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Global solvability and boundedness to a attraction–repulsion model with logistic source

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Abstract

In this paper, we deal with an attraction–repulsion model with a logistic source as follows:

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w) + \mu u^q(1-u) & \text{in } Q,\\ v_t = \Delta v - \alpha_1 v + \beta_1 u & \text{in } Q,\\ w_t = \Delta w - \alpha_2 w + \beta_2 u & \text{in } Q,\end{cases}$$

where $Q = \Omega\times\mathbb{R}^+$, $\Omega\subset\mathbb{R}^3$ is a bounded domain. We mainly focus on the influence of logistic damping on the global solvability of this model. In dimension 2, q can be equal to 1 (Math. Methods Appl. Sci. 39(2):289–301, 2016). In dimension 3, we derive that the problem admits a global bounded solution when $q > \frac{8}{7}$. In fact, we transfer the difficulty of estimation to the logistic term through iterative methods, thus, compared to the results in (J. Math. Anal. Appl. 2:448 2017; Z. Angew. Math. Phys. 73(2):1–25 2022) in dimension 3, our results do not require any restrictions on the coefficients.

Mathematics Subject Classification: Primary 35K45; 92C17; secondary 35A01; 35Q92; 35B35

Keywords: Chemotaxis; Global solvability; Boundedness; Logistic source; Attraction–repulsion

1 Introduction

Chemotaxis refers to the directional movement of cells or organisms towards chemical stimuli along a concentration gradient, which plays an essential role in various biological processes such as wound healing, cancer invasion, and avoidance of predators [4]. The chemotaxis model was first proposed by Keller and Segel in [5] as follows:

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v), & x\in\Omega, t>0,\\ v_t = \Delta v - v + u, & x\in\Omega, t>0,\end{cases} \tag{1.1}$$

which describes the cellular slime mold move towards a higher concentration of a chemical signal. Since it was proposed, it has attracted interest of a large number of scholar's and they obtained many results about boundedness or blow-up of solutions [6–11]. In a one-dimensional bounded domain, (1.1) admits a global bounded solution [6]. But in dimension two, the solution of (1.1) may blow-up, that is, when $\|u_0\|_{L^1} < 4\pi$, the classical solution is global and bounded [7], and when $\|u_0\|_{L^1} \geq 4\pi$, $\|u_0\|_{L^1} \notin \{4k\pi | k \in \mathbb{N}\}$, the blow-up of the solution may occur in finite or infinite time [8, 9]. In dimension N ($N \geq 3$), a small critical mass condition alone is not enough to prevent the blow-up of the solution [10, 11].

However, cellular slime mold might experience proliferation or death during the process of directional movement. Since then, Mimura and Tsujikawa [12] introduced a generalized K–S model which considers the proliferation and death of bacteria, that is,

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + f(u), & x \in \Omega, t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega, t > 0, \end{cases} \tag{1.2}$$

where $\tau \geq 0$, $f(u)$ represents the proliferation and death of bacteria, which we call a logistic term. Actually, $\tau = 0$ is a simplified form, which represents the case when chemicals move much faster than bacteria [13]. When $\tau = 0$, we assume that $f \in C^1([0,\infty))$, and it satisfies $f(s) \leq c - \mu s^2$ for all $s \geq 0$, $f(s) > 0$ if $0 < s < 1$, and $f(s) < 0$ if $s > 1$, with some positive constants c, μ. Tello and Winkler [14] derived that the model (1.2) admits a unique global bounded classical solution when either $N \leq 2$ or $\mu > \frac{N-2}{N}\chi$. When $\tau > 0$, in 2001, Osaki and Yagi [15] proved the model (1.2) admits a global bounded classical solution in $\mathbb{R}^1$, and then in 2002, they also derived similar results in $\mathbb{R}^2$ [16, 17]. In 2010, Winkler [18] extended their results to arbitrary space dimensions with the condition $\mu > \mu_0$ for some $\mu_0 > 0$. For other results related to (1.2), please refer to [19–22]. In addition, related mathematical models which describe the chemotaxis phenomenon were widely studied, such as the chemotaxis–haptotaxis model [23, 24], chemotaxis–fluid model [25, 26], attraction–repulsion chemotaxis model [27], and so on.

To describe the formation of senile plaques in Alzheimer disease (AD), Luca et al. [28] proposed a attraction–repulsion chemotaxis system in 2003. The interesting aspect of this model is that it includes both chemoattraction and chemorepulsion, and the model read as

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w) + f(u), & \text{in } Q, \\ \tau_1 v_t = \Delta v - \alpha_1 v + \beta_1 u, & \text{in } Q, \\ \tau_2 w_t = \Delta w - \alpha_2 w + \beta_2 u, & \text{in } Q, \\ \frac{\partial u}{\partial \mathbf{n}}|_{\partial\Omega} = \frac{\partial v}{\partial \mathbf{n}}|_{\partial\Omega} = \frac{\partial w}{\partial \mathbf{n}}|_{\partial\Omega} = 0, & \\ u(x,0) = u_0(x), \quad \tau_1 v(x,0) = v_0(x), \quad \tau_2 w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where $Q = \Omega \times \mathbb{R}^+$, Ω is a bounded domain; $\frac{\partial}{\partial \mathbf{n}}$ represent the derivative with respect to the outer normal of $\partial\Omega$; u, v, w represent the concentration of microglia, the concentration of chemoattractant and the concentration of chemorepellent, respectively; χ and ξ are chemotactic coefficients; $f(u)$ represents the proliferation or death of cells; $\alpha_i, \beta_i > 0$ ($i = 1, 2$) are rates of production and decay of the chemicals, respectively. This model attracted a large number of scholars, and they obtained many results of this model. First, we introduce the results if $f(u) \equiv 0$. In [29], Tao and Wang derived that the model (1.3) admits a global classical solution in higher dimensions if repulsion prevails over attraction ($\xi\alpha_2 - \chi\alpha_1 > 0$) for the case $\tau_1 = \tau_2 = 0$; they also proved that the model (1.3) admits a global classical solution in $\mathbb{R}^2$ if repulsion prevails over attraction for the case $\tau_1 = \tau_2 = 1$ which needs the smallness assumption on the initial data u_0. Then Jin [30] removed the smallness assumption on u_0 in [29], and proved that model (1.3) admits a unique nonnegative classical solution when $\tau_1 = \tau_2 = 1$ in $\mathbb{R}^2$. The global solvability of the model (1.3) for the case $\xi\alpha_2 - \chi\alpha_1 = 0$ in $\mathbb{R}^2$ was established in [31]. The results in [32] indicated that even for the case $\xi\alpha_2 - \chi\alpha_1 < 0$, (1.3) admits a global classical solution in $\mathbb{R}^2$ for the parabolic–parabolic system.

For the case $f(u) \not\equiv 0$, there are also many results. When $\tau_1 = \tau_2 = 0$, $f(u) \le \lambda - \mu u^p$, Li and Zhao [33] proved that the model (1.3) admits a unique global bounded classical solution if

$$\begin{cases} N \ge 1, & \xi\beta_2 > \chi\beta_1, \quad \text{and} \quad p \ge 1, \\ N \ge 2, & \xi\beta_2 = \chi\beta_1, \quad \text{and} \quad p > \frac{1}{2}(\sqrt{N^2+2N} - N + 2), \\ N \ge 2, & \xi\beta_2 < \chi\beta_1, \quad \text{and} \quad p > 2 \quad \text{or} \quad p = 2 \quad \text{and} \quad \mu > \frac{N-2}{N}(\chi\beta_1 - \xi\beta_2), \\ N = 1, & \xi\beta_2 \le \chi\beta_1, \quad \text{and} \quad p \ge 1. \end{cases}$$

The results indicated that the model (1.3) admits a global bounded classical solution under the weak logistic damping in the case $\xi\beta_2 > \chi\beta_1$, but when $\xi\beta_2 < \chi\beta_1$, the logistic damping must be stronger. Xu and Zheng [34] improved the the results in [33] for the case $\xi\beta_2 = \chi\beta_1$ by proving that a weaker restriction $p > \frac{2N+2}{N+2}$ is sufficient to ensure the global boundedness of solutions. When $\tau_1 = \tau_2 = 0, f(u) \le \mu u(1-u)$, Zhang and Li [35] proved that the model (1.3) admits a unique global bounded classical solution if one of the following assumptions holds:

(a) $\chi\alpha_1 - \xi\alpha_2 \le \mu$,

(b) $N \le 2$,

(c) $\dfrac{N-2}{N}(\chi\alpha_1 - \xi\alpha_2) < \mu, \quad N \ge 3.$

When $\tau_1 = \tau_2 = 1, f(u) \le \lambda u - \mu u^p$ with $p \ge 1$, $\xi\beta_2 = \chi\beta_1$, Wang, Zhuang, and Zheng [36] proved that the model (1.3) admits a global bounded classical solution if

$$N \le 3, \quad \text{or}$$
$$p > p_N := \min\left\{\frac{N+2}{4}, \frac{N\sqrt{N^2+6N+17} - N^2 - 3N + 4}{4}\right\}, \quad \text{with } N \ge 2.$$

In [2], Li et al. proved that when $\tau_1 = \tau_2 = 1, f(u) = u(1-\mu u^p)$ with $p \ge 1$, the model (1.3) admits a global bounded classical solution if $\alpha_i (i = 1, 2) \ge \frac{1}{2}$, $\mu \ge \max\{(\frac{41}{2}\chi\beta_1 + 9\xi\beta_2)^p, (9\chi\beta_1 + \frac{41}{2}\xi\beta_2)^p\}$ in $\mathbb{R}^3$. In 2022, Ren and Liu [3] improved their results in [2] by avoiding any restriction on α_i $(i = 1, 2)$.

In this paper, we consider the following attraction–repulsion model:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w) + \mu u^q (1-u), & \text{in } Q, \\ v_t = \Delta v - \alpha_1 v + \beta_1 u, & \text{in } Q, \\ w_t = \Delta w - \alpha_2 w + \beta_2 u, & \text{in } Q, \\ \frac{\partial u}{\partial \mathbf{n}}|_{\partial\Omega} = \frac{\partial v}{\partial \mathbf{n}}|_{\partial\Omega} = \frac{\partial w}{\partial \mathbf{n}}|_{\partial\Omega} = 0, & \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

where $Q = \Omega \times \mathbb{R}^+$, $\Omega \subset \mathbb{R}^3$ is a bounded domain. All the coefficients χ, ξ, μ, α_1, α_2, β_1, β_2 are positive. Actually, the model (1.4) admits a global classical solution when $q \geq 1$ in $\mathbb{R}^2$ [1], which requires no restrictions on the coefficients, and the logistic damping is weak. Our results indicate that the model (1.4) admits a global classical solution when $q > \frac{8}{7}$ in dimension 3. Compared to the results in [2, 3], our results do not require any restrictions on the coefficients. In fact, we transferred the difficulty of estimation to the logistic term. Our proof is mainly divided into two parts, that is, we consider the case $q \geq 2$ and $\frac{8}{7} < q < 2$. When $\frac{8}{7} < q < 2$, we use the iterative method to improve the regularity of u.

Then we give the assumptions of this paper:

$$\begin{cases} u_0 \in C^0(\overline{\Omega}), \quad u_0 \geq 0, \quad \text{and} \quad u_0 \not\equiv 0, \\ v_0 \in W^{1,\infty}(\Omega), \quad v_0 \geq 0, \\ w_0 \in W^{1,\infty}(\Omega), \quad w_0 \geq 0. \end{cases} \tag{H}$$

Our main results read as follows:

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Assume that* (H) *holds and $q > \frac{8}{7}$. Then for any $\chi, \xi, \mu, \alpha_1, \alpha_2, \beta_1, \beta_2 > 0$,* (1.4) *admits a unique global bounded classical solution which satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leq C \quad \text{for all } t > 0,$$

where C is independent of t.

2 Preliminaries

In this paper, for the convenience of writing, we denote $\|\cdot\|_{L^p} = \|\cdot\|_{L^p(\Omega)}$. Since all the estimates of v and w are almost the same except for the coefficients, we show the details for v, and just list the results for w without any proof. Next, we will give some lemmas, which will be used throughout this paper.

Lemma 2.1 ([37]) *Let $T > 0$, $\tau \in (0,T)$, $\delta \geq 0$, $a > 0$, $b \geq 0$, and suppose that $f : [0,T) \to [0,\infty)$ is absolutely continuous and satisfies*

$$f'(t) + a f^{1+\delta}(t) \leq h(t), \quad t \in \mathbb{R},$$

where $h \geq 0$, $h(t) \in L^1_{\text{loc}}([0,T))$, and

$$\int_{t-\tau}^{t} h(s)\,ds \leq b, \quad \text{for all } t \in [\tau, T).$$

Then

$$\sup_{t\in(0,T)} f(t) + a \sup_{t\in(\tau,T)} \int_{t-\tau}^{t} f^{1+\delta}(s)\,ds \le b + 2\max\left\{f(0) + b + a\tau, \frac{b}{a\tau} + 1 + 2b + 2a\tau\right\}.$$

Lemma 2.2 ([38]) *Assume $u_0 \in W^{2,p}(\Omega)$, and $f \in L^p_{\mathrm{loc}}([0,+\infty); L^p(\Omega))$ with*

$$\sup_{t\in(\tau,+\infty)} \int_{t-\tau}^{t} \|f\|_{L^p}^p\,ds \le A,$$

where $\tau > 0$ is a fixed constant. Then the following problem:

$$\begin{cases} u_t - \alpha\Delta u + \beta u = f(x,t), \\ \frac{\partial u}{\partial \boldsymbol{n}}|_{\partial\Omega} = 0, \\ u(x,0) = u_0(x) \end{cases}$$

admits a unique solution $u \in L^p_{\mathrm{loc}}([0,+\infty); W^{2,p}(\Omega))$, $u_t \in L^p_{\mathrm{loc}}([0,+\infty); L^p(\Omega))$ with

$$\sup_{t\in(\tau,+\infty)} \int_{t-\tau}^{t} \left(\|u\|_{W^{2,p}}^p + \|u_t\|_{L^p}^p\right) ds \le C(A,\alpha,\beta)\frac{e^{p\tau}}{e^{\frac{p}{2}\tau} - 1} + C(\alpha,\beta)e^{\frac{p}{2}\tau}\|u_0\|_{W^{2,p}}^p,$$

where $C(A,\alpha,\beta)$ and $C(\alpha,\beta)$ are constants independent of τ.

Remark 2.1 In this paper, we fix $\tau = \min\{1, \frac{T_{\max}}{2}\} \le 1$. Thus, all the constants in this paper are independent of τ. In fact, if $\tau = 1$, it is easy to see that the constants in Lemma 2.2 can be fixed. While if $\tau < 1$, it implies that $T_{\max} < 2$, all the $\int_{t-\tau}^{t} \|\cdot\|\,ds$ can be replaced by $\int_0^{T_{\max}} \|\cdot\|\,ds$.

Lemma 2.3 ([23]) *Assume that Ω is bounded and let $\omega \in C^2(\overline{\Omega})$ satisfy $\frac{\partial\omega}{\partial\boldsymbol{n}}|_{\partial\Omega} = 0$, where $\boldsymbol{n}$ is the outward unit normal vector to the boundary $\partial\Omega$. Then we have*

$$\frac{\partial|\nabla\omega|^2}{\partial\boldsymbol{n}} \le 2\kappa|\nabla\omega|^2, \quad \text{on } \partial\Omega,$$

where $\kappa > 0$ is an upper bound for the curvatures of Ω.

3 Main results

Using a fixed point argument similar to that in [39], we obtain the following local existence result of classical solution to (1.4).

Lemma 3.1 *Let $\Omega \subset \mathbb{R}^N$, $N \ge 1$ be a bounded domain with a smooth boundary. Assume that u_0, v_0, w_0 satisfy (H). Then there exists $T_{\max} \in (0,+\infty]$ such that the problem (1.4) admits a unique nonnegative classical solution $(u,v,w) \in C^0(\overline{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max}))$. Moreover, either $T_{\max} = \infty$, or*

$$\lim_{t\to T_{\max}} \left(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)}\right) = \infty, \quad \text{if } T_{\max} < \infty.$$

By Lemma 3.1, we see that in order to prove the global existence of a classical solution, we assume that $T_{\max} < \infty$, and only need to show the boundedness of $\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)}$ for any $t \in (0, T_{\max})$.

It is not difficult to obtain the following lemma.

Lemma 3.2 *Let (u,v,w) be the solution of (1.4), and assume (H) holds. Then we have*

$$\sup_{t\in(0,T_{\max})} \|u(\cdot,t)\|_{L^1} + \mu \sup_{t\in(\tau,T_{\max})} \int_{t-\tau}^t \int_\Omega u^{q+1}\,dx\,ds \le C, \tag{3.1}$$

$$\sup_{t\in(\tau,T_{\max})} \int_{t-\tau}^t \|v(\cdot,s)\|_{W^{2,q+1}}^{q+1}\,ds \le C, \tag{3.2}$$

$$\sup_{t\in(\tau,T_{\max})} \int_{t-\tau}^t \|w(\cdot,s)\|_{W^{2,q+1}}^{q+1}\,ds \le C, \tag{3.3}$$

$$\sup_{t\in(0,T_{\max})} \|v(\cdot,t)\|_{H^1}^2 + \sup_{t\in(\tau,T_{\max})} \int_{t-\tau}^t \|v(\cdot,t)\|_{H^2}^2\,ds \le C, \tag{3.4}$$

$$\sup_{t\in(0,T_{\max})} \|w(\cdot,t)\|_{H^1}^2 + \sup_{t\in(\tau,T_{\max})} \int_{t-\tau}^t \|w(\cdot,t)\|_{H^2}^2\,ds \le C, \tag{3.5}$$

where the constants C at most depend on χ, μ, ξ, α_1, α_2, β_1, β_2, u_0, v_0, w_0, but are independent of $T_{\max}$ and τ.

Proof By a direct integration to the first equation of (1.4), we have

$$\frac{d}{dt}\int_\Omega u\,dx + \mu\int_\Omega u^{q+1}\,dx = \mu\int_\Omega u^q\,dx \le \frac{\mu}{2}\int_\Omega u^{q+1}\,dx + C_1,$$

which means

$$\frac{d}{dt}\int_\Omega u\,dx + \frac{\mu}{2}\int_\Omega u^{q+1}\,dx \le C_1.$$

Since Ω is a bounded domain, it is easy to see that $(\int_\Omega u\,dx)^{q+1} \le \int_\Omega u^{q+1}\,dx$ for any $q \ge 1$, and then, by Lemma 2.1, we obtain (3.1).

Using (3.1) and Lemma 2.2, we obtain (3.2). Similarly, we obtain (3.3).

Multiplying the second equation of (1.4) by v and $-\Delta v$, respectively, integrating them over Ω, and using Young's inequality, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega v^2\,dx + \int_\Omega |\nabla v|^2\,dx + \alpha_1\int_\Omega v^2\,dx &= \beta_1\int_\Omega uv\,dx \\
&\le \frac{1}{2}\alpha_1\int_\Omega v^2\,dx + C_2\int_\Omega u^2\,dx, \\
\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v|^2\,dx + \int_\Omega |\Delta v|^2\,dx + \alpha_1\int_\Omega |\nabla v|^2\,dx &= -\beta_1\int_\Omega u\Delta v \\
&\le \frac{1}{2}\int_\Omega |\Delta v|^2\,dx + C_3\int_\Omega u^2\,dx.
\end{aligned}$$

Combining the above two inequalities yields

$$\frac{d}{dt}\int_\Omega \left(v^2+|\nabla v|^2\right)dx+\int_\Omega \left(v^2+|\nabla v|^2+|\Delta v|^2\right)dx\le C_4\int_\Omega u^2\,dx$$

and then, by (3.1) and Lemma 2.1, we obtain (3.4). Similarly, we have (3.5). □

Through the above lemma, we have the following results.

Lemma 3.3 *Let (u,v,w) be the solution of (1.4), and assume (H) holds. Then for any $r\ge 2$, $R>1$, $p>1$, we have*

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^p\,dx+\frac{p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2\,dx+\frac{\mu p}{2}\int_\Omega u^{p+q}\,dx\\&\quad\le C\int_\Omega\left(|\nabla v|^{\frac{2(p+q)}{q}}+|\nabla w|^{\frac{2(p+q)}{q}}\right)dx+C,\end{aligned}\tag{3.6}$$

$$\begin{aligned}&\frac{1}{r}\frac{d}{dt}\int_\Omega|\nabla v|^r\,dx+\frac{3}{4}\int_\Omega|\nabla v|^{r-2}\left|D^2v\right|^2dx+\frac{r-2}{2}\int_\Omega|\nabla v|^{r-2}\left(\nabla|\nabla v|\right)^2dx\\&\qquad+\frac{3}{4}\alpha_1\int_\Omega|\nabla v|^r\,dx\\&\quad\le\frac{1}{4}\alpha_1\|\nabla v\|_{L^{R(r-2)}}^{R(r-2)}+C\|u\|_{L^{\frac{2R}{R-1}}}^{\frac{2R}{R-1}}+C,\end{aligned}\tag{3.7}$$

$$\begin{aligned}&\frac{1}{r}\frac{d}{dt}\int_\Omega|\nabla w|^r\,dx+\frac{3}{4}\int_\Omega|\nabla w|^{r-2}\left|D^2w\right|^2dx+\frac{r-2}{2}\int_\Omega|\nabla w|^{r-2}\left(\nabla|\nabla w|\right)^2dx\\&\qquad+\frac{3}{4}\alpha_2\int_\Omega|\nabla w|^r\,dx\\&\quad\le\frac{1}{4}\alpha_2\|\nabla w\|_{L^{R(r-2)}}^{R(r-2)}+C\|u\|_{L^{\frac{2R}{R-1}}}^{\frac{2R}{R-1}}+C,\end{aligned}\tag{3.8}$$

where C at most depend on χ, μ, ξ, α_1, α_2, β_1, β_2, u_0, v_0, w_0, but are independent of $T_{\max}$ and τ.

Proof For any $p>1$, multiplying the first equation of (1.4) by pu^{p-1}, and integrating it over Ω, for any $q>1$, we have

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^p\,dx+p(p-1)\int_\Omega u^{p-2}|\nabla u|^2\,dx+\mu p\int_\Omega u^{p+q}\,dx\\&\quad=\chi p(p-1)\int_\Omega u^{p-1}\nabla u\nabla v\,dx-\xi p(p-1)\int_\Omega u^{p-1}\nabla u\nabla w\,dx+\mu p\int_\Omega u^{p+q-1}\,dx\\&\quad\le\frac{p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2\,dx+C_5\int_\Omega u^p\left(|\nabla v|^2+|\nabla w|^2\right)dx+\frac{\mu p}{4}\int_\Omega u^{p+q}\,dx+C_6\\&\quad\le\frac{p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2\,dx+C_7\int_\Omega\left(|\nabla v|^{\frac{2(p+q)}{q}}+|\nabla w|^{\frac{2(p+q)}{q}}\right)dx\\&\qquad+\frac{\mu p}{2}\int_\Omega u^{p+q}\,dx+C_8,\end{aligned}$$

and then rearranging it we obtain (3.6).

Applying ∇ to the second equation of (1.4), and multiplying the resulting equation by $|\nabla v|^{r-2}\nabla v$ (for any $r \geq 2$), by a direct calculation, it is easy to see that $|\nabla v|^{r-2}\nabla v \cdot \nabla\Delta v = \frac{1}{2}|\nabla v|^{r-2}\Delta|\nabla v|^2 - |\nabla v|^{r-2}|D^2 v|^2$. Now integrating the result over Ω and combining with Lemma 2.3, we have

$$\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\int_\Omega |\nabla v|^r\,dx + \int_\Omega |\nabla v|^{r-2}\left|D^2 v\right|^2 dx \\
&\quad + (r-2)\int_\Omega |\nabla v|^{r-2}\left(\nabla|\nabla v|\right)^2 dx + \alpha_1\int_\Omega |\nabla v|^r\,dx \\
&= \beta_1\int_\Omega |\nabla v|^{r-2}\nabla v\nabla u\,dx + \frac{1}{2}\int_{\partial\Omega}\frac{\partial(|\nabla v|^2)}{\partial n}|\nabla v|^{r-2}\,dx \\
&= -\beta_1\int_\Omega u|\nabla v|^{r-2}\Delta v\,dx - \beta_1(r-2)\int_\Omega u|\nabla v|^{r-3}\nabla v\nabla|\nabla v|\,dx \\
&\quad + \frac{1}{2}\int_{\partial\Omega}\frac{\partial(|\nabla v|^2)}{\partial n}|\nabla v|^{r-2}\,dx \\
&\leq \frac{1}{4}\int_\Omega |\nabla v|^{r-2}\left|D^2 v\right|^2 dx + \frac{r-2}{4}\int_\Omega |\nabla v|^{r-2}\left(\nabla|\nabla v|\right)^2 dx \\
&\quad + C_9\int_\Omega |\nabla v|^{r-2}u^2\,dx + \kappa\int_{\partial\Omega}|\nabla v|^r\,ds.
\end{aligned}$$

Rearranging the above inequality, for any $R > 1$, we have

$$\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\int_\Omega |\nabla v|^r\,dx + \frac{3}{4}\int_\Omega |\nabla v|^{r-2}\left|D^2 v\right|^2 dx \\
&\quad + \frac{3(r-2)}{4}\int_\Omega |\nabla v|^{r-2}\left(\nabla|\nabla v|\right)^2 dx + \alpha_1\int_\Omega |\nabla v|^r\,dx \\
&\leq C_9\int_\Omega |\nabla v|^{r-2}u^2\,dx + \kappa\int_{\partial\Omega}|\nabla v|^r\,ds \\
&\leq \frac{1}{4}\alpha_1\|\nabla v\|_{L^{R(r-2)}}^{R(r-2)} + C_{10}\|u\|_{L^{\frac{2R}{R-1}}}^{\frac{2R}{R-1}} + \kappa\int_{\partial\Omega}|\nabla v|^r\,ds.
\end{aligned} \tag{3.9}$$

It is easy to see that for any $\delta > 0$ and $f \in L^{q_1}\cap L^{q_2}$, we have

$$\|f\|_{L^{r'}} \leq \|f\|_{L^{q_1}}^{\alpha}\|f\|_{L^{q_2}}^{1-\alpha} \leq \delta\|f\|_{L^{q_1}} + C_\delta\|f\|_{L^{q_2}}, \tag{3.10}$$

where $1 \leq q_1 < q_2$, $r' \in [q_1, q_2]$, $\alpha = \frac{q_1(q_2-r')}{r'(q_2-q_1)}$. Using the boundary trace embedding inequalities [40], (3.4) and (3.10), we obtain

$$\begin{aligned}
\kappa\int_{\partial\Omega}|\nabla v|^r\,ds &\leq \varepsilon\left\|\nabla\left(|\nabla v|^{\frac{r}{2}}\right)\right\|_{L^2}^2 + C_\varepsilon\left\||\nabla v|^{\frac{r}{2}}\right\|_{L^1}^2 \\
&\leq \varepsilon\left\|\nabla\left(|\nabla v|^{\frac{r}{2}}\right)\right\|_{L^2}^2 + \frac{1}{4}\alpha_1\|\nabla v\|_{L^r}^r + C(\varepsilon,\alpha_1)\|\nabla v\|_{L^1}^r \\
&\leq \varepsilon\left\|\nabla\left(|\nabla v|^{\frac{r}{2}}\right)\right\|_{L^2}^2 + \frac{1}{4}\alpha_1\|\nabla v\|_{L^r}^r + C(\varepsilon,\alpha_1,\Omega)\|\nabla v\|_{L^2}^r \\
&\leq \varepsilon\left\|\nabla\left(|\nabla v|^{\frac{r}{2}}\right)\right\|_{L^2}^2 + \frac{1}{4}\alpha_1\|\nabla v\|_{L^r}^r + C_{11}.
\end{aligned}$$

Now taking $\varepsilon = \frac{r-2}{r^2}$ and combining this inequality with (3.9), we obtain (3.7). □

(I) Estimates for $q \geq 2$.

Lemma 3.4 *Let (u, v, w) be the solution of (1.4), and assume (H) holds. If $N = 3$, $q \geq 2$, then for any $p > 1$, we have*

$$\sup_{t\in(0,T_{\max})} \|u(\cdot,t)\|_{L^p}^p + \sup_{t\in(\tau,T_{\max})} \int_{t-\tau}^t \|\nabla(u^{\frac{p}{2}})\|_{L^2}^2 \, ds \leq C_p, \tag{3.11}$$

$$\sup_{t\in(0,T_{\max})} \|v(\cdot,t)\|_{W^{1,\infty}} \leq C, \tag{3.12}$$

$$\sup_{t\in(0,T_{\max})} \|w(\cdot,t)\|_{W^{1,\infty}} \leq C, \tag{3.13}$$

where C_p depends only on p, Ω, u_0, and C depend on Ω, u_0. All of them are independent of τ and $T_{\max}$.

Proof Recalling (3.1), and taking $\frac{2R}{R-1} = q+1$, $r = q+1$ in (3.7) and (3.8), then using Lemma 2.1, we obtain

$$\sup_{t\in(0,T_{\max})} \|\nabla v(\cdot,t)\|_{L^{q+1}}^{q+1} + \sup_{t\in(\tau,T_{\max})} \int_{t-\tau}^t \int_\Omega \left(\nabla\left(|\nabla v|^{\frac{q+1}{2}}\right)\right)^2 dx\, ds \leq C_{12}, \tag{3.14}$$

$$\sup_{t\in(0,T_{\max})} \|\nabla w(\cdot,t)\|_{L^{q+1}}^{q+1} + \sup_{t\in(\tau,T_{\max})} \int_{t-\tau}^t \int_\Omega \left(\nabla\left(|\nabla w|^{\frac{q+1}{2}}\right)\right)^2 dx\, ds \leq \overline{C_{12}}. \tag{3.15}$$

For any $p > 1$, multiplying by u^{p-1} the first equation of (1.4), and integrating the result over Ω, we have

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p\,dx + \mu\int_\Omega u^{q+p}\,dx + (p-1)\int_\Omega u^{p-2}|\nabla u|^2\,dx \\
&\quad = \chi(p-1)\int_\Omega u^{p-1}\nabla u\nabla v\,dx - \xi(p-1)\int_\Omega u^{p-1}\nabla u\nabla w\,dx + \mu\int_\Omega u^{p+q-1}\,dx \\
&\quad \leq \frac{p-1}{2}\int_\Omega u^{p-2}|\nabla u|^2\,dx + C_{13}\int_\Omega u^p\left(|\nabla v|^2 + |\nabla w|^2\right)dx + \frac{\mu}{2}\int_\Omega u^{q+p}\,dx + C_{14}.
\end{aligned}$$

Now rearranging it, we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u^p\,dx + \frac{1}{2}p(p-1)\int_\Omega u^{p-2}|\nabla u|^2\,dx + \frac{1}{2}\mu p\int_\Omega u^{p+q}\,dx \\
&\quad \leq C_{13}\int_\Omega u^p\left(|\nabla v|^2 + |\nabla w|^2\right)dx + C_{14}.
\end{aligned} \tag{3.16}$$

Using (3.1), (3.14), and Gagliardo–Nirenberg interpolation inequality [34], for any $q > 2$, we have

$$\begin{aligned}
C_{13}\int_\Omega u^p|\nabla v|^2\,dx &= C_{13}\left\|u^{\frac{p}{2}}|\nabla v|\right\|_{L^2}^2 \\
&\leq C_{13}\left\|u^{\frac{p}{2}}\right\|_{L^{\frac{2(q+1)}{q-1}}}^2 \|\nabla v\|_{L^{q+1}}^2 \\
&\leq C_{14}\left\|u^{\frac{p}{2}}\right\|_{L^2}^{\frac{2(q-2)}{q+1}} \left\|\nabla\left(u^{\frac{p}{2}}\right)\right\|_{L^2}^{\frac{6}{q+1}} + C_{15}\|u\|_{L^1}^p
\end{aligned}$$

$$\leq \varepsilon \left\| \nabla \left(u^{\frac{p}{2}}\right) \right\|_{L^2}^2 + C_\varepsilon \left\| u^{\frac{p}{2}} \right\|_{L^2}^2 + C_{16}$$
$$\leq \varepsilon \left\| \nabla \left(u^{\frac{p}{2}}\right) \right\|_{L^2}^2 + \frac{1}{4}\mu p \int_\Omega u^{p+q}\,dx + C_\varepsilon.$$

Similarly, we have

$$C_{13} \int_\Omega u^p |\nabla w|^2\,dx \leq \overline{\varepsilon} \left\| \nabla \left(u^{\frac{p}{2}}\right) \right\|_{L^2}^2 + \frac{1}{4}\mu p \int_\Omega u^{p+q}\,dx + \overline{C_\varepsilon}.$$

Taking $\varepsilon, \overline{\varepsilon} = \frac{p-1}{2p}$, and using the above two inequalities in (3.16), we obtain

$$\frac{d}{dt} \int_\Omega u^p\,dx + \frac{1}{4} p(p-1) \int_\Omega u^{p-2} |\nabla u|^2\,dx + \frac{1}{4}\mu p \int_\Omega u^{p+q}\,dx \leq C_{17}.$$

And then by Lemma 2.1, we obtain (3.11).

Now we consider the case for $q = 2$. Taking $p = 3$ in (3.16), we obtain

$$\frac{d}{dt} \int_\Omega u^3\,dx + 2 \int_\Omega u |\nabla u|^2\,dx + \frac{3}{2}\mu \int_\Omega u^5\,dx$$
$$\leq C_{18} \int_\Omega u^3 \left(|\nabla v|^2 + |\nabla w|^2\right) dx + C_{19}$$
$$\leq \frac{3}{4}\mu \int_\Omega u^5\,dx + C_{20} \int_\Omega \left(|\nabla v|^5 + |\nabla w|^5\right) dx + C_{19}.$$

By Gagliardo–Nirenberg interpolation inequality and (3.14), we have

$$\|\nabla v\|_{L^5}^5 = \left\| |\nabla v|^{\frac{3}{2}} \right\|_{L^{\frac{10}{3}}}^{\frac{10}{3}}$$
$$\leq C_\varepsilon \left\| |\nabla v|^{\frac{3}{2}} \right\|_{L^2}^{\frac{4}{3}} \left\| \nabla \left(|\nabla v|^{\frac{3}{2}}\right) \right\|_{L^2}^2 + C_\varepsilon \|\nabla v\|_{L^3}^5 \leq C_\varepsilon \left(1 + \left\| \nabla \left(|\nabla v|^{\frac{3}{2}}\right) \right\|_{L^2}^2\right).$$

Similarly, we get

$$\|\nabla w\|_{L^5}^5 \leq \overline{C_\varepsilon} \left(1 + \left\| \nabla \left(|\nabla w|^{\frac{3}{2}}\right) \right\|_{L^2}^2\right).$$

Now combining these above three inequalities, we have

$$\frac{d}{dt} \int_\Omega u^3\,dx + 2 \int_\Omega u |\nabla u|^2\,dx + \frac{3}{4}\mu \int_\Omega u^5\,dx$$
$$\leq C_\varepsilon \left(1 + \left\| \nabla \left(|\nabla v|^{\frac{3}{2}}\right) \right\|_{L^2}^2\right) + \overline{C_\varepsilon} \left(1 + \left\| \nabla \left(|\nabla w|^{\frac{3}{2}}\right) \right\|_{L^2}^2\right) + C_{19}.$$

Using (3.14), (3.15), and Lemma 2.1, we obtain

$$\sup_{t \in (0, T_{\max})} \left\| u(\cdot, t) \right\|_{L^3}^3 + \sup_{t \in (\tau, T_{\max})} \int_{t-\tau}^t \left\| \nabla \left(u^{\frac{3}{2}}\right) \right\|_{L^2}^2 ds \leq C. \tag{3.17}$$

By Duhamel's principle, the second equation of (1.4) can be expressed as follows:

$$v(t) = e^{-\alpha_1 t} e^{t\Delta} v_0 + \beta_1 \int_0^t e^{-\alpha_1 (t-s)} e^{\alpha_1 (t-s)\Delta} u(s)\,ds,$$

where $\{e^{t\Delta}\}_{t\geq 0}$ represents the Neumann heat semigroup in Ω; for more details about the theory of Neumann heat semigroups, please refer to [10, 41, 42]. Then for any $r \in (1, +\infty)$, $t \in (0, T_{\max})$, we have

$$\begin{aligned}
&\left\|\nabla v(\cdot,t)\right\|_{L^r} \\
&\quad\leq e^{-\alpha_1 t}\|\nabla v_0\|_{L^r} + C_{21}\int_0^t e^{-\alpha_1(t-s)}\left[\alpha_1(t-s)\right]^{-\frac{3}{2}(\frac{1}{3}-\frac{1}{r})-\frac{1}{2}}\left\|u(s)\right\|_{L^3}\,ds \\
&\quad\leq e^{-\alpha_1 t}\|\nabla v_0\|_{L^r} + \sup_{s\in(0,T_{\max})}\left\|u(s)\right\|_{L^3}\int_0^\infty e^{-s}s^{-1+\frac{3}{2r}}\,ds \leq C_{22},
\end{aligned}$$

thus, we obtain

$$C_{18}\int_\Omega u^P|\nabla v|^2\,dx \leq C_{23}\|u\|_{L^{p+q}}^p\|\nabla v\|^2_{L^{\frac{2(p+q)}{q}}} \leq C_{24}\|u\|_{L^{p+q}}^p \leq \frac{1}{4}\mu p\int_\Omega u^{p+q}\,dx + C_{25}.$$

Similarity, we have

$$\overline{C_{18}}\int_\Omega u^P|\nabla w|^2\,dx \leq \frac{1}{4}\mu p\int_\Omega u^{p+q}\,dx + \overline{C_{25}},$$

and then, using the above two inequalities in (3.16) and taking advantage of Lemma 2.1, we have (3.11).

Similar to the proof above, and by (3.11), for any $t \in (0, T_{\max})$, we also have

$$\begin{aligned}
\left\|v(\cdot,t)\right\|_{L^\infty} &\leq e^{-\alpha_1 t}\|v_0\|_{L^\infty} + C_{26}\int_0^t e^{-\alpha_1(t-s)}\left[\alpha_1(t-s)\right]^{-\frac{3}{2}\cdot\frac{1}{3}}\left\|u(s)\right\|_{L^3}\,ds \\
&\leq e^{-\alpha_1 t}\|v_0\|_{L^\infty} + \sup_{s\in(0,T_{\max})}\left\|u(s)\right\|_{L^3}\int_0^\infty e^{-s}s^{-\frac{1}{2}}\,ds \leq C_{27}
\end{aligned}$$

and

$$\begin{aligned}
\left\|\nabla v(\cdot,t)\right\|_{L^\infty} &\leq e^{-\alpha_1 t}\|\nabla v_0\|_{L^\infty} + C_{28}\int_0^t e^{-\alpha_1(t-s)}\left[\alpha_1(t-s)\right]^{-\frac{1}{4}\cdot\frac{3}{2}-\frac{1}{2}}\left\|u(s)\right\|_{L^4}\,ds \\
&\leq e^{-\alpha_1 t}\|\nabla v_0\|_{L^\infty} + C_{29}\sup_{s\in(0,T_{\max})}\left\|u(s)\right\|_{L^4}\int_0^\infty e^{-s}s^{-\frac{7}{8}}\,ds \leq C_{30}.
\end{aligned}$$

The estimate of w is similar that of v, so we have (3.13). The proof is complete. □

(II) Estimates for $q < 2$.

Lemma 3.5 *Let (u, v, w) be the solution of (1.4). Assume (H) holds, and $a_n + q < 5$ with $q < 2$. If*

$$\int_{t-\tau}^t\int_\Omega u^{a_n+q}\,dx\,ds \leq C_n,$$

then for any $r < \frac{3(a_n+q)}{5-(a_n+q)}$, we have

$$\sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^t\int_\Omega|\nabla v|^{\frac{5}{3}r}\,dx\,ds \leq C_n(r), \tag{3.18}$$

$$\sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^{t}\int_{\Omega}|\nabla w|^{\frac{5}{3}r}\,dx\,ds\le C_n(r),\tag{3.19}$$

and for any $p+q<\frac{5q}{2}\cdot\frac{a_n+q}{5-(a_n+q)}$, *we have*

$$\sup_{t\in(0,T_{\max})}\int_{\Omega}u^p\,dx+\sup_{t\in(0,T_{\max})}\int_{t-\tau}^{t}\left(\left\|\nabla\left(u^{\frac{p}{2}}\right)\right\|_{L^2}^2+\|u\|_{L^{p+q}}^{p+q}\right)ds\le C_{n+1}(p),\tag{3.20}$$

where $C_n(r)$, $C_{n+1}(p)$ *are independent of* τ *and* $T_{\max}$, *which only depend on* p, r, n, u_0, v_0, *and* Ω.

Proof Taking $r=a_n+q$, $R=\frac{a_n+q}{a_n+q-2}$ in (3.7), where $\{a_n\}$ is positive with $a_1=1$, we have

$$\begin{aligned}&\frac{1}{a_n+q}\frac{d}{dt}\int_{\Omega}|\nabla v|^{a_n+q}\,dx+\frac{a_n+q-2}{2}\int_{\Omega}|\nabla v|^{a_n+q-2}\left(\nabla|\nabla v|\right)^2dx+\frac{1}{2}\alpha_1\int_{\Omega}|\nabla v|^{a_n+q}\,dx\\&\quad\le C_{31}\|u\|_{L^{a_n+q}}^{a_n+q}+C_{32}.\end{aligned}$$

Then by Lemma 2.1, we obtain

$$\sup_{t\in(0,T_{\max})}\left\|\nabla v(\cdot,t)\right\|_{L^{a_n+q}}^{a_n+q}+\sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^{t}\int_{\Omega}\left(\nabla\left(|\nabla v|^{\frac{a_n+q}{2}}\right)\right)^2dx\,ds\le C_{33}.\tag{3.21}$$

Moreover, we select a nonnegative sequence $\{r_k\}$ with $r_{k+1}=2+\frac{5(a_n+q-2)}{3(a_n+q)}r_k$. Obviously, r_k is monotonically increasing. Next, we prove that if

$$\sup_{t\in(0,T_{\max})}\left\|\nabla v(\cdot,t)\right\|_{L^{r_k}}^{r_k}+\sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^{t}\int_{\Omega}\left(\nabla\left(|\nabla v|^{\frac{r_k}{2}}\right)\right)^2dx\,ds\le C\tag{3.22}$$

then

$$\sup_{t\in(0,T_{\max})}\left\|\nabla v(\cdot,t)\right\|_{L^{r_{k+1}}}^{r_{k+1}}+\sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^{t}\int_{\Omega}\left(\nabla\left(|\nabla v|^{\frac{r_{k+1}}{2}}\right)\right)^2dx\,ds\le C.\tag{3.23}$$

By Gagliardo–Nirenberg interpolation inequality, we have

$$\begin{aligned}\left\||\nabla v|^{\frac{r_k}{2}}\right\|_{L^{\frac{10}{3}}}^{\frac{10}{3}}&\le C_{34}\left\||\nabla v|^{\frac{r_k}{2}}\right\|_{L^2}^{\frac{4}{3}}\left\|\nabla\left(|\nabla v|^{\frac{r_k}{2}}\right)\right\|_{L^2}^2+C_{35}\left\||\nabla v|^{\frac{r_k}{2}}\right\|_{L^2}^{\frac{10}{3}}\\&\le C_{36}\left(1+\left\|\nabla\left(|\nabla v|^{\frac{r_k}{2}}\right)\right\|_{L^2}^2\right),\end{aligned}$$

which means

$$\sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^{t}\left\|\nabla v(\cdot,s)\right\|_{\frac{5}{3}r_k}^{\frac{5}{3}r_k}ds=\sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^{t}\left\|\left|\nabla v(\cdot,s)\right|^{\frac{r_k}{2}}\right\|_{\frac{10}{3}}^{\frac{10}{3}}ds\le C_{37}.\tag{3.24}$$

Recalling (3.7) and taking $\frac{2R}{R-1}=a_n+q$, $(r-2)R=\frac{5}{3}r_k$, that is, $R=\frac{a_n+q}{a_n+q-2}$, $r=2+\frac{5(a_n+q-2)}{3(a_n+q)}r_k$, we then have

$$\frac{1}{r}\frac{d}{dt}\int_{\Omega}|\nabla v|^r\,dx+\frac{r-2}{2}\int_{\Omega}|\nabla v|^{r-2}\left(\nabla|\nabla v|\right)^2dx+\frac{3}{4}\alpha_1\int_{\Omega}|\nabla v|^r\,dx$$

$$\leq \frac{1}{4}\left\|\nabla v(\cdot,s)\right\|_{\frac{5}{3}r_k}^{\frac{5}{3}r_k} + C_{38}\|u\|_{L^{a_n+q}}^{a_n+q} + C_{39}.$$

Now (3.23) is obtained by Lemma 2.1. Note that $r_{k+1} - \frac{3(a_n+q)}{5-(a_n+q)} = \frac{5}{3} \cdot \frac{a_n+q-2}{a_n+q}(r_k - \frac{3(a_n+q)}{5-(a_n+q)})$, where $0 < \frac{5}{3} \cdot \frac{a_n+q-2}{a_n+q} < 1$, which means $\{r_k - \frac{3(a_n+q)}{5-(a_n+q)}\}$ is monotonically decreasing, so r_k goes to $\frac{3(a_n+q)}{5-(a_n+q)}$, thus (3.18) holds. Recalling (3.8) and taking $\frac{2(p+q)}{q} = \frac{5}{3}r$, similarly we can obtain (3.19) through (3.8), and thus (3.20) holds for any $p+q < \frac{5q}{2} \cdot \frac{a_n+q}{5-(a_n+q)}$. □

Lemma 3.6 *Let (u,v,w) be the solution of (1.4), assume (H) holds, and $\frac{8}{7} < q < 2$. Then for any $p \in (1,+\infty)$, we have*

$$\sup_{t\in(0,T_{\max})}\left\|u(\cdot,t)\right\|_{L^p}^p + \sup_{t\in(\tau,T_{\max})}\int_{t-\tau}^t \left\|\nabla\left(u^{\frac{p}{2}}\right)\right\|_{L^2}^2 ds \leq C_p, \tag{3.25}$$

$$\sup_{t\in(0,T_{\max})}\left\|v(\cdot,t)\right\|_{W^{1,\infty}} \leq C, \tag{3.26}$$

$$\sup_{t\in(0,T_{\max})}\left\|w(\cdot,t)\right\|_{W^{1,\infty}} \leq C, \tag{3.27}$$

where C are independent of τ and $T_{\max}$.

Proof Letting $A_{n+1} + q = \frac{5q}{2} \cdot \frac{A_n+q}{5-(A_n+q)}$ with $A_1 = 1$, we see that

$$\frac{A_2+q}{A_1+q} = \frac{5q}{2} \cdot \frac{1}{5-(A_1+q)} > 1,$$

$$\vdots$$

$$\frac{A_{n+1}+q}{A_n+q} = \frac{5q}{2} \cdot \frac{1}{5-(A_n+q)} > \cdots > \frac{5q}{2} \cdot \frac{1}{5-(A_1+q)} > 1.$$

Since $q > \frac{8}{7}$, it indicates that $\{A_{n+1}+q\}$ is monotonically increasing. Thus there exists $M = M(q)$ such that $A_M + q > 5$. Then by Lemma 3.5, there exists $p_M + q \geq 5$ such that (3.20) holds. Since $q < 2$, we have

$$\|u\|_{L^3} \leq C.$$

Then similar to the proof in Lemma 3.4 for the case $q = 2$, we have (3.25), (3.26), and (3.27). □

Proof of Theorem 1.1 Combining Lemmas 3.4 and 3.6, we see that for any $q > \frac{8}{7}$,

$$\sup_{t\in(0,T_{\max})}\left\|v(\cdot,t)\right\|_{W^{1,\infty}} \leq C, \tag{3.28}$$

$$\sup_{t\in(0,T_{\max})}\left\|w(\cdot,t)\right\|_{W^{1,\infty}} \leq C. \tag{3.29}$$

Next, we use the standard Moser's iterative technique to prove the boundedness of $\|u(\cdot,t)\|_{L^\infty}$.

Multiplying the first equation of (1.4) by pu^{p-1} (for any $p \geq 2$), and by (3.28), (3.29), we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u^p\,dx + p(p-1)\int_\Omega u^{p-2}|\nabla u|^2\,dx + \mu p\int_\Omega u^{p+q}\,dx \\
&= \chi p(p-1)\int_\Omega u^{p-1}\nabla u\nabla v\,dx - \xi p(p-1)\int_\Omega u^{p-1}\nabla u\nabla w\,dx + \mu p\int_\Omega u^{p+q-1}\,dx \\
&\leq \frac{1}{2}p(p-1)\int_\Omega u^{p-2}|\nabla u|^2\,dx \\
&\quad + \frac{1}{2}\max\{\chi^2,\xi^2\}p(p-1)\int_\Omega u^p\left(|\nabla v|^2 + |\nabla w|^2\right)dx + \mu p\int_\Omega u^{p+q-1}\,dx \\
&\leq \frac{1}{2}p(p-1)\int_\Omega u^{p-2}|\nabla u|^2\,dx + Cp^2\int_\Omega u^p\,dx + \frac{1}{2}\mu p\int_\Omega u^{p+q}\,dx,
\end{aligned}$$

which means

$$\frac{d}{dt}\int_\Omega u^p\,dx + \frac{1}{2}p(p-1)\int_\Omega u^{p-2}|\nabla u|^2\,dx + \int_\Omega u^p\,dx \leq Cp^2\int_\Omega u^p\,dx, \tag{3.30}$$

where C is independent of p. By Gagliardo–Nirenberg interpolation inequality and Young's inequality, we have

$$\begin{aligned}
Cp^2\int_\Omega u^p\,dx &= Cp^2\left\|u^{\frac{p}{2}}\right\|_{L^2}^2 \\
&\leq C_{40}p^2\left\|u^{\frac{p}{2}}\right\|_{L^1}^{\frac{4}{5}}\left\|\nabla\left(u^{\frac{p}{2}}\right)\right\|_{L^2}^{\frac{6}{5}} + C_{41}p^2\left\|u^{\frac{p}{2}}\right\|_{L^1}^2 \\
&\leq \left\|\nabla\left(u^{\frac{p}{2}}\right)\right\|_{L^2}^2 + C_{42}p^5\left\|u^{\frac{p}{2}}\right\|_{L^1}^2,
\end{aligned}$$

where C_{40}, C_{41}, C_{42} are independent of p, and $\frac{1}{2}p(p-1) > \frac{1}{4}p^2$ since $p \geq 2$. Thus, we have

$$\frac{d}{dt}\|u\|_{L^p}^p + \|u\|_{L^p}^p \leq Cp^{N+2}\|u\|_{L^{\frac{p}{2}}}^{2\cdot\frac{p}{2}}, \tag{3.31}$$

where C is independent of p. Taking $p_j = 2p_{j-1}$ with $p_1 = 2$, $Q_j = \max\{\sup_{t\in(0,T_{\max})}\|u(\cdot,t)\|_{L^{p_j}}, \|u_0\|_{L^\infty}\}$, replacing p, $\frac{p}{2}$ by p_j, p_{j-1} in (3.31), and by a direct calculation, for any $j \geq 2$, we obtain

$$Q_j \leq C^{\frac{1}{p_j}}p_j^{\frac{5}{p_j}}Q_{j-1} = C^{\frac{1}{2^j}}2^{\frac{5j}{2^j}}Q_{j-1}.$$

Then for any $n > 2$, we have

$$Q_n \leq C^{\sum_{j=2}^n\frac{1}{2^j}}2^{\sum_{j=2}^n\frac{5j}{2^j}}Q_1.$$

Letting $n \to \infty$, we obtain

$$\sup_{t\in(0,T_{\max})}\left\|u(\cdot,t)\right\|_{L^\infty} \leq C^{\sum_{j=2}^\infty\frac{1}{2^j}}2^{\sum_{j=2}^\infty\frac{5j}{2^j}}\sup_{t\in(0,T_{\max})}\left\|u(\cdot,t)\right\|_{L^2}.$$

It is easy to see that $\sum_{j=2}^{\infty}\frac{1}{2^j}$ and $\sum_{j=2}^{\infty}\frac{5j}{2^j}$ are convergent. Recalling Lemmas 3.4 and 3.6, we have

$$\sup_{t\in(0,T_{\max})}\|u(\cdot,t)\|_{L^\infty}\leq C.$$

Combining this with (3.28), (3.29), we complete the proof of Theorem 1.1. □

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