}
&{}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(t,a)\right] \\
&= \frac{1}{k\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha)}\int_a^t e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(s))}\big(\psi(t)-\psi(s)\big)^{\frac{\alpha}{k}-1}e^{\frac{\rho-1}{k\rho}(\psi(s)-\psi(a))}\big(\psi(s)-\psi(a)\big)^{\frac{\omega}{k}-1}\psi'(s)ds \\
&= \frac{e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(a))}}{k\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha)}\int_a^t \big(\psi(t)-\psi(s)\big)^{\frac{\alpha}{k}-1}\big(\psi(s)-\psi(a)\big)^{\frac{\omega}{k}-1}\psi'(s)ds.
\end{aligned} \tag{27}$$

Changing the variable $(\psi(t)-\psi(a))z = \psi(s)-\psi(a)$ with (12)–(15), equation (27) can be rewritten as

$$\begin{aligned}
{}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(t,a)\right] &= \frac{e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(a))}}{k\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha)}\int_a^t (\psi(t)-\psi(a))^{\frac{\alpha}{k}-1} \\
&\qquad \times\left(1-\frac{\psi(s)-\psi(a)}{\psi(t)-\psi(a)}\right)^{\frac{\alpha}{k}-1}\big(\psi(s)-\psi(a)\big)^{\frac{\omega}{k}-1}\psi'(s)ds. \\
&= \frac{{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha+\omega}{k}-1}(t,a)}{\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha)}\left(\frac{1}{k}\int_0^1 (1-z)^{\frac{\alpha}{k}-1}z^{\frac{\omega}{k}-1}dz\right) \\
&= \frac{\Gamma_k(\omega)}{\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha+\omega)}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha+\omega}{k}-1}(t,a).
\end{aligned}$$

The proof of property (ii) is similarly processed. Next, we will prove property (iii), while (iv) is analogous. By applying (2) and Definition 11, it follows that

$$\begin{aligned}
{}_{k}\mathfrak{D}^{n,\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(t,a)\right] &= {}_{k}\mathfrak{D}^{n-1,\rho;\psi}\left({}_{k}\mathfrak{D}^{\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(t,a)\right]\right) \\
&= {}_{k}\mathfrak{D}^{n-1,\rho;\psi}\bigg[(1-\rho)\left(e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\frac{\omega}{k}-1}\right) \\
&\qquad +\frac{k\rho}{\psi'(t)}\cdot\frac{d}{dt}\left(e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\frac{\omega}{k}-1}\right)\bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \rho(\omega-k)_k\mathfrak{D}^{n-1,\rho;\psi}\left[e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\frac{\omega-k}{k}-1}\right]\\
&= \rho(\omega-k)\left({}_k\mathfrak{D}^{n-1,\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega-k}{k}-1}(t,a)\right]\right).
\end{aligned}
$$

Repeating the above process, it follows that

$$
\begin{aligned}
{}_k\mathfrak{D}^{n,\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(t,a)\right] &= \rho^n(\omega-k)(\omega-2k)\cdots(\omega-(n-2)k)\ {}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega-nk}{k}-1}(t,a)\\
&= \frac{\rho^n\Gamma_k(\omega)}{\Gamma_k(\omega-nk)}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega-nk}{k}-1}(t,a).
\end{aligned}
$$

Finally, we will prove property (v). By using Definition 11, we have

$$
{}^{RL}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(t,a)\right] = {}_k\mathfrak{D}^{n,\rho;\psi}\left({}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(t,a)\right]\right). \tag{28}
$$

By using properties (i) and (iii), equation (28) can be written as

$$
\begin{aligned}
{}^{RL}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(t,a)\right] &= \frac{\Gamma_k(\omega)}{\rho^{\frac{nk-\alpha}{k}}\Gamma_k(\omega+nk-\alpha)}\left({}_k\mathfrak{D}^{n,\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega+nk-\alpha}{k}-1}(t,a)\right]\right)\\
&= \frac{\rho^{\frac{\alpha}{k}}\Gamma_k(\omega)}{\Gamma_k(\omega-\alpha)}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega-\alpha}{k}-1}(t,a).
\end{aligned}
$$

The proof of relation (vi) is the same. □

Lemma 6 *Let* $\alpha\in\mathbb{C}$, $Re(\alpha)>0$, $k>0$, $\rho\in(0,1]$, *and* $\nu=\frac{\rho k}{1+\rho(k-1)}$. *Then we have*

(i) ${}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}f(t) = \left(\frac{\nu}{k\rho}\right)^{\frac{\alpha}{k}}{}_a\mathcal{I}^{\frac{\alpha}{k},\nu;\psi}f(t)$ and $\mathcal{I}^{\alpha,\rho;\psi}_{b,k}f(t) = \left(\frac{\nu}{k\rho}\right)^{\frac{\alpha}{k}}\mathcal{I}^{\frac{\alpha}{k},\nu;\psi}_{b}f(t)$;

(ii) ${}_k\mathfrak{D}^{n,\rho;\psi}f(t) = [1+\rho(k-1)]^n\left(\mathfrak{D}^{n,\nu;\psi}f(t)\right)$ and ${}_k\mathfrak{D}^{n,\rho;\psi}_{\ominus}f(t) = [1+\rho(k-1)]^n\left(\mathfrak{D}^{n,\nu;\psi}_{\ominus}f(t)\right)$;

(iii) ${}^{RL}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}f(t) = \left(\frac{\nu}{\rho k}\right)^{-\frac{\alpha}{k}}\left({}^{RL}_{a}\mathfrak{D}^{\frac{\alpha}{k},\nu;\psi}f(t)\right)$ and ${}^{RL}\mathfrak{D}^{\alpha,\rho;\psi}_{b,k}f(t) = \left(\frac{\nu}{\rho k}\right)^{-\frac{\alpha}{k}}\left({}^{RL}\mathfrak{D}^{\frac{\alpha}{k},\nu;\psi}_{b}f(t)\right)$.

Proof We will prove relation (i) by directly proving only the left-sided case, while the right-sided case is similar. Using (13) and Definition 10 yields

$$
\begin{aligned}
{}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}f(t) &= \frac{1}{k\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha)}\int_a^t {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-1}(t,s)\psi'(s)f(s)ds\\
&= \frac{1}{k^{\frac{\alpha}{k}}\rho^{\frac{\alpha}{k}}\Gamma\left(\frac{\alpha}{k}\right)}\int_a^t {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-1}(t,s)\psi'(s)f(s)ds\\
&= \left(\frac{\nu}{k\rho}\right)^{\frac{\alpha}{k}}\left({}_a\mathcal{I}^{\frac{\alpha}{k},\nu;\psi}f(t)\right).
\end{aligned}
$$

Next, we will show relation (ii). By (16) with $\nu=\frac{\rho k}{1+\rho(k-1)}$, one has

$$
\mathfrak{D}^{\nu;\psi}f(t) = (1-\nu)f(t)+\nu\frac{f'(t)}{\psi'(t)} = \frac{1}{1+\rho\,(k-1)}{}_k\mathfrak{D}^{\rho;\psi}f(t),
$$

which implies that

$$
{}_k\mathfrak{D}^{\rho;\psi}f(t) = [1+\rho(k-1)]\ \mathfrak{D}^{\nu;\psi}f(t).
$$

Repeating the process, it follows that

$$
\begin{aligned}
{}_{k}\mathfrak{D}^{n,\rho;\psi} f(t) &= \Big(\underbrace{[1+\rho\,(k-1)]\,\mathfrak{D}^{\nu;\psi}\cdots[1+\rho\,(k-1)]\,\mathfrak{D}^{\nu;\psi}}_{n-\text{times}} f \Big)(t) \\
&= [1+\rho\,(k-1)]^{n}\,\mathfrak{D}^{n,\nu;\psi} f(t).
\end{aligned}
$$

The proof of the right-sided case of property (*ii*) is similar. Finally, we will show relation (*iii*). From properties (13), relation (*ii*), and Definition 11, it becomes

$$
\begin{aligned}
{}^{RL}_{a,k}\mathfrak{D}^{\,\alpha,\rho;\psi} f(t) &= \frac{{}_{k}\mathfrak{D}^{n,\rho;\psi}}{k\rho^{\frac{nk-\alpha}{k}}\Gamma_k(nk-\alpha)} \int_a^t {}^{\rho}_{k}\Psi_{\psi}^{\frac{nk-\alpha}{k}-1}(t,s)\psi'(s)f(s)ds \\
&= \frac{{}_{k}\mathfrak{D}^{n,\rho;\psi}}{k\rho^{\frac{nk-\alpha}{k}}k^{\frac{nk-\alpha}{k}-1}\Gamma\left(\frac{nk-\alpha}{k}\right)} \int_a^t {}^{\rho}_{k}\Psi_{\psi}^{n-\frac{\alpha}{k}-1}(t,s)\psi'(s)f(s)ds \\
&= \frac{[1+\rho(k-1)]^{n}\,\mathfrak{D}^{n,\nu;\psi}}{\rho^{\frac{nk-\alpha}{k}}k^{\frac{nk-\alpha}{k}}\Gamma\left(n-\frac{\alpha}{k}\right)} \int_a^t e^{\frac{\nu-1}{\nu}(\psi(t)-\psi(s))}(\psi(t)-\psi(s))^{n-\frac{\alpha}{k}-1}\psi'(s)f(s)ds \\
&= \left(\frac{\nu}{\rho k}\right)^{n-\frac{\alpha}{k}} [1+\rho(k-1)]^{n}\,\mathfrak{D}^{n,\nu;\psi}\left({}_{a}\mathcal{I}^{\,n-\frac{\alpha}{k},\nu;\psi} f(t)\right) \\
&= \left(\frac{\nu}{\rho k}\right)^{-\frac{\alpha}{k}} {}^{RL}_{a}\mathfrak{D}^{\frac{\alpha}{k},\nu;\psi} f(t).
\end{aligned}
$$

The proof of the right-sided case of relation (*iii*) is the same process. □

The semi-group properties of the (k,ψ)-PFIO are shown below.

Lemma 7 *Let $\alpha_i \in \mathbb{C}$, $Re(\alpha_i) > 0$, $i = 1,2$, $k > 0$, and $\rho \in (0,1]$. Then we have*

(i) ${}_{a,k}\mathcal{I}^{\,\alpha_2,\rho;\psi}\left({}_{a,k}\mathcal{I}^{\,\alpha_1,\rho;\psi} f(t)\right) = {}_{a,k}\mathcal{I}^{\,\alpha_1+\alpha_2,\rho;\psi} f(t) = {}_{a,k}\mathcal{I}^{\,\alpha_1,\rho;\psi}\left({}_{a,k}\mathcal{I}^{\,\alpha_2,\rho;\psi} f(t)\right)$.

(ii) $\mathcal{I}_{b,k}^{\alpha_2,\rho;\psi}\left(\mathcal{I}_{b,k}^{\alpha_1,\rho;\psi} f(t)\right) = \mathcal{I}_{b,k}^{\alpha_1+\alpha_2,\rho;\psi} f(t) = \mathcal{I}_{b,k}^{\alpha_1,\rho;\psi}\left(\mathcal{I}_{b,k}^{\alpha_2,\rho;\psi} f(t)\right)$.

Proof By directly presenting, we will show only relation (*i*), but (*ii*) is a similar process. From Definition 10 and (13), we get

$$
\begin{aligned}
&{}_{a,k}\mathcal{I}^{\,\alpha_2,\rho;\psi}\left({}_{a,k}\mathcal{I}^{\,\alpha_1,\rho;\psi} f(t)\right) \\
&= \frac{1}{k\rho^{\frac{\alpha_2}{k}}\Gamma_k(\alpha_2)} \int_a^t e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(\tau))}(\psi(t)-\psi(\tau))^{\frac{\alpha_2}{k}-1}\psi'(\tau) \\
&\quad \times \left(\frac{1}{k\rho^{\frac{\alpha_1}{k}}\Gamma_k(\alpha_1)} \int_a^\tau e^{\frac{\rho-1}{k\rho}(\psi(\tau)-\psi(s))}(\psi(\tau)-\psi(s))^{\frac{\alpha_1}{k}-1}\psi'(s)f(s)ds \right) d\tau \\
&= \frac{1}{k^2\rho^{\frac{\alpha_1+\alpha_2}{k}}\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)} \int_a^t \int_a^\tau e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(s))}\,(\psi(t)-\psi(\tau))^{\frac{\alpha_2}{k}-1} \\
&\quad \times (\psi(\tau)-\psi(s))^{\frac{\alpha_1}{k}-1}\,\psi'(\tau)\psi'(s)f(s)ds d\tau \\
&= \frac{1}{k^2\rho^{\frac{\alpha_1+\alpha_2}{k}}\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)} \int_a^t e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(s))}\psi'(s) \\
&\quad \times \left[\int_s^t (\psi(t)-\psi(\tau))^{\frac{\alpha_2}{k}-1}(\psi(\tau)-\psi(s))^{\frac{\alpha_1}{k}-1}\psi'(\tau)d\tau \right] f(s)ds.
\end{aligned}
$$

Setting $(\psi(t)-\psi(s))z=\psi(\tau)-\psi(s)$, it follows that

$$
\begin{aligned}
&{}_{a,k}\mathcal{I}^{\alpha_2,\rho;\psi}\left({}_{a,k}\mathcal{I}^{\alpha_1,\rho;\psi}f(t)\right)\\
&=\frac{1}{k^2\rho^{\frac{\alpha_1+\alpha_2}{k}}\Gamma_k(\alpha_1)\Gamma_k(\alpha_2)}\int_a^t e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(s))}(\psi(t)-\psi(s))^{\frac{\alpha_2}{k}+\frac{\alpha_1}{k}-1}\psi'(s)\\
&\times\left[\int_0^1 z^{\frac{\alpha_1}{k}-1}(1-z)^{\frac{\alpha_2}{k}-1}dz\right]f(s)ds\\
&=\frac{1}{k\rho^{\frac{\alpha_1+\alpha_2}{k}}\Gamma_k(\alpha_1+\alpha_2)}\int_{a^+}^t {}_k^\rho\Psi_\psi^{\frac{\alpha_1+\alpha_2}{k}-1}(t,s)\psi'(s)f(s)ds\\
&={}_{a,k}\mathcal{I}^{\alpha_1+\alpha_2,\rho;\psi}f(t).
\end{aligned}
$$

The proof is done. □

Lemma 8 *Assume that f is an integrable on $\mathcal{J}$, $k>0$, $\eta\in\mathbb{C}$, $Re(\eta)>0$, $\rho\in(0,1]$, and $0\le m<[Re(\eta)]+1$. Then we obtain the following relations*:

$$
{}_k\mathfrak{D}^{m,\rho;\psi}\left({}_{a,k}\mathcal{I}^{\eta,\rho;\psi}f(t)\right)={}_{a,k}\mathcal{I}^{\eta-mk,\rho;\psi}f(t),\tag{29}
$$

$$
{}_k\mathfrak{D}_\ominus^{m,\rho;\psi}\left(\mathcal{I}_{b,k}^{\eta,\rho;\psi}f(t)\right)=\mathcal{I}_{b,k}^{\eta-mk,\rho;\psi}f(t).\tag{30}
$$

Proof We will prove relation (29) by directly showing, while (30) is similar. By using Definition 10, (19), and (16), we get

$$
\begin{aligned}
{}_k\mathfrak{D}^{m,\rho;\psi}\left({}_{a,k}\mathcal{I}^{\eta,\rho;\psi}f(t)\right)&={}_k\mathfrak{D}^{m,\rho;\psi}\left[\frac{1}{k\rho^{\frac{\eta}{k}}\Gamma_k(\eta)}\int_a^t {}_k^\rho\Psi_\psi^{\frac{\eta}{k}-1}(t,s)\psi'(s)f(s)ds\right]\\
&={}_k\mathfrak{D}^{m-1,\rho;\psi}\left[{}_k\mathfrak{D}^{\rho;\psi}\left(\frac{1}{k\rho^{\frac{\eta}{k}}\Gamma_k(\eta)}\int_a^t {}_k^\rho\Psi_\psi^{\frac{\eta}{k}-1}(t,s)\psi'(s)f(s)ds\right)\right]\\
&=\frac{1}{k\rho^{\frac{\eta}{k}}\Gamma_k(\eta)}{}_k\mathfrak{D}^{m-1,\rho;\psi}\left[(1-\rho)\int_a^t {}_k^\rho\Psi_\psi^{\frac{\eta}{k}-1}(t,s)\psi'(s)f(s)ds\right.\\
&\quad\left.+\frac{k\rho}{\psi'(t)}\cdot\frac{d}{dt}\left(\int_a^t {}_k^\rho\Psi_\psi^{\frac{\eta}{k}-1}(t,s)\psi'(s)f(s)ds\right)\right]\\
&=\frac{1}{k\rho^{\frac{\alpha}{k}-1}\Gamma_k(\eta-k)}{}_k\mathfrak{D}^{m-1,\rho;\psi}\left[\int_a^t {}_k^\rho\Psi_\psi^{\frac{\eta-k}{k}-1}(t,s)\psi'(s)f(s)ds\right]\\
&={}_k\mathfrak{D}^{m-1,\rho;\psi}\left({}_{a,k}\mathcal{I}^{\eta-k,\rho;\psi}f(t)\right).
\end{aligned}
$$

Repeating m-times the same process, it follows that

$$
{}_k\mathfrak{D}^{m,\rho;\psi}\left({}_{a,k}\mathcal{I}^{\eta,\rho;\psi}f(t)\right)={}_{a,k}\mathcal{I}^{\eta-mk,\rho;\psi}f(t).
$$

The proof is done. □

Lemma 9 *Let $\omega, \eta \in \mathbb{C}$, $Re(\eta) > 0$, $Re(\omega) > 0$, $k > 0$, $\rho \in (0,1]$, $m \in \mathbb{N}$, $Re(\eta) > Re(\omega)$, and $n = \lfloor Re(\omega)/k \rfloor + 1$. Then we obtain the following relations*:

$$ {}^{RL}_{a,k}\mathfrak{D}^{\omega,\rho;\psi}\left({}_{a,k}\mathcal{I}^{\eta,\rho;\psi}f(t)\right) = {}_{a,k}\mathcal{I}^{\eta-\omega,\rho;\psi}f(t), \tag{31} $$

$$ {}^{RL}\mathfrak{D}^{\omega,\rho;\psi}_{b,k}\left(\mathcal{I}^{\eta,\rho;\psi}_{b,k}f(t)\right) = \mathcal{I}^{\eta-\omega,\rho;\psi}_{b,k}f(t). \tag{32} $$

Proof We will prove relation (31) by directly showing, while (32) is similar. From Definition 11, Lemma 7, and relation (29), we have

$$ \begin{aligned} {}^{RL}_{a,k}\mathfrak{D}^{\omega,\rho;\psi}\left({}_{a,k}\mathcal{I}^{\eta,\rho;\psi}f(t)\right) &= {}_{k}\mathfrak{D}^{n,\rho;\psi}\left({}_{a,k}\mathcal{I}^{nk-\omega,\rho;\psi}\left({}_{a,k}\mathcal{I}^{\eta,\rho;\psi}f(t)\right)\right) \\ &= {}_{k}\mathfrak{D}^{n,\rho;\psi}\left({}_{a,k}\mathcal{I}^{nk-\omega+\eta,\rho;\psi}f(t)\right) \\ &= {}_{a,k}\mathcal{I}^{\eta-\omega,\rho;\psi}f(t). \end{aligned} $$

The proof of property (32) is similar. The proof is done. □

Lemma 10 *Let $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, $k > 0$, and $\rho \in (0,1]$, then*

(i) $$ {}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}\left({}^{RL}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}f(t)\right) = f(t) - \sum_{i=1}^{n}\frac{{}^{\rho}_{k}\Psi^{\frac{\alpha}{k}-i}_{\psi}(t,a)}{\rho^{\frac{\alpha-ki}{k}}\Gamma_k(\alpha+k-ki)}\left[{}_{k}\mathfrak{D}^{n-i,\rho;\psi}\left({}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\psi}f(a^{+})\right)\right]; $$

(ii) $$ \mathcal{I}^{\alpha,\rho;\psi}_{b,k}\left({}^{RL}\mathfrak{D}^{\alpha,\rho;\psi}_{b,k}f(t)\right) = f(t) - \sum_{i=1}^{n}\frac{{}^{\rho}_{k}\Psi^{\frac{\alpha}{k}-i}_{\psi}(b,t)}{\rho^{\frac{\alpha-ki}{k}}\Gamma_k(\alpha+k-ki)}\left[{}_{k}\mathfrak{D}^{n-i,\rho;\psi}_{\ominus}\left(\mathcal{I}^{nk-\alpha,\rho;\psi}_{b,k}f(b^{-})\right)\right]. $$

Proof From property (29) in Lemma 8 with $m = 1$, we obtain

$$ {}_{k}\mathfrak{D}^{\rho;\psi}\left({}_{a,k}\mathcal{I}^{k,\rho;\psi}h(t)\right) = h(t). \tag{33} $$

By using (33) with $h(t)$ replaced by $h(t) = {}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}\left({}^{RL}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}f(t)\right)$, we have

$$ \begin{aligned} &{}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}\left({}^{RL}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}f(t)\right) \\ &= {}_{k}\mathfrak{D}^{\rho;\psi}\left({}_{a,k}\mathcal{I}^{k,\rho;\psi}\left({}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}\left({}^{RL}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}f(t)\right)\right)\right) \\ &= {}_{k}\mathfrak{D}^{\rho;\psi}\left({}_{a,k}\mathcal{I}^{k+\alpha,\rho;\psi}\left({}^{RL}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}f(t)\right)\right) \\ &= {}_{k}\mathfrak{D}^{\rho;\psi}\left(\frac{1}{k\rho^{\frac{k+\alpha}{k}}\Gamma_k(k+\alpha)}\int_a^t {}^{\rho}_{k}\Psi^{\frac{\alpha}{k}}_{\psi}(t,s)\psi'(s)\,{}^{RL}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}f(s)ds\right). \end{aligned} \tag{34} $$

By direct computation with integration by parts technique, we get

$$ \begin{aligned} &\frac{1}{k\rho^{\frac{k+\alpha}{k}}\Gamma_k(k+\alpha)}\int_a^t {}^{\rho}_{k}\Psi^{\frac{\alpha}{k}}_{\psi}(t,s)\psi'(s)\left({}^{RL}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}f(s)\right)ds \\ &= \frac{1}{k\rho^{\frac{k+\alpha}{k}}\Gamma_k(k+\alpha)}\int_a^t {}^{\rho}_{k}\Psi^{\frac{\alpha}{k}}_{\psi}(t,s)\psi'(s)\left[{}_{k}\mathfrak{D}^{n,\rho;\psi}\left({}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\psi}f(s)\right)\right]ds \\ &= \frac{1}{k\rho^{\frac{k+\alpha}{k}}\Gamma_k(k+\alpha)}\int_a^t {}^{\rho}_{k}\Psi^{\frac{\alpha}{k}}_{\psi}(t,s)\psi'(s){}_{k}\mathfrak{D}^{\rho;\psi}\left[{}_{k}\mathfrak{D}^{n-1,\rho;\psi}\left({}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\psi}f(s)\right)\right]ds \end{aligned} $$

$$
\begin{aligned}
&= \frac{1}{k\rho^{\frac{k+\alpha}{k}}\Gamma_k(k+\alpha)}\left\{\int_a^t {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}}(t,s)\psi'(s)(1-\rho)\Big[{}_k\mathfrak{D}^{n-1,\rho;\psi}\Big({}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\psi}f(s)\Big)\Big]ds\right.\\
&\left.\quad + k\rho\int_a^t {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}}(t,s)\frac{d}{ds}\Big[{}_k\mathfrak{D}^{n-1,\rho;\psi}\Big({}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\psi}f(s)\Big)\Big]ds\right\}\\
&= \frac{1}{k\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha)}\int_a^t {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-1}(t,s)\psi'(s)\Big[{}_k\mathfrak{D}^{n-1,\rho;\psi}\Big({}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\psi}f(s)\Big)\Big]ds\\
&\quad - \frac{{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}}(t,a)}{\rho^{\frac{\alpha}{k}}\Gamma_k(k+\alpha)}\Big[{}_k\mathfrak{D}^{n-1,\rho;\psi}\big({}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\psi}f(a^+)\big)\Big].
\end{aligned}
$$

Repeating the procedure of integration by parts at nth-step, we obtain

$$
\begin{aligned}
&\frac{1}{k\rho^{\frac{k+\alpha}{k}}\Gamma_k(k+\alpha)}\int_a^t {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}}(t,s)\psi'(s)\Big({}^{RL}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}f(s)\Big)ds\\
&= \frac{1}{\rho^{\frac{\alpha}{k}-n+1}k\Gamma_k(\alpha-nk+k)}\int_a^t {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-n}(t,s)\psi'(s)\Big({}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\psi}f(s)\Big)ds\\
&\quad - \sum_{i=1}^{n}\frac{{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-i+1}(t,a)}{\rho^{\frac{\alpha+k-ki}{k}}\Gamma_k(\alpha+2k-ki)}\Big[{}_k\mathfrak{D}^{n-i,\rho;\psi}\Big({}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\psi}f(a^+)\Big)\Big]\\
&= \frac{1}{k\rho}\int_a^t e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(s))}\psi'(s)f(s)ds\\
&\quad - \sum_{i=1}^{n}\frac{{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-i+1}(t,a)}{\rho^{\frac{\alpha+k-ki}{k}}\Gamma_k(\alpha+2k-ki)}\big[{}_k\mathfrak{D}^{n-i,\rho;\psi}\big({}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\psi}f(a^+)\big)\big].
\end{aligned}
\tag{35}
$$

Substituting (35) into (34), we have

$$
\begin{aligned}
&{}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}\Big({}^{RL}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}f(t)\Big)\\
&= {}_k\mathfrak{D}^{\rho;\psi}\left[\frac{1}{k\rho}\int_a^t e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(s))}\psi'(s)f(s)ds\right.\\
&\left.\quad - \sum_{i=1}^{n}\frac{{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-i+1}(t,a)}{\rho^{\frac{\alpha+k-ki}{k}}\Gamma_k(\alpha+2k-ki)}\Big[{}_k\mathfrak{D}^{n-i,\rho;\psi}\Big({}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\psi}f(a^+)\Big)\Big]\right].
\end{aligned}
\tag{36}
$$

By applying property (*iii*) in Lemma 5 and Lemma 8, relation (36) can be rewritten as follows:

$$
\begin{aligned}
{}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}\Big({}^{RL}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}f(t)\Big) &= {}_k\mathfrak{D}^{\rho;\psi}\left[\frac{1}{k\rho}\int_a^t e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(s))}\psi'(s)f(s)ds\right]\\
&\quad - \sum_{i=1}^{n}\frac{{}_k\mathfrak{D}^{n-i,\rho;\psi}\Big({}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\psi}f(a^+)\Big)}{\rho^{\frac{\alpha+k-ki}{k}}\Gamma_k(\alpha+2k-ki)}{}_k\mathfrak{D}^{\rho;\psi}\Big[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-i+1}(t,a)\Big]\\
&= f(t) - \sum_{i=1}^{n}\frac{{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-i}(t,a)}{\rho^{\frac{\alpha-ki}{k}}\Gamma_k(\alpha+k-ki)}\Big[{}_k\mathfrak{D}^{n-i,\rho;\psi}\Big({}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\psi}f(a^+)\Big)\Big].
\end{aligned}
$$

The proof is done. The process of the proof in relation (*ii*) is the same. □

3.2 (k,ψ)-Caputo proportional fractional derivative operator and properties

Definition 12 ((k,ψ)-Caputo proportional fractional derivative operator ((k,ψ)-Caputo-PFDO)) Let $\alpha\in\mathbb{C}$, $Re(\alpha)>0$, $k>0$, $\rho\in[0,1]$, $f\in\mathcal{C}^n(\mathcal{J},\mathbb{R})$, $\psi(t)\in\mathcal{C}^n(\mathcal{J},\mathbb{R})$ with $\psi'(t)\neq 0$, and $n\in\mathbb{N}$ such that $n=\lfloor Re(\alpha)/k\rfloor+1$. Then the left-sided and right-sided (k,ψ)-Caputo-PFDO of order α of f are defined by, respectively,

$$\begin{aligned}
{}^{C}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}f(t) &= {}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\psi}\Big({}_{k}\mathfrak{D}^{n,\rho;\psi}f(t)\Big)\\
&= \frac{1}{\rho^{\frac{nk-\alpha}{k}}k\Gamma_k(nk-\alpha)}\int_a^t {}^{\rho}_{k}\Psi_{\psi}^{\frac{nk-\alpha}{k}-1}(t,s)\psi'(s)\Big({}_{k}\mathfrak{D}^{n,\rho;\psi}f(s)\Big)ds,\\
{}^{C}\mathfrak{D}^{\alpha,\rho;\psi}_{b,k}f(t) &= \mathcal{I}^{nk-\alpha,\rho;\psi}_{b,k}\Big({}_{k}\mathfrak{D}^{n,\rho;\psi}_{\ominus}f(t)\Big)\\
&= \frac{1}{\rho^{\frac{nk-\alpha}{k}}k\Gamma_k(nk-\alpha)}\int_t^b {}^{\rho}_{k}\Psi_{\psi}^{\frac{nk-\alpha}{k}-1}(s,t)\psi'(s)\Big({}_{k}\mathfrak{D}^{n,\rho;\psi}_{\ominus}f(s)\Big)ds.
\end{aligned}$$

Lemma 11 *Let α, $\omega\in\mathbb{C}$, $Re(\alpha)\geq 0$, $Re(\omega)>0$, $k>0$, $\rho\in(0,1]$ such that $Re(\omega)/k>-1$. Then*

(i) ${}^{C}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}\Big[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(t,a)\Big]=\frac{\rho^{\frac{\alpha}{k}}\Gamma_k(\omega)}{\Gamma_k(\omega-\alpha)}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega-\alpha}{k}-1}(t,a)$;

(ii) ${}^{C}\mathfrak{D}^{\alpha,\rho;\psi}_{b,k}\Big[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(b,t)\Big]=\frac{\rho^{\frac{\alpha}{k}}\Gamma_k(\omega)}{\Gamma_k(\omega-\alpha)}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega-\alpha}{k}-1}(b,t)$.

In particular, for $m=0,1,\dots,n-1$, we have

$${}^{C}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}\Big[{}^{\rho}_{k}\Psi_{\psi}^{m}(t,a)\Big]=0 \qquad \textit{and} \qquad {}^{C}\mathfrak{D}^{\alpha,\rho;\psi}_{b,k}\Big[{}^{\rho}_{k}\Psi_{\psi}^{m}(b,t)\Big]=0.$$

Proof By applying Definition 12 and properties (*i*), (*iii*) in Lemma 5, we have

$$\begin{aligned}
{}^{C}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}\Big[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(t,a)\Big] &= {}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\psi}\Big({}_{k}\mathfrak{D}^{n,\rho;\psi}\Big[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(t,a)\Big]\Big)\\
&= \frac{\rho^n\Gamma_k(\omega)}{\Gamma_k(\omega-nk)}\Big({}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\psi}\Big[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega-nk}{k}-1}(t,a)\Big]\Big)\\
&= \frac{\rho^{\frac{\alpha}{k}}\Gamma_k(\omega)}{\Gamma_k(\omega-\alpha)}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega-\alpha}{k}-1}(t,a).
\end{aligned}$$

The proof of property (*ii*) is similar. □

Lemma 12 *Let $\alpha\in\mathbb{C}$, $Re(\alpha)>0$, $k>0$, $\rho\in(0,1]$, and $\nu=\frac{\rho k}{1+\rho(k-1)}$. Then*

(i) ${}^{C}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}f(t)=\Big(\frac{\nu}{k\rho}\Big)^{-\frac{\alpha}{k}}\,{}^{C}_{a}\mathfrak{D}^{\frac{\alpha}{k},\nu;\psi}f(t)$;

(ii) ${}^{C}\mathfrak{D}^{\alpha,\rho;\psi}_{b,k}f(t)=\Big(\frac{\nu}{k\rho}\Big)^{-\frac{\alpha}{k}}\,{}^{C}\mathfrak{D}^{\frac{\alpha}{k},\nu;\psi}_{b}f(t)$.

Proof By applying property (*ii*) in Lemma 6 and Definition 12, it follows that

$$\begin{aligned}
{}^{C}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}f(t) &= {}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\psi}\Big({}_{k}\mathfrak{D}^{n,\rho;\psi}f(t)\Big)\\
&= \Big(\frac{\nu}{k\rho}\Big)^{\frac{nk-\alpha}{k}}[1+\rho(k-1)]^n\Big[{}_{a}\mathcal{I}^{n-\frac{\alpha}{k},\nu;\psi}\Big(\mathfrak{D}^{n,\nu;\psi}f(t)\Big)\Big]\\
&= \Big(\frac{\nu}{k\rho}\Big)^{-\frac{\alpha}{k}}\,{}^{C}\mathfrak{D}^{\frac{\alpha}{k},\nu;\psi}_{a^+}f(t).
\end{aligned}$$

The proof of property (*ii*) is similar. The proof is completed. □

Lemma 13 *Let $k, \omega, \eta \in \mathbb{C}$, $Re(\omega) \geq 0$, $Re(\eta) > 0$, $\rho \in (0,1]$, $n = \lfloor Re(\omega)/k \rfloor + 1$, and $Re(\eta) > nk$. Then we obtain the following relations*:

(*i*) ${}^{C}_{a,k}\mathfrak{D}^{\omega,\rho;\psi}\left({}_{a,k}\mathcal{I}^{\eta,\rho;\psi} f(t)\right) = {}_{a,k}\mathcal{I}^{\eta-\omega,\rho;\psi} f(t)$;

(*ii*) ${}^{C}\mathfrak{D}^{\omega,\rho;\psi}_{b,k}\left(\mathcal{I}^{\eta,\rho;\psi}_{b,k} f(t)\right) = \mathcal{I}^{\eta-\omega,\rho;\psi}_{b,k} f(t)$.

Proof We omitted the proof since this lemma is similar to the proof in Lemma 8. □

Lemma 14 *Let $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, $k > 0$, $Re(\alpha)/k \in (n-1,n]$, $n \in \mathbb{N}$, and $\rho \in (0,1]$. Then*

(*i*) ${}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}\left({}^{C}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi} f(t)\right) = f(t) - \sum_{i=1}^{n} \frac{{}^{\rho}_{k}\Psi^{n-i}_{\psi}(t,a)}{(\rho k)^{n-i}(n-i)!} {}_{k}\mathfrak{D}^{n-i,\rho;\psi} f(a)$;

(*ii*) $\mathcal{I}^{\alpha,\rho;\psi}_{b,k}\left({}^{C}\mathfrak{D}^{\alpha,\rho;\psi}_{b,k} f(t)\right) = f(t) - \sum_{i=1}^{n} \frac{{}^{\rho}_{k}\Psi^{n-i}_{\psi}(b,t)}{(\rho k)^{n-i}(n-i)!} {}_{k}\mathfrak{D}^{n-i,\rho;\psi}_{\ominus} f(b)$.

Proof From property (*i*) in Lemma 8 with $m = 1$, we obtain

$$
{}_{k}\mathfrak{D}^{\rho;\psi}\left({}_{a,k}\mathcal{I}^{k,\rho;\psi} h(t)\right) = h(t). \tag{37}
$$

Using equation (37) and $h(t) = {}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}\left({}^{C}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi} f(t)\right)$ implies that

$$
\begin{aligned}
{}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}\left({}^{C}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi} f(t)\right) &= {}_{k}\mathfrak{D}^{\rho;\psi}\left({}_{a,k}\mathcal{I}^{k,\rho;\psi}\left({}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}\left({}^{C}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi} f(t)\right)\right)\right) \\
&= {}_{k}\mathfrak{D}^{\rho;\psi}\left({}_{a,k}\mathcal{I}^{k+nk,\rho;\psi}\left({}_{k}\mathfrak{D}^{n,\rho;\psi} f(t)\right)\right) \\
&= {}_{k}\mathfrak{D}^{\rho;\psi}\left(\frac{1}{k\rho^{n+1}\Gamma_k(nk+k)} \int_a^t {}^{\rho}_{k}\Psi^{n}_{\psi}(t,s)\psi'(s)\left({}_{k}\mathfrak{D}^{n,\rho;\psi} f(s)\right) ds\right).
\end{aligned}
$$

By using integration by parts, we have

$$
\begin{aligned}
&\frac{1}{k\rho^{n+1}\Gamma_k(nk+k)} \int_a^t {}^{\rho}_{k}\Psi^{n}_{\psi}(t,s)\psi'(s)\left({}_{k}\mathfrak{D}^{n,\rho;\psi} f(s)\right) ds \\
&= \frac{1}{k\rho^{n+1}\Gamma_k(nk+k)} \int_a^t {}^{\rho}_{k}\Psi^{n}_{\psi}(t,s)\psi'(s)\left[{}_{k}\mathfrak{D}^{\rho;\psi}\left({}_{k}\mathcal{D}^{n-1,\rho;\psi} f(s)\right)\right] ds \\
&= \frac{1}{k\rho^{n+1}\Gamma_k(nk+k)}\left[(1-\rho)\int_a^t {}^{\rho}_{k}\Psi^{n}_{\psi}(t,s)\psi'(s)\left({}_{k}\mathfrak{D}^{n-1,\rho;\psi} f(s)\right) ds\right. \\
&\qquad \left. + k\rho \int_a^t {}^{\rho}_{k}\Psi^{n}_{\psi}(t,s)\frac{d}{ds}\left({}_{k}\mathfrak{D}^{n-1,\rho;\psi} f(s)\right) ds\right] \\
&= \frac{1}{k\rho^{n+1}\Gamma_k(nk+k)}\left[-k\rho\, {}^{\rho}_{k}\Psi^{n}_{\psi}(t,s)\left({}_{k}\mathfrak{D}^{n-1,\rho;\psi} f(a)\right)\right. \\
&\qquad \left. + nk\rho \int_a^t {}^{\rho}_{k}\Psi^{n-1}_{\psi}(t,s)\psi'(s)\left({}_{k}\mathfrak{D}^{n-1,\rho;\psi} f(s)\right) ds\right].
\end{aligned}
$$

Repeating the procedure of integration by parts at nth-step yields that

$$
\begin{aligned}
{}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}\left({}^{C}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi} f(t)\right) &= {}_{k}\mathfrak{D}^{\rho;\psi}\left[\frac{1}{\rho k}\int_a^t e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(s))}\psi'(s) f(s) ds\right. \\
&\qquad \left. - \sum_{i=1}^{n} \frac{{}^{\rho}_{k}\Psi^{n+1-i}_{\psi}(t,a)}{(\rho k)^{n+1-i}(n+1-i)!}\left({}_{k}\mathfrak{D}^{n-i,\rho;\psi} f(a)\right)\right].
\end{aligned}
$$

By using equation (16), Lemma 5, and Lemma 8, we have

$$
\begin{aligned}
&{}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}\left({}^{C}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}f(t)\right)\\
&= f(t)-\sum_{i=1}^{n}\frac{\rho\Gamma_k\left(nk+(2-i)k\right)\,{}^{\rho}_{k}\Psi_{\psi}^{n-i}(t,a)}{(\rho k)^{n+1-i}\Gamma_k\left(nk+(1-i)k\right)(n+1-i)!}\left({}_{k}\mathfrak{D}^{n-i,\rho;\psi}f(a)\right)\\
&= f(t)-\sum_{i=1}^{n}\frac{{}^{\rho}_{k}\Psi_{\psi}^{n-i}(t,a)}{(\rho k)^{n-i}(n-i)!}\left({}_{k}\mathfrak{D}^{n-i,\rho;\psi}f(a)\right).
\end{aligned}
$$

The proof of property (*ii*) is similar. The proof is completed. □

3.3 (k,ψ)-Hilfer proportional fractional derivative operators and properties

Definition 13 ((k,ψ)-Hilfer proportional fractional derivative operator ((k,ψ)-Hilfer-PFDO)) Let $\alpha\in\mathbb{C}$, $Re(\alpha)>0$, $k>0$, $\rho\in(0,1]$, $\beta\in[0,1]$, $f\in\mathcal{C}^n(\mathcal{J},\mathbb{R})$, $\psi(t)\in\mathcal{C}^n(\mathcal{J},\mathbb{R})$ with $\psi'(t)\neq 0$ so that $n=\lfloor Re(\alpha)/k\rfloor+1$, $n\in\mathbb{N}$. Then the left-sided and right-sided (k,ψ)-Hilfer-PFDO of order α and types β of f are defined by, respectively,

$$
{}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi}f(t)={}_{a,k}\mathcal{I}^{\,\beta(nk-\alpha),\rho;\psi}\left({}_{k}\mathfrak{D}^{n,\rho;\psi}\left({}_{a,k}\mathcal{I}^{\,(1-\beta)(nk-\alpha),\rho;\psi}f(t)\right)\right),\tag{38}
$$

$$
{}^{H}\mathfrak{D}^{\alpha,\beta,\rho;\psi}_{b,k}f(t)=\mathcal{I}^{\beta(nk-\alpha),\rho;\psi}_{b,k}\left({}_{k}\mathfrak{D}^{n,\rho;\psi}_{\ominus}\left(\mathcal{I}^{(1-\beta)(nk-\alpha),\rho;\psi}_{b,k}f(t)\right)\right).\tag{39}
$$

Lemma 15 *Let $\alpha,\omega\in\mathbb{C}$, $Re(\alpha)\geq 0$, $Re(\omega)>0$, $k>0$, $\rho\in(0,1]$ so that $Re(\omega)/k>-1$. Hence*

(*i*) ${}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(t,a)\right]=\frac{\rho^{\frac{\alpha}{k}}\Gamma_k(\omega)}{\Gamma_k(\omega-\alpha)}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega-\alpha}{k}-1}(t,a)$;

(*ii*) ${}^{H}\mathfrak{D}^{\alpha,\beta,\rho;\psi}_{b,k}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(b,t)\right]=\frac{\rho^{\frac{\alpha}{k}}\Gamma_k(\omega)}{\Gamma_k(\omega-\alpha)}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega-\alpha}{k}-1}(b,t)$.

In particular, for $m=0,1,\dots,n-1$, we have

$$
{}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{m}(t,a)\right]=0\quad\textit{and}\quad {}^{H}\mathfrak{D}^{\alpha,\beta,\rho;\psi}_{b,k}\left[{}^{\rho}_{k}\Psi_{\psi}^{m}(b,t)\right]=0.
$$

Proof Definition 13, property (*i*) in Lemma 5, and property (*i*) in Lemma 8 imply that

$$
\begin{aligned}
&{}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(t,a)\right]\\
&={}_{a,k}\mathcal{I}^{\beta(nk-\alpha),\rho;\psi}\left({}_{k}\mathfrak{D}^{n,\rho;\psi}\left({}_{a,k}\mathcal{I}^{(1-\beta)(nk-\alpha),\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(t,a)\right]\right)\right)\\
&=\frac{\Gamma_k(\omega)}{\rho^{\frac{(1-\beta)(nk-\alpha)}{k}}\Gamma_k(\omega+(1-\beta)(nk-\alpha))}{}_{a,k}\mathcal{I}^{\beta(nk-\alpha),\rho;\psi}\left({}_{k}\mathfrak{D}^{n,\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega+(1-\beta)(nk-\alpha)}{k}-1}(t,a)\right]\right)\\
&=\frac{\rho^{n}\Gamma_k(\omega)}{\rho^{\frac{(1-\beta)(nk-\alpha)}{k}}\Gamma_k(\omega+(1-\beta)(nk-\alpha)-nk)}\left({}_{a,k}\mathcal{I}^{\beta(nk-\alpha),\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega-\alpha-\beta(nk-\alpha)}{k}-1}(t,a)\right]\right)\\
&=\frac{\rho^{\frac{\alpha}{k}}\Gamma_k(\omega)}{\Gamma_k(\omega-\alpha)}\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega-\alpha}{k}-1}(t,a).
\end{aligned}
$$

The proof of property (*ii*) is similar. □

Lemma 16 *Let $\alpha\in\mathbb{C}$, $Re(\alpha)>0$, $k>0$, $\rho\in(0,1]$, and $\upsilon=\frac{\rho k}{1+\rho(k-1)}$. Then*

(i) ${}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi}f(t)=\left(\frac{\upsilon}{k\rho}\right)^{-\frac{\alpha}{k}}\,{}^{H}_{a}\mathfrak{D}^{\frac{\alpha}{k},\beta,\upsilon;\psi}f(t)$;

(ii) ${}^{H}\mathfrak{D}_{b,k}^{\alpha,\beta,\rho;\psi}f(t) = \left(\frac{\upsilon}{k\rho}\right)^{-\frac{\alpha}{k}} {}^{H}\mathfrak{D}_{b}^{\frac{\alpha}{k},\beta,\upsilon;\psi}f(t)$.

Proof Definition 13, properties (*i*) and (*iii*) in Lemma 6 yield that

$$\begin{aligned}
{}_{a,k}^{H}\mathfrak{D}^{\alpha,\beta,\rho;\psi}f(t) &= {}_{a,k}\mathcal{I}^{\beta(nk-\alpha),\rho;\psi}\left({}_{k}\mathfrak{D}^{n,\rho;\psi}\left({}_{a,k}\mathcal{I}^{(1-\beta)(nk-\alpha),\rho;\psi}f(t)\right)\right) \\
&= \left(\frac{\upsilon}{k\rho}\right)^{-\frac{\alpha}{k}} {}_{a}\mathcal{I}^{\frac{\beta(nk-\alpha)}{k},\upsilon;\psi}\left(\mathfrak{D}^{n,\upsilon;\psi}\left({}_{a}\mathcal{I}^{(1-\beta)\left(n-\frac{\alpha}{k}\right),\upsilon;\psi}f(t)\right)\right) \\
&= \left(\frac{\upsilon}{k\rho}\right)^{\frac{-\alpha}{k}} {}_{a}^{H}\mathfrak{D}^{\frac{\alpha}{k},\beta,\upsilon;\psi}f(t).
\end{aligned}$$

The proof of property (*ii*) is similar. □

Lemma 17 *Let* ω, $\eta \in \mathbb{C}$, $Re(\omega) \geq 0$, $Re(\eta) > 0$, $k > 0$, $\rho \in (0,1]$, $n = \lfloor Re(\omega)/k \rfloor + 1$, *and* $Re(\eta) > nk$. *Then we obtain the following relations*:

(*i*) ${}_{a,k}^{H}\mathfrak{D}^{\omega,\beta,\rho;\psi}\left({}_{a,k}\mathcal{I}^{\eta,\rho;\psi}f(t)\right) = {}_{a,k}\mathcal{I}^{\eta-\omega,\rho;\psi}f(t)$;

(*ii*) ${}^{H}\mathfrak{D}_{b,k}^{\omega,\beta,\rho;\psi}\left(\mathcal{I}_{b,k}^{\eta,\rho;\psi}f(t)\right) = \mathcal{I}_{b,k}^{\eta-\omega,\rho;\psi}f(t)$.

Proof We omitted the proof because the process is similar to that in Lemma 8. □

Lemma 18 *Let* $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, $k > 0$, $\rho \in (0,1]$, *and* $\eta = Re(\alpha) + \beta(nk - Re(\alpha))$. *Then*

(*i*) ${}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}\left({}_{a,k}^{H}\mathfrak{D}^{\alpha,\beta,\rho;\psi}f(t)\right) = {}_{a,k}\mathcal{I}^{\eta,\rho;\psi}\left({}_{a,k}^{RL}\mathfrak{D}^{\eta,\rho;\psi}f(t)\right)$;

(*ii*) $\mathcal{I}_{b,k}^{\alpha,\rho;\psi}\left({}^{H}\mathfrak{D}_{b,k}^{\alpha,\beta,\rho;\psi}f(t)\right) = \mathcal{I}_{b,k}^{\eta,\rho;\psi}\left({}^{RL}\mathfrak{D}_{b,k}^{\eta,\rho;\psi}f(t)\right)$.

Proof By applying Definition 13 and property (*i*) in Lemma 7, we have

$$\begin{aligned}
{}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}\left({}_{a,k}^{H}\mathfrak{D}^{\alpha,\beta,\rho;\psi}f(t)\right) &= {}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}\left({}_{a,k}\mathcal{I}^{\beta(nk-\alpha),\rho;\psi}\left({}_{k}\mathfrak{D}^{n,\rho;\psi}\left({}_{a,k}\mathcal{I}^{(1-\beta)(nk-\alpha),\rho;\psi}f(t)\right)\right)\right) \\
&= {}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}\left({}_{a,k}\mathcal{I}^{\eta-\alpha,\rho;\psi}\left({}_{a,k}^{RL}\mathfrak{D}^{\eta,\rho;\psi}f(t)\right)\right) \\
&= {}_{a,k}\mathcal{I}^{\eta,\rho;\psi}\left({}_{a,k}^{RL}\mathfrak{D}^{\eta,\rho;\psi}f(t)\right).
\end{aligned}$$

Using property (*i*) in Lemma 10 yields that

$$\begin{aligned}
&{}_{a,k}\mathcal{I}^{\alpha,\rho;\psi}\left({}_{a,k}^{H}\mathfrak{D}^{\alpha,\beta,\rho;\psi}f(t)\right) \\
&\quad = f(t) - \sum_{i=1}^{n}\frac{{}_{k}^{\rho}\Psi_{\psi}^{\frac{\eta}{k}-i}(t,a)}{\rho^{\frac{\eta-ki}{k}}\Gamma_k(\eta+k-ki)}\left[{}_{k}\mathfrak{D}^{n-i,\rho;\psi}\left({}_{a,k}\mathcal{I}^{nk-\eta,\rho;\psi}f(a^{+})\right)\right].
\end{aligned}$$

The proof is completed. □

4 ψ-Laplace transform and uniqueness result

4.1 ψ-Laplace transform for the (k,ψ)-Hilfer-PFOs

In this subsection, we investigate some basic properties of (k,ψ)-HPFOs, which are proposed below.

Definition 14 ([43]) Assume that $f : [a,\infty) \to \mathbb{R}$ is a real-valued function, $\psi \in \mathcal{C}([a,\infty),\mathbb{R})$ with $\psi'(t) > 0$. Then the ψ-Laplace transform of f is given by

$$\mathcal{L}_\psi\{f(t)\} = \int_a^\infty e^{-s(\psi(s)-\psi(a))} f(t)\psi'(t)dt, \quad \forall s, \tag{40}$$

where the right-hand side of the above equation is valid.

Definition 15 ([43]) A function $f : [0,\infty) \to \mathbb{R}$ is said to be of ψ-exponential order if there are nonnegative numbers c, M, and T so that $|f(t)| \le Me^{c\psi(t)}$ for all $t \ge T$.

Theorem 1 ([43]) *Let $f \in \mathcal{C}_{\alpha,\psi}(\mathcal{J},\mathbb{R})$ and $\psi(t)$-exponential order so that $f^{[1]}(t)$ is piecewise continuous on $\mathcal{J}$. Then the ψ-Laplace transform of $f^{[1]}(t)$ exists, where*

$$\mathcal{L}_\psi\left\{f^{[1]}(t)\right\} = s\mathcal{L}_\psi\{f(t)\} - f(a), \qquad f^{[1]}(t) = \frac{f'(t)}{\psi'(t)}.$$

Now, the generalized convolution integral is given as follows.

Corollary 1 ([43]) *Assume $f \in \mathcal{C}^{n-1}_{\alpha,\psi}(\mathcal{J},\mathbb{R})$ so that $f^{[i]}$, $i = 0,1,2,\dots,n-1$, are of ψ-exponential order and $f^{[n]}$ is a piecewise continuous function on $\mathcal{J}$. Then the generalized Laplace transform of $f^{[n]}(t)$ exists where*

$$\mathcal{L}_\psi\left\{f^{[n]}(t)\right\} = s^n\mathcal{L}_\psi\left\{f(t)\right\} - \sum_{i=0}^{n-1} s^{n-i-1} f^{[i]}(a).$$

Next, we provide some details about the Mittag-Leffler ($\mathbb{ML}$) functions, which are used to studying the theory of fractional calculus.

Lemma 19 ([44]) *Assume $z \in \mathbb{C}$, $\alpha > 0$, and $\beta > 0$. Then the $\mathbb{ML}$ functions for one parameter and two parameters are provided by, respectively,*

$$\mathbb{E}_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(k\alpha+1)} \quad \text{and} \quad \mathbb{E}_{\alpha,\alpha+\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma((k+1)\alpha+\beta)}.$$

If $\mathbb{E}_\alpha(\cdot)$ and $\mathbb{E}_{\alpha,\beta}(\cdot)$ are two nonnegative functions, we have the following properties:

$$\mathbb{E}_\alpha(z) := \mathbb{E}_{\alpha,1}(z) \le 1 \quad \text{and} \quad \mathbb{E}_{\alpha,\beta}(z) \le \frac{1}{\Gamma(\beta)}, \quad \forall z < 0, \tag{41}$$

with $\mathbb{E}_\alpha(0) = 1$ and $\mathbb{E}_{\alpha,\beta}(0) = 1/\Gamma(\beta)$.

The ψ-Laplace transforms of basic functions were defined as in Lemma 20.

Lemma 20 ([43]) *Let $\alpha,\beta \in \mathbb{C}$, $Re(\alpha) > 0$, $Re(\beta) > 0$, and $\left|\frac{\lambda}{s^\alpha}\right| < 1$. Then we have the following properties*:

(*i*) $\mathcal{L}_\psi\{1\} = \frac{1}{s}$, $s > 0$;
(*ii*) $\mathcal{L}_\psi\left\{(\psi(t)-\psi(a))^\beta\right\} = \frac{\Gamma(\beta+1)}{s^{\beta+1}}$, $s > 0$;
(*iii*) $\mathcal{L}_\psi\left\{e^{\lambda(\psi(t)-\psi(a))}\right\} = \frac{1}{s-\lambda}$, $s > \lambda$;

(*iv*) $\mathcal{L}_{\psi}\left\{e^{\lambda(\psi(t)-\psi(a))}f(t)\right\} = \mathcal{L}_{\psi}\{f(t)\}(s-\lambda)$;
(*v*) $\mathcal{L}_{\psi}\left\{(\psi(t)-\psi(a))^{\beta-1}\mathbb{E}_{\alpha,\beta}\left(\lambda(\psi(t)-\psi(a))^{\alpha}\right)\right\} = \frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}$.

The ψ-convolution integral operator of two functions is given below.

Definition 16 ([43]) Assume that f and g are two piecewise continuous functions on $\mathcal{J}$ and of ψ-exponential order. The ψ-convolution of f and g is given by

$$\left(f *_{\psi} g\right)(t) = \int_a^t f(s)g\left(\psi^{-1}\left(\psi(t)+\psi(a)-\psi(s)\right)\right)\psi'(s)ds,$$

where $(f *_{\psi} g)(t) = (g *_{\psi} f)(t)$.

Definition 17 ([43]) Assume that f and g are two piecewise continuous functions on $\mathcal{J}$ and of ψ-exponential order. Then we have the following relation:

$$\mathcal{L}_{\psi}\{f(t) *_{\psi} g(t)\} = \mathcal{L}_{\psi}\{f(t)\}\mathcal{L}_{\psi}\{g(t)\}.$$

Next, the ψ-Laplace transforms of (k,ψ)-PFDO, (k,ψ)-PFIO, and (k,ψ)-HPFDO are provided as follows.

Theorem 2 *Assume that $f \in \mathcal{C}_{\alpha,\psi}(\mathcal{J},\mathbb{R})$ and is of ψ-exponential order such that ${}_k\mathfrak{D}^{\rho;\psi}$ is piecewise continuous on $\mathcal{J}$. Then the ψ-Laplace transform of ${}_k\mathfrak{D}^{\rho;\psi}$ is defined by*

$$\mathcal{L}_{\psi}\left\{{}_k\mathfrak{D}^{\rho;\psi}f(t)\right\} = (1-\rho+k\rho s)\mathcal{L}_{\psi}\{f(t)\} - k\rho f(a).$$

Proof By using equation (16), Theorem 1, and Definition 17, it follows that

$$\begin{aligned}\mathcal{L}_{\psi}\left\{{}_k\mathfrak{D}^{\rho;\psi}f(t)\right\} = \mathcal{L}_{\psi}\left\{(1-\rho)f(t) + k\rho\frac{f'(t)}{\psi'(t)}\right\} &= (1-\rho)\mathcal{L}_{\psi}\{f(t)\} + k\rho\mathcal{L}_{\psi}\left\{\frac{f'(t)}{\psi'(t)}\right\}\\ &= (1-\rho)\mathcal{L}_{\psi}\{f(t)\} + k\rho\Big[s\mathcal{L}_{\psi}\{f(t)\} - f(a)\Big]\\ &= (1-\rho+k\rho s)\mathcal{L}_{\psi}\{f(t)\} - k\rho f(a).\end{aligned}$$

The proof is completed. □

Corollary 2 *Assume $f \in \mathcal{C}_{\psi}^{n-1}([a,\infty),\mathbb{R})$ is such that $f^{[i]}$, $i = 1,2,\dots,n-1$, are of ψ-exponential order on $\mathcal{J}$ and $f^{[n]}$ is a piecewise continuous function on $\mathcal{J}$. Then the ψ-Laplace transform of ${}_k\mathfrak{D}^{n,\rho;\psi}$ is given by*

$$\mathcal{L}_{\psi}\left\{{}_k\mathfrak{D}^{n,\rho;\psi}f(t)\right\} = (1-\rho+k\rho s)^n\mathcal{L}_{\psi}\{f(t)\} - k\rho\sum_{i=0}^{n-1}(1-\rho+k\rho s)^{n-1-i}\Big({}_k\mathfrak{D}^{i,\rho;\psi}f(a)\Big).$$

Proof It is easy to show by mathematical induction. We omit the proof. □

Now, we will show the only the ψ-Laplace transform of the left-sided (k,ψ)-PFIO.

Theorem 3 *Assume $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, and $\rho \in (0,1]$. Suppose that f is a piecewise continuous function on $[a,t]$ and of ψ-exponential order. Then we have*

$$\mathcal{L}_\psi \left\{ {}_{a,k}\mathcal{I}^{\alpha,\rho;\psi} f(t) \right\} = \frac{\mathcal{L}_\psi\{f(t)\}}{(k\rho s - \rho + 1)^{\frac{\alpha}{k}}}.$$

Proof By applying Definition 10, Definition 17, and Lemma 20, it follows that

$$\begin{aligned}
\mathcal{L}_\psi \left\{ {}_{a,k}\mathcal{I}^{\alpha,\rho;\psi} f(t) \right\} &= \frac{1}{k\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha)} \mathcal{L}_\psi \left\{ f *_\psi e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(a))} (\psi(t)-\psi(a))^{\frac{\alpha}{k}-1} \right\} \\
&= \frac{1}{k\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha)} \mathcal{L}_\psi\{f(t)\} \mathcal{L}_\psi \left\{ e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(a))} (\psi(t)-\psi(a))^{\frac{\alpha}{k}-1} \right\} \\
&= \frac{1}{k\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha)} \mathcal{L}_\psi\{f(t)\} \mathcal{L}_\psi \left\{ (\psi(t)-\psi(a))^{\frac{\alpha}{k}-1} \right\} \left(s - \frac{\rho-1}{k\rho} \right) \\
&= \frac{\Gamma\left(\frac{\alpha}{k}\right)}{k\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha)\left(s - \frac{\rho-1}{k\rho}\right)^{\frac{\alpha}{k}}} \mathcal{L}_\psi\{f(t)\} \\
&= \frac{\mathcal{L}_\psi\{f(t)\}}{(k\rho s - \rho + 1)^{\frac{\alpha}{k}}}.
\end{aligned}$$

The proof is completed. □

Next, the ψ-Laplace transform of the left-sided (k,ψ)-RL-PFDO is analyzed.

Theorem 4 *Assume $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, $Re(\alpha)/k \in (n-1,n)$, $k > 0$, $\rho \in (0,1]$, $f \in \mathcal{AC}^n_{\alpha,\psi}(\mathcal{J},\mathbb{R})$, $\psi \in \mathcal{C}^n(\mathcal{J},\mathbb{R})$ so that $\psi' > 0$, and ${}_{a,k}\mathcal{I}^{nk-\alpha-ik,\rho;\psi} f$ is of ψ-exponential order, $i = 0,1,\dots,n-1$, $n \in \mathbb{N}$. Then*

$$\begin{aligned}
\mathcal{L}_\psi \left\{ {}^{RL}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi} f(t) \right\} &= (1-\rho+k\rho s)^{\frac{\alpha}{k}} \mathcal{L}_\psi\{f(t)\} \\
&\quad -k\rho \sum_{i=0}^{n-1} (1-\rho+k\rho s)^{n-1-i} \left({}_{a,k}\mathcal{I}^{nk-\alpha-ik,\rho;\psi} f(a^+) \right).
\end{aligned}$$

Proof Using Definition 11, Corollary 2, and Theorem 3 implies that

$$\begin{aligned}
&\mathcal{L}_\psi \left\{ {}^{RL}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi} f(t) \right\} \\
&= \mathcal{L}_\psi \left\{ {}_k\mathcal{D}^{n,\rho;\psi} \left({}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\psi} f(t) \right) \right\} \\
&= (1-\rho+k\rho s)^n \mathcal{L}_\psi \left\{ {}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\psi} f(t) \right\} \\
&\quad -k\rho \sum_{i=0}^{n-1} (1-\rho+k\rho s)^{n-1-i} \left[{}_k\mathfrak{D}^{i,\rho;\psi} \left({}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\psi} f(a) \right) \right] \\
&= \frac{(1-\rho+k\rho s)^n}{(k\rho s-\rho+1)^{\frac{nk-\alpha}{k}}} \mathcal{L}_\psi\{f(t)\} - k\rho \sum_{i=0}^{n-1} (1-\rho+k\rho s)^{n-1-i} \left({}_{a,k}\mathcal{I}^{nk-\alpha-ik,\rho;\psi} f(a) \right) \\
&= (1-\rho+k\rho s)^{\frac{\alpha}{k}} \mathcal{L}_\psi\{f(t)\} - k\rho \sum_{i=0}^{n-1} (1-\rho+k\rho s)^{n-1-i} \left({}_{a,k}\mathcal{I}^{nk-\alpha-ik,\rho;\psi} f(a) \right).
\end{aligned}$$

The proof is done. □

Next, the ψ-Laplace transform of the left-sided (k,ψ)-Caputo-PFDO is proved.

Theorem 5 *Let $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, $Re(\alpha)/k \in (n-1,n)$, $k > 0$, $\rho \in (0,1]$, $f \in \mathcal{AC}^n_{\alpha,\psi}(\mathcal{J},\mathbb{R})$, $\psi \in \mathcal{C}^n(\mathcal{J},\mathbb{R})$ so that $\psi' > 0$, and ${}_k\mathfrak{D}^{i,\rho;\psi} f$ is of ψ-exponential order, $i = 0,1,\dots,n-1$, $n \in \mathbb{N}$. Then*

$$\mathcal{L}_\psi\left\{{}^{C}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi} f(t)\right\} = (1-\rho+k\rho s)^{\frac{\alpha}{k}}\left[\mathcal{L}_\psi\{f(t)\} - k\rho\sum_{i=0}^{n-1}(1-\rho+k\rho s)^{-i-1}\left({}_k\mathfrak{D}^{i,\rho;\psi} f(a)\right)\right].$$

Proof Using Definition 12, Corollary 2, and Theorem 3 yields that

$$\begin{aligned}
\mathcal{L}_\psi\left\{{}^{C}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi} f(t)\right\} &= \mathcal{L}_\psi\left\{{}_{a,k}\mathcal{I}^{nk-\alpha,\rho;\psi}\left({}_k\mathfrak{D}^{n,\rho;\psi} f(t)\right)\right\}\\
&= \frac{1}{(k\rho s-\rho+1)^{\frac{nk-\alpha}{k}}}\,\mathcal{L}_\psi\left\{{}_k\mathfrak{D}^{n,\rho;\psi} f(t)\right\}\\
&= \frac{1}{(k\rho s-\rho+1)^{\frac{nk-\alpha}{k}}}\Bigg[(1-\rho+k\rho s)^n\mathcal{L}_\psi\{f(t)\}\\
&\quad -k\rho\sum_{i=0}^{n-1}(1-\rho+k\rho s)^{n-1-i}\left({}_k\mathfrak{D}^{i,\rho;\psi} f(a)\right)\Bigg]\\
&= (1-\rho+k\rho s)^{\frac{\alpha}{k}}\mathcal{L}_\psi\{f(t)\} - k\rho\sum_{i=0}^{n-1}(1-\rho+k\rho s)^{\frac{\alpha}{k}-1-i}\left({}_k\mathfrak{D}^{i,\rho;\psi} f(a)\right).
\end{aligned}$$

The proof is finished. □

This result studies the ψ-Laplace transform of the left-sided (k,ψ)-Hilfer-PFDO.

Theorem 6 *Assume $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, $Re(\alpha)/k \in (n-1,n)$, $k > 0$, $\rho \in (0,1]$, $f \in \mathcal{AC}^n_{\alpha,\psi}(\mathcal{J},\mathbb{R})$, $\psi \in \mathcal{C}^n(\mathcal{J},\mathbb{R})$ so that $\psi'(t) > 0$, and ${}_{a,k}\mathcal{I}^{(1-\beta)(nk-\alpha),\rho;\psi} f$ is of ψ-exponential order, $i = 0,1,\dots,n-1$, $n \in \mathbb{N}$. Then we have*

$$\begin{aligned}
\mathcal{L}_\psi\left\{{}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi} f(t)\right\} &= (1-\rho+k\rho s)^{\frac{\alpha}{k}}\,\mathcal{L}_\psi\{f(t)\} - k\rho\sum_{i=0}^{n-1}(1-\rho+k\rho s)^{\frac{\alpha\beta}{k}+n(1-\beta)-1-i}\\
&\quad\times\left[{}_k\mathfrak{D}^{i,\rho;\psi}\left({}_{a,k}\mathcal{I}^{(1-\beta)(nk-\alpha),\rho;\psi} f(a)\right)\right].
\end{aligned}$$

Proof By applying Definition 13 and Theorem 3, we have

$$\begin{aligned}
&\mathcal{L}_\psi\left\{{}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi} f(t)\right\}\\
&= \mathcal{L}_\psi\left\{{}_{a,k}\mathcal{I}^{\beta(nk-\alpha),\rho;\psi}\left({}_k\mathfrak{D}^{n,\rho;\psi}\left({}_{a,k}\mathcal{I}^{(1-\beta)(nk-\alpha),\rho;\psi} f(t)\right)\right)\right\}\\
&= \frac{1}{(1-\rho+k\rho s)^{\frac{\beta(nk-\alpha)}{k}}}\,\mathcal{L}_\psi\left\{{}_k\mathfrak{D}^{n,\rho;\psi}\left({}_{a,k}\mathcal{I}^{(1-\beta)(nk-\alpha),\rho;\psi} f(t)\right)\right\}.
\end{aligned} \tag{42}$$

From Corollary 2 and Theorem 3, we obtain

$$\mathcal{L}_\psi\left\{{}_k\mathfrak{D}^{n,\rho;\psi}\left({}_{a,k}\mathcal{I}^{(1-\beta)(nk-\alpha),\rho;\psi} f(t)\right)\right\}$$

$$
\begin{aligned}
&= (1-\rho+k\rho s)^{n}\,\mathcal{L}_{\psi}\left\{{}_{a,k}\mathcal{I}^{(1-\beta)(nk-\alpha),\rho;\psi}f(t)\right\}\\
&\quad -\rho\sum_{i=0}^{n-1}(1-\rho+k\rho s)^{n-1-i}\Big[{}_{k}\mathfrak{D}^{i,\rho;\psi}\left({}_{a,k}\mathcal{I}^{(1-\beta)(nk-\alpha),\rho;\psi}f(a)\right)\Big]\\
&= \frac{(1-\rho+k\rho s)^{n}}{(1-\rho+k\rho s)^{\frac{(1-\beta)(nk-\alpha)}{k}}}\,\mathcal{L}_{\psi}\{f(t)\}\\
&\quad -k\rho\sum_{i=0}^{n-1}(1-\rho+k\rho s)^{n-1-i}\Big[{}_{k}\mathfrak{D}^{i,\rho;\psi}\left({}_{a,k}\mathcal{I}^{(1-\beta)(nk-\alpha),\rho;\psi}f(a)\right)\Big]. \qquad (43)
\end{aligned}
$$

Substituting (43) into (42), we obtain the following result:

$$
\begin{aligned}
\mathcal{L}_{\psi}\left\{{}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi}f(t)\right\} &= \frac{1}{(1-\rho+k\rho s)^{\frac{\beta(nk-\alpha)}{k}}}\Bigg(\frac{(1-\rho+k\rho s)^{n}}{(1-\rho+k\rho s)^{\frac{(1-\beta)(nk-\alpha)}{k}}}\,\mathcal{L}_{\psi}\{f(t)\}\\
&\quad -k\rho\sum_{i=0}^{n-1}(1-\rho+k\rho s)^{n-1-i}\Big[{}_{k}\mathfrak{D}^{i,\rho;\psi}\left({}_{a,k}\mathcal{I}^{(1-\beta)(nk-\alpha),\rho;\psi}f(a)\right)\Big]\Bigg)\\
&= (1-\rho+k\rho s)^{\frac{\alpha}{k}}\mathcal{L}_{\psi}\{f(t)\} - k\rho\sum_{i=0}^{n-1}\frac{(1-\rho+k\rho s)^{n-1-i}}{(1-\rho+k\rho s)^{\frac{\beta(nk-\alpha)}{k}}}\\
&\quad \times\Big[{}_{k}\mathfrak{D}^{i,\rho;\psi}\left({}_{a,k}\mathcal{I}^{(1-\beta)(nk-\alpha),\rho;\psi}f(a)\right)\Big].
\end{aligned}
$$

The proof is done. □

This part considers the Cauchy-type problem in the context of (k,ψ)-Hilfer-PFDO.

Example 1 The Cauchy-type initial value problem under the (k,ψ)-Hilfer-PFDO

$$
\begin{cases}
{}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi}u(t) = \lambda(k\rho)^{\frac{\alpha}{k}}u(t) + f(t),\ \alpha/k\in(n-1,n),\ k>0,\ \rho\in(0,1],\ \lambda<0,\\
\lim\limits_{t\to a^{+}}{}_{a,k}\mathcal{I}^{(1-\beta)(nk-\alpha)-ik,\rho;\psi}u(t) = c_i,\ c_i\in\mathbb{R},\ \beta\in[0,1],\ i=0,1,\dots,n-1,\ n\in\mathbb{N},
\end{cases} \qquad (44)
$$

has the solution $u(t)$, where

$$
\begin{aligned}
u(t) &= (k\rho)^{-\frac{\alpha}{k}}\int_{a}^{t}{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-1}(t,s)\mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}\left(\lambda(\psi(t)-\psi(s))^{\frac{\alpha}{k}}\right)\psi'(s)f(s)ds\\
&\quad +\sum_{i=0}^{n-1}\frac{c_i{}^{\rho}_{k}\Psi_{\psi}^{i+\frac{\beta(nk-\alpha)+\alpha}{k}-n}(t,a)}{(k\rho)^{i+\frac{\beta(nk-\alpha)+\alpha}{k}-n}}\mathbb{E}_{\frac{\alpha}{k},1+i+\frac{\beta(nk-\alpha)+\alpha}{k}-n}\left(\lambda(\psi(t)-\psi(a))^{\frac{\alpha}{k}}\right). \qquad (45)
\end{aligned}
$$

By using the ψ-Laplace transform to the first equation of (44), we obtain the result

$$
\mathcal{L}_{\psi}\left\{{}^{H}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}u(t)\right\} = \mathcal{L}_{\psi}\left\{(k\rho)^{\frac{\alpha}{k}}\lambda u(t)\right\} + \mathcal{L}_{\psi}\{f(t)\}. \qquad (46)
$$

From Theorem 6, equation (46) can be written as

$$
\left[(1-\rho+k\rho s)^{\frac{\alpha}{k}} - (k\rho)^{\frac{\alpha}{k}}\lambda\right]\mathcal{L}_{\psi}\{u(t)\} = \mathcal{L}_{\psi}\left\{f(t)\right\} + k\rho\sum_{i=0}^{n-1}\frac{c_i(1-\rho+k\rho s)^{n-1-i}}{(1-\rho+k\rho s)^{\frac{\beta(nk-\alpha)}{k}}}.
$$

Then

$$u(t) = \mathcal{L}_{\psi}^{-1}\left\{ \sum_{i=0}^{n-1} \frac{c_i\left(s-\frac{\rho-1}{k\rho}\right)^{n-1-i-\frac{\beta(nk-\alpha)}{k}}}{\left(\left(s-\frac{\rho-1}{k\rho}\right)^{\frac{\alpha}{k}}-\lambda\right)(k\rho)^{i+\frac{\beta(nk-\alpha)+\alpha}{k}-n}} \right\} + \mathcal{L}_{\psi}^{-1}\left\{ \frac{(k\rho)^{-\frac{\alpha}{k}}}{\left(s-\frac{\rho-1}{k\rho}\right)^{\frac{\alpha}{k}}-\lambda} \mathcal{L}_{\psi}\{f(t)\} \right\}. \tag{47}$$

From Lemma 20, it follows that

$$\mathcal{L}_{\psi}\left\{ e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{i+\frac{\beta(nk-\alpha)+\alpha}{k}-n}\mathbb{E}_{\frac{\alpha}{k},1+i+\frac{\beta(nk-\alpha)+\alpha}{k}-n}\left(\lambda(\psi(t)-\psi(a))^{\frac{\alpha}{k}}\right)\right\} = \frac{\left(s-\frac{\rho-1}{k\rho}\right)^{n-1-i-\frac{\beta(nk-\alpha)}{k}}}{\left(s-\frac{\rho-1}{k\rho}\right)^{\frac{\alpha}{k}}-\lambda}$$

and

$$\mathcal{L}_{\psi}\left\{ e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\frac{\alpha}{k}-1}\mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}\left(\lambda(\psi(t)-\psi(a))^{\frac{\alpha}{k}}\right)\right\} = \frac{1}{\left(s-\frac{\rho-1}{k\rho}\right)^{\frac{\alpha}{k}}-\lambda}.$$

Equation (47) can be computed as

$$\begin{aligned}
&u(t)\\
&= \mathcal{L}_{\psi}^{-1}\left\{(k\rho)^{-\frac{\alpha}{k}}\mathcal{L}_{\psi}\left\{ e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\frac{\alpha}{k}-1}\mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}\left(\lambda(\psi(t)-\psi(a))^{\frac{\alpha}{k}}\right)\right\}\mathcal{L}_{\psi}\{f(t)\}\right\}\\
&+\mathcal{L}_{\psi}^{-1}\left\{\sum_{i=0}^{n-1}\left[(k\rho)^{n-i-\frac{\beta(nk-\alpha)+\alpha}{k}}\left({}_{a,k}\mathcal{I}^{(1-\beta)(nk-\alpha)-ik,\rho;\psi}u(a)\right)\right.\right.\\
&\times\mathcal{L}_{\psi}\left\{e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{i+\frac{\beta(nk-\alpha)+\alpha}{k}-n}\right.\\
&\left.\left.\left.\times\mathbb{E}_{\frac{\alpha}{k},1+i+\frac{\beta(nk-\alpha)+\alpha}{k}-n}\left(\lambda(\psi(t)-\psi(a))^{\frac{\alpha}{k}}\right)\right\}\right]\right\}\\
&= \mathcal{L}_{\psi}^{-1}\left\{(k\rho)^{-\frac{\alpha}{k}}\mathcal{L}_{\psi}\left\{ e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{\frac{\alpha}{k}-1}\mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}\left(\lambda(\psi(t)-\psi(a))^{\frac{\alpha}{k}}\right)*f(t)\right\}\right\}\\
&+\sum_{i=0}^{n-1}\left[e^{\frac{\rho-1}{k\rho}(\psi(t)-\psi(a))}(\psi(t)-\psi(a))^{i+\frac{\beta(nk-\alpha)+\alpha}{k}-n}\mathbb{E}_{\frac{\alpha}{k},1+i+\frac{\beta(nk-\alpha)+\alpha}{k}-n}\left(\lambda(\psi(t)-\psi(a))^{\frac{\alpha}{k}}\right)\right.\\
&\left.\times(k\rho)^{n-i-\frac{\beta(nk-\alpha)+\alpha}{k}}\left({}_{a,k}\mathcal{I}^{(1-\beta)(nk-\alpha)-ik,\rho;\psi}u(a)\right)\right]\\
&= (k\rho)^{-\frac{\alpha}{k}}\int_a^t {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-1}(t,s)\mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}\left(\lambda(\psi(t)-\psi(s))^{\frac{\alpha}{k}}\right)\psi'(s)f(s)ds\\
&+\sum_{i=0}^{n-1}\left[c_i\,{}^{\rho}_{k}\Psi_{\psi}^{i+\frac{\beta(nk-\alpha)+\alpha}{k}-n}(t,a)\mathbb{E}_{\frac{\alpha}{k},1+i+\frac{\beta(nk-\alpha)+\alpha}{k}-n}\left(\lambda(\psi(t)-\psi(a))^{\frac{\alpha}{k}}\right)(k\rho)^{n-i-\frac{\beta(nk-\alpha)+\alpha}{k}}\right].
\end{aligned}$$

Equation (45) is obtained.

4.2 Uniqueness result

This section studies the existence and uniqueness of solutions of the proposed problem (1). By applying the ψ-Laplace transform to the proposed problem (1), it follows that

$$\begin{aligned} u(t) &= (k\rho)^{-\frac{\alpha}{k}} \int_a^t {}_k^\rho\Psi_\psi^{\frac{\alpha}{k}-1}(t,s)\mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}\left(\lambda(\psi(t)-\psi(s))^{\frac{\alpha}{k}}\right)\psi'(s)f(s,u(s))ds \\ &\quad + \sum_{i=0}^{n-1} \frac{\theta_i {}_k^\rho\Psi_\psi^{i+\frac{\beta(nk-\alpha)+\alpha}{k}-n}(t,a)}{(k\rho)^{i+\frac{\beta(nk-\alpha)+\alpha}{k}-n}} \mathbb{E}_{\frac{\alpha}{k},1+i+\frac{\beta(nk-\alpha)+\alpha}{k}-n}\left(\lambda(\psi(t)-\psi(a))^{\frac{\alpha}{k}}\right). \end{aligned} \tag{48}$$

Next, we will establish the existence and uniqueness result for the proposed problem (1) by using Picard's iterative technique, see [45] for more details.

Theorem 7 *Suppose that there is a positive constant M so that $\sup_{t\in\mathcal{J}}\{f(t,u(t))\} \le M$, and there is a positive number γ so that $|f(t,u_1(t))-f(t,u_2(t))| \le \gamma\,|u_1(t)-u_2(t)|$, $\forall t\in\mathcal{J}$, $u_i \in \mathcal{C}^n(\mathcal{J},\mathbb{R})$, $i=1,2$. Then there is one and only one solution $u(t)$ of the proposed problem (1) on $\mathcal{J}$ provided that*

$$\gamma\left(\frac{(\psi(T)-\psi(a))^{\frac{\alpha}{k}}}{\alpha\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha)}\right) < 1. \tag{49}$$

Proof It is easy to present that the proposed problem (1) has a solution corresponding to the solution of equation (48). Firstly, we define

$$u_0(t) = \sum_{i=0}^{n-1} \frac{\theta_i {}_k^\rho\Psi_\psi^{i+\frac{\beta(nk-\alpha)+\alpha}{k}-n}(t,a)}{(k\rho)^{i+\frac{\beta(nk-\alpha)+\alpha}{k}-n}} \mathbb{E}_{\frac{\alpha}{k},1+i+\frac{\beta(nk-\alpha)+\alpha}{k}-n}\left(\lambda(\psi(t)-\psi(a))^{\frac{\alpha}{k}}\right), \tag{50}$$

$$u_j(t) = u_0(t) + (k\rho)^{-\frac{\alpha}{k}} \int_a^t {}_k^\rho\Psi_\psi^{\frac{\alpha}{k}-1}(t,s)\mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}\left(\lambda(\psi(t)-\psi(s))^{\frac{\alpha}{k}}\right)\psi'(s)f(s,u_{j-1}(s))ds, \tag{51}$$

where $j\in\mathbb{N}$. Clearly, the term $u_j(t) = u_0(t)+\sum_{l=1}^{j}[u_l(t)-u_{l-1}(t)]$ is a partial sum of the series term $u_0(t)+\sum_{l=1}^{\infty}[u_l(t)-u_{l-1}(t)]$. The goal is to demonstrate that a sequence $\{u_j(t)\}$ converges to $u(t)$. By using mathematical induction, for $t\in\mathcal{J}$, it follows that

$$\|u_j-u_{j-1}\| = M\gamma^{j-1}\left(\frac{(\psi(T)-\psi(a))^{\frac{\alpha}{k}}}{\alpha\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha)}\right)^j, \qquad i\in\mathbb{N}. \tag{52}$$

Using (50)–(51) and property (i) in Lemma 5 implies that

$$\begin{aligned} \|u_1-u_0\| &= \left\|(k\rho)^{-\frac{\alpha}{k}} \int_a^t {}_k^\rho\Psi_\psi^{\frac{\alpha}{k}-1}(t,s)\mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}\left(\lambda(\psi(t)-\psi(s))^{\frac{\alpha}{k}}\right)\psi'(s)f(s,u_0(s))ds\right\| \\ &\le \frac{M(k\rho)^{-\frac{\alpha}{k}}}{\Gamma\left(\frac{\alpha}{k}\right)}\left\|\int_a^t (\psi(t)-\psi(s))^{\frac{\alpha}{k}-1}\psi'(s)ds\right\| \\ &\le M\frac{(\psi(T)-\psi(a))^{\frac{\alpha}{k}}}{\alpha\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha)}. \end{aligned}$$

Then, for $j=1$, we have that inequality (52) is satisfied.

Next, we will show that inequality (52) is satisfied for $j = r$. Since

$$\begin{aligned}
&\|u_{r+1} - u_r\| \\
&= (k\rho)^{-\frac{\alpha}{k}} \bigg\| \int_a^t {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-1}(t,s)\mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}\Big(\lambda(\psi(t)-\psi(s))^{\frac{\alpha}{k}}\Big)\psi'(s)f(s,u_r(s))ds \\
&\quad - \int_a^t {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-1}(t,s)\mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}\Big(\lambda(\psi(t)-\psi(s))^{\frac{\alpha}{k}}\Big)\psi'(s)f(s,u_{r-1}(s))ds\bigg\| \\
&\leq \gamma(k\rho)^{-\frac{\alpha}{k}} \bigg\| \int_a^t {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-1}(t,s)\mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}\Big(\lambda(\psi(t)-\psi(s))^{\frac{\alpha}{k}}\Big)\psi'(s)\,|u_r(s)-u_{r-1}(s)|\,ds\bigg\| \\
&\leq M\gamma^r\left(\frac{(\psi(T)-\psi(a))^{\frac{\alpha}{k}}}{\alpha\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha)}\right)^r(k\rho)^{-\frac{\alpha}{k}}\bigg\| \int_a^t {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-1}(t,s)\mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}\Big(\lambda(\psi(t)-\psi(s))^{\frac{\alpha}{k}}\Big)\psi'(s)ds\bigg\| \\
&\leq M\gamma^r\left(\frac{(\psi(T)-\psi(a))^{\frac{\alpha}{k}}}{\alpha\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha)}\right)^{r+1},
\end{aligned}$$

it implies that inequality (52) is satisfied for $j = r + 1$. By using mathematical induction, (52) is satisfied for each $j \in \mathbb{N}$, and for all $t \in \mathcal{J}$, we get

$$\sum_{j=1}^{\infty}\|u_j - u_{j-1}\| \leq M\sum_{j=1}^{\infty}\left[\gamma^{j-1}\left(\frac{(\psi(T)-\psi(a))^{\frac{\alpha}{k}}}{\alpha\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha)}\right)^j\right]. \tag{53}$$

From assumption (49), the right-hand side of (53) is convergent. Therefore, the term $\sum_{j=1}^{\infty}\|u_j - u_{j-1}\|$ is also convergent, which implies that $u_0 + \sum_{l=1}^{\infty}\|u_l - u_{l-1}\|$ converges. By setting $u^* = u_0 + \sum_{l=1}^{\infty}\|u_l - u_{l-1}\|$, we obtain

$$\left\|u_j - u^*\right\| \to 0 \qquad \text{as} \qquad j \to \infty, \tag{54}$$

which yields that the solution of the proposed problem (1) exists. From (54), one has

$$\left\|f(\cdot,u_{j-1}(\cdot)) - f(\cdot,u^*(\cdot))\right\| \leq \gamma\left\|u_{j-1} - u^*\right\| \to 0 \quad \text{as} \quad j \to \infty.$$

Hence,

$$\lim_{j\to\infty} f(t,u_{j-1}(t)) = f(t,u(t)). \tag{55}$$

Taking the limit $j \to \infty$ in (51) and using (55), it follows that

$$\begin{aligned}
u(t) &= (k\rho)^{-\frac{\alpha}{k}}\int_a^t {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-1}(t,s)\mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}\left(\lambda(\psi(t)-\psi(s))^{\frac{\alpha}{k}}\right)\psi'(s)f(s,u(s))ds \\
&\quad + \sum_{i=0}^{n-1}\frac{\theta_i {}^{\rho}_{k}\Psi_{\psi}^{i+\frac{\beta(nk-\alpha)+\alpha}{k}-n}(t,a)}{(k\rho)^{i+\frac{\beta(nk-\alpha)+\alpha}{k}-n}}\,\mathbb{E}_{\frac{\alpha}{k},1+i+\frac{\beta(nk-\alpha)+\alpha}{k}-n}\left(\lambda(\psi(t)-\psi(a))^{\frac{\alpha}{k}}\right),
\end{aligned}$$

which shows a solution of the proposed problem (1).

Finally, we will show that problem (1) has a unique solution. Assume that $u(t)$ and $\tilde{u}(t)$ are the solution of problem (1), we obtain that

$$\begin{aligned}\|u-\tilde{u}\| &= (k\rho)^{-\frac{\alpha}{k}} \sup_{t\in\mathcal{J}} \bigg| \int_a^t {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-1}(t,s)\mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}\Big(\lambda(\psi(t)-\psi(s))^{\frac{\alpha}{k}}\Big)\psi'(s)f(s,u(s))ds \\ &\quad - \int_a^t {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-1}(t,s)\mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}\Big(\lambda(\psi(t)-\psi(s))^{\frac{\alpha}{k}}\Big)\psi'(s)f(s,\tilde{u}(s))ds\bigg| \\ &\le \gamma(k\rho)^{-\frac{\alpha}{k}} \int_a^t {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-1}(t,s)\mathbb{E}_{\frac{\alpha}{k},\frac{\alpha}{k}}\Big(\lambda(\psi(t)-\psi(s))^{\frac{\alpha}{k}}\Big)\psi'(s)\left|u(s)-\tilde{u}(s)\right|ds \\ &\le \gamma\left(\frac{(\psi(T)-\psi(a))^{\frac{\alpha}{k}}}{\alpha\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha)}\right)\|u-\tilde{u}\|.\end{aligned}$$

Hence, it follows from assumption (49) that $\|u-\tilde{u}\| = 0$, that is, $u(t) = \tilde{u}(t)$. □

5 Some examples

This section gives two examples of the proposed problem (1) to show the theoretical main results.

Example 2 Consider the following Cauchy-type initial value problem under the (k,ψ)-Hilfer-PFDO:

$$\begin{cases} {}^{H}_{0,\frac{8}{10}}\mathfrak{D}^{\ln 2,\frac{2}{3},\frac{17}{20};\frac{e^{2t-9}}{5}}u(t) = \dfrac{3t^2+2t}{\ln(2t+5)} - 6\left(\dfrac{17}{25}\right)^{\frac{10\ln 2}{8}} u(t) \\ \qquad\qquad\qquad + \dfrac{9e^{-2t}}{2(t^2+3t+1)}\cdot\dfrac{|u(t)|}{5+2|u(t)|}, \\ \lim_{t\to 0^+} {}_{0,\frac{8}{10}}\mathcal{I}^{\frac{4-5\ln 2}{15},\frac{17}{20};\frac{e^{2t-9}}{5}}u(t) = \dfrac{4}{3}. \end{cases} \tag{56}$$

From the proposed problem (56), we obtain that $\alpha = \ln 2$, $\beta = 2/3$, $\rho = 17/20$, $k = 8/10$, $\psi(t) = \exp(2t-9)/5$, $\lambda = -6$, $a = 0$, $T = 5$, $\theta_0 = 4/3$. Then

$$f(t,u(t)) = \frac{3t^2+2t}{\ln(2t+5)} + \frac{9e^{-2t}}{2(t^2+3t+1)}\cdot\frac{|u(t)|}{5+2|u(t)|}.$$

For $u_i \in \mathbb{R}$, $i = 1,2$, and $t \in [0,5]$, we can compute that $|f(t,u_1(t)) - f(t,u_2(t))| \le (9/10)|u_1(t)-u_2(t)|$. The assumption in Theorem 7 is satisfied with $\gamma = 9/10$. Hence,

$$\gamma\left(\frac{(\psi(T)-\psi(a))^{\frac{\alpha}{k}}}{\alpha\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha)}\right) \approx 0.7797850224 < 1.$$

Since all conditions in Theorem 7 are satisfied, the proposed problem (56) has a unique solution.

Table 1 List of special cases of the (k,ψ)-Hilfer-PFDO

${}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi}$		$0<\rho<1$		$\rho=1$	
$\psi(t)$	β	$k>0$	$k=1$	$k>0$	$k=1$
$\psi(t)$	0	(k,ψ)-RL-PFDO	ψ-RL-PFDO [40, 41]	(k,ψ)-RL-FDO [21]	ψ-RL-FDO [3]
	1	(k,ψ)-Caputo-PFDO	ψ-Caputo-PFDO [40, 41]	(k,ψ)-Caputo-FDO [21]	ψ-Caputo-FDO [46]
	β	(k,ψ)-Hilfer-PFDO	ψ-Hilfer-PFDO [9]	(k,ψ)-Hilfer-FDO [21]	ψ-Hilfer-FDO [8]
t	0	k-RL-PFDO	RL-PFDO [6]	k-RL-FDO [47]	RL-FDO [3]
	1	k-Caputo-PFDO	Caputo-PFDO [6]	k-Caputo-FDO [21]	Caputo-FDO [3]
	β	k-Hilfer-PFDO	Hilfer-PFDO [7]	k-Hilfer-FDO [21]	Hilfer-FDO [4]
t^{μ}	0	k-RL-Katugampola-PFDO	RL-Katugampola-PFDO [9]	k-RL-Katugampola-FDO [21]	RL-Katugampola-FDO [48]
	1	k-Caputo-Katugampola-PFDO	Caputo-Katugampola-PFDO [9]	k-Caputo-Katugampola-FDO [21]	Caputo-Katugampola-FDO [48, 49]
	β	k-Hilfer-Katugampola-PFDO	Hilfer-Katugampola-PFDO [9]	k-Hilfer-Katugampola-FDO [21]	Hilfer-Katugampola-FDO [50]
$\log t$	0	k-RL-Hadamard-PFDO	RL-Hadamard-PFDO [9]	k-RL-Hadamard-FDO [21]	RL-Hadamard-FDO [3]
	1	k-Caputo-Hadamard-PFDO	Caputo-Hadamard-PFDO [9]	k-Caputo-Hadamard-FDO [21]	Caputo-Hadamard-FDO [3]
	β	k-Hilfer-Hadamard-PFDO	Hilfer-Hadamard-PFDO [9]	k-Hilfer-Hadamard-FDO [21]	Hilfer-Hadamard-FDO [51]
e^{t}	0	k-RL-Exponential-PFDO	RL-Exponential-PFDO [9]	k-RL-Exponential-FDO [21]	RL-Exponential-FDO [52]
	1	k-Caputo-Exponential-PFDO	Caputo-Exponential-PFDO [9]	k-Caputo-Exponential-FDO [21]	Caputo-Exponential-FDO [52]
	β	k-Hilfer-Exponential-PFDO	Hilfer-Exponential-PFDO [9]	k-Hilfer-Exponential-FDO [21]	Hilfer-Exponential-FDO [21]

Example 3 Consider the following Cauchy-type initial value problem under the (k,ψ)-Hilfer-PFDO:

$$\begin{cases} {}^{H}_{0,\frac{9}{10}}\mathfrak{D}^{\sqrt{3},\frac{4}{5},\frac{7}{10};\ln(2t+3)}u(t) = \dfrac{\cos(t^2+2t)}{e^{3t+2}} - 3\left(\dfrac{63}{100}\right)^{\frac{10\sqrt{3}}{9}} u(t) \\ \qquad\qquad + \dfrac{4\sin(t^2+2t)}{5+\ln(3t+1)} \cdot \dfrac{|u(t)|}{2+|u(t)|}, \\ \lim_{t\to 0^+} {}_{0,\frac{9}{10}}\mathcal{I}^{\frac{9-5\sqrt{3}}{25},\frac{7}{10};\ln(2t+3)}u(t) = \dfrac{7}{10}, \\ \lim_{t\to 0^+} {}_{\frac{9}{10}}\mathfrak{D}^{\frac{7}{10};\ln(2t+3)}\left({}_{0,\frac{9}{10}}\mathcal{I}^{\frac{9-5\sqrt{3}}{25},\frac{7}{10};\ln(2t+3)}u(t)\right) = \dfrac{6}{5}. \end{cases} \tag{57}$$

From the proposed problem (57), we obtain that $\alpha = \sqrt{3}$, $\beta = 4/5$, $\rho = 7/10$, $k = 9/10$, $\psi(t) = \ln(2t+3)$, $\lambda = -3$, $a = 0$, $T = 3$, $\theta_0 = 7/10$, and $\theta_1 = 6/5$. Then

$$f(t,u(t)) = \frac{\cos(t^2+2t)}{e^{3t+2}} + \frac{4\sin(t^2+2t)}{5+\ln(3t+1)} \cdot \frac{|u|}{2+|u|}.$$

For $u_i \in \mathbb{R}$, $i = 1,2$, and $t \in [0,3]$, we can compute that $|f(t,u_1(t)) - f(t,u_2(t))| \le \frac{2}{5}|u_1(t) - u_2(t)|$. The assumption in Theorem 7 is satisfied with $\gamma = 2/5$. Hence,

$$\gamma\left(\frac{(\psi(T)-\psi(a))^{\frac{\alpha}{k}}}{\alpha\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha)}\right) \approx 0.6245508422 < 1.$$

Since all conditions in Theorem 7 are satisfied, the proposed problem (57) has a unique solution.

6 Conclusion

In this study, we proposed the most generalized version of the Hilfer derivative operator, which is (k,ψ)-Hilfer-PFDO, and developed certain essential properties. The ψ-Laplace transform has shown to be an excellent tool for investigating the characteristics of the proposed operators and solving the Cauchy-type problem with an initial condition under the (k,ψ)-Hilfer-PFDO. In addition, the existence and uniqueness result for the higher-order initial value problem under (k,ψ)-Hilfer-PFDO has been established by using Picard's iterative technique. To demonstrate the usefulness of our results, we provided some examples that illustrated the new extensions.

Finally, we argued that the obtained results are novel and generalize previous ones from the literature. It is important to note that the proposed operator combines the current ones in terms of (k,ψ)-$\mathbb{RL}$-PFDO and (k,ψ)-Caputo-PFDO, respectively. For different functions ψ and the different parameters β, ρ, and k, the proposed operator reduced different types of fractional derivative operators, which were previously constructed. The details can be shown in Table 1. This achievement can be viewed as an advance in the qualitative part of extended fractional calculus. Furthermore, the obtained results are a major motivator for academics and researchers to study this form of extended fractional calculus. Then, we will concentrate our efforts on applying these extension fractional operators to real-world problems and researching novel features and inequalities associated with such operators in the future. Especially, an intriguing challenge is to provide an extension of Gronwall's inequality using the (k,ψ)-proportional fractional integral and to investigate the existence and uniqueness of the initial/boundary value problems using the (k,ψ)-Hilfer proportional fractional derivative. On the other hand, it appears that the (k,ψ)-Hilfer proportional fractional operator may be generalized by simply taking the variable order $\alpha(t)$ and type $\beta(t) \in (0,1]$.

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Author contributions
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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

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References
1. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, Yverdon (1993)
2. Podlubny, I.: Fractional Differential Equations. Academic Press, New York (1999)
3. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Application of Fractional Differential Equations. North Holland Mathematics Studies, vol. 204. Amsterdam, (2006)
4. Hilfer, R.: Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000)
5. Katugampola, U.N.: New approach to a generalized fractional integral. Appl. Math. Comput. **218**, 860–865 (2011)
6. Jarad, F., Abdeljawad, T., Alzabut, J.: Generalized fractional derivatives generated by a class of local proportional derivatives. Eur. Phys. J. Spec. Top. **226**, 3457–3471 (2017)
7. Ahmed, I., Kumam, P., Jarad, F., Borisut, P., Jirakitpuwapat, W.: On Hilfer generalized proportional fractional derivative. Adv. Differ. Equ. **1**, 1–18 (2020)
8. Vanterler, J., Sousa, D.C., de Oliveira, E.C.: On the ψ-Hilfer fractional derivative. Commun. Nonlinear Sci. Numer. Simul. **60**, 72–91 (2018)
9. Mallah, I., Ahmed, I., Akgul, A., Jarad, F., Alha, S.: On ψ-Hilfer generalized proportional fractional operators. AIMS Math. **7**(1), 82–103 (2022)
10. Osler, T.J.: The fractional derivative of composite function. SIAM J. Math. Anal. **1**(2), 288–293 (1970)
11. Atangana, A., Secer, A.: A note on fractional order derivatives and table of fractional derivatives of some special functions. In: Abstract and Applied Analysis, vol. 2013. Hindawi, (2013)
12. Atangana, A.: Derivative with a New Parameter: Theory, Methods and Applications. Academic Press, San Diego (2015)
13. Atangana, A., Koca, I.: New direction in fractional differentiation. Math. Nat. Sci. **1**, 18–25 (2017)
14. Sadek, L.A.: Cotangent fractional derivative with the application. Fractal Fract. **7**(6), 444 (2023)
15. Sadek, L.A., Lazǎr, T.A.: On Hilfer cotangent fractional derivative and a particular class of fractional problems. AIMS Math. **8**(12), 28334–28352 (2023)
16. Diaz, R., Teruel, C.: *q*, *k*-Generalized gamma and beta functions. J. Nonlinear Math. Phys. **12**, 118–134 (2005)
17. Diaz, R., Pariguan, E.: On hypergeometric functions and Pochhammer *k*-symbol. Divulg. Mat. **15**, 179–192 (2007)
18. Diaz, R., Ortiz, C., Pariguan, E.: On the *k*-gamma *q*-distribution. Cent. Eur. J. Math. **8**, 448–458 (2010)
19. Mubeen, S., Habibullah, G.M.: *k*-fractional integrals and application. Int. J. Contemp. Math. Sci. **7**(2), 89–94 (2012)
20. Romero, L.G., Luque, L.L., Dorrego, G.A., Cerutti, R.A.: On the *k*-Riemann-Liouville fractional derivative. Int. J. Contemp. Math. Sci. **8**(1), 41–51 (2013)

21. Kucche, K.D., Mali, A.D.: On the nonlinear (k,ψ)-Hilfer fractional differential equations. Chaos Solitons Fractals **152**, 111335 (2021)
22. Aljaaidi, T.A., Pachpatte, D.B., Abdo, M.S., Botmart, T., Ahmad, H., Almalahi, M.A., Redhwan, S.S.: (k,ψ)-Proportional fractional integral Pólya-Szegö and Grüss-type inequalities. Fractal Fract. **5**, 172 (2021)
23. Hilfer, R.: Experimental evidence for fractional time evolution in glass forming materials. Chem. Phys. **284**, 399–408 (2002)
24. Ali, I., Malik, N.A.: Hilfer fractional advection-diffusion equations with power-law initial condition; a numerical study using variational iteration method. Comput. Math. Appl. **68**, 1161–1179 (2014)
25. Bulavatsky, V.M.: Mathematical modeling of fractional differential filtration dynamics based on models with Hilfer-Prabhakar derivative. Cybern. Syst. Anal. **53**, 204–216 (2017)
26. Bulavatsky, V.M.: Mathematical models and problems of fractional-differential dynamics of some relaxation filtration processes. Cybern. Syst. Anal. **54**, 727–736 (2018)
27. Yang, M., Alsaedi, A., Ahmad, B., Zhou, Y.: Attractivity for Hilfer fractional stochastic evolution equations. Adv. Differ. Equ. **2020**, 130 (2020)
28. Ledesma, C.E.T., Nyamoradi, N.: (k,ψ)-Hilfer variational problem. J. Elliptic Parabolic Equ. **8**, 681–709 (2022)
29. Qin, X., Rui, Z., Peng, W.: Fractional derivative of demand and supply functions in the cobweb economics model and Markov process. Front. Phys. **11**, 1266860 (2023)
30. Kamocki, R.: A new representation formula for the Hilfer fractional derivative and its application. J. Comput. Appl. Math. **307**, 39–45 (2016)
31. Joshi, H., Jha, B.K.: Chaos of calcium diffusion in Parkinson?s infectious disease model and treatment mechanism via Hilfer fractional derivative. Math. Model. Numer. Simul. Appl. **1**, 84–94 (2021)
32. Sadek, L.A., Akgül, A., Bataineh, A.S., Hashim, I.: A cotangent fractional Gronwall inequality with applications. AIMS Math. **9**(4), 7819–7833 (2024)
33. Sadek, L.A.: Controllability, observability, and stability of φ-conformable fractional linear dynamical systems. Asian J. Control (2024)
34. Marin, M., Hobiny, A., Abbas, I.: The effects of fractional time derivatives in porothermoelastic materials using finite element method. Mathematics **9**(14), 1606 (2021)
35. Hobiny, A., Abbas, I.: Analytical solutions of fractional bioheat model in a spherical tissue. Mech. Based Des. Struct. Mach. **49**(3), 430–439 (2021)
36. Marin, M., Seadawy, A., Vlase, S., Chirila, A.: On mixed problem in thermos-elasticity of type III for Cosserat media. J. Taibah Univ. Sci. **16**(1), 1264–1274 (2022)
37. Marin, M., Hobiny, A., Abbas, I.: Finite element analysis of nonlinear bioheat model in skin tissue due to external thermal sources. Mathematics **9**, 1459 (2021)
38. Shannon, A.G., Özkan, E.: Some aspects of interchanging difference equation orders. Notes Number Theory Discrete Math. **28**(3), 507–516 (2022)
39. Abdeljawad, T., Thabet, S.T.M., Kedim, I., Ayari, M.I., Khan, A.: Atangana-Baleanu fractional operators with respect to another function and a Gronwall-type inequality. Bound. Value Probl. **2023**, 49 (2023)
40. Jarad, F., Alqudah, M.A., Abdeljawad, T.: On more general forms of proportional fractional operators. Open Math. **18**, 167–176 (2020)
41. Jarad, F., Abdeljawad, T., Rashid, S., Hammouch, Z.: More properties of the generalized proportional fractional integrals and derivatives of a function with respect to another function. Adv. Differ. Equ. **2020**, 303 (2020)
42. Kwun, Y.C., Farid, G., Nazeer, W., Ullah, S., Kang, S.M.: Generalized Riemann-Liouville k-fractional integrals associated with Ostrowski type inequalities and error bounds of Hadamard inequalities. IEEE Access **6**, 64946–64953 (2018)
43. Jarad, F., Abdeljawad, T.: Generalized fractional derivatives and Laplace transform. Discrete Contin. Dyn. Syst., Ser. S **13**(3), 709–722 (2020)
44. Almalahi, M.A., Panchal, S.K.: Some existence and stability results for ψ-Hilfer fractional implicit differential equation with periodic conditions. J. Math. Anal. Model. **1**(1), 1–19 (2020)
45. Lyons, R., Vatsala, A., Chiquet, R.: Picard's iterative method for Caputo fractional differential equations with numerical results. Mathematics **5**(4), 65 (2017)
46. Almeida, R.: A Caputo fractional derivative of a function with respect to another function. Commun. Nonlinear Sci. Numer. Simul. **44**, 460–481 (2017)
47. Dorrego, G.A.: An alternative definition for the k-Riemann-Liouville fractional derivative. Appl. Math. Sci. **9**(10), 481–491 (2015)
48. Katugampola, U.N.: A new approach to generalized fractional derivatives. Bull. Math. Anal. Appl. **6**(4), 1–15 (2014)
49. Jarad, F., Abdeljawad, T., Baleanu, D.: On the generalized fractional derivatives and their Caputo modification. J. Nonlinear Sci. Appl. **10**(5), 2607–2619 (2017)
50. Oliveira, D.S., de Oliveira, E.C.: Hilfer-Katugampola fractional derivatives. Comput. Appl. Math. **37**, 3672–3690 (2018)
51. Hilfer, R., Luchko, Y., Tomovski, Z.: Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives. Fract. Calc. Appl. Anal. **12**(12), 299–318 (2009)
52. Ntouyas, S.K., Tariboon, J., Sawaddee, C.: Nonlocal initial and boundary value problems via fractional calculus with exponential singular kernel. J. Nonlinear Sci. Appl. **11**, 1015–1030 (2018)

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