Duan et al. *Boundary Value Problems* (2024) 2024:108
https://doi.org/10.1186/s13661-024-01917-4

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

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C^1-Regularity for subelliptic systems with drift in the Heisenberg group: the superquadratic controllable growth

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Abstract

We investigate the interior regularity to nonlinear subelliptic systems in divergence form with drift term for the case of superquadratic controllable structure conditions in the Heisenberg group. On the basis of a generalization of the $\mathcal{A}$-harmonic approximation technique, C^1-regularity is established for horizontal gradients of vector-valued solutions to the subelliptic systems with drift term. Specially, our result is optimal in the sense that in the case of Hölder continuous coefficients we directly attain the optimal Hölder exponent for the horizontal gradients of weak solutions on the regular set.

Mathematics Subject Classification: 35H20; 35B65

Keywords: Drift term; Super-quadratic structure; Controllable growth condition; Dini continuity coefficient; $\mathcal{A}$-harmonic approximation technique

1 Introduction

In this paper, our focus lies on examining the nonlinear subelliptic systems denoted as

$$-\sum_{i=1}^{2n} X_i A_i^\alpha(\xi,u,Xu) - Tu = B^\alpha(\xi,u,Xu), \quad \text{in } \Omega, \alpha = 1,2,\dots,N, \tag{1.1}$$

with the drift term Tu in the Heisenberg group, subject to the superquadratic controllable growth condition, where $\Omega \subset \mathbb{H}^n = \mathbb{R}^{2n+1}$ is a bounded domain, $A_i^\alpha(\xi,u,Xu) : \Omega \times \mathbb{R}^N \times \mathbb{R}^{2n\times N} \to \mathbb{R}^{2n\times N}$, and $B^\alpha(\xi,u,Xu) : \Omega \times \mathbb{R}^N \times \mathbb{R}^{2n\times N} \to \mathbb{R}^N$.

In the case of subelliptic systems with Hölder continuous coefficients, Wang and Niu [1] demonstrated the optimal local Hölder exponents for horizontal gradients of weak solutions to systems under the superquadratic ($m > 2$) structure condition. Additionally, Wang and Liao [2] derived the subquadratic condition ($1 < m < 2$) and established partial regularity for weak solutions to nonlinear subelliptic systems under natural growth condition in Carnot groups. For coefficients in the VMO class, refer to [3–5]. Recently, we also note that several interesting results for subelliptic problems in the Heisenberg groups have been obtained. These include critical Choquard–Kirchhoff problems [6], critical Kirchhoff equations involving p-sub-Laplacians operators [7], and reverse weighted Hardy–Littlewood–Sobolev inequalities [8].

The objective of this paper is to relax the requirements on coefficients A_i^α, which are typically assumed to have Hölder continuity in the variables (ξ, u) to the less restrictive condition of Dini continuity. Moreover, we aim to establish a partial regularity outcome with optimal estimates for the modulus of continuity concerning the horizontal derivative Xu. Specifically, we assume the continuity of A_i^α with respect to the variables (ξ, u) such that

$$\left|A_i^\alpha(\xi,u,p) - A_i^\alpha\left(\tilde{\xi},\tilde{u},p\right)\right| \le (1+|p|)^{\frac{m}{2}}\kappa(|u|)\mu\left(d\left(\xi,\tilde{\xi}\right)+|u-\tilde{u}|\right)$$

for all ξ, $\tilde{\xi} \in \Omega$ and $p \in \mathbb{R}^{2n\times N}$, where $\kappa : (0,+\infty) \to [0,+\infty)$ is nondecreasing and $\mu : (0,+\infty) \to [0,+\infty)$ is nondecreasing and concave with $\mu(0_+) = 0$. We also require that $r \to r^{-\gamma}\mu(r)$ is nonincreasing for some $\gamma \in (0,1)$ and that

$$M(r) = \int_0^r \frac{\mu(\rho)}{\rho}d\rho < \infty \quad \text{for some } r > 0. \tag{1.2}$$

Also we know that several regularity results have been established for weak solutions with Dini continuous coefficients, see [9] for the subelliptic case of superquadratic growth and [10] for the subquadratic case. For additional regularity results concerning elliptic systems and degenerate parabolic systems, readers can consult [11, 12] and the cited references therein.

However, it is worth noting the aforementioned results without any drift term. It would be intriguing to investigate whether these regularity results extend to nonlinear subelliptic systems with the drift term in the Heisenberg group. Such systems are of significant interest due to the presence of operators with drift terms, such as the Kolmogorov–Fokker–Planck operator (refer to [13]), which finds applications in physics, natural sciences, and statistical models of transmission diffusion equations.

Regarding subelliptic systems with the drift term, advancements have been made in enhancing the regularity results for weak solutions. For instance, Bramanti and Zhu [14] established L^p estimates and Schauder estimates for the nondivergent linear degenerate elliptic operator

$$\mathcal{L} = \sum_{i,j=1}^q a_{ij}(x)X_iY_j + a_0(x)X_0$$

constructed by Hörmander's vector fields, highlighting differences between equations with and without X_0. When X_0 represents the drift vector field on homogeneous groups, Hou and Niu [15] obtained weighted Sobolev–Morrey estimates for hypoelliptic operator $\mathcal{L}$. Furthermore, Du, Han, and Niu [16] provided the interior Morrey estimates and demonstrated Hölder continuity for the operator $\mathcal{L}$ with VMO coefficient. In a different approach, Austin and Tyson [17] achieved C^∞-smoothness for the operator

$$L = -\frac{1}{4}\sum_{i=1}^n\left(X_i^2 + Y_j^2\right) \pm \sqrt{3}T$$

using geometric analysis method in $\mathbb{H}^n$. Recently, Zhang and Niu [18] concluded the Hölder regularity of horizontal gradients of weak solutions for quasilinear degenerate elliptic equation with a drift term in $\mathbb{H}^n$. Then Zhang and Wang considered a discontinuous subelliptic system with drift term [19] and established the partial $C^{0,\alpha}$ Hölder regularity of weak solutions and the partial Morrey regularity of horizontal gradients for weak solutions.

So we investigate how to achieve C^1-regularity of weak solutions to nonlinear subelliptic systems with the drift term Tu in the Heisenberg group when the assumption of Hölder continuity of A_i^α is relaxed to Dini continuity. The main new aspect of this paper is the fact that we are able to deal with the general nonlinear subelliptic systems with Dini continuous coefficients with the drift Tu and the superquadratic growth $2 < m < \infty$ with respect to horizontal gradients Xu. The drift term Tu will bring us new challenges due to the lack of the prior assumption for the vertical derivative Tu. As usual, when we consider the regularity of subelliptic equations with the drift, we shall require the integrability of Tu, Xu, and X^2u, such as [18]. However, this integrability cannot be obtained by difference quotient in our situation. In this paper, we adopt a new clever method to avoid the requirement of the integrability. In fact, we employ the relationship of $T = X_iX_{n+i} - X_{n+i}X_i$ to establish suitable estimates for Tu in subtle ways. So the processing of drift terms is different from that of the processing of other terms in the system. Actually, we are going to employ a generalization of the $\mathcal{A}$-harmonic approximation technique introduced by Duzaar and Steffen [20].

Subsequently, we introduce the following precise structural assumptions for the coefficients A_i^α and B^α that are essential for our analysis throughout the paper.

(H1). The term $A_i^\alpha(\xi,u,p)$ exhibits differentiability with respect to p, and a constant L exists such that

$$\left|A^\alpha_{i,p^j_\beta}(\xi,u,p)\right| \le L\left(1+|p|^2\right)^{\frac{m-2}{2}}, \quad (\xi,u,p)\in\Omega\times\mathbb{R}^N\times\mathbb{R}^{2n\times N}, \quad m\ge 2,$$

as specified by $A^\alpha_{i,p^j_\beta}(\xi,u,p) = \frac{\partial A_i^\alpha(\xi,u,p)}{\partial p^j_\beta}$.

(H2). The term $A_i^\alpha(\xi,u,p)$ satisfies the following ellipticity condition:

$$A^\alpha_{i,p^j_\beta}(\xi,u,p)\eta_i^\alpha\eta_j^\beta \ge \lambda\left(1+|p|^2\right)^{\frac{m-2}{2}}|\eta|^2, \quad \forall \eta\in\mathbb{R}^{2n\times N},$$

where λ is a positive constant.

(H3). There exist a modulus of continuity $\mu:(0,+\infty)\to[0,+\infty)$ and a nondecreasing function $\kappa:[0,+\infty)\to[1,+\infty)$ such that

$$\left|A_i^\alpha(\xi,u,p) - A_i^\alpha\left(\tilde{\xi},\tilde{u},p\right)\right| \le (1+|p|)^{\frac{m}{2}}\kappa(|u|)\mu\left(d\left(\xi,\tilde{\xi}\right)+|u-\tilde{u}|\right). \tag{1.3}$$

Without loss of generality, we can assume $\kappa\ge 1$ and that

(μ1) μ is nondecreasing with $\mu(0_+)=0$, $\mu(1)=1$;

(μ2) μ is concave, in the proof of the regularity result we have to require that $r\to\frac{\mu(r)}{r^\gamma}$ is nonincreasing for some exponent $\gamma\in(0,1)$;

(μ3) Dini's condition $M(r)=\int_0^r\frac{\mu(\rho)}{\rho}d\rho<\infty$ for some $r>0$.

(HC) (Controllable growth condition). The term $B^\alpha(\xi,u,p)$ conforms to the following superquadratic controllable growth condition:

$$|B^\alpha(\xi,u,p)| \le a|p|^{m\left(1-\frac{1}{r}\right)} + b|u|^{r-1} + c,$$

where a, b, and c are positive constants, also, r takes the value $\frac{mQ}{Q-m}$ if $m < Q$; or falls within the range $Q \le r < +\infty$ if $m = Q$.

Furthermore, condition **(H1)** implies

$$\left|A_i^\alpha(\xi,u,p) - A_i^\alpha(\xi,u,\tilde{p})\right| \le C(L)\left(1+|p|^2+|\tilde{p}|^2\right)^{\frac{m-2}{2}}|p-\tilde{p}|,$$

and there exists a continuously nonnegative and bounded function $\omega(s,t) : [0,\infty) \times [0,\infty) \to [0,\infty)$, satisfying $\omega(s,0) = 0$ for all s. Additionally, $\omega(s,t)$ is monotonously nondecreasing in s for fixed t, is concave and monotonously nondecreasing in t for fixed s, such that for all $(\xi,u,p)\,,\left(\tilde{\xi},\tilde{u},\tilde{p}\right) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{2n\times N}$,

$$\left|A^\alpha_{i,p^j_\beta}(\xi,u,p) - A^\alpha_{i,p^j_\beta}(\xi,u,\tilde{p})\right| \le C\left(1+|p|^2+|\tilde{p}|^2\right)^{\frac{m-2}{2}}\omega\left(|p|,|p-\tilde{p}|^2\right). \tag{1.4}$$

Further, **(H2)** enables us to infer the following inequality:

$$\left(A_i^\alpha(\xi,u,p) - A_i^\alpha(\xi,u,\tilde{p})\right)(p-\tilde{p}) \ge \lambda'\left(|p-\tilde{p}|^2 + |p-\tilde{p}|^m\right), \tag{1.5}$$

with a positive constant λ'. Refer to [1] for detailed explanations.

In this paper, we adapt the method of $\mathcal{A}$-harmonic approximation to the subelliptic systems (1.1) in $\mathbb{H}^n$, aiming to establish partial regularity for weak solutions. The crucial aspect lies in establishing a certain excess decay estimate for the excess functional $\Phi(\xi_0,\rho,Xl)$. In the case $m > 2$, this functional is defined by

$$\Phi(\xi_0,\rho,Xl) = \fint_{B_\rho(\xi_0)}\left[|Xu-Xl|^2 + |Xu-Xl|^m\right]d\xi,$$

here we write

$$\fint_{B_\rho(\xi_0)} u(\xi)d\xi = \left|B_\rho(\xi_0)\right|^{-1}_{\mathbb{H}^n}\int_{B_\rho(\xi_0)} u(\xi)d\xi.$$

It is demonstrated that if $\Phi\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right)$ is sufficiently small on a ball $B_\rho(\xi_0)$, then for some fixed $\theta \in (0,\frac{1}{4}]$, there exists an excess improvement

$$\Phi\left(\xi_0,\theta\rho,(Xu)_{\xi_0,\theta\rho}\right) \le \theta^{2\tau}\Phi\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right) + K^*\left(|u_{\xi_0,\rho}|,|(Xu)_{\xi_0,\rho}|\right)\eta\left(\rho^{\frac{m}{m-1}}\right).$$

The iteration of this result leads to the excess decay estimate, which in turn implies the regularity results.

Under the set of assumptions, we are prepared to present the main regularity results.

Theorem 1 *Assume that coefficients A_i^α and B^α satisfy* **(*H*1–*H*3)**, **(*HC*)**, *and* $(\mu 1)$–$(\mu 3)$. *Let $u \in HW^{1,m}\left(\Omega, \mathbb{R}^N\right)$ be a weak solution to system* (1.1), *i.e.,*

$$\begin{aligned} &\int_\Omega A_i^\alpha(\xi, u, Xu) X_i \varphi^\alpha d\xi - \int_\Omega X_i u \cdot X_{n+i} \varphi^\alpha d\xi + \int_\Omega X_{n+i} u \cdot X_i \varphi^\alpha d\xi \\ &= \int_\Omega B^\alpha(\xi, u, Xu) \varphi^\alpha d\xi, \quad \forall \varphi \in C_0^\infty\left(\Omega, \mathbb{R}^N\right). \end{aligned} \tag{1.6}$$

Then there exists a relatively closed set $\Omega_0 \subset \Omega$ such that $u \in C^1\left(\Omega \backslash \Omega_0, \mathbb{R}^N\right)$. What is more, $\Omega_0 \subset \Sigma_1 \cup \Sigma_2$ and Haar measure $(\Omega \backslash \Omega_0) = 0$, where

$$\begin{aligned} \Sigma_1 &= \left\{\xi_0 \in \Omega : \sup_{r>0} \left(\left|u_{\xi_0,r}\right| + \left|(Xu)_{\xi_0,r}\right|\right) = \infty\right\}, \\ \Sigma_2 &= \left\{\xi_0 \in \Omega : \liminf_{r \to 0^+} \fint_{B_r(\xi_0)} \left|Xu - (Xu)_{\xi_0,r}\right|^2 d\xi > 0\right\}. \end{aligned}$$

In addition, for $\tau \in [\gamma, 1)$ and $\xi_0 \in \Omega \backslash \Omega_0$, the horizontal derivative Xu has the modulus of continuity $r \to r^\tau + M(r)$ in a neighborhood of ξ_0.

Remark 1 It is worth noting that our situation includes the subelliptic m-Laplacian system with the drift term Tu

$$-\sum_{i=1}^{2n} X_i \left(A_i^\alpha(\xi) \left(1 + |Xu|^2\right)^{\frac{m-2}{2}} X_i u^\alpha\right) - Tu = B^\alpha(\xi, u, Xu),$$

where $\xi \in \Omega$, $2 < m < \infty$. As previously mentioned, introducing a drift term adds complexity, notably in deriving appropriate estimates for second-order derivatives $X_i X_j u$ of a test function. The challenge arises due to the necessity of establishing technical results allowing for exchange. Specifically, we must leverage the relationship between the horizontal vector field X_i and the vertical vector field T ingeniously to estimate the drift term Tu accurately.

The paper's content unfolds as follows. In Sect. 2, we gather fundamental concepts and facts relevant to the Heisenberg group, encompassing horizontal Sobolev space, Poincaré-type inequalities, Jensen's inequality, properties associated with Dini continuity, and an $\mathcal{A}$-harmonic approximation lemma. Section 3 delves into demonstrating a Caccioppoli-type inequality for weak solutions under conditions **(H1–H3)**, **(HC)**, and $(\mu 1)$–$(\mu 3)$. Moving to Sect. 4, we establish partial C^1-regularity results through a series of steps. Firstly, we need to present an approximate linearization strategy outlined in Lemma 6, to apply $\mathcal{A}$-harmonic approximation technique. Subsequently, by employing the $\mathcal{A}$-harmonic approximation lemma in Sect. 2, we derive an excess improvement estimate for the functional $\Phi\left(\xi_0, \theta\rho, (Xu)_{\xi_0,\theta\rho}\right)$ under three smallness condition assumptions. Finally, the proof of Theorem 1 is provided by iteration.

2 Preliminaries

This section will offer an introduction to the Heisenberg group $\mathbb{H}^n$, define function spaces, and present some elementary estimates essential for subsequent discussions.

2.1 Introduction of the Heisenberg group $\mathbb{H}^n$

The Heisenberg group $\mathbb{H}^n$ is defined as $\mathbb{R}^{2n+1}$ endowed with the following group multiplication:

$$\left((\xi^1,t),\left(\tilde{\xi}^1,\tilde{t}\right)\right)\mapsto\left(\xi^1+\tilde{\xi}^1,t+\tilde{t}+\frac{1}{2}\sum_{i=1}^{n}\left(x_i\tilde{y}_i-\tilde{x}_iy_i\right)\right),$$

for all $\xi=\left(\xi^1,t\right)=(x_1,x_2,\dots,x_n,y_1,y_2,\dots,y_n,t)$, $\tilde{\xi}=\left(\tilde{\xi}^1,\tilde{t}\right)=(\tilde{x_1},\tilde{x_2},\dots,\tilde{x_n},\tilde{y_1},\tilde{y_2},\dots,\tilde{y_n},\tilde{t})$. This multiplication corresponds to addition in Euclidean $\mathbb{R}^{2n+1}$, its neutral element is $\mathbf{0}$, and its inverse to $\left(\tilde{\xi}^1,t\right)$ is given by $\left(-\tilde{\xi}^1,-t\right)$. Particularly, the mapping $\left(\xi,\tilde{\xi}\right)\mapsto\xi\cdot\tilde{\xi}^{-1}$ is smooth, therefore $(\mathbb{H}^n,\cdot)$ is a Lie group.

The basic vector fields corresponding to its Lie algebra can be explicitly calculated and are given by

$$X_i\equiv X_i(\xi)=\frac{\partial}{\partial x_i}-\frac{y_i}{2}\frac{\partial}{\partial t},\quad X_{n+i}\equiv X_{n+i}(\xi)=\frac{\partial}{\partial y_i}+\frac{x_i}{2}\frac{\partial}{\partial t},\quad i=1,2,\dots,n,\tag{2.1}$$

with $T\equiv T(\xi)=\frac{\partial}{\partial t}$, and note the special structure of the commutators:

$$T=[X_i,X_{n+i}]=-[X_{n+i},X_i]=X_iX_{n+i}-X_{n+i}X_i,\quad\text{else}$$
$$[X_i,X_j]=0,\quad\text{and}\quad[T,T]=[T,X_i]=0,$$

that is, $(\mathbb{H}^n,\cdot)$ is a nilpotent Lie group of step 2. For the horizontal gradient $X=(X_1,X_2,\dots,X_{2n})$, we call $X_1,X_2,\dots,X_{2n}$ horizontal vector fields and T the vertical vector field.

The homogeneous norm is defined by

$$\left|\left|(\xi^1,t)\right|\right|=\left(\left|\xi^1\right|^4+t^2\right)^{1/4},$$

and the metric induced by this homogeneous norm is given by

$$d\left(\tilde{\xi},\xi\right)=\left|\left|\xi^{-1}\cdot\tilde{\xi}\right|\right|.$$

The measure used on $\mathbb{H}^n$ is Haar measure (Lebesgue measure in $\mathbb{R}^{2n+1}$), and the volume of the homogeneous ball

$$B_R(\xi_0)=\left\{\xi\in\mathbb{H}^n:d(\xi_0,\xi)<R\right\}$$

is given by

$$|B_R(\xi_0)|=R^{2n+2}\,|B_1(\xi_0)|\triangleq\omega_nR^Q,$$

where the number

$$Q=2n+2$$

is termed the homogeneous dimension of $\mathbb{H}^n$, while ω_n represents the volume of the homogeneous ball with radius 1.

2.2 Some definitions and lemmas

Our discussion will introduce the function space and inequalities that are essential for our results. Initially, we define the horizontal Sobolev space.

Definition 1 (Horizontal Sobolev space) Let $\Omega \subset \mathbb{H}^n$ be an open set, the horizontal Sobolev space $HW^{1,m}(\Omega)(1 \leq m < \infty)$ is defined as

$$HW^{1,m}(\Omega) = \left\{u \in L^m(\Omega) \middle| X_i u \in L^m(\Omega), i = 1, 2, \dots, 2n\right\},$$

which is a Banach space under the norm

$$||u||_{HW^{1,m}(\Omega)} = ||u||_{L^m(\Omega)} + \sum_{i=1}^{2n} ||X_i u||_{L^m(\Omega)}, \tag{2.2}$$

and the space $HW_0^{1,m}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ under the norm (2.2).

Lu [21] established a Poincaré-type inequality linked to Hörmander vector fields, naturally applicable to the Heisenberg group $\mathbb{H}^n$.

Lemma 1 (Poincaré-type inequality) *Let $1 < q < Q$ with Q being the homogeneous dimension. For every $u \in HW^{1,q}\left(B_\rho(\xi_0)\right)$, there exists a positive constant C_p, the following holds*:

$$\left(\fint_{B_\rho(\xi_0)} |u(\xi) - u_{\xi_0,\rho}|^p d\xi\right)^{\frac{1}{p}} \leq C_p \rho \left(\fint_{B_\rho(\xi_0)} |Xu|^q d\xi\right)^{\frac{1}{q}}, \tag{2.3}$$

where $\xi_0 \in \Omega$ and $1 \leq p \leq \frac{qQ}{Q-q}$. And that (2.3) is valid for $p = q = m(\geq 2)$.

Lemma 2 (Lyapunov's inequality) *Assume $0 < m, q < \infty$, and $f \in L^m(\Omega) \cap L^q(\Omega)$, then for any r with $m < r < q$, we have $f \in L^r(\Omega)$ and*

$$||f||_{L^r(\Omega)} \leq ||f||_{L^m(\Omega)}^{1-\alpha} ||f||_{L^q(\Omega)}^{\alpha},$$

where $0 < \alpha < 1$ with $\frac{1}{r} = \frac{1-\alpha}{m} + \frac{\alpha}{q}$.

In fact, the conclusion can be obtained by Hölder's inequality with the exponent pair $\left(\frac{m}{(1-\alpha)r}, \frac{q}{\alpha r}\right)$.

Following [22], for technical convenience, by letting $\eta(t) = \mu^2\left(\sqrt{t}\right)$, we have the corresponding properties for η:

($\eta 1$) η is continuous, nondecreasing and $\eta(0) = 0$;

($\eta 2$) η is concave, and $r \to r^{-\gamma}\eta(r)$ is nonincreasing for some exponent $\gamma \in (0, 1)$;

($\eta 3$) For some $r < 0$,

$$H(r) := 4M^2\left(\sqrt{r}\right) = \left[\int_0^r \rho^{-1}\sqrt{\eta(\rho)}d\rho\right]^2 < \infty.$$

Changing κ by a constant, but keeping $\kappa \geq 1$, we can assume that

($\eta 4$) $\eta(1) = 1$, implying $\eta(t) \geq t$ for $t \in [0, 1]$.

We deduce from the nondecreasing nature of η that $s\eta(t) \leq t\eta(s)$ for all $0 \leq t \leq s$. Leveraging the nonincreasing property of $r \mapsto \frac{\eta(r)}{r}$ and $\eta(1) \leq 1$, we address both cases to obtain

$$s\eta(t) \leq s\eta(s) + t, \quad s \in [0,1], t > 0. \tag{2.4}$$

From (η2), we deduce for $\theta \in (0,1)$, $t > 0$, $j \in \mathbb{N} \cup \{0\}$,

$$\frac{2}{\gamma}(1-\theta^{\gamma})\sqrt{\eta\left(\theta^{2j}t\right)} = \int_{\theta^{2(j+1)}t}^{\theta^{2j}t} \tau^{\frac{\gamma}{2}-1} \frac{\sqrt{\eta(\theta^{2j}t)}}{(\theta^{2j}t)^{\frac{\gamma}{2}}} d\tau \leq \int_{\theta^{2(j+1)}t}^{\theta^{2j}t} \frac{\sqrt{\eta(\tau)}}{\tau} d\tau,$$

which implies

$$\sum_{j=0}^{\infty} \sqrt{\eta\left(\theta^{2j}t\right)} \leq \frac{\gamma}{2(1-\theta^{\gamma})} \int_0^t \frac{\sqrt{\eta(\tau)}}{\tau} d\tau = \frac{\gamma}{2(1-\theta^{\gamma})}\sqrt{H(t)}.$$

It yields particularly that $\eta(t) \leq \frac{\gamma^2}{4}H(t)$ for all $t \leq 0$, and $t \mapsto t^{-\gamma}H(t)$ is also nonincreasing.

In the sequel, we let $\rho_1(s,t) = (1+s+t)^{-1}\kappa(s+t)^{-1}$ and $K_1(s,t) = (1+t)^{2m}\kappa^4(s+t)$ for $s,t \geq 0$. Note that $\rho_1 \leq 1$ and that $s \to \rho_1(s,t)$, $t \to \rho_1(s,t)$ are nonincreasing functions.

Specifically, we can get the following $\mathcal{A}$-harmonic approximation lemma in $\mathbb{H}^n$ similarly to [23], serving as the primary tool for establishing C^1-continuity outcomes.

Lemma 3 ($\mathcal{A}$-harmonic approximation lemma) *Suppose that λ and L are fixed positive constants and $n, N \in \mathbb{N}$; for every $\varepsilon > 0$, there is a constant $\delta = \delta(n,N,\lambda,\varepsilon) \in (0,1]$ such that the following hold*:

(***I***) *Assume that $\mathcal{A} \in Bil\left(\mathbb{R}^{2n\times N}\right)$ with the properties*

$$\mathcal{A}(v,v) \geq \lambda|v|^2 \quad \textit{and} \quad \mathcal{A}(v,\bar{v}) \leq L|v||\bar{v}|, \quad v,\bar{v} \in \mathbb{R}^{2n\times N}. \tag{2.5}$$

(***II***) *For any $w \in HW^{1,2}\left(B_\rho(\xi_0),\mathbb{R}^N\right)$ and that*

$$\fint_{B_\rho(\xi_0)} |Xw|^2 d\xi \leq 1 \tag{2.6}$$

and

$$\left|\fint_{B_\rho(\xi_0)} \mathcal{A}(Xw,X\varphi)d\xi\right| \leq \delta \sup_{B_\rho(\xi_0)} |X\varphi|, \quad \forall \varphi \in C_0^1\left(B_\rho(\xi_0),\mathbb{R}^N\right), \tag{2.7}$$

there exists an $\mathcal{A}$-harmonic function h such that

$$\fint_{B_\rho(\xi_0)} |Xh|^2 d\xi \leq 1$$

and

$$\rho^{-2}\fint_{B_\rho(\xi_0)} |h-w|^2 d\xi \leq \varepsilon. \tag{2.8}$$

Now, we briefly introduce a prior estimate for weak solution u.

Lemma 4 (Prior estimate) *Let $h \in HW^{1,m}\left(B_\rho(\xi_0), \mathbb{R}^N\right)$ be any $\mathcal{A}$-harmonic function defined in $B_\rho(\xi_0)$. Then there exists $C_0 \geq 1$ such that*

$$\sup_{B_{\rho/2}(\xi_0)} \left(|u|^2 + \rho^2 |Xu|^2 + \rho^4 \left|X^2 u\right|^2\right) \leq C_0 \rho^2 \fint_{B_\rho(\xi_0)} |Xu|^2 d\xi. \tag{2.9}$$

Lemma 5 (Jensen's inequality) *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a concave function and $f(x) \in L^1(\Omega)$, then the inequality*

$$\fint_\Omega \omega(f(x))dx \leq \omega\left(\fint_\Omega f(x)dx\right)$$

holds. For the proof of this lemma, one can refer to [24].

Throughout our exposition, we claim that C is a constant, which may vary from line to line, and the dependence of C on the associated coefficients is specified within parentheses following each constant, if necessary.

3 Caccioppoli type inequality

This section is dedicated to proving a Caccioppoli-type inequality featuring a drift term under the superquadratic controllable growth condition.

Lemma 6 (Caccioppoli type inequality) *Let $u \in HW^{1,m}\left(\Omega, \mathbb{R}^N\right)$ represent a weak solution to system* (1.1)*, satisfying* ***(H1–H3)***, ***(HC)****, and* $(\mu 1)$–$(\mu 3)$*. For any $\xi_0 = \left(x_1^0, x_2^0, \ldots, x_n^0, y_1^0, y_2^0, \ldots, y_n^0, t^0\right) \in \Omega$, $u_0 \in \mathbb{R}^N$, $Xl \in \mathbb{R}^{2n\times N}$, and for every ρ such that $0 < 2\rho < \rho_1^{\frac{m}{2}}(|u_0|, |Xl|) \leq 1$ and $B_{2\rho}(\xi_0) \subset\subset \Omega$, the inequality*

$$\begin{aligned}
&\fint_{B_\rho(\xi_0)} \left[|Xu - Xl|^2 + |Xu - Xl|^m\right] d\xi \\
\leq & C_c \left[\rho^{-2} \fint_{B_{2\rho}(\xi_0)} |u - l|^2 d\xi + \rho^{-m} \fint_{B_{2\rho}(\xi_0)} |u - l|^m d\xi + K_1(|u_0|, |Xl|)\eta\left(4\rho^2\right)\right] \\
& + C_c \left[\fint_{B_{2\rho}(\xi_0)} \left(|Xu|^m + |u|^r + 1\right) d\xi\right]^{\frac{m(r-1)}{r(m-1)}}
\end{aligned}$$

holds, where C_c is constant, $l = u_0 + Xl\left(\xi^1 - \xi_0^1\right)$, $u_0 = u_{\xi_0,\rho} = \fint_{B_\rho(\xi_0)} u(\xi)d\xi$, and $\xi^1 = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ are the horizontal components of $\xi = (x_1, \ldots, x_n, y_1, \ldots, y_n, t) \in \mathbb{H}^n$.

Proof We test the subelliptic system (1.1) with the testing function $\varphi^\alpha = \phi^2 v$ with $v = u - l$, where $\phi \in C_0^\infty\left(B_{2\rho}(\xi_0)\right)$ serves as a cut-off function with $0 \leq \phi \leq 1$, $|X\phi| \leq C/\rho$, and $\phi \equiv 1$ on $B_\rho(\xi_0)$. From (2.1), we find $Xv = Xu - Xl$ and $X\varphi^\alpha = \phi^2(Xu - Xl) + 2\phi(u - l)X\phi$, which leads to

$$\begin{aligned}
&\fint_{B_{2\rho}(\xi_0)} A_i^\alpha(\xi, u, Xu)\phi^2(Xu - Xl)d\xi \\
= & -2\fint_{B_{2\rho}(\xi_0)} \phi X\phi A_i^\alpha(\xi, u, Xu)(u - l)d\xi + \fint_{B_{2\rho}(\xi_0)} B^\alpha(\xi, u, Xu)\varphi^\alpha d\xi \\
& + \fint_{B_{2\rho}(\xi_0)} X_i u \cdot X_{n+i}\varphi^\alpha d\xi - \fint_{B_{2\rho}(\xi_0)} X_{n+i} u \cdot X_i \varphi^\alpha d\xi.
\end{aligned}$$

From a testing function $\varphi^\alpha = \phi^2(u-l)$, we have

$$
\begin{aligned}
&-\fint_{B_{2\rho}(\xi_0)} A_i^\alpha(\xi,u,Xl)\phi^2(Xu-Xl)d\xi \\
=&2\fint_{B_{2\rho}(\xi_0)} \phi X\phi A_i^\alpha(\xi,u,Xl)(u-l)d\xi - \fint_{B_{2\rho}(\xi_0)} A_i^\alpha(\xi,u,Xl)X\varphi^\alpha d\xi .
\end{aligned}
$$

Note that $A_i^\alpha(\xi_0,u_0,Xl)$ is a constant, an integration by parts infers that

$$
\fint_{B_{2\rho}(\xi_0)} A_i^\alpha(\xi_0,u_0,Xl)X\varphi^\alpha d\xi = 0. \tag{3.1}
$$

We can obtain

$$
\begin{aligned}
&\fint_{B_{2\rho}(\xi_0)} \left[A_i^\alpha(\xi,u,Xu)-A_i^\alpha(\xi,u,Xl)\right]\phi^2(Xu-Xl)d\xi \\
=&2\fint_{B_{2\rho}(\xi_0)} \left[A_i^\alpha(\xi,u,Xl)-A_i^\alpha(\xi,u,Xu)\right]\phi(u-l)X\phi d\xi \\
&+\fint_{B_{2\rho}(\xi_0)} \left[A_i^\alpha(\xi,l,Xl)-A_i^\alpha(\xi,u,Xl)\right]X\varphi^\alpha d\xi \\
&+\fint_{B_{2\rho}(\xi_0)} \left[A_i^\alpha(\xi_0,u_0,Xl)-A_i^\alpha(\xi,l,Xl)\right]X\varphi^\alpha d\xi + \fint_{B_{2\rho}(\xi_0)} B^\alpha(\xi,u,Xu)\varphi^\alpha d\xi \\
&+\fint_{B_{2\rho}(\xi_0)} X_iu\cdot X_{n+i}\varphi^\alpha d\xi - \fint_{B_{2\rho}(\xi_0)} X_{n+i}u\cdot X_i\varphi^\alpha d\xi \\
\le&2C(L)\fint_{B_{2\rho}(\xi_0)} \left(1+|Xu|^2+|Xl|^2\right)^{\frac{m-2}{2}}|Xu-Xl||\phi||u-l||X\phi|d\xi \\
&+\fint_{B_{2\rho}(\xi_0)} (1+|Xl|)^{\frac{m}{2}}\kappa\left(|u_0|+2\rho|Xl|\right)\eta^{\frac12}\left(|u-l|^2\right)\phi^2|Xu-Xl|d\xi \\
&+2\fint_{B_{2\rho}(\xi_0)} (1+|Xl|)^{\frac{m}{2}}\kappa\left(|u_0|+2\rho|Xl|\right)\eta^{\frac12}\left(|u-l|^2\right)|\phi||u-l||X\phi|d\xi \\
&+\fint_{B_{2\rho}(\xi_0)} (1+|Xl|)^{\frac{m}{2}}\kappa\left(|u_0|+2\rho|Xl|\right)\eta^{\frac12}\left(4\rho^2\left(1+|Xl|^2\right)\right) \\
&\times\left[\phi^2|Xu-Xl|+2|\phi||u-l||X\phi|\right]d\xi \\
&+\fint_{B_{2\rho}(\xi_0)} \left(a|Xu|^{m\left(1-\frac1r\right)}+b|u|^{r-1}+c\right)\varphi^\alpha d\xi \\
&+\fint_{B_{2\rho}(\xi_0)} X_iu\cdot X_{n+i}\left(\phi^2(u-l)\right)d\xi - \fint_{B_{2\rho}(\xi_0)} X_{n+i}u\cdot X_i\left(\phi^2(u-l)\right)d\xi . \\
:=&J_1+J_2+J_3+J_4+J_5+J_6.
\end{aligned} \tag{3.2}
$$

The left-hand side in (3.2) can be estimated by using version (1.5), leading to

$$
\begin{aligned}
&\fint_{B_{2\rho}(\xi_0)} \left[A_i^\alpha(\xi,u,Xu)-A_i^\alpha(\xi,u,Xl)\right](Xu-Xl)\phi^2 d\xi \\
\ge&\lambda'\fint_{B_{2\rho}(\xi_0)} \left[|Xu-Xl|^2\phi^2+|Xu-Xl|^m\phi^2\right]d\xi
\end{aligned} \tag{3.3}
$$

$$\begin{aligned}
\geq&\lambda' \frac{|B_\rho(\xi_0)|_{\mathbb{H}^n}}{|B_{2\rho}(\xi_0)|_{\mathbb{H}^n}} \fint_{B_\rho(\xi_0)} \left[|Xu - Xl|^2 + |Xu - Xl|^m\right] d\xi \\
=&\lambda' 2^{-Q} \fint_{B_\rho(\xi_0)} \left[|Xu - Xl|^2 + |Xu - Xl|^m\right] d\xi .
\end{aligned}$$

Now, let us estimate each term of the right-hand side in (3.2) individually.

To estimate J_1, for a sufficiently small positive ε, employing Young's inequality and $|X\phi| < C\rho^{-1}$, we obtain

$$\begin{aligned}
J_1 \leq& C\varepsilon \fint_{B_{2\rho}(\xi_0)} |Xu - Xl|^2 |\phi|^2 d\xi + C\varepsilon^{-1}\rho^{-2} \fint_{B_{2\rho}(\xi_0)} |u - l|^2 d\xi \\
&+ C\varepsilon \fint_{B_{2\rho}(\xi_0)} \left[|Xu - Xl|^m |\phi|^{\frac{m}{m-1}}\right] d\xi + C\varepsilon^{1-m}\rho^{-m} \fint_{B_{2\rho}(\xi_0)} |u - l|^m d\xi .
\end{aligned} \tag{3.4}$$

Next, we estimate J_2. Utilizing Jensen's inequality, (2.4), and $\eta(ts) \leq t\eta(s)$ for $t \geq 1$, we get

$$\begin{aligned}
&(1 + |Xl|)^m \kappa^2(|u_0| + |Xl|) \fint_{B_{2\rho}(\xi_0)} \eta\left(|u - l|^2\right) d\xi \\
\leq&(2\rho)^{-2}\left[(1 + |Xl|)^m \kappa^2(|u_0| + |Xl|)(2\rho)^2\right] \eta\left(\fint_{B_{2\rho}(\xi_0)} \left(|u - l|^2\right) d\xi\right) \\
\leq&\frac{1}{4}\rho^{-2}\Bigg(\fint_{B_{2\rho}(\xi_0)} |u - l|^2 d\xi \\
&+ (1 + |Xl|)^m \kappa^2(|u_0| + |Xl|)(2\rho)^2 \eta\left[4\rho^2(1 + |Xl|)^m \kappa^2(|u_0| + |Xl|)\right]\Bigg) \\
\leq&\frac{1}{4}\rho^{-2} \fint_{B_{2\rho}(\xi_0)} |u - l|^2 d\xi + (1 + |Xl|)^{2m} \kappa^4(|u_0| + |Xl|)\eta\left(4\rho^2\right).
\end{aligned} \tag{3.5}$$

Note that (2.4) in the second-to-last inequality is applied with the assumption $2\rho \leq \rho_1^{\frac{m}{2}}(|u_0|, |Xl|) \leq 1$.

Using Dini's continuity condition (H3), Young's inequality, and (3.5) in J_2, we find

$$\begin{aligned}
J_2 \leq& \varepsilon \int_{B_{2\rho}(\xi_0)} \phi^4 |Xu - Xl|^2 d\xi \\
&+ \varepsilon^{-1}(1 + |Xl|)^m \kappa^2(|u_0| + |Xl|) \fint_{B_{2\rho}(\xi_0)} \eta\left(|u - l|^2\right) d\xi \\
\leq& \varepsilon \fint_{B_{2\rho}(\xi_0)} \phi^2 |Xu - Xl|^2 d\xi + \frac{1}{4}\varepsilon^{-1}\rho^{-2} \fint_{B_{2\rho}(\xi_0)} |u - l|^2 d\xi \\
&+ \varepsilon^{-1}(1 + |Xl|)^{2m} \kappa^4(|u_0| + |Xl|)\eta\left(4\rho^2\right).
\end{aligned} \tag{3.6}$$

Similarly, we can obtain

$$\begin{aligned} J_3 \le & C \fint_{B_{2\rho}(\xi_0)} |u-l|^2 |X\phi|^2 d\xi \\ & + (1+|Xl|)^m \kappa^2(|u_0|+|Xl|) \fint_{B_{2\rho}(\xi_0)} \eta\left(|u-l|^2\right) |\phi|^2 d\xi \\ \le & C\rho^{-2} \fint_{B_{2\rho}(\xi_0)} |u-l|^2 d\xi + (1+|Xl|)^{2m} \kappa^4(|u_0|+|Xl|)\eta\left(4\rho^2\right) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} J_4 \le & (1+|Xl|)^{\frac{m}{2}} \kappa(|u_0|+2\rho|Xl|) \\ & \times \fint_{B_{2\rho}(\xi_0)} \eta^{\frac{1}{2}}\left(4\rho^2\left(1+|Xl|^2\right)\right)\left[\phi^2|Xu-Xl| + 2|\phi||u-l||X\phi|\right] d\xi \\ \le & \varepsilon \fint_{B_{2\rho}(\xi_0)} |Xu-Xl|^2|\phi|^2 d\xi + C\varepsilon \fint_{B_{2\rho}(\xi_0)} |u-l|^2|X\phi|^2 d\xi \\ & + 2\varepsilon^{-1}(1+|Xl|)^m\kappa^2(|u_0|+|Xl|)\eta\left(\fint_{B_{2\rho}(\xi_0)} 4\rho^2\left(1+|Xl|^2\right) d\xi\right) \\ \le & \varepsilon \fint_{B_{2\rho}(\xi_0)} |Xu-Xl|^2\phi^2 d\xi + C\varepsilon\rho^{-2} \fint_{B_{2\rho}(\xi_0)} |u-l|^2 d\xi \\ & + 2\varepsilon^{-1}(1+|Xl|)^{m+2}\kappa^4(|u_0|+|Xl|)\eta\left(4\rho^2\right), \end{aligned} \tag{3.8}$$

where we have used the fact $\kappa \ge 1$ in the last inequality.

The term J_5 can be estimated by using the controllable growth condition **(HC)** and Hölder's inequality, which yields

$$\begin{aligned} J_5 \le & C' \fint_{B_{2\rho}(\xi_0)} \left(|Xu|^{\frac{m(r-1)}{r}} + |u|^{r-1} + 1\right)\varphi^\alpha d\xi \\ \le & C' \left(\fint_{B_{2\rho}(\xi_0)} |\varphi|^r d\xi\right)^{\frac{1}{r}} \left(\fint_{B_{2\rho}(\xi_0)} \left[|Xu|^m + |u|^r + 1\right] d\xi\right)^{\frac{r-1}{r}} \\ \le & C' \left(\fint_{B_{2\rho}(\xi_0)} \left[|Xu|^m + |u|^r + 1\right] d\xi\right)^{\frac{m(r-1)}{r(m-1)}}, \end{aligned} \tag{3.9}$$

where $C' = \max\{a, b, c\}$, and we have used the facts $|\varphi| \le \rho^2 \le 1$ and $\frac{r-1}{r} \le \frac{m(r-1)}{r(m-1)}$.

Lastly, noticing l is independent of t in the last term and thus $Tl = 0$, i.e.,

$$X_i X_{n+i} l - X_{n+i} X_i l = 0,$$

and using $|T\phi| \le \frac{1}{4}c_1\rho^{-2}$, we conclude that

$$\begin{aligned} J_6 = & \fint_{B_{2\rho}(\xi_0)} X_i u \cdot X_{n+i}\left(\phi^2(u-l)\right) d\xi - \fint_{B_{2\rho}(\xi_0)} X_{n+i} u \cdot X_i\left(\phi^2(u-l)\right) d\xi \\ = & \fint_{B_{2\rho}(\xi_0)} X_i(u-l) \cdot X_{n+i}\left(\phi^2(u-l)\right) d\xi + \fint_{B_{2\rho}(\xi_0)} X_i l \cdot X_{n+i}\left(\phi^2(u-l)\right) d\xi \end{aligned}$$

$$
\begin{aligned}
&-⨍_{B_{2\rho}(\xi_0)} X_{n+i}(u-l)\cdot X_i\left(\phi^2(u-l)\right)d\xi - ⨍_{B_{2\rho}(\xi_0)} X_{n+i}l\cdot X_i\left(\phi^2(u-l)\right)d\xi\\
&=⨍_{B_{2\rho}(\xi_0)} \phi^2 X_i(u-l)\cdot X_{n+i}(u-l)d\xi + ⨍_{B_{2\rho}(\xi_0)} \phi X_{n+i}\phi X_i\left((u-l)^2\right)d\xi\\
&-⨍_{B_{2\rho}(\xi_0)} X_{n+i}X_i l\cdot\left(\phi^2(u-l)\right)d\xi - ⨍_{B_{2\rho}(\xi_0)} \phi^2 X_{n+i}(u-l)\cdot X_i(u-l)d\xi \qquad (3.10)\\
&-⨍_{B_{2\rho}(\xi_0)} \phi X_i\phi X_{n+i}\left((u-l)^2\right)d\xi + ⨍_{B_{2\rho}(\xi_0)} X_i X_{n+i}l\cdot\left(\phi^2(u-l)\right)d\xi\\
&=⨍_{B_{2\rho}(\xi_0)} \phi X_{n+i}\phi X_i\left((u-l)^2\right)d\xi - ⨍_{B_{2\rho}(\xi_0)} \phi X_i\phi X_{n+i}\left((u-l)^2\right)d\xi\\
&=-⨍_{B_{2\rho}(\xi_0)} \phi X_i X_{n+i}\phi\cdot(u-l)^2 d\xi + ⨍_{B_{2\rho}(\xi_0)} \phi X_{n+i}X_i\phi\cdot(u-l)^2 d\xi\\
&\le\frac{1}{4}c_1\rho^{-2}⨍_{B_{2\rho}(\xi_0)}|u-l|^2 d\xi.
\end{aligned}
$$

Joining the estimates (3.3)–(3.4), (3.6)–(3.10) with (3.2), we arrive at

$$
\begin{aligned}
&\Lambda\left(⨍_{B_\rho(\xi_0)}\left[|Xu-Xl|^2+|Xu-Xl|^m\right]d\xi\right)\\
&\le\left(C\left(\varepsilon^{-1}+1+\varepsilon\right)+\frac{1}{4}\varepsilon^{-1}+\frac{1}{4}c_1\right)\rho^{-2}⨍_{B_{2\rho}(\xi_0)}|u-l|^2 d\xi\\
&+C\varepsilon^{1-m}\rho^{-m}⨍_{B_{2\rho}(\xi_0)}|u-l|^m d\xi\\
&+\left(3\varepsilon^{-1}+1\right)(1+|Xl|)^{2m}\kappa^4(|u_0|+|Xl|)\eta\left(4\rho^2\right)\\
&+C'\left(⨍_{B_{2\rho}(\xi_0)}\left[|Xu|^m+|u|^r+1\right]d\xi\right)^{\frac{m(r-1)}{r(m-1)}},
\end{aligned}
$$

here, $C'=\max\{a,b,c\}$, $\Lambda=2^{-Q}\left(\lambda'-(2+C)\varepsilon\right)$. By choosing a suitable ε such that $\Lambda>0$, we have thus shown the desired Caccioppoli-type estimates. This proves the claim. □

Remark 2 We emphasize that the function v employed in the proof significantly differs from that in [25]. Specifically, the horizontal vector fields $\{X_1,X_2,\dots,X_{2n}\}$ in the Heisenberg group $\mathbb{H}^n$ are more intricate compared to the vector fields $\left\{\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\dots,\frac{\partial}{\partial x_{2n}}\right\}$ in the Euclidean space. Here,

$$
X_i=\frac{\partial}{\partial x_i}-\frac{y_i}{2}\frac{\partial}{\partial t},\quad X_{n+i}=\frac{\partial}{\partial y_i}+\frac{x_i}{2}\frac{\partial}{\partial t},\quad i=1,2,\dots,n.
$$

Thus, the function v chosen here incorporates the horizontal affine function.

4 Optimal partial Hölder continuity for subelliptic systems

To utilize the $\mathcal{A}$-harmonic approximation lemma, our focus is directed towards a lemma that outlines a linearization strategy applicable to nonlinear subelliptic systems such as (1.1).

4.1 Linearization strategy

Lemma 7 *Assume $u \in HW^{1,m}\left(\Omega, \mathbb{R}^N\right)$ with $m > 2$ is a weak solution to (1.1) under assumptions of **(H1–H3)**, **(HC)**, and $(\mu 1)$–$(\mu 3)$. Let $B_\rho(\xi_0) \subset\subset \Omega$ with $\rho \le \rho_1^{\frac{m}{2}}(|u_0|, |Xl|)$ and for all $\varphi \in C_0^\infty\left(B_\rho(\xi_0), \mathbb{R}^N\right)$ with $\sup_{B_\rho(\xi_0)} |X\varphi| \le 1$ and $|\varphi| \le \rho^2 \le 1$, it follows that*

$$
\begin{aligned}
&\fint_{B_\rho(\xi_0)} A^\alpha_{i,p^j_\beta}(\xi_0, u_0, Xl)(Xu - Xl)X\varphi^\alpha d\xi \\
\le & C_1 \sup_{B_\rho(\xi_0)} |X\varphi| \Big[\omega^{\frac{1}{m}}\left(|Xl|, \Phi(\xi_0, \rho, Xl)\right) \Phi^{\frac{1}{2}}(\xi_0, \rho, Xl) + \Phi(\xi_0, \rho, Xl) \\
& + \Phi^{\frac{1}{2}}(\xi_0, \rho, Xl) + F(|u_0|, |Xl|)\eta^{\frac{1}{2}}\left(\rho^2\right) \Big],
\end{aligned}
$$

where $C_1 = 2^{\frac{m-2}{2}}\left(C(L) + 2C' + 5 + 2C_p\right) > 1$ and denote

$$
F(s,t) = K_1(s,t) + (1 + s + t)^{r-1}.
$$

Here, we define $v = u - l$ is an approximately $\mathcal{A}$-harmonic map and l representing a horizontal affine function.

Proof A straightforward computation leads to

$$
\begin{aligned}
\left|A^\alpha_i(\xi_0, u_0, Xu) - A^\alpha_i(\xi_0, u_0, Xl)\right| &= \int_0^1 \frac{d}{d\theta} A^\alpha_i(\xi_0, u_0, \theta Xu + (1-\theta)Xl) d\theta \\
&= \int_0^1 A^\alpha_{i,p^j_\beta}(\xi_0, u_0, \theta Xu + (1-\theta)Xl)(Xu - Xl) d\theta,
\end{aligned}
$$

where we used (3.1) over the ball $B_\rho(\xi_0)$ and (1.6).

Noting that $\fint_{B_\rho(\xi_0)} A^\alpha_i(\xi_0, u_0, Xl)X\varphi^\alpha d\xi = 0$, one has

$$
\begin{aligned}
&\fint_{B_\rho(\xi_0)} A^\alpha_{i,p^j_\beta}(\xi_0, u_0, Xl)(Xu - Xl)X\varphi^\alpha d\xi \\
\le & \fint_{B_\rho(\xi_0)} \Big[\int_0^1 \left(A^\alpha_{i,p^j_\beta}(\xi_0, u_0, Xl) - A^\alpha_{i,p^j_\beta}(\xi_0, u_0, \theta Xu + (1-\theta)Xl) \right) \\
& \times (Xu - Xl) d\theta \Big] \sup_{B_\rho(\xi_0)} |X\varphi| d\xi \\
& + \fint_{B_\rho(\xi_0)} \left|A^\alpha_i(\xi_0, u_0, Xu) - A^\alpha_i(\xi, l, Xu)\right| \sup_{B_\rho(\xi_0)} |X\varphi| d\xi \\
& + \fint_{B_\rho(\xi_0)} \left|A^\alpha_i(\xi, l, Xu) - A^\alpha_i(\xi, u, Xu)\right| \sup_{B_\rho(\xi_0)} |X\varphi| d\xi \\
& + \fint_{B_{2\rho}(\xi_0)} B^\alpha(\xi, u, Xu)\varphi^\alpha d\xi \\
& + \fint_{B_\rho(\xi_0)} X_i u \cdot X_{n+i}\varphi^\alpha d\xi - \fint_{B_\rho(\xi_0)} X_{n+i} u \cdot X_i\varphi^\alpha d\xi \\
:= & J_1' + J_2' + J_3' + J_4' + J_5',
\end{aligned}
\tag{4.1}
$$

with the obvious meaning of $J_1' - J_5'$. Now, we estimate the term J_1'. Using (1.2) and (1.4), we compute

$$\begin{aligned}
&\left|A^{\alpha}_{i,p^j_\beta}(\xi_0,u_0,Xl) - A^{\alpha}_{i,p^j_\beta}(\xi_0,u_0,\theta Xu + (1-\theta)Xl)\right|^{\frac{1}{m}+\frac{m-1}{m}} \\
\le & \left[C\left(1+|Xl|^2+|\theta(Xu-Xl)+Xl|^2\right)^{\frac{m-2}{2}}\omega\left(|Xl|,|\theta(Xu-Xl)|^2\right)\right]^{\frac{1}{m}} \\
& \times\left[L\left(1+|Xl|^2\right)^{\frac{m-2}{2}} + L\left(1+|\theta(Xu-Xl)+Xl|^2\right)^{\frac{m-2}{2}}\right]^{\frac{m-1}{m}} \\
\le & \left[C\left(1+3|Xl|^2+2|Xu-Xl|^2\right)^{\frac{m-2}{2}}\omega\left(|Xl|,|\theta(Xu-Xl)|^2\right)\right]^{\frac{1}{m}} \\
& \times\left[L\left(1+2|Xl|^2+2|Xu-Xl|^2\right)^{\frac{m-2}{2}}\right]^{\frac{m-1}{m}} \\
\le & C(L)\left(1+|Xl|^2\right)^{\frac{m-2}{2}}\left(1+|Xu-Xl|^2\right)^{\frac{m-2}{2}}\omega^{\frac{1}{m}}\left(|Xl|,|Xu-Xl|^2\right) \\
\le & C(L)\left(1+|Xl|^2\right)^{\frac{m-2}{2}}|Xu-Xl|^{m-2}\omega^{\frac{1}{m}}\left(|Xl|,|Xu-Xl|^2\right).
\end{aligned} \tag{4.2}$$

Through the utilization of (4.2), Hölder's inequality, and leveraging the concavity of $t \to \omega^2(s,t)$ along with Jensen's inequality, we ultimately reach the following expression:

$$\begin{aligned}
J_1' \le & C(L)\fint_{B_\rho(\xi_0)}\omega^{\frac{1}{m}}\left(|Xl|,|Xu-Xl|^2\right)|Xu-Xl|^{m-1}\sup_{B_\rho(\xi_0)}|X\varphi|d\xi \\
\le & C(L)\sup_{B_\rho(\xi_0)}|X\varphi|\left[\fint_{B_\rho(\xi_0)}\omega\left(|Xl|,|Xu-Xl|^2\right)\right]^{\frac{1}{m}}\left[\fint_{B_\rho(\xi_0)}|Xu-Xl|^m d\xi\right]^{\frac{m-1}{m}} \\
\le & C(L)\sup_{B_\rho(\xi_0)}|X\varphi|\omega^{\frac{1}{m}}\left(|Xl|,\fint_{B_\rho(\xi_0)}|Xu-Xl|^2 d\xi\right)\Phi^{\frac{1}{2}}(\xi_0,\rho,Xl) \\
\le & C(L)\sup_{B_\rho(\xi_0)}|X\varphi|\omega^{\frac{1}{m}}\left(|Xl|,\Phi(\xi_0,\rho,Xl)\right)\Phi^{\frac{1}{2}}(\xi_0,\rho,Xl),
\end{aligned} \tag{4.3}$$

where we have used the fact $\frac{1}{2} \le \frac{m-1}{m} \le 1$ and the assumption $\Phi(\xi_0,\rho,Xl) \le 1$.

The integral J_2' can be bounded by employing the Dini continuity condition (1.3) and the inequality $\eta(ts) \le t\eta(s)$ for $t \ge 1$, leading to

$$\begin{aligned}
J_2' \le & \sup_{B_\rho(\xi_0)}|X\varphi|\kappa(|u_0|+|Xl|)\mu\left(\rho(1+|Xl|)\right)\fint_{B_\rho(\xi_0)}(1+|Xu|)^{\frac{m}{2}}d\xi \\
\le & \sup_{B_\rho(\xi_0)}|X\varphi|\kappa(|u_0|+|Xl|)(1+|Xl|)\eta^{\frac{1}{2}}\left(\rho^2\right)\fint_{B_\rho(\xi_0)}(1+|Xl|+|Xu-Xl|)^{\frac{m}{2}}d\xi \\
\le & 2^{\frac{m-2}{2}}\sup_{B_\rho(\xi_0)}|X\varphi|\Bigg[\kappa(|u_0|+|Xl|)(1+|Xl|)^{1+\frac{m}{2}}\eta^{\frac{1}{2}}\left(\rho^2\right) \\
& +\kappa^2(|u_0|+|Xl|)(1+|Xl|)^2\eta\left(\rho^2\right)+\fint_{B_\rho(\xi_0)}|Xu-Xl|^m d\xi\Bigg] \\
\le & 2^{\frac{m-2}{2}}\sup_{B_\rho(\xi_0)}|X\varphi|\left[\Phi(\xi_0,\rho,Xl)+2\kappa^2(|u_0|+|Xl|)(1+|Xl|)^{1+\frac{m}{2}}\eta^{\frac{1}{2}}\left(\rho^2\right)\right],
\end{aligned} \tag{4.4}$$

where we utilized $\eta\left(\rho^2\right) \le \eta^{\frac{1}{2}}\left(\rho^2\right)$, which is deduced from the nondecreasing nature of function $\eta(t)$, $(\eta 4)$, and considering the assumption $\rho \le \rho_1 \le 1$.

Similarity, by utilizing the Dini continuity condition (1.3), (3.5), and the Poincaré inequality (2.3) with $p = q = 2$, we can approximate as follows:

$$
\begin{aligned}
J_3' \le& \sup_{B_\rho(\xi_0)} |X\varphi| \fint_{B_\rho(\xi_0)} \kappa(|u_0| + |Xl|)\eta^{\frac{1}{2}}\left(|u-l|^2\right)(1+|Xu|)^{\frac{m}{2}} d\xi \\
\le& 2^{\frac{m-2}{2}} \sup_{B_\rho(\xi_0)} |X\varphi| \Bigg[\fint_{B_\rho(\xi_0)} |Xu - Xl|^m d\xi + \kappa^2(|u_0| + |Xl|) \fint_{B_\rho(\xi_0)} \eta\left(|u-l|^2\right) d\xi \\
&+ \kappa(|u_0| + |Xl|)(1+|Xl|)^{\frac{m}{2}} \fint_{B_\rho(\xi_0)} \eta^{\frac{1}{2}}\left(|u-l|^2\right) d\xi \Bigg] \\
\le& 2^{\frac{m-2}{2}} \sup_{B_\rho(\xi_0)} |X\varphi| \Bigg[\Phi(\xi_0, \rho, Xl) + 2\rho^{-2} \fint_{B_\rho(\xi_0)} |u-l|^2 d\xi \\
&+ \kappa^4(|u_0| + |Xl|)\eta\left(|u-l|^2\right) + \kappa^2(|u_0| + |Xl|)(1+|Xu|)^m \eta^{\frac{1}{2}}\left(\rho^2\right) \Bigg] \\
\le& 2^{\frac{m-2}{2}} \sup_{B_\rho(\xi_0)} |X\varphi| \Bigg[\Phi(\xi_0, \rho, Xl) + 2C_p \fint_{B_\rho(\xi_0)} |Xu - Xl|^2 d\xi \\
&+ 2\kappa^4(|u_0| + |Xl|)(1+|Xl|)^m \eta^{\frac{1}{2}}\left(\rho^2\right) \Bigg] \\
\le& 2^{\frac{m-2}{2}} \sup_{B_\rho(\xi_0)} |X\varphi| \left[(1+2C_p)\Phi(\xi_0, \rho, Xl) + 2\kappa^4(|u_0| + |Xl|)(1+|Xl|)^m \eta^{\frac{1}{2}}\left(\rho^2\right) \right].
\end{aligned}
\tag{4.5}
$$

Note the fact that $\sup_{B_\rho(\xi_0)} |\varphi| \le \rho^2 \le 1$ and $(\eta 4)$, this leads to

$$
\begin{aligned}
J_4' \le& C' \fint_{B_\rho(\xi_0)} \left(|Xu|^{\frac{m(r-1)}{r}} + |u|^{r-1} + 1 \right) |\varphi| d\xi \\
\le& C' \fint_{B_\rho(\xi_0)} |Xu|^{\frac{m(r-1)}{r}} |\varphi| d\xi + C' \fint_{B_\rho(\xi_0)} |u-l|^{r-1} |\varphi| d\xi \\
&+ C'\rho^2 \left[1 + (|u_0| + \rho|Xl|)^{r-1} \right] \\
\le& C' \left(\fint_{B_\rho(\xi_0)} |Xu - Xl|^m d\xi \right)^{\frac{r-1}{r}} \left(\fint_{B_\rho(\xi_0)} |\varphi|^r d\xi \right)^{\frac{1}{r}} \\
&+ C' \left(\fint_{B_\rho(\xi_0)} |Xl|^m d\xi \right)^{\frac{r-1}{r}} \left(\fint_{B_\rho(\xi_0)} |\varphi|^r d\xi \right)^{\frac{1}{r}} \\
&+ C' \left(\fint_{B_\rho(\xi_0)} |Xu - Xl|^m d\xi \right)^{\frac{r-1}{m}} \left(\fint_{B_\rho(\xi_0)} |\varphi|^r d\xi \right)^{\frac{1}{r}} + C'\rho^2 \left[1 + (|u_0| + |Xl|)^{r-1} \right] \\
\le& 2C' \left(\fint_{B_\rho(\xi_0)} |Xu - Xl|^m d\xi \right)^{\frac{r-1}{r}} \left(\fint_{B_\rho(\xi_0)} |\varphi|^r d\xi \right)^{\frac{1}{r}} \\
&+ C'\rho^2 \left[1 + (|u_0| + |Xl|)^{r-1} + |Xl|^{\frac{m(r-1)}{r}} \right] \\
\le& 2C'\Phi(\xi_0, \rho, Xl) + 2C'\eta^{\frac{1}{2}}\left(\rho^2\right) \left[1 + (|u_0| + |Xl|)^{r-1} \right],
\end{aligned}
\tag{4.6}
$$

where we have used $r - 1 \ge \frac{m(r-1)}{r}$ and $\eta(s) \le 1$ for $s \in (0, 1]$.

Now, we estimate the last term J_5'. Noting $T = X_iX_{n+i} - X_{n+i}X_i$, it implies

$$\begin{aligned}
J_5' &= \fint_{B_\rho(\xi_0)} X_iu \cdot X_{n+i}\varphi d\xi - \fint_{B_\rho(\xi_0)} X_{n+i}u \cdot X_i\varphi d\xi \\
&= \fint_{B_\rho(\xi_0)} X_i(u-l) \cdot X_{n+i}\varphi d\xi + \fint_{B_\rho(\xi_0)} X_il \cdot X_{n+i}\varphi d\xi \\
&\quad - \fint_{B_\rho(\xi_0)} X_{n+i}(u-l) \cdot X_i\varphi d\xi - \fint_{B_\rho(\xi_0)} X_{n+i}l \cdot X_i\varphi d\xi \\
&= \fint_{B_\rho(\xi_0)} X_i(u-l) \cdot X_{n+i}\varphi d\xi - \fint_{B_\rho(\xi_0)} X_{n+i}(u-l) \cdot X_i\varphi d\xi \\
&\quad - \fint_{B_\rho(\xi_0)} X_{n+i}X_il \cdot \varphi d\xi + \fint_{B_\rho(\xi_0)} X_iX_{n+i}l \cdot \varphi d\xi \\
&= \fint_{B_\rho(\xi_0)} X_i(u-l) \cdot X_{n+i}\varphi d\xi - \fint_{B_\rho(\xi_0)} X_{n+i}(u-l) \cdot X_i\varphi d\xi \\
&\le \sup_{B_\rho(\xi_0)} |X\varphi| \left(\fint_{B_\rho(\xi_0)} |Xu - Xl| d\xi \right) \\
&\le \sup_{B_\rho(\xi_0)} |X\varphi| \left(\fint_{B_\rho(\xi_0)} |Xu - Xl|^2 d\xi \right)^{\frac{1}{2}} \\
&\le \sup_{B_\rho(\xi_0)} |X\varphi| \Phi^{\frac{1}{2}}(\xi_0, \rho, Xl).
\end{aligned} \tag{4.7}$$

Now, we substitute (4.3)–(4.7) into (4.1) and yield the claim with $C_1 = 2^{\frac{m-2}{2}} \big(C(L) + 2C' + 5 + 2C_p\big)$. □

4.2 Excess improvement

The proof strategy involves approximating the given solution with $\mathcal{A}$-harmonic functions, for which decay estimates are available from classical theory. This allows us to establish the improvement in excess.

Lemma 8 *Assuming the conditions of Theorem 1 are met and given a fixed $\gamma \in (0,1)$, positive constants C_2, C_3, and δ satisfy the conditions in the $\mathcal{A}$-harmonic approximation lemma. Letting $\theta \in (0,1)$ be arbitrary, we impose the following smallness conditions on the excess:*

$$\omega^{\frac{1}{m}} \left(\left|u_{\xi_0,\rho}\right| + \left|(Xu)_{\xi_0,\rho}\right|, \Phi\left(\xi_0, \rho, (Xu)_{\xi_0,\rho}\right)\right) + \Phi^{\frac{1}{2}}\left(\xi_0, \rho, (Xu)_{\xi_0,\rho}\right) \le \frac{\delta}{4}, \tag{4.8}$$

$$C_2F^2\left(\left|u_{\xi_0,\rho}\right|, \left|(Xu)_{\xi_0,\rho}\right|\right) \eta\left(\rho^2\right) \le \delta^2, \tag{4.9}$$

where $C_2 = 8C_1^2C_4$ is a positive constant, together with the radius condition

$$\rho \le \rho_1^{\frac{m}{2}} \left(1 + \left|u_{\xi_0,\rho}\right|, 1 + \left|(Xu)_{\xi_0,\rho}\right|\right), \tag{4.10}$$

then for $\tau \in [\gamma, 1)$ there holds the excess improvement estimate

$$\Phi\left(\xi_0, \theta\rho, (Xu)_{\xi_0,\theta\rho}\right) \le \theta^{2\tau} \Phi\left(\xi_0, \rho, (Xu)_{\xi_0,\rho}\right) + K^*\left(\left|u_{\xi_0,\rho}\right|, \left|(Xu)_{\xi_0,\rho}\right|\right) \eta\left(\rho^{\frac{m}{m-1}}\right),$$

where $K^*(s,t) = C_7F^2(1+s, 1+t)$.

Proof We define $w = \left[u - u_{\xi_0,\rho} - (Xu)_{\xi_0,\rho}\left(\xi^1 - \xi_0^1\right)\right]\sigma_1$, where

$$\sigma_1 = C_1^{-1}\left[\left(\frac{\delta}{4}\right)^{-2}\Phi\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right) + 4\delta^{-2}F^2\left(\left|u_{\xi_0,\rho}\right|,\left|(Xu)_{\xi_0,\rho}\right|\right)\eta\left(\rho^2\right)\right]^{-\frac{1}{2}} \tag{4.11}$$

with $C_1 > 1$ in Lemma 7. Thus we have $Xw = \sigma_1\left(Xu - (Xu)_{\xi_0,\rho}\right)$. Now we consider $B_\rho(\xi_0) \subset\subset \Omega$ such that $\rho \le \rho_1^{\frac{m}{2}}\left(\left|u_{\xi_0,\rho}\right|,\left|(Xu)_{\xi_0,\rho}\right|\right) \le 1$. It yields

$$\fint_{B_\rho(\xi_0)} |Xw|^2 d\xi = \sigma_1^2\Phi\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right) \le \left(\frac{\delta}{4}\right)^2\frac{\Phi\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right)}{C_1^2\Phi\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right)} \le \frac{1}{C_1^2} \le 1. \tag{4.12}$$

Applying Lemma 7 on $B_\rho(\xi_0)$ to u, for any $\varphi \in C_0^\infty\left(B_\rho(\xi_0),\mathbb{R}^N\right)$,

$$\begin{aligned}
&\fint_{B_\rho(\xi_0)} A^\alpha_{i,p^j_\beta}\left(\xi_0,u_0,(Xu)_{\xi_0,\rho}\right)XwX\varphi^\alpha d\xi \\
\le& C_1\sigma_1\sup_{B_\rho(\xi_0)}|X\varphi|\Bigg[\omega^{\frac{1}{m}}\left(\left|(Xu)_{\xi_0,\rho}\right|,\Phi\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right)\right)\Phi^{\frac{1}{2}}\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right) \\
&+\Phi^{\frac{1}{2}}\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right)+\Phi\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right)+F(|u_0|,|Xl|)\eta^{\frac{1}{2}}\left(\rho^2\right)\Bigg] \\
\le& \sup_{B_\rho(\xi_0)}|X\varphi|\Bigg\{\left[\left(\frac{\delta}{4}\right)\Phi^{-\frac{1}{2}}\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right)\right] \\
&\times\Bigg[\omega^{\frac{1}{m}}\left(\left|(Xu)_{\xi_0,\rho}\right|,\Phi\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right)\right)\Phi^{\frac{1}{2}}\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right) \\
&+\Phi^{\frac{1}{2}}\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right)+\Phi\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right)\Bigg] \\
&+\left[4\delta^{-2}F^2\left(\left|u_{\xi_0,\rho}\right|,\left|(Xu)_{\xi_0,\rho}\right|\right)\eta\left(\rho^2\right)\right]^{-\frac{1}{2}}F(|u_0|,|Xl|)\eta^{\frac{1}{2}}\left(\rho^2\right)\Bigg\} \\
\le& \sup_{B_\rho(\xi_0)}|X\varphi|\Bigg[\frac{\delta}{4}\omega^{\frac{1}{m}}\left(\left|(Xu)_{\xi_0,\rho}\right|,\Phi\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right)\right) \\
&+\frac{\delta}{4}+\frac{\delta}{4}\Phi^{\frac{1}{2}}\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right)+\frac{\delta}{2}\Bigg] \\
\le& \sup_{B_\rho(\xi_0)}|X\varphi|\left[\omega^{\frac{1}{m}}\left(\left|(Xu)_{\xi_0,\rho}\right|,\Phi\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right)\right)+\Phi^{\frac{1}{2}}\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right)+\frac{3\delta}{4}\right] \\
\le& \delta\sup_{B_\rho(\xi_0)}|X\varphi|,
\end{aligned} \tag{4.13}$$

Considering the smallness condition (4.8), we observe that (4.12) and (4.13) imply conditions (2.6) and (2.7) respectively in the $\mathcal{A}$-harmonic approximation lemma. Also, note that assumptions **(H1)** and **(H2)** with $u = u_{\xi_0,\rho}$ and $p = (Xu)_{\xi_0,\rho}$ imply condition (2.5). So there exists an $A^\alpha_{i,p^j_\beta}(\xi_0,u_{\xi_0,\rho},(Xu)_{\xi_0,\rho})$- harmonic function $h \in HW^{1,2}\left(B_\rho\left(\xi_0\right),\mathbb{R}^N\right)$ such that

$$\fint_{B_\rho(\xi_0)} |Xh|^2 d\xi \le 1 \tag{4.14}$$

and

$$\rho^{-2} \fint_{B_\rho(\xi_0)} |w-h|^2 d\xi \le \varepsilon. \tag{4.15}$$

Using Lemma 6 on the ball $B_{2\theta\rho}(\xi_0)$ with $u_0 = u_{\xi_0,2\theta\rho}$, $\theta \in (0, \frac{1}{4}]$ and replacing Xl by $(Xu)_{\xi_0,\rho} + \sigma_1^{-1}(Xh)_{\xi_0,2\theta\rho}$, we obtain

$$\begin{aligned}
&\int_{B_{\theta\rho}(\xi_0)} \Big(\big|Xu - \big((Xu)_{\xi_0,\rho} + \sigma_1^{-1}(Xh)_{\xi_0,2\theta\rho}\big)\big|^2 \\
&\quad + \big|Xu - \big((Xu)_{\xi_0,\rho} + \sigma_1^{-1}(Xh)_{\xi_0,2\theta\rho}\big)\big|^m \Big) d\xi \\
\le& \frac{C_c}{(\theta\rho)^2} \int_{B_{2\theta\rho}(\xi_0)} \big|u - u_{\xi_0,2\theta\rho} - \big((Xu)_{\xi_0,\rho} + \sigma_1^{-1}(Xh)_{\xi_0,2\theta\rho}\big)\left(\xi^1 - \xi_0^1\right)\big|^2 d\xi \\
&+ \frac{C_c}{(\theta\rho)^m} \int_{B_{2\theta\rho}(\xi_0)} \big|u - u_{\xi_0,2\theta\rho} - \big((Xu)_{\xi_0,\rho} + \sigma_1^{-1}(Xh)_{\xi_0,2\theta\rho}\big)\left(\xi^1 - \xi_0^1\right)\big|^m d\xi \\
&+ C_c\omega_n(2\theta\rho)^Q \left[K_1\left(\left|u_{\xi_0,2\theta\rho}\right|, \left|(Xu)_{\xi_0,\rho} + \sigma_1^{-1}(Xh)_{\xi_0,2\theta\rho}\right|\right) \eta\left((2\theta\rho)^2\right)\right] \\
&+ C_c \left[\fint_{B_\rho(\xi_0)} \left(|Xu|^m + |u|^r + 1\right) |d\xi\right]^{\frac{m(r-1)}{r(m-1)}} \\
:=& J_1'' + J_2'' + J_3'' + J_4''.
\end{aligned} \tag{4.16}$$

We see that $g(\tau) = \int_{B_{2\theta\rho}(\xi_0)} (u-\tau)^2 d\xi$ achieves its minimal value at $\tau = u_{\xi_0,2\theta\rho}$, and

$$u - \left((Xu)_{\xi_0,\rho} + \sigma_1^{-1}(Xh)_{\xi_0,2\theta\rho}\right)\left(\xi^1 - \xi_0^1\right)$$

has the mean value $u_{\xi_0,2\theta\rho}$ on the ball $B_{2\theta\rho}(\xi_0)$. Applying the definition of w, the Poincaré inequality, (4.15), (2.9), and (4.14), we derive the following estimate:

$$\begin{aligned}
&\frac{C_c}{(\theta\rho)^2} \fint_{B_{2\theta\rho}(\xi_0)} \big|u - u_{\xi_0,2\theta\rho} - \big((Xu)_{\xi_0,\rho} + \sigma_1^{-1}(Xh)_{\xi_0,2\theta\rho}\big)\left(\xi^1 - \xi_0^1\right)\big|^2 d\xi \\
\le& 4C_c(2\theta\rho)^{-2}\sigma_1^{-2} \fint_{B_{2\theta\rho}(\xi_0)} \left|w - h_{\xi_0,2\theta\rho} - (Xh)_{\xi_0,2\theta\rho}\left(\xi^1 - \xi_0^1\right)\right|^2 d\xi \\
\le& 8C_c(2\theta\rho)^{-2}\sigma_1^{-2} \fint_{B_{2\theta\rho}(\xi_0)} \left[|w-h|^2 + \left|h - h_{\xi_0,2\theta\rho} - (Xh)_{\xi_0,2\theta\rho}\left(\xi^1 - \xi_0^1\right)\right|^2\right] d\xi \\
\le& 8C_c\sigma_1^{-2} \left((2\theta)^{-Q-2}\rho^{-2} \fint_{B_\rho(\xi_0)} |w-h|^2 d\xi + C_p \fint_{B_{2\theta\rho}(\xi_0)} \left|Xh - (Xh)_{\xi_0,2\theta\rho}\right|^2 d\xi\right) \\
\le& 8C_c\sigma_1^{-2} \left((2\theta)^{-Q-2}\varepsilon + C_p^2(2\theta\rho)^2 \fint_{B_{2\theta\rho}(\xi_0)} \left|X^2h\right|^2 d\xi\right) \\
\le& 4C_c\sigma_1^{-2}\left[2^{-Q-1}\theta^{-Q-2}\varepsilon + 8C_p^2C_0\theta^2\right] \\
\le& C_cC_3(C_1\sigma_1)^{-2}\left[\theta^{-Q-2}\varepsilon + \theta^2\right],
\end{aligned} \tag{4.17}$$

where $C_3 = 4C_1^2\left(2^{-Q-1} + 8C_p^2C_0\right) > 1$, we have employed the definition of σ_1 in (4.11) with $u_0 = u_{\xi_0,2\theta\rho}$ along with the fact that

$$\fint_{B_{2\theta\rho}(\xi_0)} \left|X^2h\right| d\xi \le \sup_{B_\rho(\xi_0)} \left|X^2h\right| \le C_0\rho^{-2} \fint_{B_\rho(\xi_0)} |Xh|^2 \, d\xi \le C_0\rho^{-2}.$$

Then it follows

$$J_1'' \le C_cC_3\omega_n(2\theta\rho)^Q(C_1\sigma_1)^{-2}\left[\theta^{-Q-2}\varepsilon + \theta^2\right].$$

For $2 < m < Q$, we find $\frac{Q-m}{Qm} = \frac{1}{m^*} < \frac{1}{m} < \frac{1}{2}$. Hence, there exists $t \in [0,1)$ such that

$$\frac{1}{m} = \frac{1}{2}(1-t) + \frac{1}{m^*}t.$$

Employing Lyapunov's inequality, Young's inequality, and (4.17), then

$$\begin{aligned}
&\frac{C_c}{(\theta\rho)^m} \fint_{B_{2\theta\rho}(\xi_0)} \left|u - u_{\xi_0,2\theta\rho} - \left((Xu)_{\xi_0,\rho} + \sigma_1^{-1}(Xh)_{\xi_0,2\theta\rho}\right)\left(\xi^1 - \xi_0^1\right)\right|^m d\xi \\
\le &\frac{C_c}{(\theta\rho)^m} \left[\fint_{B_{2\theta\rho}(\xi_0)} \left|u - u_{\xi_0,2\theta\rho} - \left((Xu)_{\xi_0,\rho} + \sigma_1^{-1}(Xh)_{\xi_0,2\theta\rho}\right)\left(\xi^1 - \xi_0^1\right)\right|^2 d\xi\right]^{(1-t)\frac{m}{2}} \\
&\times \left[\fint_{B_{2\theta\rho}(\xi_0)} \left|u - u_{\xi_0,2\theta\rho} - \left((Xu)_{\xi_0,\rho} + \sigma_1^{-1}(Xh)_{\xi_0,2\theta\rho}\right)\left(\xi^1 - \xi_0^1\right)\right|^{m^*} d\xi\right]^{t\frac{m}{m^*}} \\
\le &\frac{C_c}{(\theta\rho)^m} \left\{\frac{C_3(\theta\rho)^2}{(C_1\sigma_1)^2}\left[\theta^{-Q-2}\varepsilon + \theta^2\right]\right\}^{(1-t)\frac{m}{2}} \\
&\times \left[(2C_p\theta\rho)^m \fint_{B_{2\theta\rho}(\xi_0)} \left|Xu - \left((Xu)_{\xi_0,\rho} + \sigma_1^{-1}(Xh)_{\xi_0,2\theta\rho}\right)\right|^m d\xi\right]^t \\
\le &C_c\varepsilon^{\frac{-t}{1-t}} \left\{\frac{C_3}{(C_1\sigma_1)^2}\left[\theta^{-Q-2}\varepsilon + \theta^2\right]\right\}^{\frac{m}{2}} \\
&+ C_c(2C_p)^m\varepsilon \fint_{B_{2\theta\rho}(\xi_0)} \left|Xu - \left((Xu)_{\xi_0,\rho} + \sigma_1^{-1}(Xh)_{\xi_0,2\theta\rho}\right)\right|^m d\xi \\
\le &C_cC_3^{\frac{m}{2}}\varepsilon^{\frac{-t}{1-t}}(C_1\sigma_1)^{-2}\left[\theta^{-Q-2}\varepsilon + \theta^2\right] \\
&+ C_c(2C_p)^m\varepsilon_1 \fint_{B_{\theta\rho}(\xi_0)} \left|Xu - \left((Xu)_{\xi_0,\rho} + \sigma_1^{-1}(Xh)_{\xi_0,2\theta\rho}\right)\right|^m d\xi,
\end{aligned}$$

where

$$\varepsilon_1 = \frac{\varepsilon \fint_{B_{2\theta\rho}(\xi_0)} \left|Xu - \left((Xu)_{\xi_0,\rho} + \sigma_1^{-1}(Xh)_{\xi_0,2\theta\rho}\right)\right|^m d\xi}{\fint_{B_{\theta\rho}(\xi_0)} \left|Xu - \left((Xu)_{\xi_0,\rho} + \sigma_1^{-1}(Xh)_{\xi_0,2\theta\rho}\right)\right|^m d\xi}.$$

Then we conclude

$$\begin{aligned}
J_2'' \le &C_cC_3^{\frac{m}{2}}\omega_n(2\theta\rho)^Q\varepsilon^{\frac{-t}{1-t}}(C_1\sigma_1)^{-2}\left[\theta^{-Q-2}\varepsilon + \theta^2\right] \\
&+ C_c(2C_p)^m\varepsilon_1 2^Q \int_{B_{\theta\rho}(\xi_0)} \left|Xu - \left((Xu)_{\xi_0,\rho} + \sigma_1^{-1}(Xh)_{\xi_0,2\theta\rho}\right)\right|^m d\xi.
\end{aligned}$$

Noting that the smallness conditions (4.8)–(4.9) imply

$$\sigma_1^{-2}C_4 = C_1^2 C_4 \Phi\left(\xi_0, \rho, (Xu)_{\xi_0,\rho}\right) + \frac{1}{2}C_2 F^2 \eta \delta^{-2} \le 1,$$

with $C_4 = \max\left\{C_0, (2\theta)^{-Q}\right\}$, assuming $\frac{1}{2}C_1^2 C_4 \delta^2 \le 1$ (which is not restrictive), we apply the prior estimate (2.8) for the $\mathcal{A}$-harmonic function h, then

$$\begin{aligned}\left|\sigma_1^{-1}(Xh)_{\xi_0,2\theta\rho}\right| &= \sigma_1^{-1}\sup_{B_{2\theta\rho}(\xi_0)}|Xh| \le \sigma_1^{-1}\sqrt{C_0}\left(⨍_{B_\rho(\xi_0)}|Xh|^2 d\xi\right)^{\frac{1}{2}} \\ &\le \sigma_1^{-1}\sqrt{C_0} \le 1.\end{aligned} \tag{4.18}$$

Furthermore, it follows by the Poincaré inequality

$$\begin{aligned}\left|u_{\xi_0,2\theta\rho}\right| &\le \left|u_{\xi_0,\rho}\right| + \left|u_{\xi_0,2\theta\rho} - u_{\xi_0,\rho}\right| \\ &= \left|u_{\xi_0,\rho}\right| + \left|⨍_{B_{2\theta\rho}(\xi_0)} u - (Xu)_{\xi_0,\rho}\left(\xi^1 - \xi_0^1\right) - u_{\xi_0,\rho} d\xi\right| \\ &\le \left|u_{\xi_0,\rho}\right| + (2\theta)^{-Q/2}\left(⨍_{B_\rho(\xi_0)}\left|u - (Xu)_{\xi_0,\rho}\left(\xi^1 - \xi_0^1\right) - u_{\xi_0,\rho}\right|^2 d\xi\right)^{\frac{1}{2}} \\ &\le \left|u_{\xi_0,\rho}\right| + (2\theta)^{-Q/2}\rho C_p \Phi^{\frac{1}{2}}\left(\xi_0, \rho, (Xu)_{\xi_0,\rho}\right) \\ &\le \left|u_{\xi_0,\rho}\right| + C_1^{-1}\sigma_1^{-1}C_p(2\theta)^{-Q/2} \\ &\le \left|u_{\xi_0,\rho}\right| + \sigma_1^{-1}\sqrt{C_4} \\ &\le \left|u_{\xi_0,\rho}\right| + 1,\end{aligned} \tag{4.19}$$

where we have used the definition of σ_1 in (4.11) and the fact $C_1 > C_p$.

Therefore, (4.18) and (4.19) yield

$$\begin{aligned}J_3'' &\le C_c \omega_n (2\theta\rho)^Q\left[K_1\left(1 + \left|u_{\xi_0,\rho}\right|, 1 + \left|(Xu)_{\xi_0,\rho}\right|\right)\eta\left((2\theta\rho)^2\right)\right] \\ &\le C_c \omega_n (2\theta\rho)^Q F\left(1 + \left|u_{\xi_0,\rho}\right|, 1 + \left|(Xu)_{\xi_0,\rho}\right|\right)\eta\left((2\theta\rho)^2\right).\end{aligned}$$

Using the Poincaré inequality (2.3), we have

$$\begin{aligned}&\left[⨍_{B_{2\theta\rho}(\xi_0)}\left(|Xu|^m + |u|^r + 1\right) d\xi\right]^{\frac{m(r-1)}{r(m-1)}} \\ &\le \left[2^{m-1}⨍_{B_{2\theta\rho}(\xi_0)}\left(\left|Xu - (Xu)_{\xi_0,\rho}\right|^m\right) d\xi\right]^{\frac{m(r-1)}{r(m-1)}} + \left(2^{m-1}\left|(Xu)_{\xi_0,\rho}\right|^m\right)^{\frac{m(r-1)}{r(m-1)}} \\ &\quad + \left[⨍_{B_{2\theta\rho}(\xi_0)} 2^{r-1}\left(\left|u - u_{\xi_0,\rho} - (Xu)_{\xi_0,\rho}\left(\xi^1 - \xi_0^1\right)\right|^r\right) d\xi\right]^{\frac{m(r-1)}{r(m-1)}} \\ &\quad + \left[\left(1 + 2^{r-1}\left|u_{\xi_0,\rho} + (Xu)_{\xi_0,\rho}\left(\xi^1 - \xi_0^1\right)\right|^r\right)\right]^{\frac{m(r-1)}{r(m-1)}}\end{aligned}$$

$$
\begin{aligned}
&\le C\left[(2\theta)^{-Q}\fint_{B_\rho(\xi_0)}\left(\left|Xu-(Xu)_{\xi_0,\rho}\right|^m\right)d\xi\right]^{\frac{m(r-1)}{r(m-1)}}+\left(2^{m-1}\left|(Xu)_{\xi_0,\rho}\right|^m\right)^{\frac{m(r-1)}{r(m-1)}}\\
&\quad+C\left[(2\theta)^{-Q}\fint_{B_\rho(\xi_0)}\left(\left|Xu-(Xu)_{\xi_0,\rho}\right|^m\right)d\xi\right]^{\frac{r-1}{m-1}}\\
&\quad+\left[\left(1+2^{r-1}\rho\left|u_{\xi_0,\rho}+(Xu)_{\xi_0,\rho}\right|^r\right)\right]^{\frac{m(r-1)}{r(m-1)}}\\
&\le C\left[(2\theta)^{-Q}\Phi\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right)\right]^{\frac{m(r-1)}{r(m-1)}}+C\left(1+\left|u_{\xi_0,\rho}\right|+\left|(Xu)_{\xi_0,\rho}\right|\right)^{r\cdot\frac{m(r-1)}{r(m-1)}}\\
&\le C\left(2+\left|u_{\xi_0,\rho}\right|+\left|(Xu)_{\xi_0,\rho}\right|\right)^{\frac{m(r-1)}{m-1}},
\end{aligned}
$$

where we utilize the fact $(2\theta)^{-Q}\Phi(\xi_0,\rho,(Xu)_{\xi_0,\rho})\le 1$ in the last inequality, implied by the assumption $\sigma_1^{-2}C_4\le 1$ with $C_4=\max\left\{C_0,(2\theta)^{-Q}\right\}$. Considering

$$
\frac{m(r-1)}{m-1}+\frac{m^2(r-1)}{r(m-1)}\le\frac{2m(r-1)}{m-1},
$$

we have

$$
J_4''\le C\omega_n(2\theta\rho)^Q F^{\frac{m}{r(m-1)}}\left(1+\left|u_{\xi_0,\rho}\right|,\left|(Xu)_{\xi_0,\rho}\right|\right)\eta\left((2\theta\rho)^{\frac{m}{m-1}}\right),
$$

where we have used $\frac{Qm(r-1)}{r(m-1)}\le Q+\frac{m}{m-1}$ and $(2\theta\rho)^{\frac{m}{m-1}}\le\eta\left((2\theta\rho)^{\frac{m}{m-1}}\right)$.

Combining J_1'',J_2'',J_3'',J_4'' with (4.16), we obtain

$$
\begin{aligned}
&\int_{B_{\theta\rho}(\xi_0)}\left|Xu-\left((Xu)_{\xi_0,\rho}+\sigma_1^{-1}(Xh)_{\xi_0,2\theta\rho}\right)\right|^2d\xi\\
&\quad+\left(1-(2C_p)^mC_c\varepsilon_12^Q\right)\int_{B_{\theta\rho}(\xi_0)}\left|Xu-\left((Xu)_{\xi_0,\rho}+\sigma_1^{-1}(Xh)_{\xi_0,2\theta\rho}\right)\right|^m d\xi\\
&\le C_cC_3^{\frac{m}{2}}\omega_n(2\theta\rho)^Q(C_1\sigma_1)^{-2}\left[\varepsilon^{\frac{-t}{1-t}}+1\right]\left[\theta^{-Q-2}\varepsilon+\theta^2\right]\\
&\quad+C_c\omega_n(2\theta\rho)^QF^2\left(\left|u_{\xi_0,\rho}\right|,1+\left|(Xu)_{\xi_0,\rho}\right|\right)\eta\left((2\theta\rho)^2\right)\\
&\quad+C\omega_n(2\theta\rho)^QF^{\frac{m}{r(m-1)}}\left(1+\left|u_{\xi_0,\rho}\right|,\left|(Xu)_{\xi_0,\rho}\right|\right)\eta\left((2\theta\rho)^{\frac{m}{m-1}}\right)\\
&:=C_5\omega_n(2\theta\rho)^Q\left[\theta^{-Q-2}\varepsilon+\theta^2\right](C_1\sigma_1)^{-2}\\
&\quad+C\omega_n(2\theta\rho)^QF^2\left(1+\left|u_{\xi_0,\rho}\right|,1+\left|(Xu)_{\xi_0,\rho}\right|\right)\eta\left(\rho^{\frac{m}{m-1}}\right),
\end{aligned}
$$

where $C_5=C_cC_3^{\frac{m}{2}}\left[\varepsilon^{\frac{-t}{1-t}}+1\right]>1$.

Selecting a suitable small $\varepsilon_1>0$ such that $1-(2C_p)^mC_c\varepsilon_12^Q>0$ and considering the smallness condition (4.10) implies

$$
\rho\le\rho_1\left(|u_{\xi_0,2\theta\rho}|,\left|(Xu)_{\xi_0,\rho}+\sigma_1^{-1}(Xh)_{\xi_0,2\theta\rho}\right|\right),
$$

as seen in (4.18) and (4.19), we conclude

$$\begin{aligned}
&\Phi\left(\xi_0,\theta\rho,(Xu)_{\xi_0,\theta\rho}\right)\\
=&\fint_{B_{\theta\rho}(\xi_0)}\left[\left|Xu-(Xu)_{\xi_0,\theta\rho}\right|^2+\left|Xu-(Xu)_{\xi_0,\theta\rho}\right|^m\right]d\xi\\
\le&\fint_{B_{\theta\rho}(\xi_0)}\left[\left|Xu-\left((Xu)_{\xi_0,\rho}+\sigma_1^{-1}(Xh)_{\xi_0,2\theta\rho}\right)\right|^2+\left|Xu-\left((Xu)_{\xi_0,\rho}+\sigma_1^{-1}(Xh)_{\xi_0,2\theta\rho}\right)\right|^m\right]d\xi\\
\le&C_5\frac{2^Q(\theta^{-Q-2}\varepsilon+\theta^2)}{\left(1-(2C_p)^mC_c\varepsilon_1 2^Q\right)}\Bigg[\left(\frac{\delta}{4}\right)^{-2}\Phi\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right)\\
&+4\delta^{-2}\eta\left(\rho^2\right)F^2\left(|u_{\xi_0,\rho}|,|(Xu)_{\xi_0,\rho}|\right)\Bigg]\\
&+CF^2\left(\left|u_{\xi_0,\rho}\right|,1+\left|(Xu)_{\xi_0,\rho}\right|\right)\eta\left(\rho^{\frac{m}{m-1}}\right)\\
\le&C_6\left[\theta^{-Q-2}\varepsilon+\theta^2\right]\left[\Phi\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right)+\left(\frac{\delta}{4}\right)^2 4\delta^{-2}\eta\left(\rho^2\right)F^2\left(|u_{\xi_0,\rho}|,|(Xu)_{\xi_0,\rho}|\right)\right]\\
&+C_6F^2\left(\left|u_{\xi_0,\rho}\right|,1+\left|(Xu)_{\xi_0,\rho}\right|\right)\eta\left(\rho^{\frac{m}{m-1}}\right),
\end{aligned}$$

where $C_6=\frac{C_5 2^Q}{\left[1-(2C_p)^mC_c\varepsilon_1 2^Q\right]}\left(\frac{\delta}{4}\right)^{-2}>1$.

Given $\tau\in[\gamma,1)$, we choose $\theta\in(0,\frac{1}{4}]$ sufficiently small to guarantee $2C_6\theta^2\le\theta^{2\tau}$, and set $\varepsilon=\theta^{Q+4}$. Consequently,

$$\begin{aligned}
&\Phi\left(\xi_0,\theta\rho,(Xu)_{\xi_0,\theta\rho}\right)\\
\le&\theta^{2\tau}\Phi\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right)+\left(\frac{1}{2}C_6\theta^2+C_6\right)F^2\left(1+\left|u_{\xi_0,\rho}\right|,1+\left|(Xu)_{\xi_0,\rho}\right|\right)\eta\left(\rho^{\frac{m}{m-1}}\right)\\
\le&\theta^{2\tau}\Phi\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right)+C_7F^2\left(1+\left|u_{\xi_0,\rho}\right|,1+\left|(Xu)_{\xi_0,\rho}\right|\right)\eta\left(\rho^{\frac{m}{m-1}}\right)\\
:=&\theta^{2\tau}\Phi\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right)+K^*\left(\left|u_{\xi_0,\rho}\right|,\left|(Xu)_{\xi_0,\rho}\right|\right)\eta\left(\rho^{\frac{m}{m-1}}\right),
\end{aligned}$$

where $C_7=\left(\frac{1}{2}C_6\theta^2+C_6\right)>1$ and $K^*(s,t)=C_7F^2(1+s,1+t)$. Thus we conclude Lemma 8. □

4.3 Iteration and Proof of Theorem 1

For $T>0$, there exists $\Phi_0(T)>0$ (depending on Q, N, λ, L, τ and ω) such that

$$\omega^{\frac{1}{m}}\left(2T,2\Phi_0(T)\right)+2\Phi_0^{\frac{1}{2}}(T)\le\frac{\delta}{4} \tag{4.20}$$

and

$$2\left(1+\sqrt{C_p}\right)\sqrt{\Phi_0(T)}\le\theta^{Q/2}\left(1-\theta^{\tau}\right)T, \tag{4.21}$$

with $\Phi_0(T)$ from (4.20) and (4.21), we choose $\rho_0(T)\in(0,1]$ (depending on Q, N, λ, L, τ, ω, η and κ) such that

$$\rho_0(T)\le\rho_1^{\frac{m}{2}}(1+2T,1+2T), \tag{4.22}$$

$$C_2F^2(2T,2T)\eta\left(\rho_0(T)^2\right) \le \delta^2, \tag{4.23}$$

$$K_0(T)\eta\left(\rho_0(T)^2\right) \le \left(\theta^{2\gamma} - \theta^{2\tau}\right)\Phi_0(T), \tag{4.24}$$

and

$$2(1+C_p)K_0(T)H\left(\rho_0(T)^2\right) \le \theta^Q(1-\theta^\gamma)^2\left(\theta^{2\gamma} - \theta^{2\tau}\right)T^2, \tag{4.25}$$

where $K_0(T) := K^*(2T,2T)$.

By applying the proof method of Lemma 5.1 in [22] and conditions (4.20)–(4.25), Lemma 9 can be proven. It suffices to complete the proof of Theorem 1 once we obtain Lemma 9.

Lemma 9 *We assert that for some $T_0 > 0$ and $B_\rho(\xi_0,\rho) \subset\subset \Omega$, we have*

(1) $|u_{\xi_0,\rho}| + |(Xu_{\xi_0,\rho})| \le T_0$;

(2) $\rho \le \rho_0(T_0)$;

(3) $\Phi\left(\xi_0,\rho,(Xu)_{\xi_0,\rho}\right) \le \Phi(T_0)$.

Then conditions (4.8)–(4.10) hold for the balls $B_{\theta^j\rho}(\xi_0)$ for $j \in \mathbb{N}\cup\{0\}$. Additionally, the existence of $\lim \Lambda_{\xi_0} = \lim_{j\to\infty}(Xu)_{\xi_0,\theta^j\rho}$ is guaranteed, and the estimate

$$⨍_{B_\rho(\xi_0)} |Xu - \Lambda_{\xi_0}|^2 d\xi \le C_8\left((r/\rho)^{2\tau}\,\Phi(\xi_0,\rho,(Xu)_{\xi_0,\rho}) + H\left(r^2\right)\right)$$

holds for $0 < r \le \rho$ with a constant $C_8 = C_8(Q,N,\lambda,L,\tau,T_0)$.

Proof The proof closely resembles that of Lemma 5.1 in [22]. We omit it here. □

Acknowledgements
The authors express their gratitude to the referees for their thorough review of the manuscript and for providing valuable suggestions.

Author contributions
The authors declare that they contribute to the paper equally, they all joined in the work of analysis, calculation and organizing the paper. All authors reviewed the manuscript.

Funding
The research is supported by the National Natural Science Foundation of China (No. 12061010), Natural Science Foundation of Jiangxi Province (No. S2024ZRZDL0326) and Science and Technology Planning Project of Jiangxi Province (No. GJJ2201204).

Data availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

Received: 8 June 2024 Accepted: 28 August 2024 Published online: 11 September 2024

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