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New results on fractional advection–dispersion equations

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Abstract

In this paper, a class of fractional Sturm–Liouville advection–dispersion equations with instantaneous and noninstantaneous impulses is considered, in particular, the nonlinearities discussed here include Caputo fractional derivatives. Since the nonlinear terms contain fractional derivatives, this problem does not directly have variational structure, we need to combine critical point theory and an iterative method to deal with such problems. Finally, the existence of at least one nontrivial solution is proved by the mountain pass theorem and the iterative method. At the same time, an example is given to illustrate the main result.

Mathematics Subject Classification: 34B15; 34B24; 34B37; 34K10

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1 Introduction

In this paper, we study the following class of fractional advection–dispersion equations whose nonlinear terms contain Caputo fractional derivatives, and the equations have inhomogeneous Sturm–Liouville boundary conditions, instantaneous and noninstantaneous impulses:

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{1}{2}\,{}_0D_t^{-\beta}(u'(t))+\dfrac{1}{2}\,{}_tD_T^{-\beta}(u'(t))\right)=\lambda f_j(t,u(t),{}_0^cD_t^{\alpha}u(t)),\ t\in(s_j,t_{j+1}],\ j=0,1,\ldots,m,\\ a\left(\dfrac{1}{2}\,{}_0D_t^{-\beta}(u'(0))+\dfrac{1}{2}\,{}_tD_T^{-\beta}(u'(0))\right)-bu(0)=A,\\ c\left(\dfrac{1}{2}\,{}_0D_t^{-\beta}(u'(T))+\dfrac{1}{2}\,{}_tD_T^{-\beta}(u'(T))\right)+du(T)=B,\\ \Delta\left(\dfrac{1}{2}\,{}_0D_t^{-\beta}(u'(t_j))+\dfrac{1}{2}\,{}_tD_T^{-\beta}(u'(t_j))\right)=\mu I_j(u(t_j)),\ j=1,\ldots,m,\\ {}_0D_t^{-\beta}(u'(t))+{}_tD_T^{-\beta}(u'(t))={}_0D_t^{-\beta}(u'(t_j^+))+{}_tD_T^{-\beta}(u'(t_j^+)),\ t\in(t_j,s_j],\ j=1,\ldots,m,\\ {}_0D_t^{-\beta}(u'(s_j^-))+{}_tD_T^{-\beta}(u'(s_j^-))={}_0D_t^{-\beta}(u'(s_j^+))+{}_tD_T^{-\beta}(u'(s_j^+)),\ j=1,\ldots,m, \end{cases} \tag{1.1}$$

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where $0 \leq \beta < 1$, $\alpha = 1 - \frac{\beta}{2}$, and $\frac{1}{2} < \alpha \leq 1$. $\lambda > 0$ and $\mu > 0$ are two parameters, $a, b, c, d > 0$, A and B are real numbers. ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ denote the left and right Riemann–Liouville fractional integrals of order β, respectively. ${}_0^cD_t^{\alpha}$ denotes the left Caputo fractional derivatives of order α. $0 = s_0 < t_1 < s_1 < t_2 < s_2 < \cdots < t_m < s_m < t_{m+1} = T$. $f_j \in C((s_j, t_{j+1}] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $F_j(t, u, y) = \int_0^u f_j(t, s, y)\mathrm{d}s$. The instantaneous impulses $I_j \in C(\mathbb{R}, \mathbb{R})$ start to change suddenly at the points t_j, and the noninstantaneous impulses continue during the finite intervals $(t_j, s_j]$ for $j = 1, \ldots, m$. Besides,

$$\begin{aligned}\Delta\left(\frac{1}{2}\,{}_0D_t^{-\beta}(u'(t_j)) + \frac{1}{2}\,{}_tD_T^{-\beta}(u'(t_j))\right) = &\left(\frac{1}{2}\,{}_0D_t^{-\beta}(u'(t_j^+)) + \frac{1}{2}\,{}_tD_T^{-\beta}(u'(t_j^+))\right)\\ &- \left(\frac{1}{2}\,{}_0D_t^{-\beta}(u'(t_j^-)) + \frac{1}{2}\,{}_tD_T^{-\beta}(u'(t_j^-))\right),\end{aligned}$$

where

$$\begin{aligned}{}_0D_t^{-\beta}(u'(s_j^\pm)) + {}_tD_T^{-\beta}(u'(s_j^\pm)) &= \lim_{t \to s_j^\pm}\left({}_0D_t^{-\beta}(u'(t)) + {}_tD_T^{-\beta}(u'(t))\right),\\ {}_0D_t^{-\beta}(u'(t_j^\pm)) + {}_tD_T^{-\beta}(u'(t_j^\pm)) &= \lim_{t \to t_j^\pm}\left({}_0D_t^{-\beta}(u'(t)) + {}_tD_T^{-\beta}(u'(t))\right).\end{aligned}$$

The emergence of the fractional advection–dispersion equation can effectively solve the problem that classical second-order convection–diffusion equation cannot accurately simulate the anomalous diffusion phenomenon, so the fractional advection–dispersion equation is widely applied in the anomalous diffusion phenomena, such as groundwater and soil pollution, porous media, fluid mechanics, polymer and nuclear magnetic resonance (see [5, 14, 27]). Not only that, the fractional advection–dispersion equation has also been extensively used in the simulation of turbulent flow, chaotic dynamics of classical conservative systems, and other physical phenomena (see [1, 3, 22]). There have been some related studies (see [2, 4, 9, 24, 29]), for example, reference [11] considered the following symmetric fractional advection–dispersion equation with Dirichlet boundary value condition:

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\,{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}\,{}_tD_T^{-\beta}(u'(t))\right) + \lambda\nabla F(t, u(t)) = 0, \ a.e.\ t \in [0, T],\\ u(0) = u(T) = 0,\end{cases}$$

where $0 \leq \beta < 1$, $\lambda > 0$. When the nonlinear term did not meet the Ambrosetti–Rabinowitz condition, the authors gave the existence of solutions of the above equation through the minimization principle. However, the authors did not consider the impact of impulses.

Regarding the instantaneous and noninstantaneous impulses, the most prominent feature of instantaneous impulse is that it can more deeply and accurately reflect the changing laws of things and fully consider the impact of instantaneous sudden changes on the state. What is more, we also need to point out that the noninstantaneous impulse proposed by Hernández in 2013 can successfully solve the problem that instantaneous impulse cannot be used to simulate the evolution process of phenomena such as dynamics (see [12]). For related research work, please refer to literature [16, 31]; especially, in the literature [30], the authors studied the following fractional noninstantaneous impulse differential equations:

$$\begin{cases} {}_tD_T^{\alpha}({}_0^cD_t^{\alpha}u(t)) = f_i(t,u(t)), \ t \in (s_i,t_{i+1}], \ i = 0,1,\ldots,n, \\ \Delta({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u(t_i))) = I_i(u(t_i)), \ i = 1,2,\ldots,n, \\ {}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) = {}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u(t_i^+)), \ t \in (t_i,s_i], \ i = 1,\ldots,n, \\ {}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u(s_i^-)) = {}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u(s_i^+)), \ i = 1,2,\ldots,n, \\ u(0) = u(T) = 0, \end{cases}$$

where $\frac{1}{2} < \alpha \leq 1$, and they got the existence of at least one solution of this boundary value problem by using the minimization principle. Note that the authors considered the Dirichlet condition here.

Last but not least, the Sturm–Liouville problem arose in the Fourier treatment of heat conduction. Later, Sturm and Liouville generalized the Fourier method, which formed the famous Sturm–Liouville theory. And the Sturm–Liouville problem plays an important role in heat conduction of uniform thin tube of finite length, axial and torsional vibration of rod, and microwave transmission (see [13, 18]). Based on the above, the Sturm–Liouville problem has also come into the attention of scholars in recent years (see [21, 23]). Reference [8] focused on the existence and uniqueness of solutions to the Sturm–Liouville problem with Hilfer fractional differentiation based on Banach's fixed point theorem and analyzed the behavior of the solutions. Later, the author used the Laplace–Adomian decomposition method to study the series solutions of fractional Sturm–Liouville equations with singular and nonsingular kernels, respectively (see [7]). And [28] discussed the following symmetric fractional advection–dispersion equation with only homogeneous Sturm–Liouville boundary value condition:

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{1}{2}\,{}_0D_t^{-\beta}(u'(t)) + \dfrac{1}{2}\,{}_tD_T^{-\beta}(u'(t))\right) = \lambda f(u(t)), \ a.e.\ t \in [0,T], \\ au(0) - b\left(\dfrac{1}{2}\,{}_0D_t^{-\beta}u'(0) + \dfrac{1}{2}\,{}_tD_T^{-\beta}u'(0)\right) = 0, \\ cu(T) + d\left(\dfrac{1}{2}\,{}_0D_t^{-\beta}u'(T) + \dfrac{1}{2}\,{}_tD_T^{-\beta}u'(T)\right) = 0, \end{cases}$$

where $0 \leq \beta < 1$, $a,c > 0$, $b,d \geq 0$, and $\lambda > 0$. The existence of infinitely many solutions to this boundary value problem was obtained by using the Ricceri generalized variational principle when $f : \mathbb{R} \to \mathbb{R}$ was an almost everywhere continuous function.

Compared with the above excellent work, the nonlinear terms in the boundary value problem (BVP as an abbreviation) (1.1) contain fractional derivatives, which means that BVP (1.1) has no direct variational structure, which makes the treatment of this kind of problem not only rely on the critical point theory, but also be combined with the iterative method. Because the research process is relatively complicated, there are a few studies on the existence of solutions of fractional differential equations with fractional derivatives in nonlinear terms (see [6, 10]). Recently, reference [19] considered the following p-Laplacian fractional differential boundary value problem with Dirichlet condition:

$$\begin{cases} {}_tD_T^{\alpha}\left(\frac{1}{\omega(t)^{p-2}}\phi_p(\omega(t)_0D_t^{\alpha}u(t))\right) + \lambda u(t) = f(t, u(t), {}_0^cD_t^{\alpha}u(t)) + h(u(t)), \\ u(0) = u(T) = 0, \ a.e.\ t \in [0, T], \end{cases}$$

where $\frac{1}{p} < \alpha \leq 1$, $p \geq 2$ and $\lambda \geq 0$. $\omega \in L^{\infty}[0, T]$, $\phi_p(s) = |s|^{p-2}s$, $f \in C([0, T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and $h : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. The authors proved that there was at least one solution to the above problem via the mountain pass theorem. On the basis of this article, the authors used the same method to investigate the existence of solutions for fractional (p, q)-Laplacian differential systems with nonlinear terms containing fractional derivatives, instantaneous impulses, and Dirichlet conditions (see [20]).

Inspired by the above research background and the existing research work, this article investigates BVP (1.1). The research on fractional advection–dispersion equations has been ongoing, but to our knowledge, there is no work that studies the nonlinear terms of fractional advection–dispersion equations that include fractional derivatives, let alone systems with nonhomogeneous Sturm–Liouville conditions and noninstantaneous impulse conditions. Note that when the coefficients and constant terms in the nonhomogeneous Sturm–Liouville condition are selected as 0 and 1, the Sturm–Liouville condition will degenerate into the Dirichlet boundary value condition, which means that the Sturm–Liouville condition is more general. In these respects, the problem studied in this paper is new and necessary. What is more, as for the assumptions, the assumptions of the impulse terms in this paper are weaker than the corresponding parts in reference [6], which is a highlight of this paper.

2 Preliminaries

For convenience, in this section we remind the readers of the relevant definitions and properties of fractional calculus.

Definition 2.1 [17, 25] Let u be a function defined on $[0, T]$. The left and right Riemann–Liouville fractional integrals of order $0 < \gamma \leq 1$ for the function u denoted by ${}_0D_t^{-\gamma}u(t)$ and ${}_tD_T^{-\gamma}u(t)$, respectively, are defined by

$${}_0D_t^{-\gamma}u(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}u(s)\mathrm{d}s$$

and

$${}_tD_T^{-\gamma}u(t) = \frac{1}{\Gamma(\gamma)}\int_t^T (s-t)^{\gamma-1}u(s)\mathrm{d}s,$$

provided the right-hand sides are pointwise defined on $[0, T]$, where $\Gamma > 0$ is the gamma function.

Definition 2.2 [17, 25] Let u be a function defined on $[0, T]$. The left and right Riemann–Liouville fractional derivatives of order $0 < \gamma \leq 1$ for the function u denoted by ${}_0D_t^{\gamma}u(t)$

and ${}_tD_T^{\gamma}u(t)$, respectively, are defined by

$$ {}_0D_t^{\gamma}u(t) = \frac{\mathrm{d}}{\mathrm{d}t}\,{}_0D_t^{\gamma-1}u(t) = \frac{1}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t (t-s)^{-\gamma}u(s)\mathrm{d}s\right) $$

and

$$ {}_tD_T^{\gamma}u(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\,{}_tD_T^{\gamma-1}u(t) = \frac{-1}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_t^T (s-t)^{-\gamma}u(s)\mathrm{d}s\right), $$

where $t \in [0,T]$.

Definition 2.3 [17, 25] Let $u \in AC([0,T],\mathbb{R}^N)$. Then the left and right Caputo fractional derivatives of order $0<\gamma\le 1$ for the function u denoted by ${}_0^cD_t^{\gamma}u(t)$ and ${}_t^cD_T^{\gamma}u(t)$, respectively, are defined by

$$ {}_0^cD_t^{\gamma}u(t) = {}_0D_t^{\gamma-1}u'(t) = \frac{1}{\Gamma(1-\gamma)}\int_0^t (t-s)^{-\gamma}u'(s)\mathrm{d}s $$

and

$$ {}_t^cD_T^{\gamma}u(t) = -\,{}_tD_T^{\gamma-1}u'(t) = \frac{-1}{\Gamma(1-\gamma)}\int_t^T (s-t)^{-\gamma}u'(s)\mathrm{d}s, $$

where $t \in [0,T]$.

Property 2.1 [17] Let u be continuous for a.e. $t \in [0,T]$, the left and right Riemann–Liouville fractional integral operators have the following properties:

$$ {}_0D_t^{-\gamma_1}({}_0D_t^{-\gamma_2}u(t)) = {}_0D_t^{-\gamma_1-\gamma_2}u(t) \ \ \textit{and} \ \ {}_tD_T^{-\gamma_1}({}_tD_T^{-\gamma_2}u(t)) = {}_tD_T^{-\gamma_1-\gamma_2}u(t), \ \ \gamma_1,\gamma_2>0. $$

Property 2.2 [17] If $u \in L^p([0,T],\mathbb{R}^N)$, $v \in L^q([0,T],\mathbb{R}^N)$ and $p\ge 1$, $q \ge 1$, $\frac{1}{p}+\frac{1}{q}\le 1+\gamma$ or $p \ne 1$, $q\ne 1$, $\frac{1}{p}+\frac{1}{q} = 1+\gamma$. Then

$$ \int_0^T [{}_0D_t^{-\gamma}u(t)]v(t)\mathrm{d}t = \int_0^T [{}_tD_T^{-\gamma}v(t)]u(t)\mathrm{d}t, \ \gamma>0. $$

Property 2.3 [17] If $0<\gamma\le 1$ and $u \in AC([0,T],\mathbb{R}^N)$, then

$$ {}_0D_t^{-\gamma}({}_0^cD_t^{\gamma}u(t)) = u(t)-u(0) \ \ \textit{and} \ \ {}_tD_T^{-\gamma}({}_t^cD_T^{\gamma}u(t)) = u(t)-u(T). $$

According to Definition 2.3 and Property 2.1, we know that

$$ \frac{1}{2}\,{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}\,{}_tD_T^{-\beta}(u'(t)) = \frac{1}{2}\,{}_0D_t^{\alpha-1}({}_0^cD_t^{\alpha}u(t)) - \frac{1}{2}\,{}_tD_T^{\alpha-1}({}_t^cD_T^{\alpha}u(t)), \tag{2.1} $$

where $\alpha = 1-\frac{\beta}{2}$ and $\frac{1}{2}<\alpha\le 1$.

Let $L^p([0,T],\mathbb{R})(1\le p<\infty)$ and $C([0,T],\mathbb{R})$ be the p-Lebesgue space and a continuous function space, respectively, with the norms

$$ \|u\|_{L^p} = \left(\int_0^T |u(t)|^p\mathrm{d}t\right)^{\frac{1}{p}}, \quad u \in L^p([0,T],\mathbb{R}) $$

and

$$\|u\|_\infty = \max_{t\in[0,T]} |u(t)|, \quad u \in C([0,T],\mathbb{R}).$$

Definition 2.4 Let $\frac{1}{2} < \alpha \le 1$ and $1 \le p < \infty$. The fractional derivative space $E^{\alpha,p}$ is defined as the closure of $C^\infty([0,T],\mathbb{R})$, that is, $E^{\alpha,p} = \overline{C^\infty([0,T],\mathbb{R})}$ with the norm

$$\|u\|_{\alpha,p} = \left(\int_0^T |{}_0^cD_t^\alpha u(t)|^p \mathrm{d}t + \int_0^T |u(t)|^p \mathrm{d}t \right)^{\frac{1}{p}}.$$

It is obvious that $E^{\alpha,p}$ is the space of functions $u(t) \in L^p([0,T],\mathbb{R})$ with an α order Caputo fractional derivative ${}_0^cD_t^\alpha u(t) \in L^p([0,T],\mathbb{R})$. According to [15], the space $E^{\alpha,p}$ with $p \in (1,\infty)$ is a reflexive and separable Banach space. What is more, for the convenience of writing, when $p = 2$, we mark $E^{\alpha,2}$ as X.

For the convenience of readers, we will review Hölder's inequality and Young's inequality here.

Hölder's inequality: If $u \in L^p([0,T],\mathbb{R}^N)$, $v \in L^q([0,T],\mathbb{R}^N)$, $p \ge 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\int_0^T |u(t)v(t)| \mathrm{d}t \le \left(\int_0^T |u(t)|^p \mathrm{d}t \right)^{\frac{1}{p}} \left(\int_0^T |v(t)|^q \mathrm{d}t \right)^{\frac{1}{q}}.$$

Young's inequality: If x and y are nonnegative real numbers, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$xy \le \Upsilon x^p + C(\Upsilon) y^q,$$

where $C(\Upsilon) = \frac{(p\Upsilon)^{-\frac{q}{p}}}{q}$.

Lemma 2.1 [28] *If $\frac{1}{2} < \alpha \le 1$, then for any $u \in X$ we have*

$$-\cos \pi\alpha \int_0^T |{}_0^cD_t^\alpha u(t)|^2 \mathrm{d}t \le -\int_0^T ({}_0^cD_t^\alpha u(t))({}_t^cD_T^\alpha u(t)) \mathrm{d}t \le \frac{1}{-\cos \pi\alpha} \int_0^T |{}_0^cD_t^\alpha u(t)|^2 \mathrm{d}t.$$

Lemma 2.2 *The norm $\|u\|_{\alpha,2}$ in X is equivalent to*

$$\|u\| = \left(-\int_0^T ({}_0^cD_t^\alpha u(t))({}_t^cD_T^\alpha u(t)) \mathrm{d}t + \frac{b}{a}(u(0))^2 + \frac{d}{c}(u(T))^2 \right)^{\frac{1}{2}}. \tag{2.2}$$

Combining Property 2.3, Lemma 2.1, and Hölder's inequality, the following lemma can be derived.

Lemma 2.3 *There is a continuous and compact embedding $X \hookrightarrow C([0,T],\mathbb{R})$. And there exists a constant $\Lambda > 0$ such that*

$$\|u\|_\infty \le \Lambda \|u\|$$

for $u \in X$, where $\|u\|$ is defined by (2.2).

Lemma 2.4 [15] *Assume that* $\frac{1}{2} < \alpha \leq 1$ *and the sequence* $\{u_n\}$ *weakly converges to* u *in* X, *that is,* $u_n \rightharpoonup u$ *in* X. *Then* $u_n \to u$ *in* $C([0,T],\mathbb{R})$, *that is,*

$$\|u_n - u\|_\infty \to 0$$

as $n \to \infty$.

It follows from the boundary conditions of BVP (1.1), integration by parts, and (2.1) that we can define the weak solution u of BVP (1.1) as follows.

Definition 2.5 A function $u \in X$ is called the weak solution of BVP (1.1) if u satisfies the following equation:

$$\begin{aligned}
&-\frac{1}{2}\int_0^T \left[({}_0^cD_t^\alpha u(t))({}_t^cD_T^\alpha v(t)) + ({}_t^cD_T^\alpha u(t))({}_0^cD_t^\alpha v(t))\right]\mathrm{d}t + \frac{b}{a}u(0)v(0) + \frac{d}{c}u(T)v(T) \\
&+ \frac{A}{a}v(0) - \frac{B}{c}v(T) + \mu\sum_{j=1}^m I_j(u(t_j))v(t_j) \\
&- \lambda\sum_{j=0}^m \int_{s_j}^{t_{j+1}} f_j(t,u(t),\, {}_0^cD_t^\alpha u(t))v(t)\mathrm{d}t = 0,\ \forall v \in X.
\end{aligned}$$

Since there is no way to define the energy functional for BVP (1.1) directly, for each $u \in X$, we first fix any $\omega \in X$ and define $J_\omega : X \to \mathbb{R}$ as shown below

$$\begin{aligned}
J_\omega(u) = &-\frac{1}{2}\int_0^T ({}_0^cD_t^\alpha u(t))({}_t^cD_T^\alpha u(t))\mathrm{d}t + \frac{b}{2a}(u(0))^2 + \frac{d}{2c}(u(T))^2 + \frac{A}{a}u(0) - \frac{B}{c}u(T) \\
&+ \mu\sum_{j=1}^m \int_0^{u(t_j)} I_j(s)\mathrm{d}s - \lambda\sum_{j=0}^m \int_{s_j}^{t_{j+1}} F_j(t,u(t),\, {}_0^cD_t^\alpha \omega(t))\mathrm{d}t \\
=&\frac{1}{2}\|u\|^2 + \frac{A}{a}u(0) \\
&- \frac{B}{c}u(T) + \mu\sum_{j=1}^m \int_0^{u(t_j)} I_j(s)\mathrm{d}s - \lambda\sum_{j=0}^m \int_{s_j}^{t_{j+1}} F_j(t,u(t),\, {}_0^cD_t^\alpha \omega(t))\mathrm{d}t.
\end{aligned} \tag{2.3}$$

Then, according to the existing conditions, the Fréchet derivative of J_ω at point $u \in X$ can be obtained as

$$\begin{aligned}
J'_\omega(u)v = &-\frac{1}{2}\int_0^T \left[({}_0^cD_t^\alpha u(t))({}_t^cD_T^\alpha v(t)) + ({}_t^cD_T^\alpha u(t))({}_0^cD_t^\alpha v(t))\right]\mathrm{d}t \\
&+ \frac{b}{a}u(0)v(0) + \frac{d}{c}u(T)v(T) \\
&+ \frac{A}{a}v(0) - \frac{B}{c}v(T) + \mu\sum_{j=1}^m I_j(u(t_j))v(t_j) \\
&- \lambda\sum_{j=0}^m \int_{s_j}^{t_{j+1}} f_j(t,u(t),\, {}_0^cD_t^\alpha \omega(t))v(t)\mathrm{d}t,\ \forall v \in X.
\end{aligned} \tag{2.4}$$

It follows from Definition 2.5 that if $u \in X$ is a solution of $J'_u(u)v = 0$, then u is a weak solution of BVP (1.1). And, by [31], the weak solution of BVP (1.1) is its classical solution.

Lemma 2.5 [26] *Let E be a real Banach space and $J \in C(E,\mathbb{R})$ satisfy the Palais–Smale condition ((PS)-condition for short). Suppose that*

(1) $J(0) = 0$;

(2) *There exist $\rho > 0$ and $\vartheta > 0$ such that $J(u_0) \geq \vartheta$ for all $u_0 \in E$ with $\|x_0\| = \rho$;*

(3) *There exists $u_1 \in E$ with $\|u_1\| \geq \rho$ such that $J(u_1) < \vartheta$.*

Then $z = \inf_{h\in\bar{\Delta}} \max_{t\in[0,1]} J(h(t)) \geq \vartheta$ is a critical value of J, where

$$\bar{\Delta} = \{h \in C([0,1],E) | h(0) = u_0, h(1) = u_1\}.$$

3 Main results

Now, we make the following hypotheses.

(H_1) For $j = 1,\dots,m$, there exist some constants L_j, $\bar{L}_j$, M_j, $\bar{M}_j > 0$ and $0 \leq \tau_j$, $\bar{\tau}_j < 1$ such that

$$-L_j|y|^{\tau_j} - M_j \leq |I_j(y)| \leq \bar{L}_j|y|^{\bar{\tau}_j} + \bar{M}_j \ \ for\ y \in \mathbb{R}.$$

(H_2) For $j = 1,\dots,m$, there exist some constants $R_j > 0$ such that

$$|I_j(x) - I_j(y)| \leq R_j|x - y|, \ \ for\ x,y \in [-C^*,C^*],$$

where $C^* = \Lambda C_1$, Λ is defined by Lemma 2.3, C_1 will be given later.

(H_3) For $j = 0,1,\dots,m$ and $t \in (s_j,t_{j+1}]$, there exist some constants K_1, K_2, N_2, $N_3 \geq 0$, $N_1 > 0$, η, σ, $\varpi > 1$, $0 < \iota < 2$, $\delta > 0$, $0 < \zeta < 1$ such that

$$f_j(t,x,y) \leq K_1|x|^{\eta} + K_2|x|^{\sigma}|y|^{\iota} \ \ for\ |x| \leq \delta,\ y \in \mathbb{R},$$

$$f_j(t,x,y) \geq N_1 x^{\varpi} - N_2|y|^{\zeta} - N_3 \ \ for\ x \geq 0,\ y \in \mathbb{R}.$$

(H_4) For $j = 0,1,\dots,m$ and $t \in (s_j,t_{j+1}]$, there exist some constants P_1, P_2, $P_3 \geq 0$, $\theta > 2$, $0 < \xi$, $\varrho < 2$ such that

$$f_j(t,x,y)x - \theta F_j(t,x,y) \geq -P_1|x|^{\xi} - P_2|y|^{\varrho} - P_3 \ \ for\ x,y \in \mathbb{R}.$$

(H_5) For $j = 0,1,\dots,m$ and $t \in (s_j,t_{j+1}]$, there exist some constants Q_1, $Q_2 > 0$ such that

$$|f_j(t,x_1,y_1) - f_j(t,x_2,y_2)| \leq Q_1|x_1 - x_2| + Q_2|y_1 - y_2|, \ \ for\ x_1,x_2 \in [-C^*,C^*],\ y_1,y_2 \in \mathbb{R}.$$

For the convenience of writing, we first define a few notations, as shown below.

$$O_1 = \frac{\delta^2}{2\Lambda^2} - \frac{|A|\delta}{a} - \frac{|B|\delta}{c}, \ \ O_2 = \sum_{j=1}^{m}\left(\frac{L_j\delta^{\tau_j+1}}{\tau_j + 1} + M_j\delta\right),$$

$$O_3 = \frac{K_1 T\delta^{\eta+1}}{\eta + 1} + \frac{\lambda K_2\delta^{\sigma+1}}{\sigma + 1}\frac{T^{\frac{2-\iota}{2}}C_1^{\iota}}{(-\cos\pi\alpha)^{\frac{\iota}{2}}}.$$

Theorem 3.1 *If $O_1 > 0$ and $\mu O_2 + \lambda O_3 < O_1$, suppose that $(H_1)-(H_5)$ hold, then BVP (1.1) admits at least one nontrivial solution.*

Proof The whole proof process is divided into three steps.

Step 1: Prove that J_ω satisfies the (PS)-condition.

Suppose that $\{u_n\} \subset X$ is a sequence with $\{J_\omega(u_n)\}$ bounded and $\lim_{n\to\infty} J'_\omega(u_n) = 0$. Then, for any fixed $\omega \in X$ with $\|\omega\| \le C_1$, according to Lemma 2.3, (H_1), and (H_4), it follows from (2.3) and (2.4) that

$$
\begin{aligned}
&\theta J_\omega(u_n) - J'_\omega(u_n)u_n \\
=&(\frac{\theta}{2}-1)\|u_n\|^2 + \frac{A(\theta-1)}{a}u_n(0) - \frac{B(\theta-1)}{c}u_n(T) \\
&+\theta\mu\sum_{j=1}^{m}\int_0^{u_n(t_j)} I_j(s)\mathrm{d}s - \mu\sum_{j=1}^{m} I_j(u_n(t_j))u_n(t_j) \\
&-\lambda\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}\left(\theta F_j(t,u_n(t),\,{}_0^cD_t^\alpha\omega(t)) - f_j(t,u_n(t),\,{}_0^cD_t^\alpha\omega(t))u_n(t)\right)\mathrm{d}t \\
\ge&(\frac{\theta}{2}-1)\|u_n\|^2 - \frac{|A|(\theta-1)\Lambda}{a}\|u_n\| - \frac{|B|(\theta-1)\Lambda}{c}\|u_n\| \\
&-\theta\mu\sum_{j=1}^{m}\left(\frac{L_j\Lambda^{\tau_j+1}}{\tau_j+1}\|u_n\|^{\tau_j+1} + M_j\Lambda\|u_n\|\right) \\
&-\mu\sum_{j=1}^{m}\left(\bar{L}_j\Lambda^{\bar{\tau}_j+1}\|u_n\|^{\bar{\tau}_j+1} + \bar{M}_j\Lambda\|u_n\|\right) \\
&-\lambda\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}\left(P_1|u_n(t)|^\xi + P_2|{}_0^cD_t^\alpha\omega(t)|^\varrho + P_3\right)\mathrm{d}t \\
\ge&(\frac{\theta}{2}-1)\|u_n\|^2 - \left(\frac{|A|(\theta-1)\Lambda}{a} + \frac{|B|(\theta-1)\Lambda}{c} - \theta\mu\sum_{j=1}^{m}M_j\Lambda - \mu\sum_{j=1}^{m}\bar{M}_j\Lambda\right)\|u_n\| \\
&-\theta\mu\sum_{j=1}^{m}\frac{L_j\Lambda^{\tau_j+1}}{\tau_j+1}\|u_n\|^{\tau_j+1} - \mu\sum_{j=1}^{m}\bar{L}_j\Lambda^{\bar{\tau}_j+1}\|u_n\|^{\bar{\tau}_j+1} - \lambda P_1 T\Lambda^\xi\|u_n\|^\xi \\
&-\lambda P_2\int_0^T |{}_0^cD_t^\alpha\omega(t)|^\varrho\mathrm{d}t - \lambda P_3 T.
\end{aligned}
$$

Note the assumptions of $1 \le \tau_j + 1$, $\bar{\tau}_j + 1 < 2$, $\theta > 2$, and $0 < \xi < 2$ and the fact that $J_\omega(u_n)$ is bounded and $\lim_{n\to\infty} J'_\omega(u_n)u_n = 0$, so we get that $\{u_n\}$ is bounded in X.

On one hand, since $\{u_n\}$ is bounded in X and X is a reflexive space, there exists a subsequence of $\{u_n\}$, still record this subsequence as $\{u_n\}$, such that $u_n \rightharpoonup u$ in X. Then, based on Lemma 2.4, we get $u_n \to u$ in $C([0,T],\mathbb{R})$. Thereby,

$$
\begin{aligned}
&< J'_\omega(u_n) - J'_\omega(u), u_n - u > \to 0, \\
&\left(I_j(u_n(t_j)) - I_j(u(t_j))\right)(u_n(t_j) - u(t_j)) \to 0, \\
&\left(f_j(t,u_n(t),\,{}_0^cD_t^\alpha\omega(t)) - f_j(t,u(t),\,{}_0^cD_t^\alpha\omega(t))\right)(u_n(t) - u(t)) \to 0
\end{aligned}
$$

as $n \to \infty$. On the other hand, in view of (2.4), we have

$$\begin{aligned}\|u_n - u\|^2 =& < J'_\omega(u_n) - J'_\omega(u), u_n - u > -\mu \sum_{j=1}^{m} \Big(I_j(u_n(t_j)) - I_j(u(t_j)) \Big)(u_n(t_j) - u(t_j)) \\ & + \lambda \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} \Big(f_j(t, u_n(t), {}_0^cD_t^\alpha \omega(t)) - f_j(t, u(t), {}_0^cD_t^\alpha \omega(t)) \Big)(u_n(t) - u(t))\mathrm{d}t \\ & \to 0\end{aligned}$$

as $n \to \infty$. Combining the above two aspects, we can get $u_n \to u$ in X. In summary, J_ω satisfies the (PS)-condition.

Step 2: Prove that J_ω admits a critical point by the mountain pass theorem.

Let $0 < \rho = \frac{\delta}{\Lambda}$ and $\|u\| = \rho$, then from Lemma 2.3 one has $\|u\|_\infty \le \Lambda\|u\| = \Lambda\rho = \delta$ for $u \in X$, where δ is given as (H_3). Further, notice that (2.3), (H_1), and (H_3), using Hölder's inequality and Lemma 2.1, we can get

$$\begin{aligned}J_\omega(u(t)) \ge& \frac{1}{2}\|u\|^2 - \frac{|A|\Lambda}{a}\|u\| - \frac{|B|\Lambda}{c}\|u\| - \mu \sum_{j=1}^{m} \left(\frac{L_j\Lambda^{\tau_j+1}}{\tau_j+1}\|u\|^{\tau_j+1} + M_j\Lambda\|u\| \right) \\ & - \lambda \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} \left(\frac{K_1}{\eta+1}|u(t)|^{\eta+1} + \frac{K_2}{\sigma+1}|u(t)|^{\sigma+1}|{}_0^cD_t^\alpha \omega(t)|^\iota \right)\mathrm{d}t \\ \ge& \frac{1}{2}\|u\|^2 - \frac{|A|\Lambda}{a}\|u\| - \frac{|B|\Lambda}{c}\|u\| - \mu \sum_{j=1}^{m} \left(\frac{L_j\Lambda^{\tau_j+1}}{\tau_j+1}\|u\|^{\tau_j+1} + M_j\Lambda\|u\| \right) \\ & - \frac{\lambda K_1 T\Lambda^{\eta+1}}{\eta+1}\|u\|^{\eta+1} - \frac{\lambda K_2\Lambda^{\sigma+1}\|u\|^{\sigma+1}}{\sigma+1} \int_0^T |{}_0^cD_t^\alpha \omega(t)|^\iota \mathrm{d}t \\ \ge& \frac{\delta^2}{2\Lambda^2} - \frac{|A|\delta}{a} - \frac{|B|\delta}{c} - \mu \sum_{j=1}^{m} \left(\frac{L_j\delta^{\tau_j+1}}{\tau_j+1} + M_j\delta \right) \\ & - \lambda \left(\frac{K_1 T\delta^{\eta+1}}{\eta+1} + \frac{K_2\delta^{\sigma+1}T^{\frac{2-\iota}{2}}C_1^\iota}{(\sigma+1)(-\cos\pi\alpha)^{\frac{\iota}{2}}} \right).\end{aligned}$$

Recall that we assume $O_1 > 0$ and $\mu O_2 + \lambda O_3 < O_1$, so we chose ρ small enough so that $J_\omega(u) \ge \vartheta > 0$ for $\|u\| = \rho$.

Next, let us define $\bar{u}_0(t) = \frac{u_0(t)}{\|u_0\|} \in X$ and $\|\bar{u}_0\| = 1$. Hence, for any $\chi > 0$, by (2.3), (H_3), and Lemma 2.3, we get

$$\begin{aligned}J_\omega(\chi\bar{u}_0) \le& \frac{\chi^2}{2}\|\bar{u}_0\|^2 + \frac{|A|\Lambda\chi}{a}\|\bar{u}_0\| + \frac{|B|\Lambda\chi}{c}\|\bar{u}_0\| \\ & + \mu \sum_{j=1}^{m} \left(\frac{\bar{L}_j\Lambda^{\bar{\tau}_j+1}\chi^{\bar{\tau}_j+1}}{\bar{\tau}_j+1}\|\bar{u}_0\|^{\bar{\tau}_j+1} + \bar{M}_j\Lambda\chi\|\bar{u}_0\| \right) \\ & - \lambda \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} \left(\frac{N_1\chi^{\varpi+1}}{\varpi+1}|\bar{u}_0(t)|^{\varpi+1} - N_2\chi|u_0(t)||{}_0^cD_t^\alpha \omega(t)|^\zeta - N_3\chi|u_0(t)| \right)\mathrm{d}t \\ \le& \frac{\chi^2}{2}\|\bar{u}_0\|^2 + \frac{|A|\Lambda\chi}{a}\|\bar{u}_0\| + \frac{|B|\Lambda\chi}{c}\|\bar{u}_0\|\end{aligned}$$

$$
\begin{aligned}
&+\mu\sum_{j=1}^{m}\left(\frac{\bar{L}_j\Lambda^{\bar{\tau}_j+1}\chi^{\bar{\tau}_j+1}}{\bar{\tau}_j+1}\|\bar{u}_0\|^{\bar{\tau}_j+1}+\bar{M}_j\Lambda\chi\|\bar{u}_0\|\right)\\
&-\frac{\lambda N_1\chi^{\varpi+1}}{\varpi+1}\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}|\bar{u}_0(t)|^{\varpi+1}\mathrm{d}t+\lambda N_2\chi\Lambda\|\bar{u}_0\|\int_0^T|{}_0^cD_t^\alpha\omega(t)|^\zeta\mathrm{d}t\\
&+\lambda N_3T\chi\Lambda\|\bar{u}_0\|\\
&=\frac{\chi^2}{2}+\left(\frac{|A|\Lambda}{a}+\frac{|B|\Lambda}{c}+\mu\sum_{j=1}^{m}\bar{M}_j\Lambda+\lambda N_2\Lambda\int_0^T|{}_0^cD_t^\alpha\omega(t)|^\zeta\mathrm{d}t+\lambda N_3T\Lambda\right)\chi\\
&+\mu\sum_{j=1}^{m}\frac{\bar{L}_j\Lambda^{\bar{\tau}_j+1}}{\bar{\tau}_j+1}\chi^{\bar{\tau}_j+1}-\frac{\lambda N_1}{\varpi+1}\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}|\bar{u}_0(t)|^{\varpi+1}\mathrm{d}t\chi^{\varpi+1}.
\end{aligned}
\tag{3.1}
$$

It follows from $\varpi+1>2$ and $1\le\bar{\tau}_j+1<2$ that $J_\omega(\chi\bar{u}_0(t))\to-\infty$ as $\chi\to\infty$, which means that there exists $\chi_0>0$ large enough such that $J_\omega(\chi_0\bar{u}_0)<0$ with $\|\chi_0\bar{u}_0\|>\rho$.

Finally, combined with $J_\omega(0)=0$, according to the mountain pass theorem, it can be inferred that there is a point $\bar{u}\in X$ such that $J'_\omega(\bar{u})=0$ and $J_\omega(\bar{u})\ge\vartheta>0$.

Step 3: Construct the sequence $\{u_n\}\subset X$ such that $u_n\to u^*$ in X, and u^* is the solution to BVP (1.1).

Suppose a sequence $\{u_n\}\subset X$ satisfying $J'_{u_{n-1}}(u_n)=0$ and $J_{u_{n-1}}(u_n)\ge\vartheta>0$ with $\|u_n\|\le C_1$ for all $n\in\mathbb{N}$. For certain $u_1\in X$ with $\|u_1\|\le C_1$. From Step 2, it can be seen that there is $u_2\in X$ such that $J'_{u_1}(u_2)=0$ and $J_{u_1}(u_2)\ge\vartheta>0$.

And then we are going to prove that $\|u_2\|\le C_1$. As a matter of fact, by (3.1) and Lemma 2.1, we have

$$
\begin{aligned}
J_{u_1}(u_2)&\le\max_{0\le\chi<\infty}J_{u_1}(\chi\bar{u}_0)\\
&\le\max_{0\le\chi<\infty}\left[\frac{\chi^2}{2}+\frac{|A|\Lambda\chi}{a}+\frac{|B|\Lambda\chi}{c}+\mu\sum_{j=1}^{m}\left(\frac{\bar{L}_j\Lambda^{\bar{\tau}_j+1}\chi^{\bar{\tau}_j+1}}{\bar{\tau}_j+1}+\bar{M}_j\Lambda\chi\right)\right.\\
&\quad\left.-\frac{\lambda N_1\chi^{\varpi+1}}{\varpi+1}\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}|\bar{u}_0(t)|^{\varpi+1}\mathrm{d}t+\lambda N_2\chi\Lambda\int_0^T|{}_0^cD_t^\alpha u_1(t)|^\zeta\mathrm{d}t+\lambda N_3T\chi\Lambda\right]\\
&\le\max_{0\le\chi<\infty}\left[\frac{\chi^2}{2}+\frac{|A|\Lambda\chi}{a}+\frac{|B|\Lambda\chi}{c}+\mu\sum_{j=1}^{m}\left(\frac{\bar{L}_j\Lambda^{\bar{\tau}_j+1}\chi^{\bar{\tau}_j+1}}{\bar{\tau}_j+1}+\bar{M}_j\Lambda\chi\right)\right.\\
&\quad\left.-\frac{\lambda N_1\chi^{\varpi+1}}{\varpi+1}\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}|\bar{u}_0(t)|^{\varpi+1}\mathrm{d}t+\frac{\lambda N_2\chi\Lambda T^{\frac{2-\zeta}{2}}C_1^\zeta}{(-\cos\pi\alpha)^{\frac{\zeta}{2}}}+\lambda N_3T\chi\Lambda\right]\\
&\le\max_{0\le\chi<\infty}\left[\frac{1}{2}\chi^2+C_2\chi+C_3\chi^{\bar{\tau}_j+1}-C_4\chi^{\varpi+1}\right],
\end{aligned}
$$

where

$$
C_2=\frac{|A|\Lambda}{a}+\frac{|B|\Lambda}{c}+\mu\sum_{j=1}^{m}\bar{M}_j\Lambda+\frac{\lambda N_2\Lambda T^{\frac{2-\zeta}{2}}C_1^\zeta}{(-\cos\pi\alpha)^{\frac{\zeta}{2}}}+\lambda N_3T\Lambda,
$$

$$C_3=\mu\sum_{j=1}^{m}\frac{\bar{L}_j\Lambda^{\bar{\tau}_j+1}}{\bar{\tau}_j+1},\ \ C_4=\frac{\lambda N_1}{\varpi+1}\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}|\bar{u}_0(t)|^{\varpi+1}\mathrm{d}t.$$

Define $G(\chi)=\frac{1}{2}\chi^2+C_2\chi+C_3\chi^{\bar{\tau}_j+1}-C_4\chi^{\varpi+1}$, then we discuss $G(\chi)$ in two cases.

Case 1: When $0\le\chi\le 1$, we have $G(\chi)\le\frac{1}{2}+C_2+C_3=C_5$.

Case 2: When $1\le\chi<\infty$, one has $G(\chi)\le(\frac{1}{2}+C_2+C_3)\chi^2-C_4\chi^{\varpi+1}$ for $1\le\bar{\tau}_j+1<2$ and $\varpi+1>2$. In this case, let $H(\chi)=(\frac{1}{2}+C_2+C_3)\chi^2-C_4\chi^{\varpi+1}$, then if $H'(\chi)=2(\frac{1}{2}+C_2+C_3)\chi-C_4(\varpi+1)\chi^{\varpi}=0$, it can be calculated that $H(\bar{\chi})=\max_{1\le\chi<\infty}H(\chi)=C_6$, when $\chi=\bar{\chi}=\left(\frac{2(\frac{1}{2}+C_2+C_3)}{C_4(\varpi+1)}\right)^{\frac{1}{\varpi-1}}$. In conclusion, $J_{u_1}(u_2)\le\max\{C_5,C_6\}=D$.

On the other hand, by (H_4), (2.3), and (2.4), combining Lemma 2.1 and Lemma 2.3, we get

$$\begin{aligned}
&(\frac{\theta}{2}-1)\|u_2\|^2\\
=&\theta J_{u_1}(u_2)-J'_{u_1}(u_2)u_2-\frac{A(\theta-1)}{a}u_2(0)+\frac{B(\theta-1)}{c}u_2(T)-\theta\mu\sum_{j=1}^{m}\int_0^{u_2(t_j)}I_j(s)\mathrm{d}s\\
&+\mu\sum_{j=1}^{m}I_j(u_2(t_j))u_2(t_j)\\
&+\lambda\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}\Big(\theta F_j(t,u_2(t),{}_0^cD_t^\alpha u_1(t))-f_j(t,u_2(t),{}_0^cD_t^\alpha u_1(t))u_2(t)\Big)\mathrm{d}t\\
\le&\theta D+\frac{|A|(\theta-1)\Lambda}{a}\|u_2\|+\frac{|B|(\theta-1)\Lambda}{c}\|u_2\|+\theta\mu\sum_{j=1}^{m}\Big(\frac{L_j\Lambda^{\tau_j+1}}{\tau_j+1}\|u_2\|^{\tau_j+1}+M_j\Lambda\|u_2\|\Big)\\
&+\mu\sum_{j=1}^{m}\Big(\bar{L}_j\Lambda^{\bar{\tau}_j+1}\|u_2\|^{\tau_j+1}+\bar{M}_j\Lambda\|u_2\|\Big)\\
&+\lambda\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}\Big(P_1|u_2(t)|^{\xi}+P_2|{}_0^cD_t^\alpha u_1(t)|^{\varrho}+P_3\Big)\mathrm{d}t\\
\le&\theta D+\Big(\frac{|A|(\theta-1)\Lambda}{a}+\frac{|B|(\theta-1)\Lambda}{c}+\theta\mu\sum_{j=1}^{m}M_j\Lambda+\mu\sum_{j=1}^{m}\bar{M}_j\Lambda\Big)\|u_2\|\\
&+\theta\mu\sum_{j=1}^{m}\frac{L_j\Lambda^{\tau_j+1}}{\tau_j+1}\|u_2\|^{\tau_j+1}\\
&+\mu\sum_{j=1}^{m}\bar{L}_j\Lambda^{\bar{\tau}_j+1}\|u_2\|^{\bar{\tau}_j+1}+\lambda P_1T\Lambda^{\xi}\|u_2\|^{\xi}+\lambda P_2\int_0^T|{}_0^cD_t^\alpha u_1(t)|^{\varrho}\mathrm{d}t+\lambda P_3T\\
\le&\theta D+D_1\|u_2\|+\sum_{j=1}^{m}D_2\|u_2\|^{\tau_j+1}+\sum_{j=1}^{m}D_3\|u_2\|^{\bar{\tau}_j+1}+D_4\|u_2\|^{\xi}+\frac{\lambda P_2T^{\frac{2-\varrho}{2}}C_1^{\varrho}}{(-\cos\pi\alpha)^{\frac{\varrho}{2}}}+\lambda P_3T\\
\le&\theta D+D_1^*+D_2^*+D_3^*+D_4^*+\frac{\lambda P_2T^{\frac{2-\varrho}{2}}C_1^{\varrho}}{(-\cos\pi\alpha)^{\frac{\varrho}{2}}}+\lambda P_3T+\frac{2\theta-4}{5}\|u_2\|^2,
\end{aligned}\tag{3.2}$$

where

$$D_1=\frac{|A|(\theta-1)\Lambda}{a}+\frac{|B|(\theta-1)\Lambda}{c}+\theta\mu\sum_{j=1}^{m}M_j\Lambda+\mu\sum_{j=1}^{m}\bar{M}_j\Lambda,$$
$$D_2=\frac{\theta\mu L_j\Lambda^{\tau_j+1}}{\tau_j+1},\ \ D_3=\mu\bar{L}_j\Lambda^{\bar{\tau}_j+1},\ \ D_4=\lambda P_1T\Lambda^{\xi},$$

and

$$D_1\|u_2\|\le\frac{5}{2(\theta-2)}D_1^2+\frac{\theta-2}{10}\|u_2\|^2,$$
$$D_2\|u_2\|^{\tau_j+1}\le\frac{1-\tau_j}{2}\left(\frac{5m(1+\tau_j)}{\theta-2}\right)^{\frac{1+\tau_j}{1-\tau_j}}D_2^{\frac{2}{1-\tau_j}}+\frac{\theta-2}{10m}\|u_2\|^2,$$
$$D_3\|u_2\|^{\bar{\tau}_j+1}\le\frac{1-\bar{\tau}_j}{2}\left(\frac{5m(1+\bar{\tau}_j)}{\theta-2}\right)^{\frac{1+\bar{\tau}_j}{1-\bar{\tau}_j}}D_3^{\frac{2}{1-\bar{\tau}_j}}+\frac{\theta-2}{10m}\|u_2\|^2,$$
$$D_4\|u_2\|^{\xi}\le\frac{2-\xi}{2}\left(\frac{5\xi}{\theta-2}\right)^{\frac{\xi}{2-\xi}}D_4^{\frac{2}{2-\xi}}+\frac{\theta-2}{10}\|u_2\|^2$$

are obtained by Young's inequality. Besides,

$$D_1^*=\frac{5}{2(\theta-2)}D_1^2,$$
$$D_2^*=\sum_{j=1}^{m}\frac{1-\tau_j}{2}\left(\frac{5m(1+\tau_j)}{\theta-2}\right)^{\frac{1+\tau_j}{1-\tau_j}}D_2^{\frac{2}{1-\tau_j}},$$
$$D_3^*=\sum_{j=1}^{m}\frac{1-\bar{\tau}_j}{2}\left(\frac{5m(1+\bar{\tau}_j)}{\theta-2}\right)^{\frac{1+\bar{\tau}_j}{1-\bar{\tau}_j}}D_3^{\frac{2}{1-\bar{\tau}_j}},$$
$$D_4^*=\frac{2-\xi}{2}\left(\frac{5\xi}{\theta-2}\right)^{\frac{\xi}{2-\xi}}D_4^{\frac{2}{2-\xi}}.$$

Arrange both sides of (3.2) to get

$$\|u_2\|^2\le\frac{10}{\theta-2}\left(\theta D+D_1^*+D_2^*+D_3^*+D_4^*+\frac{\lambda P_2T^{\frac{2-\varrho}{2}}C_1^{\varrho}}{(-\cos\pi\alpha)^{\frac{\varrho}{2}}}+\lambda P_3T\right).$$

Take $C_1=\left(\frac{10}{\theta-2}(\theta D+D_1^*+D_2^*+D_3^*+D_4^*+\frac{\lambda P_2T^{\frac{2-\varrho}{2}}C_1^{\varrho}}{(-\cos\pi\alpha)^{\frac{\varrho}{2}}}+\lambda P_3T)\right)^{\frac{1}{2}}$ to get $\|u_2\|\le C_1$. Thus, according to this processing method, it is natural to prove that $\|u_n\|\le C_1$ for every $n\in\mathbb{N}$. Since $\|u_n\|\le C_1$, $\|u_n\|_\infty\le\Lambda C_1=C^*$ can be obtained by Lemma 2.3.

According to the above, we know that there exists a subsequence of $\{u_n\}$, still record this subsequence as $\{u_n\}$, such that $u_n\rightharpoonup u^*$ in X. Then, based on Lemma 2.4, we get $u_n\to u^*$ in $C([0,T],\mathbb{R})$. And then we have to prove that $u_n\to u^*$ in X.

Proof by contradiction. Suppose that $\{u_n\}$ diverges in X, i.e., there exists $\epsilon_0>0$ for any $N>0$ such that for all $n>N$ we have $\|u_{n+1}-u_n\|\ge\epsilon_0$.

In view of $J'_{u_n}(u_{n+1})(u_{n+1}-u_n)=0$ and $J'_{u_{n-1}}(u_n)(u_{n+1}-u_n)=0$, based on (H_2), (H_5), and Lemma 2.1, one can get

$$
\begin{aligned}
&\|u_{n+1}-u_n\|^2\\
&=(J'_{u_n}(u_{n+1})-J'_{u_{n-1}}(u_n))(u_{n+1}-u_n)-\mu\sum_{j=1}^{m}\Big(I_j(u_{n+1}(t_j))-I_j(u_n(t_j))\Big)(u_{n+1}(t_j)-u_n(t_j))\\
&\quad+\lambda\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}\Big(f_j(t,u_{n+1}(t),{}_0^cD_t^\alpha u_n(t))-f_j(t,u_n(t),{}_0^cD_t^\alpha u_{n-1}(t))\Big)(u_{n+1}(t)-u_n(t))\mathrm{d}t\\
&\le\mu\sum_{j=1}^{m}R_j|u_{n+1}(t_j)-u_n(t_j)|^2+\lambda\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}Q_1|u_{n+1}(t)-u_n(t)|^2\mathrm{d}t\\
&\quad+\lambda\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}Q_2|{}_0^cD_t^\alpha u_n(t)-{}_0^cD_t^\alpha u_{n-1}(t)||u_{n+1}(t)-u_n(t)|\mathrm{d}t\\
&\le 2C^*\mu\sum_{j=1}^{m}R_j\|u_{n+1}-u_n\|_\infty+2\lambda Q_1TC^*\|u_{n+1}-u_n\|_\infty\\
&\quad+2\lambda Q_2C_1\left(\frac{T}{-\cos\pi\alpha}\right)^{\frac12}\|u_{n+1}-u_n\|_\infty.
\end{aligned}
$$

Simplify the above formula and arrange it to get

$$
\|u_{n+1}-u_n\|_\infty\ge Z\|u_{n+1}-u_n\|^2\ge Z\epsilon_0^2,
$$

where $Z=2C^*\mu\sum_{j=1}^{m}R_j+2\lambda Q_1TC^*+2\lambda Q_2C_1\left(\frac{T}{-\cos\pi\alpha}\right)^{\frac12}$. Thus, there is $\epsilon_0'>0$ for any $N>0$ such that for all $n>N$ ě we get $\|u_{n+1}-u_n\|_\infty\ge\epsilon_0'$, which is in contradiction with $u_n\to u^*$ in $C([0,T],\mathbb{R})$ as $n\to\infty$. In other words, we can get $u_n\to u^*$ in X as $n\to\infty$, and then

$$
\begin{aligned}
&({}_0^cD_t^\alpha u_n(t))({}_t^cD_T^\alpha v(t))\to({}_0^cD_t^\alpha u^*(t))({}_t^cD_T^\alpha v(t)),\ \ ({}_t^cD_T^\alpha u_n(t))({}_0^cD_t^\alpha v(t))\\
&\qquad\to({}_t^cD_T^\alpha u^*(t))({}_0^cD_t^\alpha v(t)),\\
&\frac{b}{a}u_n(0)v(0)\to\frac{b}{a}u^*(0)v(0),\ \ \frac{d}{c}u_n(T)v(T)\to\frac{d}{c}u^*(T)v(T),\\
&\sum_{j=1}^{m}I_j(u_n(t_j))v(t_j)\to\sum_{j=1}^{m}I_j(u^*(t_j))v(t_j),\ \ f_j(t,u_n(t),{}_0^cD_t^\alpha u_{n-1}(t))v(t)\\
&\qquad\to f_j(t,u^*(t),{}_0^cD_t^\alpha u^*(t))v(t)
\end{aligned}
$$

as $n\to\infty$. Combining the fact that $J'_{u_{n-1}}(u_n)v=0$ for all $v\in X$ gives $J'_{u^*}(u^*)v=0$ for all $v\in X$, which implies that u^* is a weak solution of BVP (1.1). Similarly, it can be proved that $\lim_{n\to\infty}J_{u_{n-1}}(u_n)=J_{u^*}(u^*)$, and it follows from $J_{u_{n-1}}(u_n)\ge\vartheta>0$ that $J_{u^*}(u^*)\ge\vartheta>0$, which indicates that u^* is a nontrivial classical solution of BVP (1.1). □

4 Examples

Example 4.1 Let $\beta = \frac{1}{3}$, $\alpha = 1 - \frac{\beta}{2} = \frac{5}{6}$, $T = m = 1$, $\lambda = \mu = 2$, $a = b = 1$, $c = d = 3$, and $A = B = 2$. Consider the following fractional boundary value problem:

$$\begin{cases} -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\,{}_0D_t^{-\frac{1}{3}}(u'(t)) + \frac{1}{2}\,{}_tD_1^{-\frac{1}{3}}(u'(t))\right) = 2f_j(t, u(t), {}_0^cD_t^{\frac{5}{6}}u(t)),\ t \in (s_j, t_{j+1}],\ j = 0, 1, \\ \left(\frac{1}{2}\,{}_0D_t^{-\frac{1}{3}}(u'(0)) + \frac{1}{2}\,{}_tD_1^{-\frac{1}{3}}(u'(0))\right) - u(0) = 2, \\ 3\left(\frac{1}{2}\,{}_0D_t^{-\frac{1}{3}}(u'(1)) + \frac{1}{2}\,{}_tD_1^{-\frac{1}{3}}(u'(1))\right) + 3u(1) = 2, \\ \Delta\left(\frac{1}{2}\,{}_0D_t^{-\frac{1}{3}}(u'(t_1)) + \frac{1}{2}\,{}_tD_1^{-\frac{1}{3}}(u'(t_1))\right) = 2I_1(u(t_1)), \\ {}_0D_t^{-\frac{1}{3}}(u'(t)) + {}_tD_1^{-\frac{1}{3}}(u'(t)) = {}_0D_t^{-\frac{1}{3}}(u'(t_1^+)) + {}_tD_1^{-\frac{1}{3}}(u'(t_1^+)),\ t \in (t_1, s_1], \\ {}_0D_t^{-\frac{1}{3}}(u'(s_1^-)) + {}_tD_1^{-\frac{1}{3}}(u'(s_1^-)) = {}_0D_t^{-\frac{1}{3}}(u'(s_1^+)) + {}_tD_1^{-\frac{1}{3}}(u'(s_1^+)), \end{cases} \tag{4.1}$$

where $0 = s_0 < t_1 = \frac{1}{3} < s_1 = \frac{2}{3} < t_2 = 1$. Let $I_1(u) = |u|^{\frac{1}{2}}$, there exist $L_1 = \bar{L}_1 = 2$, $M_1 = \bar{M}_1 = 1$, and $\tau_1 = \bar{\tau}_1 = \frac{1}{2}$ such that (H_1) holds. And we easily know that (H_2) holds with $R_1 = 1$. Choose $F_j(t,x,y) = (1+t)x^6 + t^2x^3\sin^2 y + 2|\cos t|$, so $f_j(t,x,y) = 6(1+t)x^5 + 3t^2x^2\sin^2 y$. When $K_1 = 12$, $K_2 = 3$, $N_1 = 6$, $N_2 = N_3 = 0$, $\delta = 2$, $\eta = 5$, $\sigma = 2$, $\iota = 1$, $\varpi = 5$ and $\zeta = \frac{1}{2}$, (H_3) holds. Moreover, (H_4) and (H_5) hold for $\theta = 6$, $P_1 = P_2 = 0$, $P_3 = 12$, $\xi = \varrho = 1$, $Q_1 = 60(C^*)^4 + 6C^*$, and $Q_2 = 12(C^*)^2$. Thus, based on Theorem 3.1, BVP (4.1) has at least one nontrivial solution.

5 Conclusion

This article considers a class of fractional advection–dispersion equations with Sturm–Liouville conditions and instantaneous, noninstantaneous impulses, where the nonlinear term includes fractional Caputo derivatives, i.e., BVP (1.1). Since BVP (1.1) does not have a direct variational structure, after defining the function space, we construct a convergent sequence through iterative methods and combine the mountain pass theorem to ensure that the limit of the sequence is the solution of BVP (1.1). It should be pointed out that Sturm–Liouville conditions in BVP (1.1) are more general than the Dirichlet condition, and the assumptions set in this article are looser. Therefore, the work done in this article fills a gap in the research field of fractional advection–dispersion equations.

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