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Sinc-Galerkin method and a higher-order method for a 1D and 2D time-fractional diffusion equations

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Abstract

In this article, a new numerical algorithm for solving a 1-dimensional (1D) and 2-dimensional (2D) time-fractional diffusion equation is proposed. The Sinc-Galerkin scheme is considered for spatial discretization, and a higher-order finite difference formula is considered for temporal discretization. The convergence behavior of the methods is analyzed, and the error bounds are provided. The main objective of this paper is to propose the error bounds for 2D problems by using the Sinc-Galerkin method. The proposed method in terms of convergence is studied by using the characteristics of the Sinc function in detail with optimal rates of exponential convergence for 2D problems. Some numerical experiments validate the theoretical results and present the efficiency of the proposed schemes.

Keywords: Sinc-Galerkin method; Time-fractional diffusion equation; Stability; Convergence

1 Introduction

In recent years, fractional differential equations (FDEs) have been used in various fields of science, engineering, physics, and chemistry. The time-fractional diffusion equations were used in the super-diffusive flow process, description of fractional random walk, etc. Many researchers have considered some efficient and reliable numerical methods to solve some kinds of FDEs. The FDEs have been discussed by many numerical methods, such as compact difference method [1], finite element method [2], weakly Galerkin finite element method [3], spectral method [4], hp-versions discontinuous Galerkin methods [5], and piecewise quadratic reconstructions [6]. Many researchers have proposed numerical methods to solve this problem in dealing with the weakly singular kernel in FDEs, for example, Lv and Xu studied a higher-order finite difference method and spectral method in [7]. The nonuniform meshes were used [8] and [9]. The two-dimensional fractional diffusion-wave equation was studied by many researchers, such as [10] and [11]. In our article, we will apply a higher-order finite difference method to approximate the 1D and 2D fractional derivative problems with the rates of convergence $\mathcal{O}(3-\alpha)$ (α is the fractional derivative order in time).

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As the exponential convergence rate can be derived, the Sinc methods have been considered to deal with the integro-differential equation in various fields in recent years. On the one hand, the Sinc collocation is used to deal with some problems by many researchers [12–17]. The Sinc-collocation method is studied for solving a partial integro-differential equation with a weakly singular kernel both in temporal discretization and spatial discretization by Xu in the paper [18]. He demonstrates that the rates of convergence are both exponential in time and space. In [19], Babaei et al. proposed a discrete scheme for a 2D variable-order fractional integro-partial differential equation using piecewise linear interpolation in time and the Sinc collocation method in space. In [20], Yang et al. investigated a space-time Sinc collocation method for solving a 1D fourth-order partial integro-differential equation with a weakly singular kernel. On the other hand, some researchers have paid attention to the Sinc-Galerkin method in [21–26]. In [27], the authors used the L1 formula in time and the Sinc-Galerkin method in space to solve a 1D time fractional convection-diffusion equation. The numerical solution is $2-\alpha$ order in time and exponential rate order in space. In our present paper, we can achieve $3-\alpha$ order using a higher-order finite difference method in time.

As the Sinc-Galerkin method and the higher-order finite difference are efficient methods, we will combine these two methods to get the discrete scheme of the 1D and 2D fractional diffusion equation with $0<\alpha<1$. This is the first paper to study the 2D fractional diffusion problem using the Sinc-Galerkin scheme, and we will show the detailed derivation of convergent rate for 2D problems. First, we will give some definitions and abilities regarding the Sinc interpolation in Sect. 2. Second, the discrete schemes for 1D and 2D problems are given in Sect. 3. Then, we will show the theoretical analysis of the discrete scheme in time and space in Sect. 4. In this part, we will demonstrate that the convergence is $3-\alpha$ order in time and exponential convergence rate in space. At last, we will use some 1D and 2D numerical examples to demonstrate the theoretical analysis in Sect. 5. The conclusion is introduced in Sect. 6.

2 Introduction of the sinc interpolation

The Sinc function is defined by:

$$\mathrm{Sinc}(x)=\begin{cases}\frac{\sin(\pi x)}{\pi x}, & x\neq 0\\ 1, & x=0.\end{cases}$$

We denote the Sinc basis function by:

$$S(k,h)(x)=\mathrm{Sinc}(\frac{x-kh}{h}), k=0,\pm1,\pm2,\dots,$$

so the Sinc function at the interpolating points $x_j=jh$ is given by

$$S(k,h)(jh)=\delta_{kj}^{(0)}=\begin{cases}1, & k=j\\ 0, & k\neq j.\end{cases}$$

Definition 1 [28] If f is a function defined on R, let $h>0$ and define the series

$$C(f,h)(x)=\sum_{j=-\infty}^{\infty} f(jh)\mathrm{Sinc}(\frac{x-jh}{h}).$$

The function $C(f,h)(x)$ is called the Whittaker cardinal expansion of f whenever the series converges.

Let $\mathcal{D}$ be a domain in the $w = u + iv$ plane with boundary points $a \neq b$. We consider the conformal map $z = \phi(w)$ to be a one-to-one conformal map of $\mathcal{D}$ onto the infinite strip

$$\mathcal{D}_S = \{z \in \mathcal{C}, z = x + iy, |y| < d\},$$

where $\phi(a) = -\infty$ and $\phi(b) = \infty$. To obtain a good result, we always choose the conformal map $z = \phi(\omega) = \ln(\frac{\omega-a}{b-\omega})$, which carries a eye-shaped region in the complex plane

$$\mathcal{D}_E = \{\omega \in \mathcal{C}, |\arg(\frac{\omega - a}{b - \omega})| < d < \pi/2\},$$

onto $\mathcal{D}_s$. There are many properties derived by Stenger in [29]. We refer readers to [29] for more properties.

Definition 2 Let $B(\mathcal{D})$ be the set of all analytic functions in $\mathcal{D}$, which satisfy for some constant α with $0 \leq \alpha < 1$,

$$\int_{\psi(x+L)} |F(w)dw| = O(|x|^{\alpha}), x \to \pm\infty,$$

where $L = iy : |y| < d$. For γ a simple closed contour in $\mathcal{D}$, we define

$$N(F,D) \equiv \lim_{\gamma \to \partial D} \int_{\gamma} |F(w)dw| < \infty.$$

The following lemma will be used for inducing the discrete scheme:

Lemma 1 [28] *Let ϕ be the conformal one-to-one mapping of the simple connected domain $\mathcal{D}$ onto $\mathcal{D}_S$, then we can get the following formulas*:

$$\delta_{kj}^{(0)} = [S(k,h) \circ \phi(x)]\Big|_{x=x_j} = \begin{cases} 1, & k = j \\ 0, & k \neq j. \end{cases} \tag{1}$$

$$\delta_{kj}^{(1)} = h\frac{d}{d\phi}[S(k,h) \circ \phi(x)]\Big|_{x=x_j} = \begin{cases} 0, & k = j \\ \frac{(-1)^{(j-k)}}{j-k}, & k \neq j. \end{cases} \tag{2}$$

$$\delta_{kj}^{(2)} = h^2\frac{d^2}{d\phi^2}[S(k,h) \circ \phi(x)]\Big|_{x=x_j} = \begin{cases} -\frac{\pi^2}{3}, & k = j \\ \frac{-2(-1)^{(j-k)}}{(j-k)^2}, & k \neq j. \end{cases} \tag{3}$$

It is easy to see that $|\delta_{kj}^{(2)}| \leq \pi^2/3$, $|\delta_{kj}^{(1)}| \leq 1$, and $|\delta_{kj}^{(0)}| \leq 1$.

The following theorems will be used to demonstrate the convergence order for the discrete scheme in space:

Theorem 1 [28] *Let $F \in B(\mathcal{D})$ and $h > 0$. Let ϕ be a one-to-one conformal map of the domain $\mathcal{D}$ to $\mathcal{D}_S$. Let $\psi = \phi^{-1}$, $x_j = \psi(jh) = \psi(jh)$ and $\Gamma = \psi(\Re)$. Then,*

$$\eta(F) \equiv \int_a^b F(x)dx - h\sum_{j=-\infty}^{\infty} \frac{F(x_j)}{\phi'(x_j)} = \frac{i}{2}\lim_{\gamma \to \partial\mathcal{D}} \int_{\gamma} \frac{F(x)\kappa(\phi,h)(x)}{\sin(\pi\phi(x)/h)}dx,$$

where i is the imaginary unit, and

$$|\,\kappa(\phi,h)(x)\,|\,|_{x\in\partial D}\equiv|\,\exp[\frac{i\pi\phi(x)}{h}sgn(\mathrm{Im}(\phi(x)))]\,|\,|_{x\in\partial D}=e^{-\pi d/h}.$$

Moreover,

$$|\eta(F)|\leq\frac{N(F,D)}{2\sinh(\pi d/h)}e^{-\pi d/h}.$$

Theorem 2 [28] *Let ϕ be a one-to-one conformal map of the simple connected domain $\mathcal{D}$ to $\mathcal{D}_S$. Then, for $d,h>0$,*

$$\begin{aligned}
&\Big|\frac{S(k,h)\circ\phi(w)}{\sin(\pi\phi(w)/h)}\Big|\Big|_{w\in\partial\mathcal{D}}\leq\frac{h}{\pi d}\equiv C_0(h,d),\\
&\Big|\frac{\frac{d}{d\phi}S(k,h)\circ\phi(w)}{\sin(\pi\phi(w)/h)}\Big|\Big|_{w\in\partial\mathcal{D}}\leq\frac{\pi d+h\tanh(\pi d/h)}{\pi d^2\tanh(\pi d/h)}\equiv C_1(h,d),\\
&\Big|\frac{\frac{d^2}{d\phi^2}S(k,h)\circ\phi(w)}{\sin(\pi\phi(w)/h)}\Big|\Big|_{w\in\partial\mathcal{D}}\leq\frac{2}{d}C_1(h,d)+\frac{\pi}{hd}\equiv C_2(h,d).
\end{aligned}$$

Furthermore, for $p=0,1,2$, there exist constants R_p such that

$$C_p(h,d)\leq R_p h^{1-p}.$$

Theorem 3 [28] *Let ϕ be a conformal one-to-one map of the simple connected domain $\mathcal{D}$ onto $\mathcal{D}_S$. Assume that $\phi(a)=-\infty$, $\phi(b)=+\infty$, and let $x_k=\psi(kh)$, $uw\in B(\mathcal{D})$, $u(S(k,h)\circ\phi w)'\in B(\mathcal{D})$, $u(S(k,h)\circ\phi w)''\in B(\mathcal{D})$, and $B_T=0$, which was defined as* (4.11) *in* [26], *then*

$$\Big|\int_a^b uw[S(k,h)\circ\phi](x)dx-h\frac{uw}{\phi'}(x_k)\Big|\leq\frac{C_0(h,d)}{2}N(uw,\mathcal{D})e^{-\pi d/h},$$

and

$$\begin{aligned}
&\Big|\int_a^b u''w[S(k,h)\circ\phi](x)dx-h\sum_{j=-\infty}^{\infty}u(x_j)[\frac{\delta_{kj}^{(2)}}{h^2}(\phi'w)(x_j)\\
&\quad+\frac{\delta_{kj}^{(1)}}{h}(\frac{\phi''}{\phi'}w+2w')(x_j)]-h\frac{uw''}{\phi'}(x_k)\Big|\\
&\leq\frac{1}{2}[C_2(h,d)N(u(\phi')^2w,\mathcal{D})+C_1(h,d)N(u[w\phi''+2\phi'w'],\mathcal{D})\\
&\quad+C_0(h,d)N(uw'',\mathcal{D})]e^{-\pi d/h}.
\end{aligned}$$

Theorem 4 [28] *Let ϕ be a conformal one-to-one map of the simple connected domain $\mathcal{D}$ onto $\mathcal{D}_S$. Assume that $\phi(a)=-\infty$, $\phi(b)=+\infty$, and let $x_k=\psi(kh)$. Further, assume that exist these positive constants α, β, and K so that*

$$\big|F(x)\big|\leq K\begin{cases}\exp(-\alpha|\phi(x)|), & x\in\Gamma_a\\ \exp(-\beta|\phi(x)|), & x\in\Gamma_b,\end{cases}$$

where $F = uw,\ u\phi' w,\ u(\frac{\phi''}{\phi'}w + 2w')$, $\Gamma_a = \{\xi \in \Gamma : \phi(\xi) = x \in (-\infty, 0)\}$ *and* $\Gamma_b = \{\xi \in \Gamma : \phi(\xi) = x \in [0, \infty)\}$. *Make the selections* $N = [|\frac{\alpha}{\beta}M + 1|]$ *and* $h = \sqrt{\frac{\pi d}{\alpha M}}$.

(1) *If* $uw \in B(\mathcal{D})$, *then*

$$\Big| \int_a^b uw[S(k,h)\circ\phi](x)dx - h\frac{uw}{\phi'}(x_k)\Big| \le L_0 M^{-1/2}\exp(-(\pi d\alpha M)^{1/2}),$$

where $L_0 \equiv \frac{R_0}{2}N(uw,\mathcal{D})\pi d/\alpha^{1/2}$ *is a constant depending on* $u,\ w$, *and* d.

(2) *If* $u(S(k,h)\circ\phi w)'' \in B(\mathcal{D})$, *then*

$$\begin{aligned}&\Big| \int_a^b u'' w[S(k,h)\circ\phi](x)dx - h\sum_{j=-M}^{N} u(x_j)[\frac{\delta_{kj}^{(2)}}{h^2}(\phi' w)(x_j) + \frac{\delta_{kj}^{(1)}}{h}(\frac{\phi''}{\phi'}w + 2w')(x_j)]\\ &\quad - h\frac{uw''}{\phi'}(x_k)\Big|\\ &\le L_1' M\exp(-(\pi d\alpha M)^{1/2}),\end{aligned}$$

where $L_1' \equiv \frac{R_0}{2}(\frac{\pi d}{\alpha})^{1/2}N(uw'',\mathcal{D}) + R_1 N(u[w\phi'' + 2\phi' w'],\mathcal{D}) + R_2(\frac{\alpha}{\pi d})^{1/2}N(u(\phi')^2 w,\mathcal{D}) + K(1/\alpha + 1/\beta)(\frac{\alpha\pi}{3d} + \sqrt{\frac{\alpha}{\pi d}})$.

3 The discrete scheme

In this paper, we consider the following time-fractional diffusion equation:

$${}_0^C\mathrm{D}_t u(\boldsymbol{x},t) = \Delta u(\boldsymbol{x},t) + f(\boldsymbol{x},t),\ (\boldsymbol{x},t)\in\Omega\times(0,T),\ \Omega\equiv[a,b]\ or\ \Omega\equiv[a,b]\times[c,d], \quad (4)$$

where $\boldsymbol{x} = x$ or $\boldsymbol{x} = (x,y)$, $\Delta = \partial_x^2$ or $\Delta = \partial_x^2 + \partial_y^2$. The initial condition is

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}),\ \ \boldsymbol{x}\in\Omega, \quad (5)$$

and the boundary conditions are

$$u(\boldsymbol{x},t) = 0, \boldsymbol{x}\in\partial\Omega \qquad t > 0. \quad (6)$$

The fractional integrals (or Riemann-Liouville integrals) with order $\mathfrak{a}$ in equation (4) is defined as

$${}_0^C\mathrm{D}_t u(\boldsymbol{x},t) = \frac{1}{\Gamma(1-\mathfrak{a})}\int_0^t \frac{\partial u(\boldsymbol{x},s)}{\partial s}(t-s)^{-\mathfrak{a}}ds, \qquad 0 < \mathfrak{a} < 1. \quad (7)$$

3.1 The discrete scheme in time

Considering the case $T = 1$, we denote the temporal uniform step to be $\Delta t = 1/N$, and $t_n = n\Delta t$. For the first term of equation (4), we use a higher-order finite difference method to get the discrete scheme just as the semi-discrete scheme, which is presented in [7]. At the first point, we have

$${}_0^C\mathrm{D}_t u(\boldsymbol{x},t_1) = \frac{1}{\Gamma(1-\mathfrak{a})}\int_0^{t_1} \frac{\partial u(\boldsymbol{x},s)}{\partial s}\frac{ds}{(t_1-s)^{\mathfrak{a}}}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(1-\alpha)} \frac{u(\boldsymbol{x},t_1)-u(\boldsymbol{x},t_0)}{\Delta t} \int_0^{t_1} \frac{ds}{(t_1-s)^{\alpha}} + r_{\Delta t}^1 \\
&= \frac{1}{\Gamma(2-\alpha)\Delta t^{\alpha}} (u(\boldsymbol{x},t_1)-u(\boldsymbol{x},t_0)) + r_{\Delta t}^1,
\end{aligned}
$$

where $r_{\Delta t}^1$ is the first-step truncation error. Then, for the remaining points, we will deal with the fractional integral term in equation (4) using the Taylor formula

$$
\begin{aligned}
H_2^j(\boldsymbol{x},\tau) = u(\boldsymbol{x},t_j) &- \frac{u(\boldsymbol{x},t_j)-u(\boldsymbol{x},t_{j-1})}{\Delta t}(t_j-\tau) \\
&- \frac{u(\boldsymbol{x},t_{j+1})-2u(\boldsymbol{x},t_j)+u(\boldsymbol{x},t_{j-1})}{\Delta t^2} \frac{(t_j-\tau)(\tau-t_{j-1})}{2},
\end{aligned}
$$

which interpolates the function u at 3 points $\{t_{j-1}, t_j, t_{j+1}\}$, i.e.,

$$
H_2^j(t_{j-1}) = u(\boldsymbol{x},t_{j-1}),\ H_2^j(t_j) = u(\boldsymbol{x},t_j),\ H_2^j(t_{j+1}) = u(\boldsymbol{x},t_{j+1}),
$$

so, for the all discrete functions $u(\boldsymbol{x},t_n)$, where $n \geq 2$, we have

$$
\begin{aligned}
{}_0^C\mathrm{D}_t u(\boldsymbol{x},t_n) &= \frac{1}{\Gamma(1-\alpha)} \Big(\sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \partial_s u(\boldsymbol{x},s) \frac{ds}{(t_n-s)^{\alpha}} + \int_{t_{n-1}}^{t_n} \partial_s u(\boldsymbol{x},s) \frac{ds}{(t_n-s)^{\alpha}} \Big) \\
&= \frac{1}{\Gamma(1-\alpha)} \Big(\sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \partial_s H_2^j(s) \frac{ds}{(t_n-s)^{\alpha}} + \int_{t_{n-1}}^{t_n} \partial_s H_2^{n-1}(s) \frac{ds}{(t_n-s)^{\alpha}} \Big) + r_{\Delta t}^n \\
&= \frac{1}{\Gamma(3-\alpha)\Delta t^{\alpha}} \Big\{ \sum_{j=1}^{n-1} (a_j u(\boldsymbol{x},t_{n-j-1}) + b_j u(\boldsymbol{x},t_{n-j}) + c_j u(\boldsymbol{x},t_{n-j+1})) \\
&\quad + \frac{\alpha}{2} u(\boldsymbol{x},t_{n-2}) - 2u(\boldsymbol{x},t_{n-1}) + \frac{4-\alpha}{2} u(\boldsymbol{x},t_n) \Big\} + r_{\Delta t}^n, 2 \leq n \leq N,
\end{aligned}
$$

where $r_{\Delta t}^n$ is the truncation error of the approximation, and

$$
\begin{aligned}
a_j &= -\frac{3}{2}(2-\alpha)(j+1)^{1-\alpha} + \frac{1}{2}(2-\alpha)j^{1-\alpha} + (j+1)^{2-\alpha} - j^{2-\alpha}, \\
b_j &= 2(2-\alpha)(j+1)^{1-\alpha} - 2(j+1)^{2-\alpha} + 2j^{2-\alpha}, \\
c_j &= -\frac{1}{2}(2-\alpha)((j+1)^{1-\alpha} + j^{1-\alpha}) + (j+1)^{2-\alpha} - j^{2-\alpha}.
\end{aligned}
$$

If we denote

$$
\begin{aligned}
L_t^{\alpha} u(t_n) = \frac{1}{\Gamma(3-\alpha)\Delta t^{\alpha}} &\Big\{ \sum_{j=1}^{n-1} (a_j u(\boldsymbol{x},t_{n-j-1}) + b_j u(\boldsymbol{x},t_{n-j}) + c_j u(\boldsymbol{x},t_{n-j+1})) \\
&+ \frac{\alpha}{2} u(\boldsymbol{x},t_{n-2}) - 2u(\boldsymbol{x},t_{n-1}) + \frac{4-\alpha}{2} u(\boldsymbol{x},t_n) \Big\}, 2 \leq n \leq N,
\end{aligned}
$$

then $r_{\Delta t}^n = {}_0^C\mathrm{D}_t u(\boldsymbol{x},t_n) - L_t^{\alpha} u(t_n) = \Delta u(\boldsymbol{x},t_n) + f(\boldsymbol{x},t_n) - L_t^{\alpha} u(t_n)$, i.e.,

$$
L_t^{\alpha} u(t_n) - \Delta u(\boldsymbol{x},t_n) = f(\boldsymbol{x},t_n) - r_{\Delta t}^n.
$$

The above reformulation motivates the following time scheme if we omit the error term $r^n_{\Delta t}$:

$$\begin{cases} L^{\mathfrak{a}}_t u^n = \Delta u^n + f^n \\ u(\boldsymbol{x}, y_0) = u_0, \end{cases}$$

where $u^n \approx u(\boldsymbol{x}, t_n)$, which is come from the semi-discrete scheme in time.

Lemma 2 [7] *For any* $\alpha \in (0,1)$, *it holds*

$$|r^1_{\Delta t}| \le C_{\mathfrak{a}} \widetilde{M}(u)\Delta t^{(2-\mathfrak{a})}, \forall \boldsymbol{x} \in \Omega,$$
$$|r^n_{\Delta t}| \le C_{\mathfrak{a}} M(u)\Delta t^{(3-\mathfrak{a})}, \forall n = 2,3,\dots,N, \forall \boldsymbol{x} \in \Omega,$$

where $C_{\mathfrak{a}}$ *depends only on* $\mathfrak{a}$, $\widetilde{M}(u) = \max_{t\in[0,T]} |\partial^2_t u(t)|$, $M(u) = \max_{t\in[0,T]} |\partial^3_t u(t)|$.

If we denote

$$\alpha_0 = \Gamma(3-\mathfrak{a})\Delta t^{\mathfrak{a}}, \widetilde{\alpha_0} = \Gamma(2-\mathfrak{a})\Delta t^{\mathfrak{a}}, \beta_0 = c_1 + 2 - \frac{\mathfrak{a}}{2},$$

then for $n \ge 4$, the semi-discrete scheme has to be changed to:

$$u^n + \alpha_0\beta_0^{-1}Au^n = \beta_0^{-1}(-(b_1+c_2-2)u^{n-1} + (-a_1-b_2-c_3-\frac{\mathfrak{a}}{2})u^{n-2} + \sum_{i=3}^{n-2}(-a_{i-1}-b_i$$
$$-c_{i+1})u^{n-i} + (-a_{n-2}-b_{n-1})u^1 - a_{n-1}u^0) + \alpha_0\beta_0^{-1}f^n.$$

When $n = 3,2,1$, the semi-discrete scheme has to be changed to:

$$u^3 + \alpha_0\beta_0^{-1}Au^3 = \beta_0^{-1}(-(b_1+c_2-2)u^2 + (-a_1-b_2-\frac{\mathfrak{a}}{2})u^1 - a_2u^0) + \alpha_0\beta_0^{-1}f^3,$$
$$u^2 + \alpha_0\beta_0^{-1}Au^2 = \beta_0^{-1}(-(b_1-2)u^1 + (-a_1-\frac{\mathfrak{a}}{2})u^0) + \alpha_0\beta_0^{-1}f^2,$$
$$u^1 + \widetilde{\alpha}_0 Au^1 = u^0 + \widetilde{\alpha}_0 f^1.$$

If we denote these labels as follows:

$$d^2_1 = -(b_1-2)\beta_0^{-1}, d^2_0 = (-a_1-\frac{\mathfrak{a}}{2})\beta_0^{-1}, d^3_2 = -(b_1+c_2-2)\beta_0^{-1},$$
$$d^3_1 = (-a_1-b_2-\frac{\mathfrak{a}}{2})\beta_0^{-1}, d^3_0 = -a_2\beta_0^{-1},$$

and for $k \ge 4$,

$$d^n_{n-1} = -(b_1+c_2-2)\beta_0^{-1}, d^n_{n-2} = (-a_1-b_2-c_3-\frac{\mathfrak{a}}{2})\beta_0^{-1},$$
$$d^n_{n-i} = (-a_{i-1}-b_i-c_{i+1})\beta_0^{-1}, (i = 3,4,\dots,n-2), d^3_1 = (-a_1-b_2-\frac{\mathfrak{a}}{2})\beta_0^{-1},$$
$$d^n_1 = -(a_{n-2}-b_{n-1})\beta_0^{-1}, d^n_0 = -a_{n-1}\beta_0^{-1},$$

then the semi-discrete scheme is changed as follows:

$$u^n + \alpha_0\beta_0^{-1}Au^n = \sum_{i=1}^{n} d_{n-i}^n u^{n-i} + \alpha_0\beta_0^{-1}f^n, \ \ 2 \le n \le N, \tag{8}$$

$$u^1 + \widetilde{\alpha}_0 Au^1 = u^0 + \widetilde{\alpha}_0 f^1. \tag{9}$$

3.2 The full-discrete scheme for 1D problem

The Sinc-Galerkin method is applied to solve the one-dimensional problem of equation (4) in this subsection, where $\boldsymbol{x}$ is chosen to be x. Denoting the approximate solution $u^n(x) \approx u_m^n(x) = \sum_{k=-M}^{N} C_k^n S_k(x)$, which is achieved from Theorem 1, where $n = 1, 2, \ldots, N+1$, and $S_k(x)$ is a dense linearly independent set of functions $S(k,h) \circ \phi(x)$ for spatial step h. Supposing the $< .,. >$ represents the inner product defined by

$$< f, g >= \int_a^b f(x)g(x)w(x)dx.$$

Let us take the inner product of both sides of equation (8). By choosing the inner product function g to be the orthogonal basis function $\{S_k\}_{k=-M}^{N}$, the equation (8) turn to be:

$$< u^n, S_k > -\alpha_0\beta_0^{-1} < u_{xx}^n, S_k >= \sum_{i=1}^{n} d_{n-i}^n < u^{n-i}, S_k > +\alpha_0\beta_0^{-1} < f^n, S_k > . \tag{10}$$

For the second term in the left-hand side of equation (10), using integrating by parts, we have

$$\begin{aligned} - < u_{xx}^n, S_k > &= -[u_x^n S_k(x)w(x)]_a^b + \int_a^b u_x^n(x)(S_k(x)w(x))'dx \\ &= -[u_x^n S_k(x)w(x)]_a^b + [u^n(x)(S_k(x)w(x))']_a^b - \int_a^b u^n(x)(S_k(x)w(x))''dx \\ &= B_T - \int_a^b u^n(S_k(x)w(x))''dx. \end{aligned} \tag{11}$$

Supposing $B_T = 0$, using Theorem 4 to the integral term in equation (11), and omitting the small errors, the above equation can be changed to

$$- < u_{xx}^n, S_k >\approx h \sum_{j=-M}^{N} \sum_{l=0}^{2} \frac{u_m^n(x_j)}{\phi'(x_j)h^l} \delta_{kj}^{(l)} g_l(x_j). \tag{12}$$

We also have

$$< F^p, S_k >\approx h \sum_{j=-M}^{N} \delta_{kj}^{(0)} \frac{F^p w}{\phi'}(x_j) = h\frac{F^p w}{\phi'}(x_k), \tag{13}$$

where

$$g_2 = -w(\phi')^2, \ g_1 = -w\phi'' - 2w'\phi', \ g_0 = -w'', \ F^p = u^p \ or \ f^p, \ p = n \ or \ n-i,$$

$x_j = \frac{a+be^{jh}}{1+e^{jh}}$ are Sinc grid points. Combing equations (12) and (13), using the formulas (1) in Lemma 1, the fully discrete scheme of equation (9) leads to:

$$h\frac{w(x_k)}{\phi'(x_k)}u_m^1(x_k) + h\widetilde{\alpha}_0\sum_{j=-M}^{N}\sum_{l=0}^{2}\frac{1}{h^l}\delta_{kj}^{(l)}\frac{g_l(x_j)}{\phi'(x_j)}u_m^1(x_j) = h\frac{w(x_k)}{\phi'(x_k)}u_m^0(x_k) + h\widetilde{\alpha}_0\frac{w(x_k)}{\phi'(x_k)}f^1(x_k),$$

and the fully discrete scheme of equation (8) leads to:

$$\begin{aligned}
&h\frac{w(x_k)}{\phi'(x_k)}u_m^n(x_k) + h\alpha_0\beta_0^{-1}\sum_{j=-M}^{N}\sum_{l=0}^{2}\frac{1}{h^l}\delta_{kj}^{(l)}\frac{g_l(x_j)}{\phi'(x_j)}u_m^n(x_j)\\
&= h\sum_{i=1}^{n}d_{n-i}^n\frac{w(x_k)}{\phi'(x_k)}u_m^{n-i}(x_k) + h\alpha_0\beta_0^{-1}\frac{w(x_k)}{\phi'(x_k)}f^n(x_k)).
\end{aligned}$$

As $\delta_{kj}^{(0)} = S_k(x_j) = 0$ for $k \neq j$, and $S_k(x_j) = 1$ for $k = j$, the approximate solution $u_m^n(x_j) = \sum_{k=-M}^{N} C_j^n S_k(x_j)$ can be written to C_k^n. Dividing each side of the above equation by h, the finally full discrete scheme of the fractional diffusion equation (4)–(6), which uses a high-order finite difference and Sinc-Galerkin methods are given as follows:

$$\frac{w(x_k)}{\phi'(x_k)}C_k^1 + \widetilde{\alpha}_0\sum_{j=-M}^{N}\sum_{l=0}^{2}\frac{1}{h^l}\delta_{kj}^{(l)}\frac{g_l(x_j)}{\phi'(x_j)}C_j^1 = \frac{w(x_k)}{\phi'(x_k)}C_k^0 + \widetilde{\alpha}_0\frac{w(x_k)}{\phi'(x_k)}f_k^1,$$

and

$$\frac{w(x_k)}{\phi'(x_k)}C_k^n + \alpha_0\beta_0^{-1}\sum_{j=-M}^{N}\sum_{l=0}^{2}\frac{1}{h^l}\delta_{kj}^{(l)}\frac{g_l(x_j)}{\phi'(x_j)}C_j^n = \sum_{i=1}^{n}d_{n-i}^n\frac{w(x_k)}{\phi'(x_k)}C_k^{n-i} + \alpha_0\beta_0^{-1}\frac{w(x_k)}{\phi'(x_k)}f_k^n,$$

where $k = -M,\dots,N$.

Substituting g_l into the above equations, we can get the approximate solution for solving the unknown coefficients C_j^n:

$$\begin{aligned}
&\frac{w(x_k)}{\phi'(x_k)}C_k^1 + \widetilde{\alpha}_0\sum_{j=-M}^{N}(-\frac{1}{h^2}\delta_{kj}^{(2)}\frac{(w(\phi')^2)(x_j)}{\phi'(x_j)}C_j^1 - \frac{1}{h}\delta_{kj}^{(1)}\frac{w(x_j)\phi''(x_j) + 2w'(x_j)\phi'(x_j)}{\phi'(x_j)}C_j^1\\
&-\delta_{kj}^{(0)}\frac{w''(x_j)}{\phi'(x_j)}C_j^1) = \frac{w(x_k)}{\phi'(x_k)}C_k^0 + \widetilde{\alpha}_0\frac{w(x_k)}{\phi'(x_k)}f_k^1,
\end{aligned}\tag{14}$$

and

$$\begin{aligned}
&\frac{w(x_k)}{\phi'(x_k)}C_k^n + \alpha_0\beta_0^{-1}\sum_{j=-M}^{N}(-\frac{1}{h^2}\delta_{kj}^{(2)}\frac{w(\phi')^2}{\phi'}C_j^n - \frac{1}{h}\delta_{kj}^{(1)}\frac{w(x_j)\phi''(x_j) + 2w'(x_j)\phi'(x_j)}{\phi'(x_j)}C_j^n\\
&-\delta_{kj}^{(0)}\frac{w''(x_j)}{\phi'(x_j)}C_j^n) = \sum_{i=1}^{n}d_{n-i}^n\frac{w(x_k)}{\phi'(x_k)}C_k^{n-i} + \alpha_0\beta_0^{-1}\frac{w(x_k)}{\phi'(x_k)}f_k^n.
\end{aligned}\tag{15}$$

To make the boundary term $B_T = 0$ and the functions F to satisfy the conditions in Theorem 4, (where $F = uw,\ u\phi' w,\ u(\frac{w\phi''}{\phi'} + 2w')$), we choose the weight function $w(x) = \frac{1}{\phi'(x)}$ in the Sinc-Galerkin inner product.

In the rest part of this section, we will consider the situation of $M = N$. Defining $\mathbf{I}^{(l)} = [\delta_{kj}^{(l)}], l = 0, 1, 2$ to be the $(2M+1)\times(2M+1)$ matrices. It is easy to see that the matrix $\mathbf{I}^{(0)}$ is an identity matrix, the matrix $\mathbf{I}^{(1)}$ is a symmetric matrix, and the matrix $\mathbf{I}^{(2)}$ is a skew-symmetric matrix. Define the $(2M+1)\times(2M+1)$ diagonal matrix as follows:

$$\mathbf{D}(g(x))_{ij} = \begin{cases} g(x_j), & k = j \\ 0, & k \neq j. \end{cases}$$

Define the column matrix $\mathbf{C}^n = (C_{-M}^n, \dots, C_M^n)^T$, $\mathbf{D}_{n-1}^n = (d_0^n, d_1^n, \dots, d_{n-1}^n)^T$ and $\mathbf{F}^n = (f_{-M}^n, \dots, f_M^n)^T$. The following matrix will be used for the solution of the unknown matrix $\mathbf{C}^n$

$$\mathbf{C}_\mathbf{1}^{\mathbf{n-1}} = \begin{pmatrix} C_{-M}^0 & C_{-M}^1 & \dots & C_{-M}^{n-1} \\ C_{-M+1}^0 & C_{-M+1}^1 & \dots & C_{-M+1}^{n-1} \\ \dots & \dots & \dots & \dots \\ C_M^0 & C_M^1 & \dots & C_M^{n-1} \end{pmatrix}.$$

By substituting $w(x) = \frac{1}{\phi'(x)}$ into Eq. (14) and Eq. (15), the discrete systems (14)–(15) can be changed to the following matrix form:

$$\begin{aligned} &[\mathbf{D}(\frac{1}{(\phi')^2}) - \widetilde{\alpha}_0\frac{1}{h^2}\mathbf{I}^{(2)} + \widetilde{\alpha}_0\frac{1}{h}\mathbf{I}^{(1)}\mathbf{D}(\frac{\phi''}{(\phi')^2}) - \widetilde{\alpha}_0\delta_{kj}^{(0)}\mathbf{D}(\frac{1}{\phi'}(\frac{1}{\phi'})'')]\mathbf{C}^1 \\ &\qquad = \mathbf{D}(\frac{1}{(\phi')^2})\mathbf{C}_\mathbf{1}^\mathbf{0} + \widetilde{\alpha}_0\mathbf{D}(\frac{1}{(\phi')^2})\mathbf{F}^1; \end{aligned}$$

$$\begin{aligned} &[\mathbf{D}(\frac{1}{(\phi')^2}) - \alpha_0\beta_0^{-1}\frac{1}{h^2}\mathbf{I}^{(2)} + \alpha_0\beta_0^{-1}\frac{1}{h}\mathbf{I}^{(1)}\mathbf{D}(\frac{\phi''}{(\phi')^2}) - \alpha_0\beta_0^{-1}\delta_{kj}^{(0)}\mathbf{D}(\frac{1}{\phi'}(\frac{1}{\phi'})'')]\mathbf{C}^n \\ &\qquad = \mathbf{D}(\frac{1}{(\phi')^2})\mathbf{C}_\mathbf{1}^{\mathbf{n-1}}\mathbf{D}_{n-1}^n + \alpha_0\beta_0^{-1}\mathbf{D}(\frac{1}{(\phi')^2})\mathbf{F}^n, \end{aligned} \tag{16}$$

where we use the equality $-\frac{\phi''}{(\phi')^2} - 2(\frac{1}{\phi'})' = (\frac{1}{\phi'})' - 2(\frac{1}{\phi'})' = -(\frac{1}{\phi'})' = \frac{\phi''}{(\phi')^2}$.

If we denote

$$\begin{aligned} \mathbf{M} = {}& \mathbf{D}(\frac{1}{(\phi')^2}) - \alpha_0\beta_0^{-1}\frac{1}{h^2}\mathbf{I}^{(2)} + \alpha_0\beta_0^{-1}\frac{1}{h}\mathbf{I}^{(1)}\mathbf{D}(\frac{\phi''}{(\phi')^2}) - \alpha_0\beta_0^{-1}\delta_{kj}^{(0)}\mathbf{D}(\frac{1}{\phi'}(\frac{1}{\phi'})'') \\ & (\textit{when } n > 1), \end{aligned}$$

or

$$\mathbf{M} = \mathbf{D}(\frac{1}{(\phi')^2}) - \widetilde{\alpha}_0\frac{1}{h^2}\mathbf{I}^{(2)} + \widetilde{\alpha}_0\frac{1}{h}\mathbf{I}^{(1)}\mathbf{D}(\frac{\phi''}{(\phi')^2}) - \widetilde{\alpha}_0\delta_{kj}^{(0)}\mathbf{D}(\frac{1}{\phi'}(\frac{1}{\phi'})'')(\textit{when } n = 1),$$

and let

$$\mathbf{R} = \mathbf{D}(\frac{1}{(\phi')^2})\mathbf{C}_\mathbf{1}^{\mathbf{n-1}}\mathbf{D}_{n-1}^n + \alpha_0\beta_0^{-1}\mathbf{D}(\frac{1}{(\phi')^2})\mathbf{F}^n(\textit{when } n > 1),$$

or $\mathbf{R} = \mathbf{D}(\frac{1}{(\phi')^2})\mathbf{C_1^0} + \widetilde{\alpha}_0\mathbf{D}(\frac{1}{(\phi')^2})\mathbf{F}^1(when\ n = 1)$, system (16) can be represented by the following brief form:

$$\mathbf{M} \cdot \mathbf{C}^n = \mathbf{R}. \tag{17}$$

We can get the numerical solution $\mathbf{C}^n$ by moving the left term $\mathbf{M}$ to the right side of the above equation.

3.3 The full-discrete scheme for 2D problem

The Sinc-Galerkin method is applied to solve the two-dimensional problem of equation (4), where $\boldsymbol{x}$ is chosen to be (x, y). The approximate solution of $u^n(x,y) \approx u_m^n(x,y) = \sum_{k=-M_x}^{N_x}\sum_{l=-M_y}^{N_y} C_{k,l}^n S_{kl}(x)$ can be achieved using Theorem 1, where $S_{kl}(x) = S_k(x)S_l^*(y) = S(k,h_x)\circ\phi_x(x)S(l,h_y)\circ\phi_y(y)$, $n = 1, 2, \ldots, N+1$. Suppose that the $\langle .,. \rangle$ represents the inner product defined by

$$\langle f,g\rangle = \int_c^d \int_a^b f(x)g(x)w(x)v(y)dxdy,$$

where the product $w(x)v(y)$ plays the role of a weight function. Let us take the inner product of both sides of equation (8). By orthogonalizing the residual for the inner product, the full discrete scheme of equation (8) is changed to:

$$\langle u^n, S_{kl}\rangle - \alpha_0\beta_0^{-1}\langle \Delta u^n, S_{kl}\rangle = \sum_{i=1}^{n} d_{n-i}^n \langle u^{n-i}, S_{kl}\rangle + \alpha_0\beta_0^{-1}\langle f^n, S_{kl}\rangle. \tag{18}$$

For the second term in the left-hand side of equation (10), using the integrating by parts, we have

$$\begin{aligned}-\langle \Delta u^n, S_{kl}\rangle &= \int_c^d \int_a^b \Delta u^n(x,y)S(k,h_x)\circ\phi_x(x)S(l,h_y)\circ\phi_y(y)w(x)v(y)dxdy\\ &= -\int_c^d \int_a^b u^n(x,y)\Delta[S(k,h_x)\circ\phi_x(x)S(l,h_y)\circ\phi_y(y)w(x)v(y)]dxdy - B_{T_2},\end{aligned}$$

where the number of B_{T_2} is given on page 196 of paper [20]. Assume that $B_{T_2} = 0$, then

$$\begin{aligned}-\langle \Delta u^n, S_{kl}\rangle &= \int_c^d \int_a^b -u^n(x,y)[S(k,h_x)\circ\phi_x(x)w(x)]''S(l,h_y)\circ\phi_y(y)v(y)dxdy\\ &\quad+ \int_c^d \int_a^b -u^n(x,y)S(k,h_x)\circ\phi_x(x)w(x)[S(l,h_y)\circ\phi_y(y)v(y)]''dxdy.\end{aligned}$$

Applying the quadrature rule (Theorem 4) to the above-iterated integrals, deleting the error terms, replacing $u(x_k, y_l)$ by C_{kl}, dividing $h_x h_y$, and using these formulas (1) to (3), the discrete sinc system can be changed to

$$\frac{w(x_k)}{\phi_x'(x_k)}\frac{v(y_l)}{\phi_y'(y_l)}C_{kl}^1 + \widetilde{\alpha}_0 \sum_{i=-M_x}^{N_x}[-\frac{1}{h_x^2}\delta_{ki}^{(2)}\phi_x'(x_i)w(x_i) - \frac{1}{h_x}\delta_{ki}^{(1)}(\frac{\phi_x''(x_i)w(x_i)}{\phi_x'(x_i)} + 2w'(x_i))$$

$$-\delta_{ki}^{(0)}\frac{w''(x_i)}{\phi_x'(x_i)}]C_{kl}^1\frac{v(y_l)}{\phi_x'(y_l)}+\widetilde{\alpha}_0\frac{w(x_k)}{\phi_x'(x_k)}C_{kl}^1\sum_{j=-M_y}^{N_y}[-\frac{1}{h_y^2}\delta_{lj}^{(2)}\phi_y'(y_j)v(y_j)-\frac{1}{h_y}\delta_{lj}^{(1)}(\frac{\phi_y''(y_j)v(y_j)}{\phi_y'(y_j)}$$

$$+2v'(y_j))-\delta_{lj}^{(0)}\frac{v''(y_j)}{\phi_y'(y_j)}]=\frac{w(x_k)}{\phi_x'(x_k)}\frac{v(y_l)}{\phi_y'(y_l)}C_{kl}^0+\widetilde{\alpha}_0\sum_{j=-M_y}^{N_y}\frac{w(x_k)}{\phi_x'(x_k)}\frac{v(y_l)}{\phi_y'(y_l)}f_{kl}^1,$$

and

$$\frac{w(x_k)}{\phi_x'(x_k)}\frac{v(y_l)}{\phi_y'(y_l)}C_{kl}^n+\alpha_0\beta_0^{-1}\sum_{k=-M_x}^{N_x}[-\frac{1}{h_x^2}\delta_{ki}^{(2)}\phi_x'(x_i)w(x_i)-\frac{1}{h_x}\delta_{ki}^{(1)}(\frac{\phi_x''(x_i)w(x_i)}{\phi_x'(x_i)}+2w'(x_i))$$

$$-\delta_{ki}^{(0)}\frac{w''(x_i)}{\phi_x'(x_i)}]C_{kl}^n\frac{v(y_l)}{\phi_x'(y_l)}+\alpha_0\beta_0^{-1}\frac{w(x_k)}{\phi_x'(x_k)}C_{kl}^n\sum_{j=-M_y}^{N_y}[-\frac{1}{h_y^2}\delta_{lj}^{(2)}\phi_y'(y_j)v(y_j)-\frac{1}{h_y}\delta_{lj}^{(1)}(\frac{\phi_y''(y_j)v(y_j)}{\phi_y'(y_j)}$$

$$+2v'(y_j))-\delta_{lj}^{(0)}\frac{v''(y_j)}{\phi_y'(y_j)}]=\sum_{i=1}^{n}d_{n-i}^n\frac{w(x_k)}{\phi_x'(x_k)}\frac{v(y_l)}{\phi_y'(y_l)}C_{kl}^{n-i}+\alpha_0\beta_0^{-1}\sum_{j=-M_y}^{N_y}\frac{w(x_k)}{\phi_x'(x_k)}\frac{v(y_l)}{\phi_y'(y_l)}f_{kl}^n, \quad (19)$$

where these points x_i, y_j are Sinc grid points $x_i=\frac{a+be^{ih}}{1+e^{ih}}$ and $y_j=\frac{a+be^{jh}}{1+e^{jh}}$ ((for $a=0, b=1$). In this paper, we choose $M_x=N_x$, $M_y=N_y$. By denoting the labels $m_x=2*M_x+1$, $m_y=2*M_y+1$, defining the notation $I^{(l)}=[\delta_{kj}^{(l)}]_{m_x\times m_y}, l=0,1,2$, $\mathbf{C}^n=[C_{kl}^n]_{m_x\times m_y}$, $\mathbf{F}^n=[f_{kl}^n]_{m_x\times m_y}$, $\phi_x(x_i)=\phi(x_i)$ and $\phi_y(y_j)=\phi(y_j)(i,j=-M_x,\dots M_x)$, the discrete scheme leads to the matrix form

$$D(\frac{w}{\phi_x'})\mathbf{C}^1D(\frac{v}{\phi_y'})+\widetilde{\alpha}_0[-\frac{1}{h_x^2}I^{(2)}D(\phi_x'w)-\frac{1}{h_x}I^{(1)}D(\frac{\phi_x''w}{\phi_x'}+2w')-I^{(0)}D(\frac{w''}{\phi_x'})]\mathbf{C}^1D(\frac{v}{\phi_y'})$$
$$+\widetilde{\alpha}_0D(\frac{w}{\phi_x'})\mathbf{C}^1[-\frac{1}{h_y^2}I^{(2)}D(\phi_y'v)-\frac{1}{h_y}I^{(1)}D(\frac{\phi_y''v}{\phi_x'}+2v')-I^{(0)}D(\frac{v''}{\phi_y'})]^T$$
$$=D(\frac{w}{\phi_x'})\mathbf{C}^0D(\frac{v}{\phi_y'})+\widetilde{\alpha}_0D(\frac{w}{\phi_x'})\mathbf{F}^1D(\frac{v}{\phi_y'}),$$

and

$$D(\frac{w}{\phi_x'})\mathbf{C}^nD(\frac{v}{\phi_y'})+\alpha_0\beta_0^{-1}[-\frac{1}{h_x^2}I^{(2)}D(\phi_x'w)-\frac{1}{h_x}I^{(1)}D(\frac{\phi_x''w}{\phi_x'}+2w')-I^{(0)}D(\frac{w''}{\phi_x'})]\mathbf{C}^nD(\frac{v}{\phi_y'})$$
$$+\alpha_0\beta_0^{-1}D(\frac{w}{\phi_x'})\mathbf{C}^n[-\frac{1}{h_y^2}I^{(2)}D(\phi_y'v)-\frac{1}{h_y}I^{(1)}D(\frac{\phi_y''v}{\phi_x'}+2v')-I^{(0)}D(\frac{v''}{\phi_y'})]^T$$
$$=\sum_{i=1}^{n}d_{n-i}^nD(\frac{w}{\phi_x'})\mathbf{C}^{n-i}D(\frac{v}{\phi_y'})+\alpha_0\beta_0^{-1}D(\frac{w}{\phi_x'})\mathbf{F}^nD(\frac{v}{\phi_y'}).$$

Premultiplying by $D(\phi_x')$ and postmultiplying by $D(\phi_y')$ yields the equivalent system

$$D(w)\mathbf{C}^1D(v)+\widetilde{\alpha}_0D(\phi_x')[-\frac{1}{h_x^2}I^{(2)}D(\phi_x'w)-\frac{1}{h_x}I^{(1)}D(\frac{\phi_x''w}{\phi_x'}+2w')-I^{(0)}D(\frac{w''}{\phi_x'})]\mathbf{C}^1D(v)$$
$$+\widetilde{\alpha}_0D(w)\mathbf{C}^1[-\frac{1}{h_y^2}I^{(2)}D(\phi_y'v)-\frac{1}{h_y}I^{(1)}D(\frac{\phi_y''v}{\phi_x'}+2v')-I^{(0)}D(\frac{v''}{\phi_y'})]^TD(\phi_y')$$
$$=D(w)[\mathbf{C}^0+\widetilde{\alpha}_0\mathbf{F}^1]D(v);$$

$$D(w)\mathbf{C}^n D(v) + \alpha_0\beta_0^{-1} D(\phi_x')[-\frac{1}{h_x^2} I^{(2)} D(\phi_x' w) - \frac{1}{h_x} I^{(1)} D(\frac{\phi_x'' w}{\phi_x'} + 2w') - I^{(0)} D(\frac{w''}{\phi_x'})]\mathbf{C}^n D(v)$$
$$+\alpha_0\beta_0^{-1} D(w)\mathbf{C}^n[-\frac{1}{h_y^2} I^{(2)} D(\phi_y' v) - \frac{1}{h_y} I^{(1)} D(\frac{\phi_y'' v}{\phi_x'} + 2v') - I^{(0)} D(\frac{v''}{\phi_y'})]^T D(\phi_y')$$
$$= D(w)[\sum_{i=1}^{n} d_{n-i}^n \mathbf{C}^{n-i} + \alpha_0\beta_0^{-1}\mathbf{F}^n] D(v).$$

Denote $A(w) = -\frac{1}{h_x^2} I^{(2)} - \frac{1}{h_x} I^{(1)} D(\frac{\phi_x''}{(\phi_x')^2} + 2\frac{w'}{\phi_x' w}) - I^{(0)} D(\frac{w''}{(\phi_x')^2 w})$. From the above equations, we present the following matrix form:

$$D(w)\mathbf{C}^1 D(v) + \widetilde{\alpha}_0[D(\phi_x')A(w)D(\phi_x')]D(w)\mathbf{C}^1 D(v) + \widetilde{\alpha}_0 D(w)\mathbf{C}^1 D(v)[D(\phi_y' v)A(v)D(\phi_y')]^T$$
$$= D(w)(\mathbf{C}^0 + \widetilde{\alpha}_0\mathbf{F}^1)D(v);$$
$$D(w)\mathbf{C}^n D(v) + \alpha_0\beta_0^{-1}[D(\phi_x')A(w)D(\phi_x')]D(w)\mathbf{C}^n D(v)$$
$$+ \alpha_0\beta_0^{-1} D(w)\mathbf{C}^n D(v)[D(\phi_y' v)A(v)D(\phi_y')]^T$$
$$= D(w)(\sum_{i=1}^{n} d_{n-i}^n \mathbf{C}^{n-i} + \alpha_0\beta_0^{-1}\mathbf{F}^n)D(v).$$

In this paper, we define $\mathbf{V}^n = D(w)\mathbf{C}^n D(v) (n \geq 1)$, $\mathbf{G}^1 = D(w)(\mathbf{C}^0 + \widetilde{\alpha}_0\mathbf{F}^1)D(v)$, and $\mathbf{G}^n = D(w)(\sum_{i=1}^{n} d_{n-i}^n \mathbf{C}^{n-i} + \alpha_0\beta_0^{-1}\mathbf{F}^n)D(v) (n > 1)$. Using the weight function $w(x)v(y) = \frac{1}{\sqrt{\phi_x'(x)\phi_y'(y)}}$ in the Sinc-Galerkin inner product, we can know $\mathbf{A}_x = D(\phi_x')A(w)D(\phi_x') = D(\phi_x')[-\frac{1}{h_x^2} I^{(2)} + D(\frac{1}{4})]D(\phi_x')$, with an analogous definition for $\mathbf{A}_y$, so the matrix form may be written in the following form

$$\mathbf{V}^1 + \widetilde{\alpha}_0\mathbf{A}_x\mathbf{V}^1 + \widetilde{\alpha}_0\mathbf{V}^1\mathbf{A}_y^T = \mathbf{G}^1; \tag{20}$$
$$\mathbf{V}^n + \alpha_0\beta_0^{-1}\mathbf{A}_x\mathbf{V}^n + \alpha_0\beta_0^{-1}\mathbf{V}^n\mathbf{A}_y^T = \mathbf{G}^n. \tag{21}$$

To solve the solution of **V**, we use the Kronecker product and concatenate the system in (20)–(21) to arrive at (see [28, Thm. A.33])

$$[I_{m_y} \otimes I_{m_x} + \widetilde{\alpha}_0 I_{m_y} \otimes \mathbf{A}_x + \widetilde{\alpha}_0\mathbf{A}_y \otimes I_{m_x}]Co(\mathbf{V}^1) = Co(\mathbf{G}^1); \tag{22}$$
$$[I_{m_y} \otimes I_{m_x} + \alpha_0\beta_0^{-1} I_{m_y} \otimes \mathbf{A}_x + \alpha_0\beta_0^{-1}\mathbf{A}_y \otimes I_{m_x}]Co(\mathbf{V}^n) = Co(\mathbf{G}^n), \tag{23}$$

where $Co(\mathbf{G}^n) = \begin{pmatrix} Co(\mathbf{G}_{i1}^n) & Co(\mathbf{G}_{i2}^n) & \dots & Co(\mathbf{G}_{im_y}^n) \end{pmatrix}^T$ is the concatenation of $\mathbf{G}^n$, which is a $m_x m_y \times 1$ vector, where $Co(\mathbf{G}_{ij}^n) = \begin{pmatrix} \mathbf{G}_{1j}^n & \mathbf{G}_{2j}^n & \dots & \mathbf{G}_{m_x j}^n \end{pmatrix}^T$ $(j = 1, 2, \dots, m_y)$. I_{m_x} is a $m_x \times m_x$ identity matrix, the same as the matrix I_{m_y}. The symbol $\otimes$ represents the Kronecker product.

By premultiplying by $D^{-1}(w)$ and postmultiplying by $D^{-1}(v)$ of $\mathbf{V}^n$, we can get the numerical solution $\mathbf{C}^n$ from the solution of $\mathbf{V}^n$.

4 The stability and convergence analysis of the discrete scheme

4.1 The stability and convergence analysis of the discrete scheme in time

The weak form of the formulation of the discrete scheme (8) with the homogeneous boundary condition $u^n \in H_0^1(\Lambda)$ is defined as:

$$(u^n, v) + \alpha_0\beta_0^{-1}(\nabla u^n, \nabla v) = \sum_{i=1}^{n} d_{n-i}^n (u^{n-i}, v) + \alpha_0\beta_0^{-1}(f^n, v), \forall v \in H_0^1(\Lambda), \tag{24}$$

where $(.,.)$ is the usual L^2-inner product, $\Lambda = [0,1]$. Just as the same derivation process in [5], the bellowing theorem is easily demonstrated:

Theorem 5 [7] *If we assume the solution of problem* (4) *has the necessary regularity and that the* $f = 0$ *to deduce the stability analysis, the semi-discretization of equation* (11) *is unconditionally stable and its solution satisfies the following inequality*:

$$\|u^n\|_0 + \sqrt{\alpha_0\beta_0^{-1}}\|\nabla u^n\|_0 \le 4\|u^0\|_0. \tag{25}$$

The following error estimate is established.

Theorem 6 [7] *Let* u *be the exact solution of equation* (4), *and let* $\{u^n\}_{n=0}^N$ *be the semi-discrete solution of* (11) *with the initial* $u^0 = u(t_0)$. *Suppose* $\partial_t^3 u \in L^\infty([0,T]; L^2(\Lambda))$, *then the following error estimate holds*:

$$\|u(t_n) - u^n\|_0 + \sqrt{\alpha_0\beta_0^{-1}}\|\nabla(u(t_n) - u^n)\|_0 \le C_{\alpha,T}\|\partial_t^3 u\|_{L^\infty(L^2)}\Delta t^{3-\alpha}, 2 \le n \le N. \tag{26}$$

4.2 The convergence analysis of the discrete scheme in space for 1D problem

Considering the fully discrete scheme (10), supposing the function $F(x)$ satisfies the condition in Theorem 4, we can get the following exponential convergence:

$$| < u^n, s_k > -h\frac{w(x_k)u(x_k)}{\phi'(x_k)}| \le L_0 M^{-1/2}\exp(-(\pi d\alpha M)^{1/2}),$$

$$\Big| < u_{xx}^n, S_k > -h\sum_{k=-M}^{N} u(x_k)[\frac{\delta_{jk}^{(2)}}{h^2}(\phi' w)(x_k) + \frac{\delta_{jk}^{(1)}}{h}(\frac{\phi''}{\phi'}w + 2w')(x_k)] - h\frac{uw''}{\phi'}(x_j)\Big|$$
$$\le L_1' M\exp(-(\pi d\alpha M)^{1/2}).$$

4.3 The convergence analysis of the discrete scheme in space for 2D problem

In this section, we will analyze the convergence rates of each term in equation (18). We will use the following lemmas to derive the convergence order.

Lemma 3 *Define the notations* $\delta_{kj}^{(s)}(s = 0,1,2)$ *as formulas* (1)–(3) *and* $\phi_x(x)$ *via* $\phi_x(x) = \ln\frac{x-a}{b-x}$, *then we have*

$$(1)\ |\sum_{i=-\infty}^{\infty}\delta_{kj}^{(0)}| = |\sum_{i=-\infty}^{\infty}[S(k,h_x)\circ\phi_x(x_i)]| = 1;$$

$$(2)\ |\sum_{i=-\infty}^{\infty}\delta_{kj}^{(1)}| < 2;$$

$$(3) \mid \sum_{i=-\infty}^{\infty} \delta_{kj}^{(2)} \mid \leq 4 + \frac{\pi^2}{3}.$$

Proof (1). The first equality can be directly obtained from [28](P.53).

(2). We change the sum $\sum_{i=-\infty}^{\infty} \delta_{kj}^{(1)}$ to two parts, which is $\sum_{i=k}^{\infty} \delta_{kj}^{(1)}$ and $\sum_{i=-\infty}^{k-1} \delta_{kj}^{(1)}$. For the first part, it is an alternating series, whose general term $\frac{1}{n} > \frac{1}{n+1}$ and $\lim_{n\to\infty} \frac{1}{n} = 0$. So according to the Leibniz law, this series is limited. Then,

$$\mid \sum_{i=k}^{\infty} \delta_{kj}^{(1)} \mid = \mid 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \mid = \ln 2 < 1.$$

And the rest part, if we choose $n = -(i-k)$, then

$$\mid \sum_{-\infty}^{i=k-1} \delta_{kj}^{(1)} \mid = \mid \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mid = \mid 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \mid = \ln 2 < 1.$$

So, $\mid \sum_{i=-\infty}^{\infty} \delta_{kj}^{(1)} \mid < 2$.

(3). Just as the demonstration of (2), the sum $\sum_{i=-\infty}^{\infty} \delta_{kj}^{(1)}$ also can be changed to two parts of $\sum_{i=k}^{\infty} \delta_{kj}^{(2)}$ and $\sum_{i=-\infty}^{k-1} \delta_{kj}^{(2)}$. The first part is also limited.

$$\mid \sum_{i=k}^{\infty} \delta_{kj}^{(2)} \mid = \mid \sum_{i=k+1}^{\infty} \frac{-2(-1)^{i-k}}{(i-k)^2} - \frac{\pi^2}{3} \mid = \mid 2 - \frac{2}{2^2} + \frac{2}{3^2} - \frac{2}{4^2} + \cdots - \frac{\pi^2}{3} \mid \leq 2 + \frac{\pi^2}{3}.$$

And the rest part, if we choose $n = -(i-k)$, then

$$\mid \sum_{-\infty}^{i=k-1} \delta_{kj}^{(2)} \mid = \mid \sum_{n=1}^{\infty} \frac{-2(-1)^{-n}}{n^2} \mid = \mid 2 - \frac{2}{2^2} + \frac{2}{3^2} - \frac{2}{4^2} + \cdots \mid \leq 2.$$

So, we get the proof. □

Lemma 4 *Let $\phi_x(x)$ be defined via $\phi_x(x) = \ln \frac{x-a}{b-x}$, then it holds that*

$$\mid \frac{1}{\phi_x'(x)} \mid < \frac{b-a}{4}.$$

Proof According to $\phi_x(x) = \ln \frac{x-a}{b-x}$, we have

$$\mid \frac{1}{\phi_x'(x)} \mid = \mid \frac{(x-a)(b-x)}{b-a} \mid = \mid \frac{-x^2 + (a+b)x - ab}{b-a} \mid = \mid \frac{-(x - \frac{a+b}{2})^2 + \frac{(b-a)^2}{4}}{b-a} \mid < \frac{b-a}{4}. \quad \square$$

It is easy to see that $\mid \frac{1}{\phi_y'(y)} \mid < \frac{d-c}{4}$.

Lemma 5 *Let $\phi_x(x)$ be defined via $\phi_x(x) = \ln \frac{x-a}{b-x}$, $w = \frac{1}{\sqrt{\phi_x'(x)}}$, $\phi_y(y) = \ln \frac{y-c}{d-y}$, $v = \frac{1}{\sqrt{\phi_y'(y)}}$, then it holds that*

$$(1) \frac{w''(x)}{\phi_x'(x)} = \frac{-1}{4}\sqrt{\phi_x'(x)}, \ \frac{\phi_x''(x)}{\phi_x'(x)} w(x) + 2w' = 0.$$

$$(2) \frac{v''(y)}{\phi_y'(y)} = \frac{-1}{4}\sqrt{\phi_y'(y)}, \ \frac{\phi_y''(y)}{\phi_y'(y)} v(y) + 2v' = 0.$$

Proof (1) According to $\phi_x(x) = \ln\frac{x-a}{b-x}$, we have

$$\phi_x'(x) = \frac{b-a}{(x-a)(b-x)},$$
$$w = \frac{1}{\sqrt{\phi_x'(x)}} = \frac{\sqrt{(x-a)(b-x)}}{\sqrt{b-a}} = e^{\frac{1}{2}\ln(x-a)+\frac{1}{2}\ln(b-x)-\frac{1}{2}\ln(b-a)},$$

and

$$\frac{\phi_x''(x)}{\phi_x'(x)}w(x) + 2w' = \frac{\phi_x''(x)}{(\phi_x'(x))^{3/2}} + 2\Big(\frac{1}{\sqrt{\phi_x'(x)}}\Big)' = \frac{\phi_x''(x)}{(\phi_x'(x))^{3/2}} + 2\Big(-\frac{1}{2}\Big)\frac{1}{(\sqrt{\phi_x'(x)})^{3/2}}\phi_x''(x) = 0,$$

then

$$w' = e^{\frac{1}{2}\ln(x-a)+\frac{1}{2}\ln(b-x)-\frac{1}{2}\ln(b-a)}\Big(\frac{1}{2}\frac{1}{(x-a)} - \frac{1}{2}\frac{1}{(b-x)}\Big),$$
$$\begin{aligned} w'' &= e^{\frac{1}{2}\ln(x-a)+\frac{1}{2}\ln(b-x)-\frac{1}{2}\ln(b-a)}\Big[\Big(\frac{1}{2}\frac{1}{(x-a)} - \frac{1}{2}\frac{1}{(b-x)}\Big)^2 - \frac{1}{2}\frac{1}{(x-a)^2} - \frac{1}{2}\frac{1}{(b-x)^2}\Big] \\ &= \frac{\sqrt{(x-a)(b-x)}}{\sqrt{b-a}}\Big[-\frac{1}{4(x-a)^2} - \frac{1}{4(b-x)^2} - \frac{1}{2(x-a)(b-x)}\Big] \\ &= \frac{-1}{4}\frac{(b-a)^{3/2}}{(x-a)^{3/2}(b-x)^{3/2}} \\ &= \frac{-1}{4}(\phi_x'(x))^{3/2}. \end{aligned}$$

The results for parameter y can be demonstrated just as above, so the proof is completed. □

Theorem 7 *Let ϕ_x and ϕ_y be conformal one-to-one maps of the simple connected domain $\mathcal{D}_x$ and $\mathcal{D}_y$, respectively, onto $\mathcal{D}_S$. $\mathcal{D}_x \times \mathcal{D}_y$ is the domain Ω. Assume that $x_k = \psi(kh_x)$, $y_l = \psi(lh_y)$, $k,l \in \mathbb{Z}$, $uF_q \in B(\mathcal{D})$, ($F_q = \sqrt{\phi_q'}(1/\phi_q')''$, $\phi_q''/\sqrt{\phi_q'}$, $(\phi_q')^{3/2}$, $(1/\sqrt{\phi_q'})''$), and $f/\sqrt{\phi_q'} \in B(\mathcal{D})$, ($q = x$ or y), $B_{T_2} = 0$, which is defined as* (4.13) *in* [26]. *To simplify the discrete scheme, we choose $w = \frac{1}{\sqrt{\phi_x'}}$, $v = \frac{1}{\sqrt{\phi_y'}}$. In this paper, we choose $\phi_x(x) = \ln\frac{x-a}{b-x}$, and $\phi_y(y) = \ln\frac{y-c}{d-y}$, then*

$$\begin{aligned} &(1).\Big|\langle u, S_{kl}\rangle - h_x h_y u(x_k, y_l)\frac{w(x_k)v(y_l)}{\phi_x'(x_k)\phi_y'(y_l)}\Big| \\ &\le \frac{(b-a)^{3/2}C_0(h_y,d_y)}{16}h_x e^{-\pi d_y/h_y}\widetilde{N}(uv, D_y) + \frac{(d-c)^{3/2}C_0(h_x,d_x)}{16}h_y e^{-\pi d_x/h_x}\widetilde{N}(uw, D_x) \\ &+ \frac{C_0(h_x,d_x)C_0(h_y,d_y)}{4}e^{-\pi d_x/h_x}e^{-\pi d_y/h_y}N(N(uw,\mathcal{D}_x)v,\mathcal{D}_y), \end{aligned}$$

$$\begin{aligned} &(2).\Big|\langle \Delta u, S_{kl}\rangle - h_x h_y \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\frac{u(x_i,y_j)}{\phi_x'(x_i)\phi_y'(y_j)}\{[S(l,h_y)\circ\phi_y(y_j)v(y_j)]I_1(x_i) \\ &+ I_1(y_j)[S(k,h_x)\circ\phi_x(x_i)w(x_i)]\}\Big| \\ &\le \frac{1}{2}h_x e^{-\pi d_y/h_y}C_0(h_y,d_y)\widetilde{N}'(u\sqrt{\phi_x'}v, D_y)\Big[\frac{1}{h_x^2}\Big(4 + \frac{\pi^2}{3}\Big) + \frac{1}{4}\Big] \end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{2}h_y e^{-\pi d_x/h_x} C_0(h_x,d_x)\widetilde{N}'(u\sqrt{\phi_y'}w, D_x)[\frac{1}{h_y^2}(4+\frac{\pi^2}{3})+\frac{1}{4}] \\
&+ \frac{(b-a)^{3/2}}{4} h_x e^{-\pi d_y/h_y}|C_2(h_y,d_y)-\frac{1}{4}C_0(h_y,d_y)|\widetilde{N}(uv'',D_y) \\
&+ \frac{(d-c)^{3/2}}{4} h_y e^{-\pi d_x/h_x}|C_2(h_x,d_x)-\frac{1}{4}C_0(h_x,d_x)|\widetilde{N}(uw'',D_x) \\
&+ C_0(h_y,d_y)e^{-\pi d_x/h_x}e^{-\pi d_y/h_y}N(N(uw'',D_x)v,D_y)|C_2(h_x,d_x)-\frac{1}{4}C_0(h_x,d_x)| \\
&+ C_0(h_x,d_x)e^{-\pi d_x/h_x}e^{-\pi d_y/h_y}N(N(uw,D_x)v'',D_y)|C_2(h_y,d_y)-\frac{1}{4}C_0(h_y,d_y)| \\
&\equiv R(\Omega),
\end{aligned}
$$

where $\widetilde{N}(u,D_y) = \int_{\partial\mathcal{D}_y} u(x_k,y)dy$, $\widetilde{N}'(u\sqrt{\phi_x'}v,D_y) = \max_i \int_{\partial\mathcal{D}_y} u(x_i,y)v(y)\sqrt{\phi_x'}(x_k)dy$, *and the same as* $\widetilde{N}(u,D_x) = \int_{\partial\mathcal{D}_x} u(x,y_l)dx$, $\widetilde{N}'(u\sqrt{\phi_y'}w,D_x) = \max_j \int_{\partial\mathcal{D}_x} u(x,y_j)w(x)\sqrt{\phi_x'}(y_l)dx$, $N(N(uw,D_x)v,D_y) = \int_{\partial\mathcal{D}_y}\int_{\partial\mathcal{D}_x} u(x,y)w(x)v(y)dxdy$,

$$
\begin{aligned}
I_1(x) &= \frac{d^2}{d\phi_x^2}[S(k,h_x)\circ\phi_x(x)](\phi_x')^2(x)w(x) + \frac{d}{d\phi_x}[S(k,h_x)\circ\phi_x(x)](\phi_x''w+2\phi_x'w')(x) \\
&+[S(k,h_x)\circ\phi_x(x)]w''(x).
\end{aligned}
$$

At the same,

$$
\begin{aligned}
I_1(y) &= \frac{d^2}{d\phi_y^2}[S(l,h_y)\circ\phi_y(y)](\phi_y')^2(y)v(y) + \frac{d}{d\phi_y}[S(l,h_y)\circ\phi_y(y)](\phi_y''v+2\phi_y'v')(y) \\
&+[S(l,h_y)\circ\phi_y(y)]v''(y).
\end{aligned}
$$

Applying Lemma 3, Lemma 4, Theorem 1, and Theorem 2 to each term of equation (18), we can prove the above theorem.

Proof (1) Utilizing the integral approximation in Theorem 1 for variable x, we have

$$
\begin{aligned}
&\big|\langle u,S_{kl}\rangle\big| = \Big|\int_c^d\int_a^b u(x,y)[S(k,h_x)\circ\phi_x(x)w(x)][S(l,h_y)\circ\phi_y(y)v(y)]dxdy\Big| \\
&= \Big|\int_c^d h_x\sum_{i=-\infty}^{\infty}\frac{u(x_i,y)[S(k,h_x)\circ\phi_x(x_i)w(x_i)]}{\phi_x'(x_i)}[S(l,h_y)\circ\phi_y(y)v(y)]dy \\
&+ \int_c^d \frac{i}{2}\int_{\partial\mathcal{D}_x}\frac{u(x,y)\kappa(\phi_x,h_x)(x)}{\sin(\pi\phi_x(x)/h_x)}[S(k,h_x)\circ\phi_x(x)w(x)]dx[S(l,h_y)\circ\phi_y(y)v(y)]dy\Big|.
\end{aligned}
$$

Then, using the inequalities about $\kappa(\phi_x,h_x)(x)$ in Theorem 1 and the first inequality about $\frac{S(k,h_x)\circ\phi_x(x)}{\sin(\pi\phi_x(x)/h_x)}$ in Theorem 2, we deduce the above equality as:

$$
\begin{aligned}
&\big|\langle u,S_{kl}\rangle\big| \le \Big|h_x\sum_{i=-\infty}^{\infty}\frac{[S(k,h_x)\circ\phi_x(x_i)w(x_i)]}{\phi_x'(x_i)}\int_c^d u(x_i,y)[S(l,h_y)\circ\phi_y(y)v(y)]dy\Big| \\
&+ \Big|\frac{C_0(h_x,d_x)}{2}e^{-\pi d_x/h_x}\int_c^d\int_{\partial\mathcal{D}_x} u(x,y)w(x)dx[S(l,h_y)\circ\phi_y(y)v(y)]dy\Big|.
\end{aligned}
$$

Then utilizing the integral approximation in Theorem 1 for variable y, we have

$$\begin{aligned}
\big|\langle u,S_{kl}\rangle\big| &\le \Big|h_xh_y\sum_{i=-\infty}^{\infty}\frac{[S(k,h_x)\circ\phi_x(x_i)w(x_i)]}{\phi_x'(x_i)}\sum_{j=-\infty}^{\infty}u(x_i,y_j)\frac{[S(l,h_y)\circ\phi_y(y_j)v(y_j)]}{\phi_y'(y_j)}\\
&+h_x\sum_{i=-\infty}^{\infty}\frac{[S(k,h_x)\circ\phi_x(x_i)w(x_i)]}{\phi_x'(x_i)}\frac{i}{2}\int_{\partial\mathcal{D}_y}\frac{u(x_i,y)\kappa(\phi_y,h_y)(y)}{\sin(\pi\phi_y(y)/h_y)}[S(l,h_y)\circ\phi_y(y)v(y)]dy\Big|\\
&+\Big|\frac{C_0(h_x,d_x)}{2}e^{-\pi d_x/h_x}h_y\sum_{j=-\infty}^{\infty}\int_{\partial\mathcal{D}_x}u(x,y_j)w(x)dx\frac{[S(l,h_y)\circ\phi_y(y_j)v(y_j)]}{\phi_y'(y_j)}\\
&+\frac{C_0(h_x,d_x)}{2}e^{-\pi d_x/h_x}\frac{i}{2}\int_{\partial\mathcal{D}_y}\int_{\partial\mathcal{D}_x}u(x,y)w(x)v(y)dx\frac{\kappa(\phi_y,h_y)(y)[S(l,h_y)\circ\phi_y(y)]}{\sin(\pi\phi_y(y)/h_y)}dy\Big|.
\end{aligned}$$

Using the inequalities in Theorem 1 and Theorem 2, we also have

$$\begin{aligned}
\big|\langle u,S_{kl}\rangle\big| &\le \Big|h_xh_y\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}u(x_i,y_j)\frac{[S(k,h_x)\circ\phi_x(x_i)w(x_i)]}{\phi_x'(x_i)}\frac{[S(l,h_y)\circ\phi_y(y_j)v(y_j)]}{\phi_y'(y_j)}\Big|\\
&+\Big|h_x\frac{C_0(h_y,d_y)}{2}e^{-\pi d_y/h_y}\sum_{i=-\infty}^{\infty}\frac{[S(k,h_x)\circ\phi_x(x_i)w(x_i)]}{\phi_x'(x_i)}\int_{\partial\mathcal{D}_y}u(x_i,y)v(y)dy\Big|\\
&+\Big|\frac{C_0(h_x,d_x)}{2}e^{-\pi d_x/h_x}h_y\sum_{j=-\infty}^{\infty}\frac{[S(l,h_y)\circ\phi_y(y_j)v(y_j)]}{\phi_y'(y_j)}\int_{\partial\mathcal{D}_x}u(x,y_j)w(x)dx\Big|\\
&+\Big|\frac{C_0(h_x,d_x)}{2}e^{-\pi d_x/h_x}\frac{C_0(h_y,d_y)}{2}e^{-\pi d_y/h_y}\int_{\partial\mathcal{D}_y}\int_{\partial\mathcal{D}_x}u(x,y)w(x)v(y)dxdy\Big|.
\end{aligned}\tag{27}$$

For the second part of the right side of equation (27), we will use the Lemma 3, Lemma 4, and the fact that $S(k,h_x)\circ\phi_x(x_i)=0$(if $i\neq k$) to demonstrate:

$$\begin{aligned}
&\Big|\sum_{i=-\infty}^{\infty}\frac{[S(k,h_x)\circ\phi_x(x_i)w(x_i)]}{\phi_x'(x_i)}\int_{\partial\mathcal{D}_y}u(x_i,y)v(y)dy\Big|\\
&\le\Big|\frac{b-a}{4}\frac{\sqrt{b-a}}{2}\int_{\partial\mathcal{D}_y}u(x_i,y)v(y)dy\Big|=\frac{(b-a)^{3/2}}{8}\widetilde{N}(uv,D_y).
\end{aligned}$$

The same as the third term of the right side of equation (27), we have:

$$\Big|\sum_{j=-\infty}^{\infty}\frac{[S(l,h_y)\circ\phi_y(y_j)v(y_j)]}{\phi_y'(y_j)}\int_{\partial\mathcal{D}_x}u(x,y_j)w(x)dx\Big|\le\frac{(d-c)^{3/2}}{8}\widetilde{N}(uw,D_x).$$

Combining these above inequalities into equation (27), we can prove the first result in Theorem 8.

(2) Using the integration by parts, and assuming $B_{T_2}=0$, we can change the second term of the left side in Eq. (18) to

$$\begin{aligned}
I&=\int_c^d\int_a^b\Delta u^n(x,y)S(k,h_x)\circ\phi_x(x)S(l,h_y)\circ\phi_y(y)w(x)v(y)dxdy\\
&=\int_c^d\int_a^b u^n(x,y)\Delta[S(k,h_x)\circ\phi(x)w(x)][S(l,h_y)\circ\phi(y)v(y)]dxdy
\end{aligned}$$

$$= \int_c^d \int_a^b u(x,y)\{I_1(x)[S(l,h_y)\circ\phi_y(y)v(y)] + [S(k,h_x)\circ\phi_x(x)w(x)]I_1(y)\}dxdy.$$

Applying the integral approximation for variable x described in Theorem 1, we have

$$\begin{aligned}
I &= \int_c^d h_x \sum_{i=-\infty}^{\infty} \frac{u(x_i,y)}{\phi_x'(x_i)} I_1(x_i)[S(l,h_y)\circ\phi_y(y)v(y)]dy \\
&+ \int_c^d h_x \sum_{i=-\infty}^{\infty} \frac{u(x_i,y)}{\phi_x'(x_i)} [S(k,h_x)\circ\phi_x(x_i)w(x_i)]I_1(y)dy \\
&+ \int_c^d \frac{i}{2}\int_{\partial\mathcal{D}_x} \frac{u(x,y)\kappa(\phi_x,h_x)(x)}{\sin(\pi\phi_x(x)/h_x)} I_1(x)dx[S(l,h_y)\circ\phi_y(y)v(y)]dy \\
&+ \int_c^d \frac{i}{2}\int_{\partial\mathcal{D}_x} \frac{u(x,y)\kappa(\phi_x,h_x)(x)}{\sin(\pi\phi_x(x)/h_x)} [S(k,h_x)\circ\phi_x(x)w(x)]dxI_1(y)dy.
\end{aligned}$$

Using Theorem 1 and Theorem 2, we can get

$$\begin{aligned}
&|I - h_x \sum_{i=-\infty}^{\infty} \int_c^d \frac{u(x_i,y)}{\phi_x'(x_i)} I_1(x_i)[S(l,h_y)\circ\phi_y(y)v(y)]dy \\
&- \int_c^d h_x \sum_{i=-\infty}^{\infty} \frac{u(x_i,y)}{\phi_x'(x_i)} [S(k,h_x)\circ\phi_x(x_i)w(x_i)]I_1(y)dy\Big| \\
&\le \Big|\frac{1}{2}e^{-\pi d_x/h_x}\int_c^d \int_{\partial\mathcal{D}_x} \frac{u(x,y)}{\sin(\pi\phi_x(x)/h_x)} I_1(x)dx[S(l,h_y)\circ\phi_y(y)v(y)]dy\Big| \\
&+\Big|\frac{1}{2}C_0(h_x,d_x)e^{-\pi d_x/h_x}\int_c^d N(uw,\mathcal{D}_x)I_1(y)dy\Big|.
\end{aligned}$$

Applying the integral approximation for variable y, we can obtain

$$\begin{aligned}
&\Big|I - h_x h_y \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} \frac{u(x_i,y_j)}{\phi_x'(x_i)\phi_y'(y_j)}\{[S(l,h_y)\circ\phi_y(y_j)v(y_j)]I_1(x_i) \\
&+I_1(y_j)[S(k,h_x)\circ\phi_x(x_i)w(x_i)]\}\Big| \\
&\le \Big|\frac{i}{2}h_x \sum_{i=-\infty}^{\infty}\int_{\partial\mathcal{D}_y} \frac{u(x_i,y)\kappa(\phi_y,h_y)(y)}{\sin(\pi\phi_y(y)/h_y)\phi_x'(x_i)}[S(l,h_y)\circ\phi_y(y)v(y)]dyI_1(x_i) \\
&+\frac{i}{2}h_x \sum_{i=-\infty}^{\infty}\int_{\partial\mathcal{D}_y} \frac{u(x_i,y)\kappa(\phi_y,h_y)(y)}{\sin(\pi\phi_y(y)/h_y)\phi_x'(x_i)} I_1(y)dy[S(k,h_x)\circ\phi_x(x_i)w(x_i)]\Big| \\
&+\Big|\frac{h_y}{2}e^{-\pi d_x/h_x}\sum_{j=-\infty}^{\infty}\frac{[S(l,h_y)\circ\phi_y(y_j)v(y_j)]}{\phi_y'(y_j)}\int_{\partial\mathcal{D}_x}\frac{u(x,y_j)}{\sin(\pi\phi_x(x)/h_x)}I_1(x)dx \\
&+\frac{i}{4}e^{-\pi d_x/h_x}\int_{\partial\mathcal{D}_y}\int_{\partial\mathcal{D}_x}\frac{u(x,y)}{\sin(\pi\phi_x(x)/h_x)}I_1(x)dx\frac{[S(l,h_y)\circ\phi_y(y)v(y)]\kappa(\phi_y,h_y)(y)}{\sin(\pi\phi_y(y)/h_y)}dy\Big| \\
&+\Big|\frac{h_y}{2}e^{-\pi d_x/h_x}C_0(h_x,d_x)\sum_{j=-\infty}^{\infty}\frac{N(u(x,y_j)w,\mathcal{D}_x)I_1(y_j)}{\phi_y'(y_j)} \\
&+\frac{i}{4}e^{-\pi d_x/h_x}C_0(h_x,d_x)\int_{\partial\mathcal{D}_y}\frac{N(uw,\mathcal{D}_x)I_1(y)\kappa(\phi_y,h_y)(y)}{\sin(\pi\phi_y(y)/h_y)}dy\Big|.
\end{aligned}$$

Applying Theorem 1 and Theorem 2 yields

$$
\begin{aligned}
&\big|I - h_x h_y \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} \frac{u(x_i,y_j)}{\phi_x'(x_i)\phi_y'(y_j)}\{[S(l,h_y)\circ\phi_y(y_j)v(y_j)]I_1(x_i)\\
&+I_1(y_j)[S(k,h_x)\circ\phi_x(x_i)w(x_i)]\}\big|\\
&\le \frac{1}{2}h_x e^{-\pi d_y/h_y}C_0(h_y,d_y)\big|\sum_{i=-\infty}^{\infty}\int_{\partial\mathcal{D}_y}\frac{u(x_i,y)v(y)}{\phi_x'(x_i)}dy I_1(x_i)\big|\\
&+\frac{1}{2}h_x e^{-\pi d_y/h_y}\big|\sum_{i=-\infty}^{\infty}\frac{[S(k,h_x)\circ\phi_x(x_i)w(x_i)]}{\phi_x'(x_i)}\int_{\partial\mathcal{D}_y}\frac{u(x_i,y)}{\sin(\pi\phi_y(y)/h_y)}I_1(y)dy\big|\\
&+\frac{1}{2}h_y e^{-\pi d_x/h_x}\big|\sum_{j=-\infty}^{\infty}\frac{[S(l,h_y)\circ\phi_y(y_j)v(y_j)]}{\phi_y'(y_j)}\int_{\partial\mathcal{D}_x}\frac{u(x,y_j)}{\sin(\pi\phi_x(x)/h_x)}I_1(x)dx\big|\\
&+\frac{1}{4}e^{-\pi d_x/h_x}e^{-\pi d_y/h_y}C_0(h_y,d_y)\big|\int_{\partial\mathcal{D}_y}\int_{\partial\mathcal{D}_x}\frac{u(x,y)v(y)}{\sin(\pi\phi_x(x)/h_x)}I_1(x)dxdy\big|\\
&+\frac{h_y}{2}e^{-\pi d_x/h_x}C_0(h_x,d_x)\big|\sum_{j=-\infty}^{\infty}\frac{N(u(x,y_j)w,\mathcal{D}_x)I_1(y_j)}{\phi_y'(y_j)}\big|\\
&+\frac{1}{4}e^{-\pi d_x/h_x}e^{-\pi d_y/h_y}C_0(h_x,d_x)\big|\int_{\partial\mathcal{D}_y}\frac{N(uw,\mathcal{D}_x)I_1(y)}{\sin(\pi\phi_y(y)/h_y)}dy\big|. \qquad (28)
\end{aligned}
$$

Next, we will analyze the error of each term in the right side of inequality (28). For the second term, we will use Lemma 3 and Lemma 5 to deduce the following error:

$$
\begin{aligned}
&\big|\int_{\partial\mathcal{D}_y}\sum_{i=-\infty}^{\infty}\frac{u(x_i,y)v(y)}{\phi_x'(x_i)}I_1(x_i)dy\big|\\
&=\big|\int_{\partial\mathcal{D}_y}\sum_{i=-\infty}^{\infty}\frac{u(x_i,y)v(y)}{\phi_x'(x_i)}\{\frac{d^2}{d\phi_x^2}[S(k,h_x)\circ\phi_x(x_i)](\phi_x')^2(x_i)w(x_i)\\
&+\frac{d}{d\phi_x}[S(k,h_x)\circ\phi_x(x_i)](\phi_x''w+2\phi_x'w')(x_i)+[S(k,h_x)\circ\phi_x(x_i)]w''(x_i)\}dy\big|\\
&=\big|\sum_{i=-\infty}^{\infty}[\frac{\delta_{ki}^{(2)}}{h_x^2}\frac{(\phi_x')^2w}{\phi_x'}(x_i)+\frac{\delta_{ki}^{(1)}}{h_x}(\frac{\phi_x''w}{\phi_x'}+2w')(x_i)\\
&+\frac{[S(k,h_x)\circ\phi_x(x_i)]w''(x_i)}{\phi_x'}]N(u(x_i,y)v,D_y)\big|\\
&\le[\frac{1}{h_x^2}(4+\frac{\pi^2}{3})+\frac{1}{4}]\widetilde{N}'(u\sqrt{\phi_x'}v,D_y). \qquad (29)
\end{aligned}
$$

Just the same as above, the sixth term can be considered:

$$
\big|\sum_{j=-\infty}^{\infty}\frac{N(u(x,y_j)w,D_x)}{\phi_y'(y_j)}I_1(y_j)dy\big|\le[\frac{1}{h_y^2}(4+\frac{\pi^2}{3})+\frac{1}{4}]\widetilde{N}'(u\sqrt{\phi_y'}w,D_x) \qquad (30)
$$

Now we consider the third and forth terms. Utilizing Lemmas 3–5 and Theorem 2, we get

$$
\begin{aligned}
&|\sum_{i=-\infty}^{\infty}\int_{\partial\mathcal{D}_y}\frac{u(x_i,y)I_1(y)[S(k,h_x)\circ\phi_x(x_i)]w(x_i)}{\sin(\pi\phi_y(y)/h_y)\phi_x'(x_i)}dy| \\
&=|\sum_{i=-\infty}^{\infty}\int_{\partial\mathcal{D}_y}\frac{u(x_i,y)[S(k,h_x)\circ\phi_x(x_i)]w(x_i)}{\phi_x'(x_i)\sin(\pi\phi_y(y)/h_y)}\{\frac{d^2}{d\phi_y^2}[S(l,h_y)\circ\phi_y(y)](\phi_y')^2(y)v(y) \\
&+\frac{d}{d\phi_y}[S(l,h_y)\circ\phi_y(y)](\phi_y''v+2\phi_y'v')(y)+[S(l,h_y)\circ\phi_y(y)]v''(y)\}dy| \\
&\le 4\frac{b-a}{4}\frac{\sqrt{b-a}}{2}|C_2(h_y,d_y)-\frac{1}{4}C_0(h_y,d_y)|\widetilde{N}(uv'',D_y),
\end{aligned}
\tag{31}
$$

and

$$
\begin{aligned}
&|\sum_{j=-\infty}^{\infty}\frac{[S(l,h_y)\circ\phi_y(y_j)v(y_j)]}{\phi_y'(y_j)}\int_{\partial\mathcal{D}_x}\frac{u(x,y_j)}{\sin(\pi\phi_x(x)/h_x)}I_1(x)dx| \\
&\le 4\frac{d-c}{4}\frac{\sqrt{d-c}}{2}|C_2(h_x,d_x)-\frac{1}{4}C_0(h_x,d_x)|\widetilde{N}(uw'',D_x).
\end{aligned}
\tag{32}
$$

The fifth and the last term will be considered as follows:

$$
\begin{aligned}
&|\int_{\partial\mathcal{D}_y}\int_{\partial\mathcal{D}_x}\frac{u(x,y)v(y)}{\sin(\pi\phi_x(x)/h_x)}I_1(x)dxdy| \\
&=|\int_{\partial\mathcal{D}_y}\int_{\partial\mathcal{D}_x}\frac{u(x,y)v(y)}{\sin(\pi\phi_x(x)/h_x)}\{\frac{d^2}{d\phi_x^2}[S(k,h_x)\circ\phi_x(x)](\phi_x')^2(x)w(x) \\
&+\frac{d}{d\phi_x}[S(k,h_x)\circ\phi_x(x)](\phi_x''w+2\phi_x'w')(x)+[S(k,h_x)\circ\phi_x(x)]w''(x)\}dxdy| \\
&\le 4N(N(uw'',D_x)v,D_y)|C_2(h_x,d_x)-\frac{1}{4}C_0(h_x,d_x)|,
\end{aligned}
\tag{33}
$$

$$
|\int_{\partial\mathcal{D}_y}\frac{N(uw,\mathcal{D}_x)I_1(y)}{\sin(\pi\phi_y(y)/h_y)}dy|\le 4N(N(uw,D_x)v'',D_y)|C_2(h_y,d_y)-\frac{1}{4}C_0(h_y,d_y)|.
\tag{34}
$$

Combining inequalities (29)–(34) into inequality (28), the proof of the second result in Theorem (8) is completed. □

Utilizing the formulas (1)–(3) yields

$$
I_1(x_i)=\frac{\delta_{ki}^{(2)}}{h_x^2}(\phi_x')^2(x_i)w(x_i)+\frac{\delta_{ki}^{(1)}}{h_x}(\phi_x''w+2\phi_x'w')(x_i)+\delta_{ki}^{(0)}w''(x_i).
$$

So, we can get

$$
\begin{aligned}
&\big|I-\{h_xh_y\sum_{i=-\infty}^{\infty}\frac{u(x_i,y_l)}{\phi_x'(x_i)\phi_y'(y_l)}v(y_l)[\frac{\delta_{ki}^{(2)}}{h_x^2}(\phi_x')^2(x_i)w(x_i)+\frac{\delta_{ki}^{(1)}}{h_x}(\phi_x''w \\
&+2\phi_x'w')(x_i)]+h_xh_y\frac{u(x_k,y_l)}{\phi_x'(x_k)\phi_y'(y_l)}v(y_l)w''(x_k)
\end{aligned}
$$

$$+h_x h_y \sum_{j=-\infty}^{\infty} \frac{u(x_k,y_j)}{\phi_x'(x_k)\phi_y'(y_j)} w(x_k)[\frac{\delta_{jl}^{(2)}}{h_y^2}(\phi_y')^2(y_j)v(y_j) + \frac{\delta_{jl}^{(1)}}{h_y}(\phi_y'' v$$

$$+2\phi_y' v')(y_j)] + h_x h_y \frac{u(x_k,y_l)}{\phi_x'(x_k)\phi_y'(y_l)} w(x_k)v''(y_l)\}\Big| \le R(\Omega).$$

The following inequalities can be directly obtained by the definition of $h_x = \frac{\sqrt{\pi d_x}}{\sqrt{\alpha_x M_x}}$, $h_y = \frac{\sqrt{\pi d_y}}{\sqrt{\alpha_y M_y}}$ and the fact that $C_p(h,d) \le R_p h^{1-p}$:

$$\begin{aligned} h_x C_0(h_y,d_y)[\frac{1}{h_x^2}(4+\frac{\pi^2}{3})+\frac{1}{4}] &\le R_0 h_y[\frac{1}{h_x}(4+\frac{\pi^2}{3})+\frac{1}{4}h_x] \\ &= R_0\frac{\sqrt{\pi d_y}}{\sqrt{\alpha_y M_y}}[\frac{\sqrt{\alpha_x M_x}}{\sqrt{\pi d_x}}(4+\frac{\pi^2}{3})+\frac{1}{4}\frac{\sqrt{\pi d_x}}{\sqrt{\alpha_x M_x}}] \\ &\le M_x M_y R_0\frac{\sqrt{\pi d_y}}{\sqrt{\alpha_y}}[\frac{\sqrt{\alpha_x}}{\sqrt{\pi d_x}}(4+\frac{\pi^2}{3})+\frac{1}{4}\frac{\sqrt{\pi d_x}}{\sqrt{\alpha_x}}]; \end{aligned} \tag{35}$$

$$\begin{aligned} h_x|C_2(h_y,d_y)-\frac{1}{4}C_0(h_y,d_y)| &\le h_x[R_2\frac{1}{h_y}+\frac{1}{4}R_0 h_y] \\ &= R_2\frac{\sqrt{\pi d_x}}{\sqrt{\alpha_x M_x}}/\frac{\sqrt{\pi d_y}}{\sqrt{\alpha_y M_y}}+\frac{1}{4}R_0\frac{\sqrt{\pi d_x}}{\sqrt{\alpha_x M_x}}\frac{\sqrt{\pi d_y}}{\sqrt{\alpha_y M_y}} \\ &\le M_x M_y(R_2\frac{\sqrt{d_x\alpha_y}}{\sqrt{\alpha_x d_y}}+\frac{1}{4}R_0\pi\frac{\sqrt{d_x d_y}}{\sqrt{\alpha_x\alpha_y}}); \end{aligned} \tag{36}$$

and

$$\begin{aligned} &C_0(h_y,d_y)|C_2(h_x,d_x)-\frac{1}{4}C_0(h_x,d_x)| \le R_0R_2\frac{h_y}{h_x}+\frac{1}{4}R_0^2h_xh_y \\ &= R_0R_2\frac{\sqrt{\pi d_y}}{\sqrt{\alpha_y M_y}}/\frac{\sqrt{\pi d_x}}{\sqrt{\alpha_x M_x}}+\frac{1}{4}R_0^2\frac{\sqrt{\pi d_x}}{\sqrt{\alpha_x M_x}}\frac{\sqrt{\pi d_y}}{\sqrt{\alpha_y M_y}} \\ &\le M_x M_y(R_0R_2\frac{\sqrt{\alpha_x d_y}}{\sqrt{\alpha_y d_x}}+\frac{1}{4}R_0^2\pi\frac{\sqrt{d_x d_y}}{\sqrt{\alpha_x\alpha_y}}). \end{aligned} \tag{37}$$

Just as above, we can get the similar bound of $h_y C_0(h_x,d_x)[\frac{1}{h_y^2}(4+\frac{\pi^2}{3})+\frac{1}{4}]$, $h_y[C_2(h_x,d_x)+\frac{1}{4}C_0(h_x,d_x)]$, and $C_0(h_x,d_x)|C_2(h_y,d_y)-\frac{1}{4}C_0(h_y,d_y)|$. Combining these above inequalities (35)–(37) into $R(\Omega)$ yields:

$$\begin{aligned} R(\Omega) \le M_x M_y\{&\frac{1}{2}\exp(-\sqrt{\alpha_y M_y \pi d_y})\widetilde{N}'(u\sqrt{\phi_x'}v,D_y) \\ &\times[R_0\frac{\sqrt{\pi d_y}}{\sqrt{\alpha_y}}(\frac{\sqrt{\alpha_x}}{\sqrt{\pi d_x}}(4+\frac{\pi^2}{3})+\frac{1}{4}\frac{\sqrt{\pi d_x}}{\sqrt{\alpha_x}})] \\ &+\frac{1}{2}\exp(-\sqrt{\alpha_x M_x \pi d_x})\widetilde{N}'(u\sqrt{\phi_y'}w,D_x)[R_0\frac{\sqrt{\pi d_x}}{\sqrt{\alpha_x}}(\frac{\sqrt{\alpha_y}}{\sqrt{\pi d_y}}(4+\frac{\pi^2}{3})+\frac{1}{4}\frac{\sqrt{\pi d_y}}{\sqrt{\alpha_y}})] \\ &+\frac{(b-a)^{3/2}}{4}\exp(-\sqrt{\alpha_y M_y \pi d_y})\widetilde{N}(uv'',D_y)[R_2\frac{\sqrt{d_x\alpha_y}}{\sqrt{\alpha_x d_y}}+\frac{1}{4}R_0\pi\frac{\sqrt{d_x d_y}}{\sqrt{\alpha_x\alpha_y}}] \end{aligned}$$

$$
\begin{aligned}
&+ \frac{(d-c)^{3/2}}{4}\exp(-\sqrt{\alpha_x M_x \pi d_x})\widetilde{N}(uw'', D_x)[R_2\frac{\sqrt{d_y\alpha_x}}{\sqrt{\alpha_y d_x}} + \frac{1}{4}R_0\pi\frac{\sqrt{d_y d_x}}{\sqrt{\alpha_y\alpha_x}}] \\
&+ \exp(-\sqrt{\alpha_x M_x \pi d_x})\exp(-\sqrt{\alpha_y M_y \pi d_y})N(N(uw'', D_x)v, D_y) \\
&\times [R_0R_2\frac{\sqrt{\alpha_x d_y}}{\sqrt{\alpha_y d_x}} + \frac{1}{4}R_0^2\pi\frac{\sqrt{d_x d_y}}{\sqrt{\alpha_x\alpha_y}}] \\
&+ \exp(-\sqrt{\alpha_x M_x \pi d_x})\exp(-\sqrt{\alpha_y M_y \pi d_y})N(N(uw, D_x)v'', D_y) \\
&\times [R_0R_2\frac{\sqrt{\alpha_y d_x}}{\sqrt{\alpha_x d_y}} + \frac{1}{4}R_0^2\pi\frac{\sqrt{d_y d_x}}{\sqrt{\alpha_x\alpha_y}}]\} \\
&\equiv M_xM_yL_2\exp(-\sqrt{\alpha_x M_x \pi d_x}) + M_xM_yL_2'\exp(-\sqrt{\alpha_y M_y \pi d_y}) \\
&+ M_xM_yL_2''\exp(-\sqrt{\alpha_x M_x \pi d_x})\exp(-\sqrt{\alpha_y M_y \pi d_y}) \\
&\equiv R'(\Omega),
\end{aligned}
$$

where

$$
\begin{aligned}
L_2 \equiv\ &\frac{1}{2}\widetilde{N}'(u\sqrt{\phi_y'}w, D_x)[R_0\frac{\sqrt{\pi d_x}}{\sqrt{\alpha_x}}(\frac{\sqrt{\alpha_y}}{\sqrt{\pi d_y}}(4+\frac{\pi^2}{3}) + \frac{1}{4}\frac{\sqrt{\pi d_y}}{\sqrt{\alpha_y}})] \\
&+\frac{(d-c)^{3/2}}{4}\widetilde{N}(uw'', D_x)[R_2\frac{\sqrt{d_y\alpha_x}}{\sqrt{\alpha_y d_x}} + \frac{1}{4}R_0\pi\frac{\sqrt{d_y d_x}}{\sqrt{\alpha_y\alpha_x}}],
\end{aligned}
$$

$$
\begin{aligned}
L_2' \equiv\ &\frac{1}{2}\widetilde{N}'(u\sqrt{\phi_x'}v, D_y)[R_0\frac{\sqrt{\pi d_y}}{\sqrt{\alpha_y}}(\frac{\sqrt{\alpha_x}}{\sqrt{\pi d_x}}(4+\frac{\pi^2}{3}) + \frac{1}{4}\frac{\sqrt{\pi d_x}}{\sqrt{\alpha_x}})] \\
&+\frac{(b-a)^{3/2}}{4}\widetilde{N}(uv'', D_y)[R_2\frac{\sqrt{d_x\alpha_y}}{\sqrt{\alpha_x d_y}} + \frac{1}{4}R_0\pi\frac{\sqrt{d_x d_y}}{\sqrt{\alpha_x\alpha_y}}],
\end{aligned}
$$

and

$$
\begin{aligned}
L_2'' \equiv\ &N(N(uw'', D_x)v, D_y)[R_0R_2\frac{\sqrt{\alpha_x d_y}}{\sqrt{\alpha_y d_x}} + \frac{1}{4}R_0^2\pi\frac{\sqrt{d_x d_y}}{\sqrt{\alpha_x\alpha_y}}] \\
&+N(N(uw, D_x)v'', D_y)[R_0R_2\frac{\sqrt{\alpha_y d_x}}{\sqrt{\alpha_x d_y}} + \frac{1}{4}R_0^2\pi\frac{\sqrt{d_y d_x}}{\sqrt{\alpha_x\alpha_y}}].
\end{aligned}
$$

The following inequality is satisfied:

$$
\begin{aligned}
\frac{h_y}{h_x}(\frac{\pi^2}{3h_x}+1) &= \frac{\sqrt{\pi d_y}}{\sqrt{\alpha_y M_y}}\frac{\pi\alpha_x M_x}{3d_x} + \frac{\sqrt{d_y}}{\sqrt{\alpha_y M_y}}\frac{\sqrt{\alpha_x M_x}}{\sqrt{d_x}} \\
&\le M_xM_y\sqrt{\frac{\pi d_y}{\alpha_y}}(\frac{\pi\alpha_x}{3d_x} + \sqrt{\frac{\alpha_x}{\pi d_x}}),
\end{aligned}
\tag{38}
$$

which will be used to demonstrate Theorem 8.

Theorem 8 *Assume that there are positive constants K, K', α_x, α_y, β_x, and β_y so that*

$$\left|F(x,y_l)\right| \le K \begin{cases} \exp(-\alpha_x|\phi_x(x)|), & x \in \Gamma_a, \\ \exp(-\beta_x|\phi_x(x)|), & x \in \Gamma_b, \end{cases}$$

where $F(x,y_l) = u(x,y_l)\phi_x'(x)w(x)v(y_l)$ or $u(x,y_l)(\frac{\phi_x''(x)}{\phi_x'(x)}w(x) + 2w'(x))v(y_l)$, and

$$\left|\bar{F}(x_k,y)\right| \le K' \begin{cases} \exp(-\alpha_y|\phi_y(y)|), & y \in \Gamma_c, \\ \exp(-\beta_y|\phi_y(y)|), & y \in \Gamma_d, \end{cases}$$

where $\bar{F}(x_k,y) = u(x_k,y)\phi_y'(y)v(y)w(x_k)$ or $u(x_k,y)(\frac{\phi_y''(y)}{\phi_y'(y)}v(y) + 2v'(y))w(x_k)$, $\Gamma_c = \{\xi \in \Gamma : \phi(\xi) = y \in (-\infty,0)\}$ and $\Gamma_d = \{\xi \in \Gamma : \phi(\xi) = y \in [0,\infty)\}$. Choosing $N_x = [|\frac{\alpha_x}{\beta_x}M_x + 1|]$, $N_y = [|\frac{\alpha_y}{\beta_y}M_y + 1|]$, $h_x = \sqrt{\frac{\pi d_x}{\alpha_x M_x}}$ and $h_y = \sqrt{\frac{\pi d_y}{\alpha_y M_y}}$, we have

$$\begin{aligned}
&\Big|\int_c^d \int_a^b u''w[S(j,h)\circ\phi](x)dx - h_x h_y \sum_{i=-M_x}^{N_x} \frac{u(x_i,y_l)}{\phi_x'(x_i)} I_1(x_i) \frac{v(y_l)}{\phi_y'(y_l)} \\
&-h_x h_y \sum_{j=-M_y}^{N_y} \frac{u(x_k,y_j)}{\phi_y'(y_j)} I_1(y_j) \frac{w(x_k)}{\phi_x'(x_k)}\Big| \le L_2'' M_x M_y \exp(-\sqrt{\alpha_x M_x \pi d_x}) \exp(-\sqrt{\alpha_y M_y \pi d_y}) \\
&+(L_2 + J)M_x M_y \exp(-\sqrt{\alpha_x M_x \pi d_x}) + (L_2' + J')M_x M_y \exp(-\sqrt{\alpha_y M_y \pi d_y}),
\end{aligned}$$

where $J \equiv K\frac{(d-c)^{3/2}}{8}(1/\alpha_x + 1/\beta_x)(\sqrt{\frac{\pi d_y}{\alpha_y}}\frac{\alpha_x \pi}{3d_x} + \sqrt{\frac{\alpha_x d_y}{\alpha_y d_x}})$, $J' \equiv K'\frac{(b-a)^{3/2}}{8}(1/\alpha_y + 1/\beta_y)(\sqrt{\frac{\pi d_x}{\alpha_x}}\frac{\alpha_y \pi}{3d_y} + \sqrt{\frac{\alpha_y d_x}{\pi d_y \alpha_x}})$.

Proof The positive proportional series $\sum_{i=M_x+1}^{\infty} e^{-\alpha_x i h_x}$ is limited, and the sum is satisfied

$$\begin{aligned}
\sum_{i=M_x+1}^{\infty} e^{-\alpha_x i h_x} &= e^{-\alpha_x(M_x+1)h_x}(1 + e^{-\alpha_x h_x} + e^{-2\alpha_x h_x} + \cdots) = e^{-\alpha_x(M_x+1)h_x} \lim_{n\to\infty} \frac{1 - e^{-n\alpha_x h_x}}{1 - e^{-\alpha_x h_x}} \\
&= e^{-\alpha_x(M_x+1)h_x}\Big(\frac{1}{1 - e^{-\alpha_x h_x}} - \lim_{n\to\infty} \frac{e^{-n\alpha_x h_x}}{1 - e^{-\alpha_x h_x}}\Big) = \frac{e^{-\alpha_x(M_x+1)h_x}}{1 - e^{-\alpha_x h_x}} \\
&= e^{-\alpha_x M_x h_x} \frac{1}{e^{-\alpha_x h_x} - 1} = e^{-\alpha_x M_x h_x} \frac{1}{1 + \alpha_x h_x + \frac{1}{2}(\alpha_x h_x)^2 + \cdots - 1} \\
&\le e^{-\alpha_x M_x h_x} \frac{1}{\alpha_x h_x}.
\end{aligned} \tag{39}$$

The following sum is conducted by the inequality $|\delta_{ij}^{(2)}| \le \frac{\pi^2}{3}$, and $|\delta_{ij}^{(1)}| \le 1$,

$$\begin{aligned}
&\Big|h_x h_y \sum_{i=-\infty}^{-M_x-1} \frac{u(x_i,y_l)}{\phi_x'(x_i)} I_1(x_i) \frac{v(y_l)}{\phi_y'(y_l)}\Big| \\
&\le \Big|h_x h_y \frac{v(y_l)}{\phi_y'(y_l)} \sum_{i=M_x+1}^{\infty} \Big[\frac{\delta_{-i,k}^{(2)}}{h_x^2}(u\phi_x' w)(x_{-i}) + \frac{\delta_{-i,k}^{(1)}}{h_x}\Big(u\Big(\frac{\phi_x''}{\phi_x'}w + 2w'\Big)(x_{-i})\Big]\Big|
\end{aligned}$$

$$\leq \Big|h_y \frac{v(y_l)}{\phi_y'(y_l)}(\frac{K\pi^2}{3h_x}+K)\sum_{i=M_x+1}^{\infty} e^{-\alpha_x i h_x}\Big|,$$

$$\leq \Big|h_y \frac{v(y_l)}{\phi_y'(y_l)}(\frac{K\pi^2}{3h_x}+K)e^{-\alpha_x M_x h_x}\frac{1}{\alpha_x h_x}\Big|.(\text{if } k\in(-M_x,N_x)), \tag{40}$$

As the same, we have $\big|h_xh_y\sum_{i=N_x+1}^{\infty}\frac{u(x_i,y_l)}{\phi_x'(x_i)}I_1(x_i)\frac{v(y_l)}{\phi_y'(y_l)}\big| \leq h_y\frac{v(y_l)}{\beta_x h_x\phi_y'(y_l)}e^{-\beta_x N_x h_x}(\frac{K\pi^2}{3h_x}+K)$, $\big|h_xh_y\sum_{j=-\infty}^{-M_y-1}\frac{u(x_k,y_j)}{\phi_y'(y_j)}I_1(y_j)\frac{v(y_j)}{\phi_y'(y_j)}\big| \leq h_x\frac{w(x_k)}{\alpha_y h_y\phi_x'(x_k)}e^{-\alpha_y M_y h_y}(\frac{K'\pi^2}{3h_y}+K')$, and $\big|h_xh_y\sum_{j=N_y+1}^{\infty}\frac{u(x_k,y_j)}{\phi_y'(y_j)}I_1(y_j)\frac{v(y_j)}{\phi_y'(y_j)}\big| \leq h_x\frac{w(x_k)}{\beta_y h_y\phi_x'(x_k)}e^{-\beta_y N_y h_y}(\frac{K'\pi^2}{3h_y}+K')$. Substituting the above inequalities (38)–(40) into the error in Theorem 8, we have

$$\begin{aligned}
&\Big|\langle\Delta u,S_{kl}\rangle-\{h_xh_y\sum_{i=-M_x}^{N_x}\sum_{j=-M_y}^{N_y}\frac{u(x_i,y_j)}{\phi_x'(x_i)\phi_y'(y_j)}([S(l,h_y)\circ\phi_y(y_j)v(y_j)]I_1(x_i)\\
&+I_1(y_j)[S(k,h_x)\circ\phi_x(x_i)w(x_i)])\}\Big|\\
&=\Big|I-h_xh_y\sum_{i=-M_x}^{N_x}\frac{u(x_i,y_l)}{\phi_x'(x_i)}I_1(x_i)\frac{v(y_l)}{\phi_y'(y_l)}-h_xh_y\sum_{j=-M_y}^{N_y}\frac{u(x_k,y_j)}{\phi_y'(y_j)}I_1(y_j)\frac{w(x_k)}{\phi_x'(x_k)}\Big|\\
&\leq\Big|h_xh_y(\sum_{i=-\infty}^{-M_x-1}+\sum_{i=N_x}^{\infty})\frac{u(x_i,y_l)}{\phi_x'(x_i)}[\frac{\delta_{ki}^{(2)}}{h_x^2}(\phi_x'w)(x_i)+\frac{\delta_{ki}^{(1)}}{h_x}((\frac{\phi_x''}{\phi_x'}w+2w')(x_i)]\frac{v(y_l)}{\phi_y'(y_l)}\\
&+h_xh_y(\sum_{j=-\infty}^{-M_y-1}+\sum_{j=N_y}^{\infty})\frac{u(x_k,y_j)}{\phi_y'(y_j)}[\frac{\delta_{jl}^{(2)}}{h_y^2}(\phi_y'v)(y_j)+\frac{\delta_{jl}^{(1)}}{h_y}(\frac{\phi_y''}{\phi_y'}v+2v')(y_j)]\frac{w(x_k)}{\phi_x'(x_k)}\Big|+R(\Omega)\\
&\leq\Big|[h_y\frac{v(y_l)}{\alpha_xh_x\phi_y'(y_l)}e^{-\alpha_xM_xh_x}+h_y\frac{v(y_l)}{\beta_xh_x\phi_y'(y_l)}e^{-\beta_xN_xh_x}](\frac{K\pi^2}{3h_x}+K)\\
&+[h_x\frac{w(x_k)}{\alpha_yh_y\phi_x'(x_k)}e^{-\alpha_yM_yh_y}+h_x\frac{w(x_k)}{\beta_yh_y\phi_x'(x_k)}e^{-\beta_yN_yh_y}](\frac{K'\pi^2}{3h_y}+K')\Big|+R'(\Omega)\\
&\leq M_xM_y\Big|K\frac{(d-c)^{3/2}}{8}\sqrt{\frac{\pi d_y}{\alpha_y}}(1/\alpha_x+1/\beta_x)(\frac{\alpha_x\pi}{3d_x}+\sqrt{\frac{\alpha_x}{\pi d_x}})e^{-\sqrt{\pi d_x\alpha_xM_x}}\\
&+K'\frac{(b-a)^{3/2}}{8}\sqrt{\frac{\pi d_x}{\alpha_x}}(1/\alpha_y+1/\beta_y)(\frac{\alpha_y\pi}{3d_y}+\sqrt{\frac{\alpha_y}{\pi d_y}})e^{-\sqrt{\pi d_y\alpha_yM_y}}\Big|+R'(\Omega).
\end{aligned}\tag{41}$$

The proof is completed. □

Remark 1 Set $u^n(\boldsymbol{x})$ to be the numerical solution of semi-discrete equation (8) using the high-order finite difference method and $u_m^n(\boldsymbol{x})$ to be the full-discrete solution of (15) and (19) using the Sinc-Galerkin method and a high-order finite difference method. Then, using the quality of $0<\alpha_0\beta^{-1}<\tilde{\alpha}$ and $\sum_{i=1}^{n} d_{n-i}^n=1$ (which is described in [7]), it is easy to see:

$$\sup_{x\in(a,b)}|u^n(x)-u_m^n(x)|\leq CM\exp(-(\pi\alpha dM)^{1/2}).$$

For the 1D problem, and

$$\begin{aligned}&\sup_{\boldsymbol{x}\in(a,b)\times(c,d)}|u^n(\boldsymbol{x})-u_m^n(\boldsymbol{x})|\\&\quad\le C_1M_xM_y\exp(-(\pi\alpha dM_x)^{1/2})+C_2M_xM_y\exp(-(\pi\alpha dM_y)^{1/2})\\&\qquad+C_3M_xM_y\exp(-(\pi\alpha dM_x)^{1/2})\exp(-(\pi\alpha dM_y)^{1/2}),\end{aligned}$$

for the 2D problem, where C, C_1, C_2, and C_3 are constants independent of M_x and M_y.

5 Numerical results

In this section, we will give some numerical examples to verify the convergence analysis. We consider the problem (4)–(6) for that $[a,b]$ and $[c,d]$ are both chosen to be $[0,1]$, and we set $d=\frac{\pi}{2}$. In this section, we select the function $\phi(x)=\ln\frac{x}{1-x}$ and $\phi(y)=\ln\frac{y}{1-y}$.

5.1 Numerical results for 1D problem

Suppose that u_j^n is denoted to be the approximate solution and $u(x_j,t_n)$ is denoted to be the exact solution. The absolute maximum error and L^2 error at Sinc grid points are taken as

$$\|e\|_\infty(M,N)=\max_{-M\le j\le M}|u(x_j,t_n)-u_j^n|,\ \ \|e\|_2(M,N)=\sqrt{\sum_{j=-M}^{M}(u(x_j,t_n)-u_j^n)^2},$$

and the following space-time convergence rates

$$order^t=log_{\frac{N_2}{N_1}}(\frac{\|e\|_2(M,N_2)}{\|e\|_2(M,N_1)}),\ order^x=log_{\frac{M_2}{M_1}}(\frac{\|e\|_2(M_2,N)}{\|e\|_2(M_1,N)}),$$

respectively, where $x_j=\frac{e^{jh}}{1+e^{jh}}$.

In this part, we choose the parameters $w=\frac{1}{\phi'(x)}$.

Example 1 We consider the following 1D nonhomogeneous problem for $f=-3t^3e^x(-x^2-3x)+\frac{18}{\Gamma(4-\alpha)}t^{(3-\alpha)}e^xx(1-x)$ and $u(x,0))=0, 0\le x\le 1$, the data are chosen so that the exact solution is

$$u(x,t)=3e^xx(1-x)t^3.$$

In this example, we can see that when $\phi(x)=\ln\frac{x}{1-x}$, $\alpha=\beta=1$. We choose $M=N$, and $d=\pi/2$, which yield $h=\frac{\pi}{\sqrt{2M}}$. The maximum absolute errors, L_2 errors, and the related convergence rates are displayed in Table 1 for different M at different α with $\Delta t=1/100$.

In Table 1, the maximum-norm errors, L_2 errors, and the spatial convergence orders with $\alpha=0.4,\ 0.7$ are presented, respectively. From Table 1, we can see that when $\Delta t=1/100$, $M=64$, and $\alpha=0.4$, the L^2 error can be 1.07194e-06, the maximum error is 2.02353e-07. The spatial convergence rates are also shown to be exponential convergence orders.

Table 2 lists the temporal errors and convergence rates at $\alpha=0.4, 0.7$, when $M=64$ is fixed at time $T=1$. It is shown that the convergence order is $3-\alpha$, which is a very high

Table 1 Errors for $\|e\|_2$, $\|e\|_\infty$ when $\mathfrak{a} = 0.4$, $\mathfrak{a} = 0.7$, $\Delta t = 1/100$ for Example 1

		$\mathfrak{a} = 0.4$			$\mathfrak{a} = 0.7$		
N	M	$\|e\|_\infty(M,N)$	$\|e\|_2(M,N)$	$order^x$	$\|e\|_\infty(M,N)$	$\|e\|_2(M,N)$	$order^x$
100	4	3.55261e-02	7.23719e-02	–	3.54749e-02	7.09014e-02	–
100	8	7.69362e-03	2.19636e-02	1.72031	7.69220e-03	2.16296e-02	1.71281
100	16	6.97755e-04	2.83983e-03	2.95124	6.97747e-04	2.80653e-03	2.94614
100	32	2.02529e-05	1.17516e-04	4.59488	2.02528e-05	1.13988e-04	4.62183
100	64	2.02353e-07	1.07194e-06	6.77649	3.22017e-06	9.56008e-06	3.57572

Table 2 Errors and orders at $\mathfrak{a} = 0.4$ and $\mathfrak{a} = 0.7$ for Example 1

	$\mathfrak{a} = 0.4$			$\mathfrak{a} = 0.7$		
M	N	$\|e\|_2(M,N)$	$order^t$	N	$\|e\|_2(M,N)$	$order^t$
64	4	4.24900e-03	–	4	1.66898e-02	–
64	8	6.63444e-04	2.67908	8	3.34640e-03	2.31828
64	16	1.04421e-04	2.66756	16	6.74620e-04	2.31047
64	32	1.63450e-05	2.67549	32	1.36144e-04	2.30894
64	64	2.52077e-06	2.69691	64	2.73293e-05	2.31661

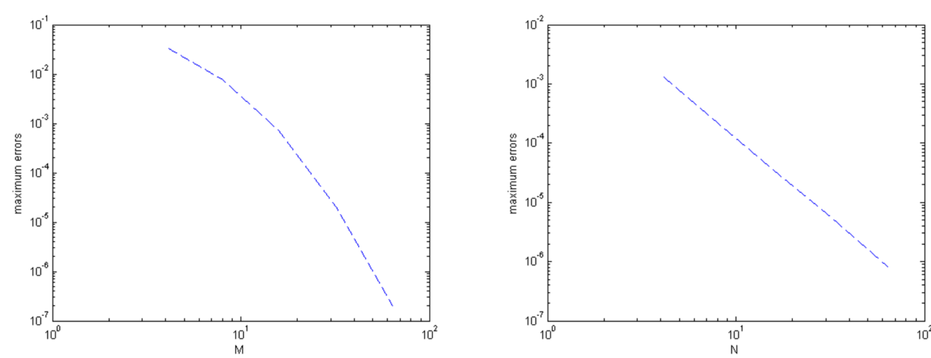

Figure 1 The spatial convergence rates (loglog) when $\mathfrak{a} = 0.4$, $N = 100$ at $T = 1$ and the temporal convergence rates (loglog) when $\mathfrak{a} = 0.4$, $M = 64$ at $T = 1$ for Example 1

order in recent years for the numerical solutions of fractional diffusion equations. These numerical solutions agree with the theoretical results. Figure 1 shows the convergence rates in space when $\mathfrak{a} = 0.4$, $\Delta t = 1/100$ at $T = 1$, and the convergence rates in time at $T = 1$, when $\mathfrak{a} = 0.4$, $M = 64$.

Example 2 The second example is chosen $f = -\frac{3}{4}t^3x^{-1/2} + \frac{6}{\Gamma(4-\mathfrak{a})}t^{(3-\mathfrak{a})}(x^{3/2} - x)$, and $u(x,0) = 0$, $0 < x < 1$ so that the exact solution is $u(x,t) = (x^{3/2} - x)t^3$. In this example, it is easy to get $\alpha = \beta = 1$ and $h = \frac{\pi}{\sqrt{2M}}$, when we choose $M = N$, and $d = \pi/2$.

Table 3 shows the errors for $\|e\|_2(M,N)$, $\|e\|_\infty(M,N)$ and the convergence order about the $\|e\|_2(M,N)$ errors in space when fixing $N = 100$. Table 4 lists the temporal errors and convergence rates at the time $T = 1$ when $\mathfrak{a} = 0.4, 0.7$. These data show the convergence rates are exponentially convergent in space and $\mathcal{O}(3 - \mathfrak{a})$ in time. Figure 2 presents the convergence rates in space when $\mathfrak{a} = 0.4$, $\Delta t = 1/100$ at $T = 1$ and the convergence rates in time at $T = 1$ when $\mathfrak{a} = 0.4$, $M = 64$.

Table 3 Errors for $\|e\|_2(M,N)$, $\|e\|_\infty(M,N)$ when $\alpha = 0.4$, $\alpha = 0.7$, $\Delta t = 1/100$ for Example 2

		$\alpha = 0.4$			$\alpha = 0.7$		
N	M	$\|e\|_\infty(M,N)$	$\|e\|_2(M,N)$	$order^x$	$\|e\|_\infty(M,N)$	$\|e\|_2(M,N)$	$order^x$
100	4	4.02973e-03	8.89992e-03	–	4.02358e-03	8.70924e-03	–
100	8	9.12983e-04	2.78249e-03	1.67741	9.12806e-04	2.73759e-03	1.66964
100	16	8.47609e-05	3.65120e-04	2.92994	8.47599e-05	3.60610e-04	2.92439
100	32	2.47960e-06	1.51728e-05	4.58881	2.47960e-06	1.47264e-05	4.61397
100	64	2.33175e-08	1.34180e-07	6.82118	3.80952e-07	1.12810e-06	3.70644

Table 4 Errors and orders at $\alpha = 0.4$ and $\alpha = 0.7$ for Example 2

	$\alpha = 0.4$			$\alpha = 0.7$		
M	N	$\|e\|_2(M,N)$	$order^t$	N	$\|e\|_2(M,N)$	$order^t$
64	4	5.01679e-04	–	4	1.97068e-03	–
64	8	7.83275e-05	2.67917	8	3.95131e-04	2.31829
64	16	1.23227e-05	2.66820	16	7.96516e-05	2.31055
64	32	1.92689e-06	2.67697	32	1.60694e-05	2.30939
64	64	2.97064e-07	2.69743	64	3.22208e-06	2.31825

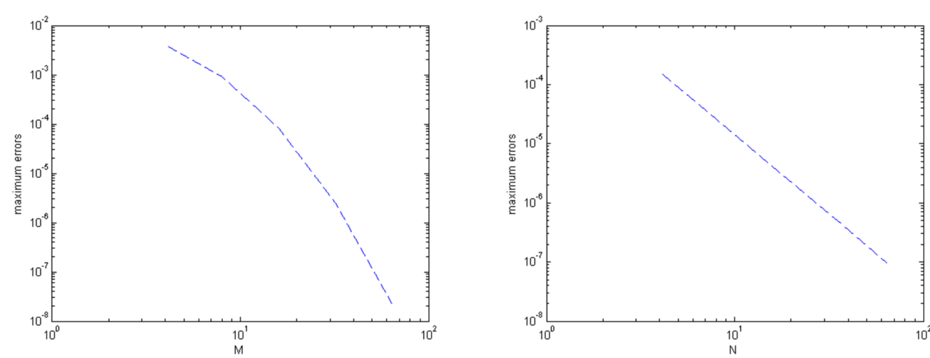

Figure 2 The spatial convergence rates (loglog) when $\alpha = 0.4$, $N = 100$ at $T = 1$ and the temporal convergence rates (loglog) when $\alpha = 0.4$, $M = 64$ at $T = 1$ for Example 2

5.2 Numerical results for 2D problem

In the case of two dimensional problem, we choose $w = \frac{1}{\sqrt{\phi'(x)}}$ and $v = \frac{1}{\sqrt{\phi'(y)}}$ to get the symmetric matrixes. For every problem, we choose $M_x = N_x$, $M_y = N_y$. In this part, we also take $d = \pi/2$.

Example 3 We consider the following 2D nonhomogeneous problem with $f(x,y,t) = -3t^3e^{x+y}[(-x^2 - 3x)y(1-y) + x(1-x)(-y^2 - 3y)] + \frac{18}{\Gamma(4-\alpha)}t^{(3-\alpha)}e^{x+y}xy(1-x)(1-y)$ and $u(x,y,0)) = 0, (x,y) \in [0,1] \times [0,1]$. The data are chosen so that the exact solution is

$$u(x,y,t) = 3e^{x+y}xy(1-x)(1-y)t^3.$$

Herein, similar to the previous analysis, we have $\alpha_x = \beta_x = \alpha_y = \beta_y = 1/2$, which yields $h_x = \frac{\pi}{\sqrt{M_x}}$ and $h_y = \frac{\pi}{\sqrt{M_y}}$.

In Table 5, fixing $N = 100$, the maximum absolute errors, L_2 errors, and the spatial convergence orders are displayed at $\alpha = 0.4$ and $\alpha = 0.7$. In Table 6, fixing $M_x = M_y = 32$, the $\|e\|_2(M,N)$ errors and the temporal convergence orders are displayed at $\alpha = 0.4$ and $\alpha = 0.7$.

Table 5 Errors for $\|e\|_2(M,N)$, $\|e\|_\infty(M,N)$ when $\alpha = 0.4$, $\alpha = 0.7$, $\Delta t = 1/100$ for Example 3

		$\alpha = 0.4$			$\alpha = 0.7$		
N	$M_x = M_y$	$\|e\|_\infty(M,N)$	$\|e\|_2(M,N)$	$order^x$	$\|e\|_\infty(M,N)$	$\|e\|_2(M,N)$	$order^x$
100	4	1.42156e-02	9.39627e-03	–	1.40344e-02	9.26449e-03	–
100	8	2.37926e-03	1.25342e-03	2.90622	2.35250e-03	1.24432e-03	2.89635
100	16	1.46364e-04	6.72258e-05	4.22071	1.42362e-04	6.63491e-05	4.22914
100	32	1.80024e-06	7.32964e-07	6.51913	5.75006e-06	3.14056e-06	4.40098

Table 6 Errors and orders at $\alpha = 0.4$ and $\alpha = 0.7$ for Example 3

	$\alpha = 0.4$			$\alpha = 0.7$		
$M_x = M_y$	N	$\|e\|_2(M,N)$	$order^t$	N	$\|e\|_2(M,N)$	$order^t$
32	4	1.42320e-03	–	4	5.50617e-03	–
32	8	2.22004e-04	2.68049	8	1.10567e-03	2.31613
32	16	3.48450e-05	2.67156	16	2.22796e-04	2.31113
32	32	5.41255e-06	2.68657	32	4.48793e-05	2.31160

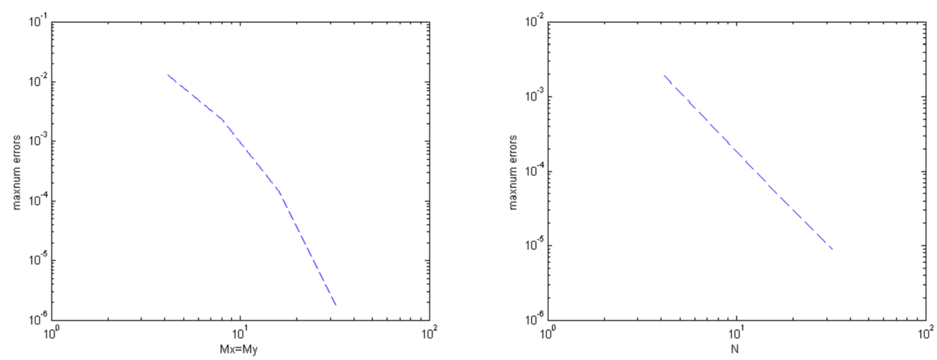

Figure 3 The spatial convergence rates (loglog) when $\alpha = 0.4$, $N = 100$ at $T = 1$ and the temporal convergence rates (loglog) when $\alpha = 0.4$, $M = 32$ at $T = 1$ for Example 3

We can also see the convergence orders are exponential in space and $\mathcal{O}(3-\alpha)$ in time. Figure 3 presents the convergence rates in space when $\alpha = 0.4$, $\Delta t = 1/100$ at $T = 1$ and the convergence rates in time at $T = 1$ when $\alpha = 0.4$, $M = 32$.

Example 4 We consider the following 2D nonhomogeneous problem for a sequence of $M_x = 4, 8, 16, 32$ $M_y = 4, 8, 16, 32$, with $f(x,y,t) = -\frac{3}{4}t^3[y^{-1/2}(x^{3/2} - x) + x^{-1/2}(y^{3/2} - y)] + \frac{6}{\Gamma(4-\alpha)}t^{(3-\alpha)}(x^{3/2} - x)(y^{3/2} - y)$ and $u(x,y,0)) = 0, (x,y) \in [0,1] \times [0,1]$. The data are chosen so that the exact solution is

$$u(x,y,t) = (x^{3/2} - x)(y^{3/2} - y)t^3.$$

It is easy to see that $\alpha_x = \beta_x = \alpha_y = \beta_y = 1/2$, which yields $h_x = \frac{\pi}{\sqrt{M_x}}$ and $h_y = \frac{\pi}{\sqrt{M_y}}$.

Maximum absolute errors and $\|e\|_2(M,N)$ errors and the related convergence orders are displayed in Table 7 for different M_x and M_y at different α with $\Delta t = 1/100$. Table 8 presents the errors in time. Figure 4 presents the convergence rates in space when $\alpha = 0.4$, $\Delta t = 1/100$ and the convergence rates in time when $\alpha = 0.4$, $M = 32$. These data show that they agree with the theoretical results.

Table 7 Errors for $\|e\|_2(M,N)$, $\|e\|_\infty(M,N)$ when $\mathfrak{a}=0.4$, $\mathfrak{a}=0.7$, $\Delta t = 1/100$ for Example 4

		$\mathfrak{a}=0.4$			$\mathfrak{a}=0.7$		
N	$M_x=M_y$	$\|e\|_\infty(M,N)$	$\|e\|_2(M,N)$	$order^x$	$\|e\|_\infty(M,N)$	$\|e\|_2(M,N)$	$order^x$
100	4	3.82411e-04	2.77539e-04	–	3.79602e-04	2.76254e-04	–
100	8	8.80720e-05	5.15086e-05	2.42980	8.73038e-05	5.12589e-05	2.43012
100	16	5.99460e-06	2.93473e-06	4.13352	5.82776e-06	2.89592e-06	4.14571
100	32	7.60582e-08	3.20636e-08	6.51615	2.42048e-07	1.30324e-07	4.47384

Table 8 Errors and orders at $\mathfrak{a}=0.4$ and $\mathfrak{a}=0.7$ for Example 4

	$\mathfrak{a}=0.4$			$\mathfrak{a}=0.7$		
$M_x=M_y$	N	$\|e\|_2(M,N)$	$order^t$	N	$\|e\|_2(M,N)$	$order^t$
32	4	5.93100e-05	–	4	2.29490e-04	–
32	8	9.25090e-06	2.68061	8	4.60821e-05	2.31615
32	16	1.45118e-06	2.67237	16	9.28488e-06	2.31125
32	32	2.24792e-07	2.69056	32	1.86956e-06	2.31218

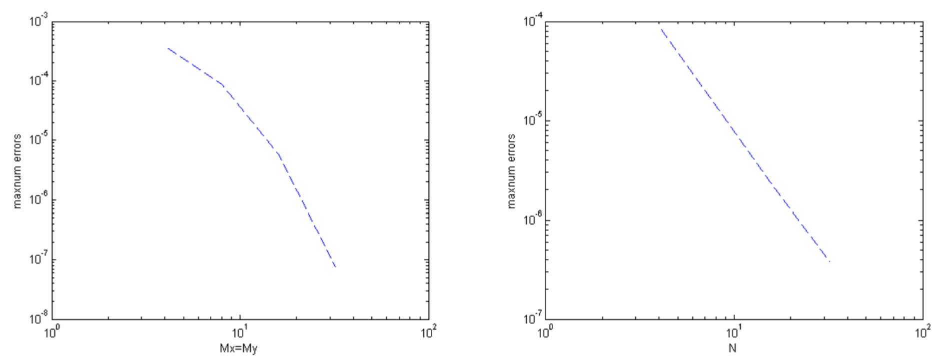

Figure 4 The spatial convergence rates (loglog) when $\mathfrak{a}=0.4$, $N=100$ at $T=1$ and the temporal convergence rates (loglog) when $\mathfrak{a}=0.4$, $M=32$ at $T=1$ for Example 4

6 Conclusion

In this work, we have introduced a Sinc-Galerkin scheme for solving a 1D and 2D time-fractional diffusion equation. The discrete scheme is based on a higher-order method in time and a Sinc-Galerkin method in space to get high convergence rates. The convergence order is $\mathcal{O}(3-\mathfrak{a})$ in time and $e^{-\sqrt{\pi\alpha dM}}$ in space. This is the first paper describing the exponentially discrete scheme order for 2D partial differential problems using the Sinc-Galerkin method. It is shown that we did not need a large mesh. Our next work is to use the Sinc-Galerkin method to study the temporal and spatial fractional equation in both time and space for 2D fractional-order problems.

Author contributions
Luo wrote the main manuscript text and all authors reviewed the manuscript.

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
This declaration is not applicable. The authors declare that they have no conflict of interest.

Competing interests
The authors declare no competing interests.

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