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Boundedness of solutions to a second-order periodic system with p-Laplacian and unbounded perturbation terms

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Abstract

The second-order periodic system with p-Laplacian and unbounded time-dependent perturbation terms is investigated. Using the principle integral method, it is shown that under certain assumptions on the unbounded and periodic terms, all solutions to the equation possess boundedness.

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Keywords: Boundedness of solutions; Periodic term; Unbounded perturbation term; Canonical transformation; Hamiltonian system

1 Introduction and main result

Consider the following second-order differential equation

$$(\varphi_p(x'))' + a\varphi_p(x^+) - b\varphi_p(x^-) + z(t)f(x) = e(t), \tag{1.1}$$

where $\varphi_p(s) = |s|^{p-2}s$ with constant $p > 2$. Variable $x \in \mathbb{R}$, $t \in \mathbb{R}$, $x^+ = max(x, 0)$, $x^- = max(-x, 0)$. a and b are positive constants ($a \neq b$) satisfying $a^{-\frac{1}{p}} + b^{-\frac{1}{p}} = 2\omega^{-1}$, ω is an irrational number, $f(x) = o(|x|)$, $z(t)$ and $e(t)$ are $2\pi_p$ periodic functions with $\pi_p = \frac{2\pi(p-1)^{\frac{1}{p}}}{p\sin\frac{\pi}{p}}$.

When $p = 2$, Eq. (1.1) is turned into

$$x'' + ax^+ - bx^- + z(t)f(x) = e(t), \quad \pi_p = \pi. \tag{1.2}$$

Provided that $z(t)f(x) = 0$, $e(t) = 1 + \gamma h(t)$ in which $h(t)$ is a suitable function, investigating the boundedness of solutions to Eq. (1.2) is very complicated. Ortega [1] proves that every solution to Eq. (1.2) is bounded if $h \in C^4(\mathbb{S}^1)$, where $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, and γ is sufficiently small. Under certain conditions on the initial data, Alonso and Ortega [2] obtain that there exists a function $e(t)$ to ensure that all solutions to Eq. (1.2) are unbounded. Ambrosio [3] establishes the boundedness to solutions to fractional relativistic Schrödinger equations. A differential inclusion system involving the $p(t)$-Laplacian is investigated in [4].

Giacomoni et al. [5] utilize the bifurcation theory to discuss the multiplicity for a strongly singular quasi-linear problem. The asymptotic properties of solutions for a second-order nonlinear discrete equation of the Emden-Fowler type are acquired in [6]. Under appropriate restrictions, Jiao et al. [7] discuss the boundedness of all solutions to Eq. (1.2) (see also [8–10]).

For $p \geq 2$, when $a^{-\frac{1}{p}} + b^{-\frac{1}{p}} = 2\omega^{-1}$, where ω^{-1} is an irrational number, Yang [11] investigates Eq. (1.1) and obtains that all the solution to Eq. (1.1) are bounded under certain assumptions. Liu [12] discusses the bounded condition for Eq. (1.1) provided that f is smooth and $\lim\limits_{x\to\pm\infty} f(x)$ is finite. Ma [13] discusses the bounded condition for Eq. (1.1) provided that f is unbounded and $z(t) = 1$.

When $p = 2$, without the assumption that $\lim\limits_{x\to\pm\infty} f(x)$ is finite, Zhang [14] has acquired the conditions to ensure that each solution of Eq. (1.1) is bounded. In this work, we will extend the result in [14] to the case $p > 2$ under the following assumptions:

$(A_1): z(t), e(t) \in C^6(\mathbb{S}^1)$, where $\mathbb{S}^1 = \mathbb{R}/2\pi_p\mathbb{Z}$.

(A_2): If $f(x) \in C^6(\mathbb{R}\setminus\{0\}) \cap \mathbb{C}^0(\mathbb{R})$, then there are two positive constants C and $\frac{1}{p-1} < \gamma < 1$, such that $|x^k f^{(k)}(x)| \leq C|x|^\gamma$, provided that $x \in \mathbb{R}\setminus\{0\}$ and $0 \leq k \leq 6$.

(A_3): There exist positive constants β_1 and β_2 such that $p\beta_1 > q\beta_2 > 0$, where positive constants p and q satisfy $\frac{1}{p} + \frac{1}{q} = 1$ and

$$xf(x) \geq \beta_1|x|^{\gamma+1}, \;\; x^2 f'(x) \leq \beta_2|x|^{\gamma+1}, \quad x \in \mathbb{R}\setminus\{0\}.$$

Here, we mention that condition (A_1) does not require $z(t) = 1$, namely, condition (A_1) is different from $z(t) = 1$ in Ma [13]. Now, we state our main conclusion.

Theorem 1.1 *Assume that $p > 2$ and $(A_1) - (A_3)$ hold and $\hat{z} = \frac{1}{2\pi_p}\int_0^{2\pi_p} z(t)dt \neq 0$. Then every solution of Eq.* (1.1) *is bounded, namely,* $\sup\limits_{t\in\mathbb{R}}(|x(t)| + |x'(t)|) < \infty$.

We set $F(x) = \int_0^x f(s)ds$. In this work, we utilize c and C to denote any positive constants (not concerning their quantity). k, l, m and n are nonnegative integers.

The structure of this work is the following: Sect. 2 presents action-angle variables, exchanging time and angle variables, and several lemmas. Section 3 provides the proof of Theorem 1.1.

2 Preliminaries

In this part, we provide several lemmas that help prove Theorem 1.1. Throughout Sect. 2, we assume that the hypotheses of Theorem 1.1 always hold.

2.1 Action-angle coordinates

Let $x' = -\omega\varphi_q(y)$, then $y = -\omega^{1-p}\varphi_p(x')$, and the equivalent form of Eq. (1.1) is the following:

$$x' = -\omega\varphi_q(y), \qquad y' = \omega[a_1\varphi_p(x^+) - b_1\varphi_p(x^-)] + \omega^{1-p}[z(t)f(x) - e(t)]$$

with the Hamiltonian function

$$H(x,y,t) = \frac{\omega}{q}|y|^q + \frac{\omega}{p}(a_1|x^+|^p + b_1|x^-|^p) + \omega^{1-p}(z(t)F(x) - e(t)x), \tag{2.1}$$

where $a_1 = \omega^{-p}a$, $b_1 = \omega^{-p}b$, a_1 and b_1 satisfy $a_1^{-\frac{1}{p}} + b_1^{-\frac{1}{p}} = 2$.

Let $sin_p(t)$ satisfy the problem

$$(\varphi_p(C'(t)))' + \varphi_p(C(t)) = 0, \quad C(0) = 0, \quad C'(0) = 1.$$

From the conclusions in [15–17], we confirm that $sin_p(t)$ is a $2\pi_p$-periodic C^2 odd function with $sin_p(\pi_p - t) = sin_p(t)$ for $t \in [0, \frac{\pi_p}{2}]$ and $sin_p(2\pi_p - t) = -sin_p(t)$ for $t \in [\pi_p, 2\pi_p]$. Moreover, for $t \in [0, \frac{\pi_p}{2}]$ and $sin'_p(t) > 0$, $sin_p(t) \in (0, (p-1)^{\frac{1}{p}})$ is implicitly given by

$$\int_0^{sin_p(t)} \frac{ds}{(1 - \frac{s^p}{p-1})^{\frac{1}{p}}} = t.$$

Suppose that $v(t)$ satisfies the initial problem

$$(\varphi_p(x'(t)))' + a_1\varphi_p(x^+) - b_1\varphi_p(x^-) = 0, \quad x(0) = (p-1)^{\frac{1}{p}}, \quad x'(0) = 0.$$

Letting $\varphi_p(v') = u$ and $q = p/(p-1) > 1$ yields

$$\frac{|u|^q}{q} + \frac{a_1|v^+|^p + b_1|v^-|^p}{p} = \frac{a_1}{q}. \tag{2.2}$$

Using (2.2), we obtain that the action-angle coordinate transformation ψ_0: $x = (d_1 r)^{\frac{1}{p}} v(\theta)$, $y = (d_1 r)^{\frac{1}{q}} u(\theta)$ with $d_1 = pa_1^{-1}$. ψ_0 is a symplectic transformation since its value of the Jacobian determinant is 1. Under ψ_0, Hamiltonian function (2.1) is transformed into

$$h(r,\theta,t) = \omega r + \omega^{1-p} z(t) F((d_1 r)^{\frac{1}{p}} v(\theta)) - \omega^{1-p} e(t)(d_1 r)^{\frac{1}{p}} v(\theta) \in \mathbb{C}^{1,1,6}(\mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{S}^1). \tag{2.3}$$

Let $\Xi = \{\theta \in \mathbb{S}^1 : v(\theta) = 0\}$. When $\theta \in \mathbb{S}^1 \backslash \Xi$ ($t \in \mathbb{S}^1$ is a parameter), we have $h(r,t,\theta) \in \mathbb{C}^6$ with respect to r.

2.2 Lemmas

Utilizing the ideas in [13, 14, 18], from conditions (A_2) and (A_3), we obtain the following conclusions.

Lemma 2.1 *For $r \gg 1, k \le 6$, it holds that*

$$|\partial_r^k F((d_1 r)^{\frac{1}{p}} v(\theta))| \le C r^{-k + \frac{\gamma+1}{p}},$$
$$|\partial_r^k f((d_1 r)^{\frac{1}{p}} v(\theta))| \le C r^{-k + \frac{\gamma}{p}},$$

in which $\theta \in \mathbb{S}^1$ provided that $k = 1$; $\theta \in \mathbb{S}^1 \setminus \Xi$ if $k \ge 2$.

Lemma 2.2 *Let*

$$\bar{F}(r) = \int_0^{2\pi_p} F((d_1 \omega^{-1} r)^{\frac{1}{p}} v(\theta)) d\theta. \tag{2.4}$$

For $r \gg 1$, the following conclusions hold

$$|\bar{F}^{(k)}(r)| \leq Cr^{-k+\frac{\gamma+1}{p}}, \quad k \leq 6,$$
$$\bar{F}'(r) \geq cr^{-1+\frac{\gamma+1}{p}}$$

and

$$\bar{F}''(r) \leq -Cr^{-2+\frac{\gamma+1}{p}}.$$

Proof For the sake of simplicity, we write $x = (d_1\omega^{-1}r)^{\frac{1}{p}}\nu(\theta)$. Using (2.4) and noticing that $\Xi \bigcap [0, 2\pi_p]$ is a finite set, we have

$$\bar{F}'(r) = \frac{1}{pr}\int_{[0,2\pi_p]\setminus\Xi} f(x)x d\theta.$$

Using condition (A3) yields

$$\bar{F}'(r) = \frac{1}{pr}\int_{[0,2\pi_p]\setminus\Xi} f(x)x d\theta \geq \frac{\beta_1}{pr}\int_{[0,2\pi_p]\setminus\Xi} |x|^{\gamma+1} d\theta = cr^{-1+\frac{\gamma+1}{p}}.$$

Differentiating (2.4) with respect to variable r, from the above analysis and condition (A3), we have

$$\begin{aligned}
\bar{F}''(r) &= \frac{1}{p^2r^2}\int_{[0,2\pi_p]\setminus\Xi} f(x)x^2 d\theta - \frac{1}{qr}\bar{F}'(r) \\
&\leq \frac{\beta_2}{p^2r^2}\int_{[0,2\pi_p]\setminus\Xi} |x|^{\gamma+1} d\theta - \frac{1}{qr}\bar{F}'(r) \\
&\leq \frac{\beta_2}{pr\beta_1}\bar{F}'(r) - \frac{1}{qr}\bar{F}'(r) \\
&= \left(\frac{\beta_2}{p\beta_1} - \frac{1}{q}\right)\frac{\bar{F}'(r)}{r} \\
&\leq \left(\frac{\beta_2}{p\beta_1} - \frac{1}{q}\right)cr^{-2+\frac{\gamma+1}{p}},
\end{aligned}$$

which finishes the proof. □

From Lemmas 2.1 and 2.2. combined with condition (A_1), we obtain that the following conclusion holds.

Lemma 2.3 *Let $h_1(r,\theta,t) = \omega^{1-p}z(t)F((d_1r)^{\frac{1}{p}}\nu(\theta)) - \omega^{1-p}e(t)(d_1r)^{\frac{1}{p}}\nu(\theta)$. For $r \gg 1, t \in \mathbb{S}^1$ then*

$$|\partial_r^k\partial_t^l h_1(r,\theta,t)| \leq cr^{-k+\frac{\gamma+1}{p}}, \tag{2.5}$$

in which $\theta \in \mathbb{S}^1$ provided that $k = 1$; $\theta \in \mathbb{S}^1 \setminus \Xi$ if $k \geq 2$.

Let

$$g(r,\theta,t)=r^{-\frac{1}{p}}h_1(r,\theta,t). \tag{2.6}$$

From Lemma 2.3, for $r\gg 1$, we have

$$|\partial_r^k\partial_t^l g(r,\theta,t)|\le cr^{-k+\frac{\gamma}{p}},\quad k+l\le 6. \tag{2.7}$$

Lemma 2.4 *For $r\gg 1, k+l\le 6$, then*

$$\begin{cases} 0<cr\le h(r,t,\theta)<Cr,\\ \partial_r h(r,t,\theta)>\frac{\omega}{2},\\ |\partial_r^k\partial_t^l h(r,t,\theta)|\le Cr^{-k+1}, \end{cases} \tag{2.8}$$

in which $\theta\in\mathbb{S}^1$ provided that $k=1$; $\theta\in\mathbb{S}^1\setminus\Xi$ if $k\ge 2$.

Proof From (2.3) and Lemma 2.1, we obtain

$$\lim_{r\to+\infty}\frac{h}{r}=\omega>0,$$

and for $r\gg 1$,

$$\frac{\partial h}{\partial r}=\omega+\omega^{1-p}z(t)\partial_r F((d_1r)^{\frac{1}{p}}\nu(\theta))-\frac{d_1}{p}\omega^{1-p}e(t)(d_1r)^{\frac{1}{p}-1}\nu(\theta)>\frac{\omega}{2},$$

which together with (2.5)–(2.7) completes the proof of (2.8). □

Lemma 2.5 [15] *Provided that function $f(x,t)$ satisfies*

$$|\partial_x^k\partial_t^l f(x,t)|\le Cx^{-k}|f(x,t)|$$

for all sufficiently large $x>0$ and all $k,l:k+l\le N$, where $N\in\mathbb{N}$. Suppose that

$$\partial_x f(x,t)\ge cx^{-1}f(x,t)>0$$

for all sufficiently large $x>0$. Then, the inverse function $g(y,t)$ of f in x satisfies

$$|\partial_y^k\partial_t^l g(y,t)|\le Cy^{-k}g(y,t)$$

for all $K+l\le N$ and all sufficiently large y.

Using Lemmas 2.3 and 2.4, for $h\gg 1, t\in\mathbb{S}^1$, we have

$$|\partial_h^k\partial_t^l r(h,t,\theta)|\le Ch^{-k+1},\quad k+l\le 6,\quad \theta\in\mathbb{S}^1\setminus\Xi. \tag{2.9}$$

Thus, we write (2.3) as

$$h(r,\theta,t)=\omega r+r^{\frac{1}{p}}g(r,\theta,t),\quad r=r(h,t,\theta). \tag{2.10}$$

In fact,[1] $\nu(t) \in C^2(\mathbb{S}^1)$ does not belong to $C^4(\mathbb{S}^1)$. We exchange the time and angle variables to prove Theorem 1.1.

2.3 Exchange of time and angle variables

Based on the conclusions in [15], the identity $rd\theta - hdt = -(hdt - rd\theta)$ guarantees that if we can solve $r = r(h,t,\theta)$ from (2.3) as a function of h, t, θ, then

$$\frac{dh}{d\theta} = -\frac{\partial r}{\partial t} r(h,t,\theta), \qquad \frac{dt}{d\theta} = \frac{\partial r}{\partial h} r(h,t,\theta), \tag{2.11}$$

i.e., Eq. (2.11) is a Hamiltonian system with a Hamiltonian function $r = r(h,t,\theta)$ in which action, angle, and time variables are h, t, and θ, respectively. The following lemma gives a more detailed description of r in (2.10) according to the magnitude of h.

Lemma 2.6 *Provided that $h \gg 1, \theta \in \mathbb{S}^1 \setminus \Xi$, $t \in \mathbb{S}^1$, it holds that*

$$r(h,t,\theta) = \omega^{-1}h - \omega^{-p} z(t) F((d_1 \omega^{-1} h)^{\frac{1}{p}} \nu(\theta)) + R(h,t,\theta), \tag{2.12}$$

where

$$|\partial_h^k \partial_t^l R(h,t,\theta)| \leq C h^{-k+max\{\gamma,\frac{1}{p}\}}, \quad k + l \leq 6. \tag{2.13}$$

Proof Using the identity (2.10) yields

$$r = \omega^{-1}h - \omega^{-1} r^{\frac{1}{p}} g(r,t,\theta). \tag{2.14}$$

Utilizing the identity (2.6) and the Taylor formula, we obtain that function $g = g(r,\theta,t)$ satisfies

$$\begin{aligned} g(r,\theta,t) &= g(\omega^{-1}h - \omega^{-1} r^{\frac{1}{p}} g, \theta, t) \\ &= g(\omega^{-1}h, \theta, t) + R_0(h,t,\theta) \\ &= (\omega^{-1}h)^{-\frac{1}{p}} \omega^{1-p} z(t) F((d_1\omega^{-1}h)^{\frac{1}{p}} \nu(\theta)) - d_1^{\frac{1}{p}} \omega^{1-p} e(t) \nu(\theta) + R_0(h,t,\theta), \end{aligned} \tag{2.15}$$

in which $R_0(h,t,\theta) = -\int_0^1 g_r'(\omega^{-1}h - s\omega^{-1} r^{\frac{1}{p}} g, \theta, t)) \omega^{-1} r^{\frac{1}{p}} g ds$.

Substituting (2.14) into (2.15), we have

$$\begin{aligned} r &= \omega^{-1}h - \omega^{-1} g(r,t,\theta)(\omega^{-1}h)^{\frac{1}{p}} (1 - h^{-1} r^{\frac{1}{p}} g)^{\frac{1}{p}} \\ &= \omega^{-1}h - \omega^{-1} g(r,t,\theta)(\omega^{-1}h)^{\frac{1}{p}} \\ &\qquad + \frac{1}{p} \omega^{-1} g(r,t,\theta)(\omega^{-1}h)^{\frac{1}{p}} \int_0^1 (1 - sh^{-1} r^{\frac{1}{p}} g)^{\frac{1}{p}-1} h^{-1} r^{\frac{1}{p}} g ds \\ &= \omega^{-1}h - \omega^{-p} z(t) F((d_1 \omega^{-1} h)^{\frac{1}{p}} \nu(\theta)) + R_1(h,t,\theta) + R_2(h,t,\theta) + R_3(h,t,\theta), \end{aligned}$$

[1] C^4 is four times continuously differentiable functions in $\mathbb{R}$ or $\mathbb{S}^1$, and C^6 is six times continuously differentiable functions in $\mathbb{R}$ or $\mathbb{S}^1$.

where

$$R_1(h,t,\theta) = \omega^{-(2+\frac{1}{p})} h^{\frac{1}{p}} \int_0^1 g_r(\omega^{-1}h - s\omega^{-1} r^{\frac{1}{p}} g, \theta, t) r^{\frac{1}{p}} g ds,$$

$$R_2(h,t,\theta) = \frac{1}{p}\omega^{-(1+\frac{1}{p})} h^{\frac{1}{p}-1} \int_0^1 (1 - sh^{-1} r^{\frac{1}{p}} g)^{\frac{1}{p}-1} r^{\frac{1}{p}} g^2 ds,$$

$$R_3(h,t,\theta) = d_1^{\frac{1}{p}} \omega^{-(p+\frac{1}{p})} v(\theta) e(t) h^{\frac{1}{p}}.$$

Direct computation gives

$$\partial_h^k \partial_t^l r^{\frac{1}{p}}(h,t,\theta) = \sum r^{\frac{1}{p}-m} \partial_h^{k_1} \partial_t^{l_1} r(h,t,\theta) \partial_h^{k_2} \partial_t^{l_2} r(h,t,\theta) \cdots \partial_h^{k_m} \partial_t^{l_m} r(h,t,\theta)$$

with $1 \le m \le k + l$, $k_1 + k_2 \cdots + k_m = k$ and $l_1 + l_2 + \cdots + l_m = l$. Using (2.9) yields

$$|\partial_h^k \partial_t^l r^{\frac{1}{p}}(h,t,\theta)| \le Ch^{-k+\frac{1}{p}}.$$

Similarly, we acquire

$$|\partial_h^k \partial_t^l g(h,t,\theta)| \le Ch^{-k+\frac{\gamma}{p}},$$

$$|\partial_h^k \partial_t^l g_r(\omega^{-1}h - s\omega^{-1} r^{\frac{1}{p}} g, t, \theta)| \le C^{-k-1+\frac{\gamma}{p}}.$$

Using $p > 2$ and the expression of R_1 yields

$$|\partial_h^k \partial_t^l R_1(h,t,\theta)| \le Ch^{-k-1+\frac{2+2\gamma}{p}} \le Ch^{-k+\gamma}.$$

Analogously, we obtain

$$|\partial_h^k \partial_t^l R_2(h,t,\theta)| \le Ch^{-k+\gamma},$$

$$|\partial_h^k \partial_t^l R_3(h,t,\theta)| \le Ch^{-k+\frac{1}{p}}.$$

Letting $R(h,t,\theta) = R_1(h,t,\theta) + R_2(h,t,\theta) + R_3(h,t,\theta)$, we obtain that inequality (2.13) holds. □

2.4 Canonical transformation

In this part, two lemmas are established to make sure that the Poincare map of the new system is close to a twist map.

Lemma 2.7 *There exists a canonical transformation* ψ_1 *of the form*: $\psi_1 : (\lambda, \varphi) \to (h, t)$

$$h = \lambda + U(\lambda, t, \theta), \quad \varphi = t + V(\lambda, t, \theta),$$

where U *and* V *are* $2\pi_p$ *periodic about* θ. *Under* ψ_1, *the Hamiltonian function* (2.12) *is transformed into*

$$r_1(\lambda, \varphi, \theta) = \omega^{-1}\lambda - \omega^{-p} \hat{z} F((d_1 \lambda \omega^{-1})^{\frac{1}{p}} v(\theta)) + \bar{R}_1(\lambda, \varphi, \theta). \tag{2.16}$$

Moreover, for $\lambda \gg 1, \theta \in \mathbb{S}^1 \setminus \Xi$, $t \in \mathbb{S}^1$, *it holds that*

$$|\partial_\lambda^k \partial_\varphi^l \bar{R}_1(\lambda,\varphi,\theta)| \le C\lambda^{-k+max\{\gamma,\frac{\gamma+1}{p}\}}, \quad k+l \le 5. \tag{2.17}$$

Proof We make a transformation $\psi_1 : (\lambda,\varphi) \to (h,t)$ implicitly given by

$$h = \lambda + \partial_t S_1(\lambda,t,\theta), \quad \varphi = t + \partial_\lambda S_1(\lambda,t,\theta) \tag{2.18}$$

with

$$S_1(\lambda,t,\theta) = \int_0^t \omega^{1-p} z_1(t) F\Big((d_1\lambda\omega^{-1})^{\frac{1}{p}}\nu(\theta)\Big)dt.$$

Under ψ_1, Hamiltonian (2.12) becomes

$$\begin{aligned} r_1(\lambda,\varphi,\theta) &= \omega^{-1}(\lambda+\partial_t S_1) - \omega^{-p}\hat{z}F\Big((d_1\omega^{-1}(\lambda+\partial_t S_1))^{\frac{1}{p}}\nu(\theta)\Big) + \partial_\theta S_1 \\ &\quad - \omega^{-p} z_1(t) F\Big((d_1\omega^{-1}(\lambda+\partial_t S_1))^{\frac{1}{p}}\nu(\theta)\Big) + R(\lambda+\partial_t S_1,t,\theta) \\ &= \omega^{-1}\lambda - \omega^{-p}\hat{z}F\Big((d_1\lambda\omega^{-1})^{\frac{1}{p}}\nu(\theta)\Big) + R_4(\lambda,\varphi,\theta) \\ &\quad + R_5(\lambda,\varphi,\theta) + R_6(\lambda,\varphi,\theta) + R_7(\lambda,\varphi,\theta), \end{aligned}$$

where $z_1(t) = z(t) - \hat{z}$ and

$$\begin{aligned} R_4 &= -\omega^p \hat{z} \int_0^1 \partial_\lambda F\Big((d_1\omega^{-1}(\lambda+\mu\partial_t S_1))^{\frac{1}{p}}\nu(\theta)\Big)\partial_t S_1 d\mu \\ &= -\frac{\hat{z}\omega^p}{p}\int_0^1 f\Big((d_1\omega^{-1}(\lambda+\mu\partial_t S_1))^{\frac{1}{p}}\nu(\theta)\Big)\nu(\theta) d_1\omega^{-1}(\lambda+\mu\partial_t S_1)^{-\frac{1}{q}}\partial_t S_1 d\mu, \end{aligned} \tag{2.19}$$

$$\begin{aligned} R_5 &= -\int_0^1 \omega^{-p} z_1(t) \partial_{d_1\omega^{-1}\lambda} F\Big((d_1\omega^{-1}(\lambda+\mu\partial_t S_1))^{\frac{1}{p}}\nu(\theta)\Big) d_1\omega^{-1}\partial_t S_1 d\mu, \\ R_6 &= \partial_\theta S_1(\lambda,t,\theta), \\ R_7 &= R(\lambda+\partial_t S_1,t,\theta). \end{aligned}$$

From Lemma 2.1, for $\lambda \gg 1$ and $k+l \le 6$, we have

$$|\partial_\lambda^k \partial_t^l S_1(\lambda,t,\theta)| \le C\lambda^{-k+\frac{\gamma+1}{p}}, \tag{2.20}$$

which together with (2.18) yields

$$\begin{cases} \frac{1}{2} < \partial_\varphi t(\lambda,\varphi,\theta) < \frac{3}{2}, \quad |\partial_\lambda t(\lambda,\varphi,\theta)| < \lambda^{-2+\frac{\gamma+1}{p}}, \\ |\partial_\lambda h(\lambda,\varphi,\theta)| \le C, \quad |\partial_\varphi h(\lambda,\varphi,\theta)| \le C\lambda^{\frac{\gamma+1}{p}}. \end{cases} \tag{2.21}$$

For $2 \le k+l \le 5$, utilizing direct calculations gives rise to

$$|\partial_\lambda^k \partial_\varphi^l h(\lambda,\varphi,\theta)| \le C\lambda^{-k+\frac{\gamma+1}{p}}, \quad |\partial_\lambda^k \partial_\varphi^l t(\lambda,\varphi,\theta)| \le C\lambda^{-k-1+\frac{\gamma+1}{p}}. \tag{2.22}$$

First, we prove $|\partial_\lambda^k \partial_\varphi^l R_4| \le C\lambda^{-k+\frac{\gamma+1}{p}}$. Direct computation gives

$$\partial_\lambda^k \partial_\varphi^l \partial_t S_1(\lambda,t,\theta) = \sum \partial_\lambda^m \partial_t^{n+1} S_1(\lambda,t,\theta)\partial_\lambda^{k_1}\partial_\varphi^{l_1} t \partial_\lambda^{k_2}\partial_\varphi^{l_2} t \cdots \partial_\lambda^{k_n}\partial_\varphi^{l_n} t$$

with $1 \le m+n \le k+l$, $m+k_1+k_2 \cdots + k_m = k$ and $l_1+l_2+\cdots+l_n = l$. Using (2.20), (2.21), and (2.22) yields

$$|\partial_\lambda^k \partial_\varphi^l \partial_t S_1| \le C\lambda^{-k+\frac{\gamma+1}{p}}.$$

In the same way, we obtain

$$|\partial_\lambda^k \partial_\varphi^l (\lambda + \mu\partial_t S_1)^{-\frac{1}{q}}| \le C\lambda^{-k-\frac{1}{q}}$$

and

$$\left|\partial_\lambda^k \partial_\varphi^l f\Big((d_1\omega^{-1}(\lambda+\mu\partial_t S_1))^{\frac{1}{p}}\nu(\theta)\Big)\right| \le C\lambda^{-k+\frac{\gamma}{p}}.$$

Noticing $0 < \frac{1}{p-1} < \gamma < 1$, from (2.19), we have $|\partial_\lambda^k \partial_\varphi^l R_4| \le C\lambda^{-k+\frac{\gamma}{p}}$. Similarly, we obtain

$$|\partial_\lambda^k \partial_\varphi^l R_i| \le C\lambda^{-k+\frac{\gamma+1}{p}}, \quad i = 5,6.$$

Applying (2.13), (2.21), and (2.22) gives rise to

$$|\partial_\lambda^k \partial_\varphi^l R_7| \le C\lambda^{-k+max\{\gamma,\frac{1}{p}\}}.$$

Set $\bar{R}_1(\lambda,\varphi,\theta) = R_4(\lambda,\varphi,\theta) + R_5(\lambda,\varphi,\theta) + R_6(\lambda,\varphi,\theta) + R_7(\lambda,\varphi,\theta)$. Hence, inequality (2.17) holds. □

Next, we eliminate the new time variable θ at the first time by constructing the transformation.

Lemma 2.8 *There exists a canonical transformation* $\psi_2 : (\lambda,\varphi) \to (\lambda,\tau)$:

$$\psi_2 : \lambda = \lambda, \quad \varphi = \tau + \partial_\lambda S_2(\lambda,\theta)).$$

Under ψ_2, *the Hamiltonian* (2.16) *is transformed into*

$$r_2(\lambda,\tau,\theta) = \omega^{-1}\lambda - \omega^{-p}\hat{z}\bar{F}(\lambda) + \bar{R}_2(\lambda,\tau,\theta). \tag{2.23}$$

The new disturbance term $\bar{R}_2$ *satisfies*

$$|\partial_\lambda^k \partial_\tau^l \bar{R}_2(\lambda,\tau,\theta)| \le C\lambda^{-k+max\{\gamma,\frac{\gamma+1}{p}\}} \tag{2.24}$$

for $k+l \le 5, \lambda \gg 1, \theta \in \mathbb{S}^1 \setminus \Xi$ *and* $t \in \mathbb{S}^1$.

Proof We choose generating function

$$S_2(\lambda,\theta) = \int_0^\theta \omega^{-p}\hat{z}[F((d_1\omega^{-1}\lambda)^{\frac{1}{p}}v(\theta)) - \bar{F}(\lambda)]d\theta.$$

Under ψ_2, then the Hamiltonian (2.16) is transformed into

$$r_2(\lambda,\tau,\theta) = r_1(\lambda,\varphi,\theta) + \partial_\theta S_2 = \omega^{-1}\lambda - \omega^{-p}\hat{z}\bar{F}(\lambda) + \bar{R}_2(\lambda,\tau,\theta),$$

where

$$\bar{R}_2(\lambda,\tau,\theta) = \bar{R}_1(\lambda,\tau + \partial_\lambda S_2,\theta). \tag{2.25}$$

Thus, inequality (2.24) is obtained from (2.17), (2.23), (2.25) and Lemma 2.2. The proof of Lemma 2.8 is finished. □

3 Proof of main result

Without loss of generality, we only need to prove Theorem 1.1 for the case $\hat{e} > 0$. For $\hat{e} < 0$, the proof is similar. For given $0 < \delta < 1$, define transformation $\psi_3 : (\lambda,\tau) \to (\nu,\tau)$ by

$$\bar{F}'(\lambda) = \delta\nu\omega^p(\hat{z})^{-1}, \quad \tau = \tau, \quad 1 \le \nu \le 4. \tag{3.1}$$

Due to $\lambda \to +\infty$, $\bar{F}'(\lambda) \to 0$, thus $\lambda \to +\infty \Leftrightarrow \delta \to 0$. For $\lambda = \lambda(\delta\nu)$, the following estimates hold.

Lemma 3.1 $c\delta^{\frac{p}{\gamma+1-p}} \le \lambda(\delta\nu) \le C\delta^{\frac{p}{\gamma+1-p}}$, $|\partial_\nu^k\lambda(\delta\nu)| \le C\lambda(\delta\nu) \quad k \le 4$.

Proof From Lemma 2.2 and (3.1), we have $c\delta^{\frac{p}{\gamma+1-p}} \le \lambda(\delta\nu) \le C\delta^{\frac{p}{\gamma+1-p}}$.

Differentiating (3.1) with respect to ν, we have $\bar{F}''(\lambda) = \omega^p\delta\hat{z}^{-1}$. Using Lemma 2.2 yields

$$|\partial_\nu\lambda| = |\frac{\omega^p\delta\hat{z}^{-1}}{\bar{F}''(\lambda)}| = |\frac{\omega^p\delta\hat{z}^{-1}\lambda}{\bar{F}''(\lambda)\lambda}| \le |\frac{\delta\lambda}{\lambda^{-1+\frac{\gamma+1}{p}}}| = |\frac{c\delta\lambda}{\bar{F}'(\lambda)}| = \frac{c\delta\lambda}{\delta\nu} \le C\lambda.$$

Taking $k(k > 1)$ order derivative about ν on both sides of (3.1), we obtain

$$\bar{F}''(\lambda)\partial_\nu^k\lambda + \sum_{s=2}^{s=k}\bar{F}^{(s+1)}\partial_\nu^{k_1}\lambda\partial_\nu^{k_2}\cdots\partial_\nu^{k_s}\lambda = 0$$

with $k_1 + k_2 + \cdots + k_s = k$. Thus,

$$\partial_\nu^k\lambda = \sum_{s=2}^{s=k}\frac{\bar{F}^{(s+1)}\partial_\nu^{k_1}\lambda\partial_\nu^{k_2}\cdots\partial_\nu^{k_s}\lambda}{\bar{F}''(\lambda)}.$$

From Lemma 2.2, using the induction methods yields

$$|\partial_\nu^k\lambda| \le C\lambda, \quad k = 2,3,4,$$

which completes the proof of Lemma 3.1. □

From the definition ψ_3, we have

$$\frac{d\nu}{d\theta} = \delta^{-1}\omega^{-p}\hat{z}\bar{F}''(\lambda)\frac{d\lambda}{d\theta} = \delta^{-1}\omega^{-p}\hat{z}\bar{F}''(\lambda)\partial_\tau \bar{R}_2(\lambda,\tau,\theta).$$

Introducing a new time variable ϑ by $\theta = -\vartheta$ yields

$$\frac{d\nu}{d\vartheta} = l_1(\nu,\tau,\vartheta,\delta), \quad \frac{d\tau}{d\vartheta} = -\omega^{-1} + \delta\nu + l_2(\nu,\tau,\vartheta,\delta), \tag{3.2}$$

where

$$l_1(\nu,\tau,\vartheta,\delta) = \delta^{-1}\omega^{-p}\hat{z}\bar{F}''(\lambda)\partial_\tau \bar{R}_2(\lambda,\tau,-\vartheta),$$
$$l_2(\nu,\tau,\vartheta,\delta) = -\partial_\lambda \bar{R}_2(\lambda,\tau,-\vartheta).$$

Lemma 3.2 *Provided that* $p > 2$, $\frac{1}{p-1} < \gamma < 1$, $0 < \delta \ll 1$, $k + l \leq 4$ *and* $\tau \in \mathbb{S}^1 \setminus \Xi(i = 1,2)$, *it holds that*

$$|\partial_\nu^k \partial_\tau^l l_i(\nu,\tau,\vartheta,\delta)| \leq C\delta^\sigma, \tag{3.3}$$

where $\sigma = \frac{p}{\gamma+1-p}(-1+\gamma) > 0$.

Proof For $k = 0$, we have

$$|\partial_\tau^l l_2| = |\partial_\lambda \partial_\tau^l \bar{R}_2(\lambda,\tau,-\vartheta)| \leq C\lambda^{-1+max\{\frac{\gamma+1}{p},\gamma\}} \leq C\delta^{\frac{p}{\gamma+1-p}(-1+max\{\frac{\gamma+1}{p},\gamma\})} \leq C\delta^\sigma.$$

Using the assumption $\gamma > \frac{1}{p-1}$ derives $\frac{1+\gamma}{p} < \gamma$. We have $|\partial_\tau^l l_2| \leq C\delta^\sigma$.

For $k > 0$, we obtain

$$\begin{aligned}|\partial_\nu^k \partial_\tau^l l_2| &= |\partial_\nu^k \partial_\tau^l \partial_\lambda \bar{R}_2(\lambda,\tau,-\vartheta)| \\ &\leq C\lambda^{-1+max\{\frac{\gamma+1}{p},\gamma\}} \\ &\leq C\delta^{\frac{p}{\gamma+1-p}(-1+max\{\frac{\gamma+1}{p},\gamma\})} \\ &\leq C\delta^\sigma.\end{aligned}$$

For l_1, we have the same estimate. The proof of Lemma 3.2 is completed. □

From Lemmas 3.1–3.2 and (3.3), we see that the solutions of (3.2) with initial value $\nu(0) = \nu_0 \in [1,2]$, $\tau(0) = \tau_0$ do exist for $0 \leq \vartheta \leq 4\pi_p$ if $\delta \ll 1$. Integrating (3.2) from 0 to $2\pi_p$, we derive that Poincaré map P in (3.2) takes the following form

$$P: \begin{cases} \tau_{2\pi_p} = \tau_0 - \omega^{-1}2\pi_p + \delta(\nu_0 + P_2(\nu_0,\tau_0,\delta)), \\ \nu_{2\pi_p} = \nu_0 + \delta P_1(\nu_0,\tau_0,\delta), \end{cases}$$

where $|\partial_{\nu_0}^k \partial_{\tau_0}^l P_i| \leq C\delta^{\sigma-1}$ for $k + l \leq 4$, $i = 1,2$.

Since P is a Poincarè map in (3.2), it is an area-preserving, and thus it possesses the intersection property in the annulus $[1,2] \times \mathbb{S}^1$. Namely, if Γ is an embedded circle in

$[1,2]\times\mathbb{S}^1$ homotopic to a circle $\nu=$ constant, then $P(\Gamma)\cap\Gamma\neq\emptyset$ (see [18]). Now, we have verified that the mapping P satisfies all the conditions of Moser's twist theorem. Hence, there exists an invariant curve Γ_δ of P surrounding $\nu_0=1$ if $\delta\ll 1$. The Γ_δ is located in ring domain $\{(\nu,\tau)|\delta<\nu<2\delta\}$. Note that $\delta\to 0\Leftrightarrow\lambda\to\infty$. The points (λ,φ,θ) satisfying $r_1(\lambda,\varphi,\theta)=r_1(\lambda,\varphi,\theta)|_{(\lambda,\varphi)\in\Gamma_\delta}$ form an invariant torus $\mathbf{T}^2_\delta$ in the extended phase space (λ,φ,θ). Thus, $\psi^{-1}(\Gamma_\delta)$ is an invariant torus for Eq. (2.1) in $(x,y,t)\in\mathbb{R}^2\times\mathbb{S}^1$, which is far away from $(0,0)$, where $\psi=\psi_1\psi_0$. The solution of Eq. (2.1) starting from inside of $\psi^{-1}(\Gamma_\delta)$ is contained inside of $\psi^{-1}(\Gamma_\delta)$. Thus, the solution of Eq. (2.1) is bounded. The proof of Theorem 1.1 is finished.

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Data availability
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Declarations

Competing interests
The authors declare no competing interests.

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