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Existence and uniqueness for a mixed fractional differential system with slit-strips conditions

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Abstract

This paper studies the existence and uniqueness of solutions for a new kind of mixed fractional differential systems with slit-strips conditions, containing Caputo-type fractional derivatives. The first result on the existence and uniqueness is based on Banach's fixed point theorem, and the second result on the existence and uniqueness of the solution is proved by using a Schaefer-type fixed point theorem. The applicability of our primary results is finally illustrated by some examples.

Keywords: Mixed fractional differential; Slit-strips condition; Fixed point; Existence and uniqueness

1 Introduction

In recent years, fractional differential equations have received increasing attention and are suitable for many complex practical problem models. Compared with integer-order operators, fractional-order operators can provide more realistic and informative mathematical modeling for many real-world phenomena, as well as their applications in various disciplines of physics and technical science [1–6]. For example, in rheology, materials science, biophysics, blood flow phenomena, control theory, wave propagation, signal and image processing, permeation, identification and fitting of experimental data [7–9], etc.

In this field, nonlinear coupled fractional differential systems have also received widespread attention [10–17]. The study of the equations involves theoretical analysis and numerical solution methods [18]. To study the well-posedness, suitable boundary conditions are essential. Common boundary conditions may lead to the ill-posedness of the problem due to the global characteristic of the fractional derivative [19–23]. To overcome these difficulties Ahmad et al. [24, 25] proposed the concept of slit-strips condition, which was applied to strip-type detectors and acoustic imaging; the integral boundary condition describes the value of an unknown function at a nonlocal point in the aperture (i.e., the boundary region outside the strip) and a finite strip of any length occupying a position on the interval $[0, 1]$. Examples of such boundary conditions include scattering from narrow slits [26], silicon strip detectors for scanning multislit X-ray imaging, acoustic impedance

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of baffle heat sinks, diffraction of adjacent elastic blades, sound field of infinitely long strips, multiple dielectric welds on conductive planes, and thermal conduction in finite regions.

Ahmad et al. [27], investigated the following slit-strips problem:

$$\begin{cases} {}^{C}D^{p}u(t) = f_1(t,u(t)), n-1 < p \le n, t \in [0,1], \\ u(0) = 0,\ u'(0) = 0, u''(0) = 0, \ldots, u^{n-2}(0) = 0, \\ u(\xi) = a_1 \int_0^{\eta} u(s)ds + a_2 \int_{\xi_1}^{1} u(s)ds, 0 < \eta < \xi < \xi_1 < 1, \end{cases}$$

where ${}^{C}D^{p}$ denote the Caputo fractional derivative of order $p, f_1 : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and a_1, a_2 are real positive constants. Then in [28], Ahmad et al. studied a coupled system of nonlinear fractional differential equations

$$\begin{cases} {}^{C}D^{\gamma}[u(t) - h_1(t,u(t),v(t))] = \theta_1(t,u(t),v(t)), t \in [0,1], 1 < \gamma \le 2, \\ {}^{C}D^{\delta}[v(t) - h_2(t,u(t),v(t))] = \theta_2(t,u(t),v(t)), t \in [0,1], 1 < \delta \le 2, \\ u(0) = 0, u(\eta) = \omega_1 \int_0^{\xi_1} v(s)ds + \omega_2 \int_{\xi_2}^{1} v(s)ds, 0 < \xi_1 < \eta < \xi_2 < 1, \\ v(0) = 0, v(\eta) = \omega_1 \int_0^{\xi_1} u(s)ds + \omega_2 \int_{\xi_2}^{1} u(s)ds, 0 < \xi_1 < \eta < \xi_2 < 1, \end{cases}$$

where ${}^{C}D^{\gamma}$ and ${}^{C}D^{\delta}$ denote the Caputo fractional derivatives of orders γ and δ, respectively, $\theta_i, h_i : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions with $h_i(0,u(0),v(0)) = 0$, $i = 1,2$, and ω_1, ω_2 are real constants.

Motivated by the work presented in [28, 29], we considered the following coupled system of mixed fractional differential system containing Caputo fractional derivatives of different orders, supplemented with slit-strips-type integral boundary conditions:

$$\begin{cases} {}^{C}D^{\alpha}_{1-} \left\{ {}^{C}D^{\beta}_{0+}[u(t) - h_1(t,u(t),v(t))] \right\} = \theta_1(t,u(t),v(t)), 0 < \beta \le 1, 1 < \alpha \le 2, \\ {}^{C}D^{p}_{1-} \left\{ {}^{C}D^{q}_{0+}[v(t) - h_2(t,v(t),u(t))] \right\} = \theta_2(t,v(t),u(t)), 0 < q \le 1, 1 < p \le 2, \\ u(0) = u(1) = 0, u(\eta) = \omega_1 \int_0^{\xi_1} v(s)ds + \omega_2 \int_{\xi_2}^{1} v(s)ds, 0 < \xi_1 < \eta < \xi_2 < 1, \\ v(0) = v(1) = 0, v(\eta) = \omega_1 \int_0^{\xi_1} u(s)ds + \omega_2 \int_{\xi_2}^{1} u(s)ds, 0 < \xi_1 < \eta < \xi_2 < 1, \end{cases} \tag{1}$$

where ${}^{C}D^{\alpha}$, ${}^{C}D^{\beta}$, ${}^{C}D^{p}$, ${}^{C}D^{q}$ denote the Caputo fractional derivative of order α, β, p, q, respectively, $\theta_1, \theta_2, h_1, h_2 : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions with $h_1(0,u(0),v(0)) = 0$, $h_2(0,v(0),u(0)) = 0$, $t \in [0,1]$, and ω_1, ω_2 are real positive constants.

The rest of the paper is organized as follows. The definitions and an auxiliary result are presented in Sect. 2. The major results for system (1) are proved in Sect. 3. The examples are presented in Sect. 4 to verify the conclusions.

2 Preliminaries

For convenience, we give a few relevant definitions [30] and a lemma.

Definition 1 The left and right Riemann–Liouville fractional integrals of order σ for a continuous function g are respectively defined as

$$I_{0+}^{\sigma}g(t)=\frac{1}{\Gamma(\sigma)}\int_0^t\frac{g(s)}{(t-s)^{1-\sigma}}ds,\quad I_{1-}^{\sigma}g(t)=\frac{1}{\Gamma(\sigma)}\int_t^1\frac{g(s)}{(s-t)^{1-\sigma}}ds,$$

where $\sigma>0$, $\Gamma(\sigma)$ is the gamma function, provided that the right-hand side is pointwise defined on $\mathbb{R}^+$.

Definition 2 The left and the right Caputo fractional derivatives of order σ of a function g are, respectively,

$${}^{C}D_{0+}^{\sigma}g(t)=\frac{1}{\Gamma(n-\sigma)}\int_0^t\frac{g^{(n)}(s)}{(t-s)^{\sigma+1-n}}ds=I_{0+}^{n-\sigma}g^{(n)}(t),\quad {}^{C}D_{1-}^{\sigma}g(t)=(-1)^nI_{1-}^{n-\sigma}g^{(n)}(t),$$

where $n=[\sigma]+1$, $t>0$, $n-1<\sigma<n$, $[\sigma]$ denotes the integer part of a real number σ.

Definition 3 For $\sigma>0$, let $g,{}^{C}D_{0+}^{\sigma}g(t),{}^{C}D_{1-}^{\sigma}g(t)\in L^1[0,1]$. Then

$$\begin{aligned}&I_{0+}^{\sigma}\,{}^{C}D_{0+}^{\sigma}g(t)=g(t)+c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1},\\&I_{1-}^{\sigma}\,{}^{C}D_{1-}^{\sigma}g(t)=g(t)+\dot{c}_0+\dot{c}_1(1-t)+\dot{c}_2(1-t)^2+\cdots+\dot{c}_{n-1}(1-t)^{n-1},\end{aligned}$$

where $c_i,\dot{c}_i\in\mathbb{R}$, $i=0,1,\dots,n-1$ ($n=[\sigma]+1$).

Lemma 1 *Let $H_i,\Theta_i\in C([0,1],\mathbb{R})$ and $H_i(0)=0$, $i=1,2$. Then the solution of the nonlinear system*

$$\begin{cases}{}^{C}D_{1-}^{\alpha}\left\{{}^{C}D_{0+}^{\beta}\left[u(t)-H_1(t)\right]\right\}=\Theta_1(t),\ t\in[0,1],\ 0<\beta\le1,1<\alpha\le2,\\{}^{C}D_{1-}^{p}\left\{{}^{C}D_{0+}^{q}\left[v(t)-H_2(t)\right]\right\}=\Theta_2(t),\ t\in[0,1],0<q\le1,\ 1<p\le2,\\u(0)=u(1)=0,\ u(\eta)=\omega_1\int_0^{\xi_1}v(s)ds+\omega_2\int_{\xi_2}^1v(s)ds,\ 0<\xi_1<\eta<\xi_2<1,\\v(0)=v(1)=0,\ v(\eta)=\omega_1\int_0^{\xi_1}u(s)ds+\omega_2\int_{\xi_2}^1u(s)ds,\ 0<\xi_1<\eta<\xi_2<1,\end{cases}\tag{2}$$

is given by

$$\begin{aligned}u(t)=&J_1(t)+g_1(t)\kappa_1J_1(1)\\&+g_2(t)\left\{-\kappa_2J_2(1)+\kappa_3\left[-J_1(\eta)+\omega_1\int_0^{\xi_1}J_2(s)ds+\omega_2\int_{\xi_2}^1J_2(s)ds\right]\right.\\&\left.+\kappa_4\left[-J_2(\eta)+\omega_1\int_0^{\xi_1}J_1(s)ds+\omega_2\int_{\xi_2}^1J_1(s)ds\right]\right\}+g_3(t)\kappa_5J_1(1)\end{aligned}\tag{3}$$

and

$$\begin{aligned}v(t)=&J_2(t)+z_1(t)\gamma_1J_2(1)\\&+z_2(t)\left\{-\gamma_2J_1(1)+\gamma_3\left[-J_1(\eta)+\omega_1\int_0^{\xi_1}J_2(s)ds+\omega_2\int_{\xi_2}^1J_2(s)ds\right]\right.\\&\left.+\gamma_4\left[-J_2(\eta)+\omega_1\int_0^{\xi_1}J_1(s)ds+\omega_2\int_{\xi_2}^1J_1(s)ds\right]\right\}+z_3(t)\gamma_5J_2(1),\end{aligned}\tag{4}$$

where

$$J_1(t) = I_{0+}^{\beta} I_{1-}^{\alpha} \Theta_1(t) + H_1(t),\ J_2(t) = I_{0+}^{q} I_{1-}^{p} \Theta_2(t) + H_2(t),$$

$$\begin{cases} g_1(t) = \dfrac{t^{\beta}}{\Lambda\Gamma(\beta+1)},\ g_2(t) = \dfrac{t^{\beta}\left[\varepsilon_1 t - \varepsilon_2(\beta+1)\right]}{\Lambda\Gamma(\beta+2)},\ g_3(t) = \dfrac{-t^{\beta+1}}{\Lambda\Gamma(\beta+2)}, \\ z_1(t) = \dfrac{-t^{q}}{\Lambda\Gamma(q+1)},\ z_2(t) = \dfrac{t^{q}\left[\varepsilon_4(q+1) - \varepsilon_3 t\right]}{\Lambda\Gamma(q+2)},\ z_3(t) = \dfrac{t^{q+1}}{\Lambda\Gamma(q+2)}, \end{cases}$$

$$\begin{cases} \kappa_1 = \varepsilon_3\left(\varepsilon_6\varepsilon_{12} - \varepsilon_8\varepsilon_{10}\right) + \varepsilon_4\left(\varepsilon_7\varepsilon_{10} - \varepsilon_6\varepsilon_{11}\right),\ \kappa_2 = \varepsilon_7\varepsilon_{12} - \varepsilon_8\varepsilon_{11}, \\ \kappa_3 = \varepsilon_4\varepsilon_{11} - \varepsilon_3\varepsilon_{12},\ \kappa_4 = \varepsilon_3\varepsilon_8 - \varepsilon_4\varepsilon_7,\ \kappa_5 = \varepsilon_3\left(\varepsilon_5\varepsilon_{12} - \varepsilon_8\varepsilon_9\right) + \varepsilon_4\left(\varepsilon_7\varepsilon_9 - \varepsilon_5\varepsilon_{11}\right), \end{cases}$$

$$\begin{cases} \gamma_1 = \varepsilon_1\left(\varepsilon_8\varepsilon_{10} - \varepsilon_6\varepsilon_{12}\right) + \varepsilon_2\left(\varepsilon_5\varepsilon_{12} - \varepsilon_8\varepsilon_9\right),\ \gamma_2 = \varepsilon_5\varepsilon_{10} - \varepsilon_6\varepsilon_9,\ \gamma_3 = \varepsilon_2\varepsilon_9 - \varepsilon_1\varepsilon_{10}, \\ \gamma_4 = \varepsilon_1\varepsilon_6 - \varepsilon_2\varepsilon_5,\ \gamma_5 = \varepsilon_1\left(\varepsilon_7\varepsilon_{10} - \varepsilon_6\varepsilon_{11}\right) + \varepsilon_2\left(\varepsilon_5\varepsilon_{11} - \varepsilon_7\varepsilon_9\right), \end{cases}$$

and

$$\begin{cases} \varepsilon_1 = \dfrac{1}{\Gamma(\beta+1)},\ \varepsilon_2 = \dfrac{1}{\Gamma(\beta+2)},\ \varepsilon_3 = \dfrac{1}{\Gamma(q+1)},\ \varepsilon_4 = \dfrac{1}{\Gamma(q+2)}, \\ \varepsilon_5 = \dfrac{\eta^{\beta}}{\Gamma(\beta+1)},\ \varepsilon_6 = \dfrac{\eta^{\beta+1}}{\Gamma(\beta+2)},\ \varepsilon_7 = -\left\{\omega_1\dfrac{\xi_1^{q+1}}{\Gamma(q+2)} + \omega_2\dfrac{1-\xi_2^{q+1}}{\Gamma(q+2)}\right\}, \\ \varepsilon_8 = -\left\{\omega_1\dfrac{\xi_1^{q+2}}{\Gamma(q+3)} + \omega_2\dfrac{1-\xi_2^{q+2}}{\Gamma(q+3)}\right\},\ \varepsilon_9 = -\left\{\omega_1\dfrac{\xi_1^{\beta+1}}{\Gamma(\beta+2)} + \omega_2\dfrac{1-\xi_2^{\beta+1}}{\Gamma(\beta+2)}\right\}, \\ \varepsilon_{10} = -\left\{\omega_1\dfrac{\xi_1^{\beta+2}}{\Gamma(\beta+3)} + \omega_2\dfrac{1-\xi_2^{\beta+2}}{\Gamma(\beta+3)}\right\},\ \varepsilon_{11} = \dfrac{\eta^{q}}{\Gamma(q+1)},\ \varepsilon_{12} = \dfrac{\eta^{q+1}}{\Gamma(q+2)}, \end{cases}$$

with the following assumption:

$$\begin{aligned} \Lambda = {} & \varepsilon_1\left[\varepsilon_3\left(\varepsilon_8\varepsilon_{10} - \varepsilon_6\varepsilon_{12}\right) + \varepsilon_4\left(\varepsilon_6\varepsilon_{11} - \varepsilon_7\varepsilon_{10}\right)\right] \\ & + \varepsilon_2\left[\varepsilon_3\left(\varepsilon_5\varepsilon_{12} - \varepsilon_8\varepsilon_9\right) + \varepsilon_4\left(\varepsilon_7\varepsilon_9 - \varepsilon_5\varepsilon_{11}\right)\right] \neq 0. \end{aligned}$$

Proof From $\Theta_i \in C([0,1],\mathbb{R}^2)$, $i = 1,2$, we get

$$\begin{aligned} u(t) &= I_{0+}^{\beta}\left(I_{1-}^{\alpha}\Theta_1(t) + c_1 + c_2 t\right) + c_3 + H_1(t) \\ &= I_{0+}^{\beta} I_{1-}^{\alpha}\Theta_1(t) + \frac{t^{\beta}}{\Gamma(\beta+1)}c_1 + \frac{t^{\beta+1}}{\Gamma(\beta+2)}c_2 + c_3 + H_1(t), \end{aligned} \tag{5}$$

$$\begin{aligned} v(t) &= I_{0+}^{q}\left(I_{1-}^{p}\Theta_2(t) + c_4 + c_5 t\right) + c_6 + H_2(t) \\ &= I_{0+}^{q} I_{1-}^{p}\Theta_2(t) + \frac{t^{q}}{\Gamma(q+1)}c_4 + \frac{t^{q+1}}{\Gamma(q+2)}c_5 + c_6 + H_2(t). \end{aligned} \tag{6}$$

Using the conditions $u(0) = v(0) = 0$, we find that $c_3 = c_6 = 0$, and thus (5) and (6) take the form

$$u(t) = I_{0+}^{\beta} I_{1-}^{\alpha}\Theta_1(t) + \frac{t^{\beta}}{\Gamma(\beta+1)}c_1 + \frac{t^{\beta+1}}{\Gamma(\beta+2)}c_2 + H_1(t),$$

$$v(t) = I_{0+}^{\beta} I_{1-}^{\alpha}\Theta_2(t) + \frac{t^{q}}{\Gamma(q+1)}c_4 + \frac{t^{q+1}}{\Gamma(q+2)}c_5 + H_2(t).$$

Using the boundary conditions $u(1) = v(1) = 0$, we obtain

$$\varepsilon_1 c_1 + \varepsilon_2 c_2 = D_1, \tag{7}$$

$$\varepsilon_3 c_4 + \varepsilon_4 c_5 = D_2, \tag{8}$$

where $D_1 = -J_1(1)$, $D_2 = -J_2(1)$.

By the coupled slit-strips-type integral boundary conditions

$$u(\eta) = \omega_1 \int_0^{\xi_1} v(s)ds + \omega_2 \int_{\xi_2}^1 v(s)ds, v(\eta) = \omega_1 \int_0^{\xi_1} u(s)ds + \omega_2 \int_{\xi_2}^1 u(s)ds$$

we obtain

$$\begin{aligned}
&I_{0+}^{\beta} I_{1-}^{\alpha} \Theta_1(\eta) + \frac{\eta^{\beta}}{\Gamma(\beta+1)} c_1 + \frac{\eta^{\beta+1}}{\Gamma(\beta+2)} c_2 + H_1(\eta) \\
&= \omega_1 \int_0^{\xi_1} \left[I_{0+}^{q} I_{1-}^{p} \Theta_2(s) + \frac{s^{q}}{\Gamma(q+1)} c_4 + \frac{s^{q+1}}{\Gamma(q+2)} c_5 + H_2(s) \right] ds \\
&\quad + \omega_2 \int_{\xi_2}^1 \left[I_{0+}^{q} I_{1-}^{p} \Theta_2(s) + \frac{s^{q}}{\Gamma(q+1)} c_4 + \frac{s^{q+1}}{\Gamma(q+2)} c_5 + H_2(s) \right] ds, \\
&I_{0+}^{q} I_{1-}^{p} \Theta_2(\eta) + \frac{\eta^{q}}{\Gamma(q+1)} c_4 + \frac{\eta^{q+1}}{\Gamma(q+2)} c_5 + H_2(\eta) \\
&= \omega_1 \int_0^{\xi_1} \left[I_{0+}^{\beta} I_{1-}^{\alpha} \Theta_1(s) + \frac{s^{\beta}}{\Gamma(\beta+1)} c_1 + \frac{s^{\beta+1}}{\Gamma(\beta+2)} c_2 + H_1(s) \right] ds \\
&\quad + \omega_2 \int_{\xi_2}^1 \left[I_{0+}^{\beta} I_{1-}^{\alpha} \Theta_1(s) + \frac{s^{\beta}}{\Gamma(\beta+1)} c_1 + \frac{s^{\beta+1}}{\Gamma(\beta+2)} c_2 + H_1(s) \right] ds.
\end{aligned}$$

Thus we get

$$\varepsilon_5 c_1 + \varepsilon_6 c_2 + \varepsilon_7 c_4 + \varepsilon_8 c_5 = D_3, \tag{9}$$

$$\varepsilon_9 c_1 + \varepsilon_{10} c_2 + \varepsilon_{11} c_4 + \varepsilon_{12} c_5 = D_4, \tag{10}$$

where

$$\begin{aligned}
D_3 &= -J_1(\eta) + \omega_1 \int_0^{\xi_1} J_2(s)ds + \omega_2 \int_{\xi_2}^1 J_2(s)ds, \\
D_4 &= -J_2(\eta) + \omega_1 \int_0^{\xi_1} J_1(s)ds + \omega_2 \int_{\xi_2}^1 J_1(s)ds.
\end{aligned}$$

Solving systems (7), (8), (9), and (10) for c_1 and c_2, we get that

$$\begin{aligned}
c_1 &= \frac{-1}{\Lambda} \left[D_1 \kappa_1 + D_2 \varepsilon_2 \kappa_2 + D_3 \varepsilon_2 \kappa_3 + D_4 \varepsilon_2 \kappa_4 \right], \\
c_2 &= \frac{-1}{\Lambda} \left[D_1 \kappa_5 + D_2 \varepsilon_1 \kappa_2 + D_3 \varepsilon_1 \kappa_3 + D_4 \varepsilon_1 \kappa_4 \right], \\
c_4 &= \frac{-1}{\Lambda} \left[D_1 \varepsilon_4 \gamma_2 + D_2 \gamma_1 + D_3 \varepsilon_4 \gamma_3 + D_4 \varepsilon_4 \gamma_4 \right], \\
c_5 &= \frac{-1}{\Lambda} \left[D_1 \varepsilon_3 \gamma_2 + D_2 \gamma_5 + D_3 \varepsilon_3 \gamma_3 + D_4 \varepsilon_3 \gamma_4 \right],
\end{aligned}$$

where Λ is given by the assumption. Substituting the values of c_1, c_2, c_4, and c_5 together with the above notations, we get solution (3)–(4).

The proof is finished. □

3 Existence and uniqueness for mixed fractional differential system

Let $X = \{u(t) \mid u(t) \in C([0,1],\mathbb{R})\}$ be the space equipped with the norm $\|u\| = \sup\{|u(t)|, t \in [0,1]\}$. Then $(X, \|\cdot\|)$ is a Banach space. Then the product space $(X \times X, \|(u,v)\|)$ is also a Banach space equipped with the norm $\|(u,v)\| = \|u\| + \|v\|$.

In view of Lemma 1, we transform the results of system (1) into a fixed point problem. We define the operator $K : X \times X \to X \times X$ by

$$K(u,v)(t) = \begin{pmatrix} K_1(u,v)(t) \\ K_2(u,v)(t) \end{pmatrix},$$

where

$$\begin{aligned}
&K_1(u,v)(t)\\
&=\hat{J}_1(t,u(t),v(t)) + g_1(t)\kappa_1\hat{J}_1(1,u(1),v(1)) + g_2(t)\Big\{-\kappa_2\hat{J}_2(1,v(1),u(1))\\
&\quad+\kappa_3\Big[-\hat{J}_1(\eta,u(\eta),v(\eta)) + \omega_1\int_0^{\xi_1}\hat{J}_2(s,v(s),u(s))ds + \omega_2\int_{\xi_2}^1\hat{J}_2(s,v(s),u(s))ds\Big]\\
&\quad+\kappa_4\Big[-\hat{J}_2(\eta,v(\eta),u(\eta)) + \omega_1\int_0^{\xi_1}\hat{J}_1(s,u(s),v(s))ds + \omega_2\int_{\xi_2}^1\hat{J}_1(s,u(s),v(s))ds\Big]\Big\}\\
&\quad+g_3(t)\kappa_5\hat{J}_1(1,u(1),v(1)),\\
&K_2(u,v)(t)\\
&=\hat{J}_2(t,v(t),u(t)) + z_1(t)\gamma_1\hat{J}_2(1,v(1),u(1)) + z_2(t)\Big\{-\gamma_2\hat{J}_1(1,u(1),v(1))\\
&\quad+\gamma_3\Big[-\hat{J}_1(\eta,u(\eta),v(\eta)) + \omega_1\int_0^{\xi_1}\hat{J}_2(s,v(s),u(s))ds + \omega_2\int_{\xi_2}^1\hat{J}_2(s,v(s),u(s))ds\Big]\\
&\quad+\gamma_4\Big[-\hat{J}_2(\eta,v(\eta),u(\eta)) + \omega_1\int_0^{\xi_1}\hat{J}_1(s,u(s),v(s))ds + \omega_2\int_{\xi_2}^1\hat{J}_1(s,u(s),v(s))ds\Big]\Big\}\\
&\quad+z_3(t)\gamma_5\hat{J}_2(1,v(1),u(1)),
\end{aligned}$$

and

$$\begin{aligned}
\hat{J}_1(t,u(t),v(t)) &= I_{0+}^{\beta}I_{1-}^{\alpha}\theta_1(t,u(t),v(t)) + h_1(t,u(t),v(t)),\\
\hat{J}_2(t,v(t),u(t)) &= I_{0+}^{q}I_{1-}^{p}\theta_2(t,v(t),u(t)) + h_2(t,v(t),u(t)).
\end{aligned}$$

Note that

$$I_{0+}^{\beta}I_{1-}^{\alpha}(1) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(m-s)^{\alpha-1}}{\Gamma(\alpha)}dmds \le \frac{t^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)},$$

where we have used the fact that $(1-s)^{\alpha} \le 1$ for $1 < \alpha \le 2$.

For convenience, we introduce the notation

$$\begin{aligned}
&\mathrm{E}_1 = \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\mathrm{E}_3,\ \mathrm{E}_2 = \frac{1}{\Gamma(p+1)\Gamma(q+1)}\mathrm{E}_4,\\
&\mathrm{E}_3 = 1+\bar{g}_1|\kappa_1| + \bar{g}_2\left[|\kappa_3| + |\kappa_4|\,|\omega_1|\,\xi_1 + |\omega_2|\,(1-\xi_2)\right] + \bar{g}_3|\kappa_5|,\\
&\mathrm{E}_4 = \bar{g}_2\left[|\kappa_2| + |\kappa_3|\,|\omega_1|\,\xi_1 + |\kappa_3|\,|\omega_2|\,(1-\xi_2) + |\kappa_4|\right],\\
&\mathrm{E}_5 = \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\mathrm{E}_7,\ \mathrm{E}_6 = \frac{1}{\Gamma(p+1)\Gamma(q+1)}\mathrm{E}_8,\\
&\mathrm{E}_7 = \bar{z}_2\left\{|\gamma_2| + |\gamma_3| + |\gamma_4|\left[|\omega_1|\,\xi_1 + |\omega_2|\,(1-\xi_2)\right]\right\},\\
&\mathrm{E}_8 = 1+\bar{z}_1|\gamma_1| + \bar{z}_2\left\{|\gamma_3|\left[|\omega_1|\,\xi_1 + |\omega_2|\,(1-\xi_2)\right] + |\gamma_4|\right\} + \bar{z}_3|\gamma_5|,
\end{aligned}\tag{11}$$

where

$$\begin{aligned}
&\bar{g}_1 = \sup_{t\in[0,1]}\left|g_1(t)\right|,\ \bar{g}_2 = \sup_{t\in[0,1]}\left|g_2(t)\right|,\ \bar{g}_3 = \sup_{t\in[0,1]}\left|g_3(t)\right|,\\
&\bar{z}_1 = \sup_{t\in[0,1]}|z_1(t)|,\ \bar{z}_2 = \sup_{t\in[0,1]}|z_2(t)|,\ \bar{z}_3 = \sup_{t\in[0,1]}|z_3(t)|.
\end{aligned}$$

Now we are ready to present our main results, that is, we prove the existence and uniqueness of system (1) via the Banach contraction mapping principle.

Theorem 1 *Let* $\theta_1,\theta_2,h_1,h_2:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ *be continuous functions, and assume that the following conditions hold*:

$(A1)$ *There exist* $\Delta_1,\Delta_2>0$ *such that*

$$\begin{aligned}
&|\theta_1(t,x_1,y_1)-\theta_1(t,x_2,y_2)| \le \Delta_1(|x_1-x_2|+|y_1-y_2|),\\
&|\theta_2(t,x_1,y_1)-\theta_2(t,x_2,y_2)| \le \Delta_2(|x_1-x_2|+|y_1-y_2|)
\end{aligned}$$

for all $t\in[0,1]$ *and* $x_i,y_i\in\mathbb{R}$, $i=1,\ 2$;

$(A2)$ *There exist* $\Pi_1,\Pi_2>0$ *such that for all* $t\in[0,1]$ *and* $x_i,y_i\in\mathbb{R}$, $i=1,2$,

$$\begin{aligned}
&|h_1(t,x_1,y_1)-h_1(t,x_2,y_2)| \le \Pi_1(|x_1-x_2|+|y_1-y_2|),\\
&|h_2(t,x_1,y_1)-h_2(t,x_2,y_2)| \le \Pi_2(|x_1-x_2|+|y_1-y_2|).
\end{aligned}$$

$(A3)$ $\varkappa := \Delta_1(E_1+E_5)+\Delta_2(E_2+E_6)+\Pi_1(E_3+E_7)+\Pi_2(E_4+E_8)<1$.

Then the boundary value problem (1) *has a unique solution on* $[0,1]$.

Proof Let

$$r > \frac{(\mathrm{E}_1+\mathrm{E}_5)\,\varrho_1 + (\mathrm{E}_2+\mathrm{E}_6)\,\varrho_2 + (\mathrm{E}_3+\mathrm{E}_7)\,\varsigma_1 + (\mathrm{E}_4+\mathrm{E}_8)\,\varsigma_2}{1-\varkappa},$$

where ϱ_1, ϱ_2, ς_1, ς_2 are constants defined as

$$\begin{aligned}
&\varrho_1 = \sup_{t\in[0,1]}|\theta_1(t,0,0)|, \qquad \varrho_2 = \sup_{t\in[0,1]}|\theta_2(t,0,0)|,\\
&\varsigma_1 = \sup_{t\in[0,1]}|h_1(t,0,0)|, \qquad \varsigma_2 = \sup_{t\in[0,1]}|h_2(t,0,0)|.
\end{aligned}$$

Consider the closed ball $B_r = \{(u,v)\in X\times X : \|(u,v)\|\le r\}$.

Step 1. We first prove that $KB_r \subset B_r$. By assumption (A1) we get

$$\begin{aligned}|\theta_1(t,u,v)| &= |\theta_1(t,u,v) - \theta_1(t,0,0) + \theta_1(t,0,0)|\\ &\leq |\theta_1(t,u,v) - \theta_1(t,0,0)| + |\theta_1(t,0,0)|\\ &\leq \Delta_1(|x(t)| + |y(t)|) + \varrho_1\\ &\leq \Delta_1(\|x\| + \|y\|) + \varrho_1 \leq \Delta_1 r + \varrho_1.\end{aligned}$$

Similarly,

$$|\theta_2(t,u,v)| \leq \Delta_2 r + \varrho_2,\ |h_1(t,u,v)| \leq \Pi_1 r + \varsigma_1,\ |h_2(t,u,v)| \leq \Pi_2 r + \varsigma_2.$$

Using the above assumptions, we obtain

$$\begin{aligned}|K_1(u,v)| \leq \sup_{t\in[0,1]} \Bigg\{ & |\hat{J}_1(t,u(t),v(t))| + |g_1(t)||\kappa_1||\hat{J}_1(1,u(1),v(1))|\\ & + |g_2(t)|\Bigg\{|\kappa_2||\hat{J}_2(1,v(1),u(1))| + |\kappa_3|\Bigg[|\hat{J}_1(\eta,u(\eta),v(\eta))|\\ & + |\omega_1|\int_0^{\xi_1} |\hat{J}_2(s,v(s),u(s))|ds + |\omega_2|\int_{\xi_2}^1 |\hat{J}_2(s,v(s),u(s))|ds\Bigg]\\ & + |\kappa_4|\Bigg[|\hat{J}_2(\eta,v(\eta),u(\eta))| + |\omega_1|\int_0^{\xi_1} |\hat{J}_1(s,u(s),v(s))|ds\\ & + |\omega_2|\int_{\xi_2}^1 |\hat{J}_1(s,u(s),v(s))|ds\Bigg]\Bigg\} + |g_3(t)||\kappa_5||\hat{J}_1(1,u(1),v(1))|\Bigg\}\\ \leq \sup_{t\in[0,1]} \Bigg\{ & \Big[I_{0+}^{\beta}I_{1-}^{\alpha}(\Delta_1 r + \varrho_1) + (\Pi_1 r + \varsigma_1)\Big] + |g_1(t)||\kappa_1|\Big[I_{0+}^{\beta}I_{1-}^{\alpha}(\Delta_1 r\\ & + \varrho_1) + (\Pi_1 r + \varsigma_1)\Big] + |g_2(t)|\{|\kappa_2|\Big[I_{0+}^{q}I_{1-}^{p}(\Delta_2 r + \varrho_2) + (\Pi_2 r + \varsigma_2)\Big]\\ & + |\kappa_3|\Bigg[\Big[I_{0+}^{\beta}I_{1-}^{\alpha}(\Delta_1 r + \varrho_1) + (\Pi_1 r + \varsigma_1)\Big] + |\omega_1|\int_0^{\xi_1}\Big[I_{0+}^{q}I_{1-}^{p}(\Delta_2 r + \varrho_2)\\ & + (\Pi_2 r + \varsigma_2)\Big]ds + |\omega_2|\int_{\xi_2}^1 \Big[I_{0+}^{q}I_{1-}^{p}(\Delta_2 r + \varrho_2) + (\Pi_2 r + \varsigma_2)\Big]ds\Bigg]\\ & + |\kappa_4|\Bigg[\Big[I_{0+}^{q}I_{1-}^{p}(\Delta_2 r + \varrho_2) + (\Pi_2 r + \varsigma_2)\Big] + |\omega_1|\int_0^{\xi_1}\Big[I_{0+}^{\beta}I_{1-}^{\alpha}(\Delta_1 r + \varrho_1)\\ & + (\Pi_1 r + \varsigma_1)\Big]ds + |\omega_2|\int_{\xi_2}^1 \Big[I_{0+}^{\beta}I_{1-}^{\alpha}(\Delta_1 r + \varrho_1) + (\Pi_1 r + \varsigma_1)\Big]ds\Bigg]\Bigg\}\\ & + |g_3(t)||\kappa_5|\Big[I_{0+}^{\beta}I_{1-}^{\alpha}(\Delta_1 r + \varrho_1) + (\Pi_1 r + \varsigma_1)\Big]\Bigg\}.\end{aligned}$$

Straightforward calculation gives

$$\begin{aligned}|K_1(u,v)| \leq & \Big[(\Delta_1 r + \varrho_1)\frac{1}{\Gamma(\alpha+1)\Gamma(\beta+1)} + (\Pi_1 r + \varsigma_1)\Big]\Bigg\{1 + \bar{g}_1|\kappa_1|\\ & + \bar{g}_2\Big[|\kappa_3| + |\kappa_4||\omega_1|\int_0^{\xi_1} 1ds + |\omega_2|\int_{\xi_2}^1 1ds\Big] + \bar{g}_3|\kappa_5|\Bigg\}\end{aligned}$$

$$
\begin{aligned}
&+\left[(\Delta_2 r+\varrho_2)\frac{1}{\Gamma(p+1)\Gamma(q+1)}+(\Pi_2 r+\varsigma_2)\right]\left\{\bar{g}_2\left[|\kappa_2|\right.\right.\\
&\left.\left.+|\kappa_3||\omega_1|\int_0^{\xi_1}1ds+|\kappa_3||\omega_2|\int_{\xi_2}^1 1ds+|\kappa_4|\right]\right\}\\
\le\;&(\Delta_1 r+\varrho_1)\mathrm{E}_1+(\Delta_2 r+\varrho_2)\mathrm{E}_2+(\Pi_1 r+\varsigma_1)\mathrm{E}_3+(\Pi_2 r+\varsigma_2)\mathrm{E}_4,
\end{aligned}
$$

so we get

$$\|K_1(u,v)\|\le(\Delta_1\mathrm{E}_1+\Delta_2\mathrm{E}_2+\Pi_1\mathrm{E}_3+\Pi_2\mathrm{E}_4)\,r+\varrho_1\mathrm{E}_1+\varrho_2\mathrm{E}_2+\varsigma_1\mathrm{E}_3+\varsigma_2\mathrm{E}_4.$$

Analogously, we find that

$$\|K_2(u,v)\|\le(\Delta_1\mathrm{E}_5+\Delta_2\mathrm{E}_6+\Pi_1\mathrm{E}_7+\Pi_2\mathrm{E}_8)\,r+\varrho_1\mathrm{E}_5+\varrho_2\mathrm{E}_6+\varsigma_1\mathrm{E}_7+\varsigma_2\mathrm{E}_8.$$

From the foregoing estimates for K_1 and K_2 we obtain

$$\|K(u,v)\|\le\varkappa r+(\mathrm{E}_1+\mathrm{E}_5)\,\varrho_1+(\mathrm{E}_2+\mathrm{E}_6)\,\varrho_2+(\mathrm{E}_3+\mathrm{E}_7)\,\varsigma_1+(\mathrm{E}_4+\mathrm{E}_8)\,\varsigma_2<r$$

for $(u,v)\in B_r$, $K(u,v)\in B_r$. Then $K(u,v)\subset B_r$.

Step 2. We show that the operator K is compact.

Let $t\in[0,1]$, $(u',v'),(u'',v'')\in X\times X$. By (A1) and (A2) it follows that

$$
\begin{aligned}
&\left\|K_1\left(u'',v''\right)-K_1\left(u',v'\right)\right\|\\
&\le\left(\left\|u''-u'\right\|+\left\|v''-v'\right\|\right)\sup_{t\in[0,1]}\left\{\left(I_{0+}^{\beta}I_{1-}^{\alpha}\Delta_1+\Pi_1\right)+\left|g_1(t)\right|\right.\\
&\quad\cdot|\kappa_1|\left(I_{0+}^{\beta}I_{1-}^{\alpha}\Delta_1+\Pi_1\right)+\left|g_2(t)\right|\left\{|\kappa_2|\left(I_{0+}^{q}I_{1-}^{p}\Delta_2+\Pi_2\right)\right.\\
&\quad+|\kappa_3|\left[\left(I_{0+}^{\beta}I_{1-}^{\alpha}\Delta_1+\Pi_1\right)+|\omega_1|\int_0^{\xi_1}\left(I_{0_+}^{q}I_{1-}^{p}\Delta_2+\Pi_2\right)ds\right.\\
&\quad\left.+|\omega_2|\int_{\xi_2}^{1}\left(I_{0_+}^{q}I_{1-}^{p}\Delta_2+\Pi_2\right)ds\right]+|\kappa_4|\left[\left(I_{0+}^{q}I_{1-}^{p}\Delta_2+\Pi_2\right)\right.\\
&\quad+|\omega_1|\int_0^{\xi_1}\left(I_{0+}^{\beta}I_{1-}^{\alpha}\Delta_1+\Pi_1\right)ds+|\omega_2|\int_{\xi_2}^{1}(I_{0+}^{\beta}I_{1-}^{\alpha}\Delta_1\\
&\quad\left.\left.\left.+\Pi_1)ds\right]\right\}+\left|g_3(t)\right||\kappa_5|\left(I_{0+}^{\beta}\alpha_{1-}^{\alpha}\Delta_1+\Pi_1\right)\right\}\\
&\le(I_{0+}^{\beta}I_{1-}^{\alpha}\Delta_1+\Pi_1)\left(\left\|u''-u'\right\|+\left\|v''-v'\right\|\right)\sup_{t\in[0,1]}\left\{1\right.\\
&\quad+\left|g_1(t)\right||\kappa_1|+\left|g_2(t)\right|\left[|\kappa_3|+|\kappa_4|\,|\omega_1|\int_0^{\xi_1}1ds+|\omega_2|\int_{\xi_2}^{1}1ds\right]\\
&\quad\left.+\left|g_3(t)\right||\kappa_5|\right\}+(I_{0+}^{q}I_{1-1}^{p}\Delta_2+\Pi_2)(\left\|u''-u'\right\|\\
&\quad+\left\|v''-v'\right\|)\sup_{t\in[0,1]}\left\{\left|g_2(t)\right|\{|\kappa_2|\right.
\end{aligned}
$$

$$+ |\kappa_3| \left[|\omega_1| \int_0^{\xi_1} 1ds + |\omega_2| \int_{\xi_2}^1 1ds \right] + |\kappa_4| \Bigg\}$$
$$\leq (\Delta_1 E_1 + \Delta_2 E_2 + \Pi_1 E_3 + \Pi_2 E_4) \left(\|u'' - u'\| + \|v'' - v'\| \right),$$

which implies that

$$\left\| K_1 \left(u'', v''\right) - K_1 \left(u', v'\right) \right\| \leq (\Delta_1 E_1 + \Delta_2 E_2 + \Pi_1 E_3 + \Pi_2 E_4) \left(\|u'' - u'\| + \|v'' - v'\| \right).$$

Likewise, we have

$$\left\| K_2 \left(u'', v''\right) - K_2 \left(u', v'\right) \right\| \leq (\Delta_1 E_5 + \Delta_2 E_6 + \Pi_1 E_7 + \Pi_2 E_8) \left(\|u'' - u'\| + \|v'' - v'\| \right).$$

From these estimates we deduce that

$$\left\| K \left(u'', v''\right) - K \left(u', v'\right) \right\| \leq \varkappa \left(\|u'' - u'\| + \|v'' - v'\| \right),$$

which shows that K is a contraction by assumption (A3), and hence it has a unique fixed point by Banach's fixed point theorem.

The proof is complete. □

Under relaxed conditions for $\theta_i, i = 1, 2$, and $h_i, i = 1, 2$, we can also prove the well-posedness of system (1). First, let us revisit Schaefer's fixed point theorem [31].

Lemma 2 (*Schaefer's fixed point theorem*). *Let X be a Banach space. Assume that $T : X \to X$ is a completely continuous operator and the set $V = \{u \in X | u = \nu Tu; 0 < \nu < 1\}$ is bounded. Then T has a fixed point in X.*

Now we prove the following result.

Theorem 2 *Let $\theta_1, \theta_2, h_1, h_2 : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous functions satisfying the condition*

(*A4*) *There exist real constants $b_j, d_j, e_j, n_j \geq 0$, $j = 0, 1, 2$, and $b_0, d_0, e_0, n_0 \neq 0$ such that for all $x_k \in \mathbb{R}$, $k = 1, 2$,*

$$|\theta_1(t, x_1, x_2)| \leq b_0 + b_1|x_1| + b_2|x_2|, \ |\theta_2(t, x_1, x_2)| \leq d_0 + d_1|x_1| + d_2|x_2|,$$
$$|h_1(t, x_1, x_2)| \leq e_0 + e_1|x_1| + e_2|x_2|, \ |h_2(t, x_1, x_2)| \leq n_0 + n_1|x_1| + n_2|x_2|.$$

Then system (1) *has at least one solution on* $[0, 1]$ *if*

$$(E_1 + E_5)b_1 + (E_2 + E_6)d_1 + (E_3 + E_7)e_1 + (E_4 + E_8)n_1 < 1$$

and

$$(E_1 + E_5)b_2 + (E_2 + E_6)d_2 + (E_3 + E_7)e_2 + (E_4 + E_8)n_2 < 1,$$

where $E_i, i = 1, 2, \ldots, 8$, are given by (11).

Proof Observe that the continuity of the functions θ_1, θ_2, h_1, h_2 implies that the operator K is continuous.

Step 1. We show that the operator K is uniformly bounded.

Let $Q \subset X \times X$ be a bounded set. Then for all $(u,v) \in Q$, there exist constants $M_i > 0$, $i = 1,2,3,4$, such that

$$|\theta_1(t,u(t),v(t))| \le M_1,\ |\theta_2(t,v(t),u(t))| \le M_2,$$
$$|h_1(t,u(t),v(t))| \le M_3,\ |h_2(t,v(t),u(t))| \le M_4.$$

For any $(u,v) \in Q$, we have

$$\begin{aligned}
\|K_1(u,v)\| \le \sup_{t\in[0,1]} \Bigg\{ & \left(I_{0+}^{\beta} I_{1-}^{\alpha} M_1 + M_3\right) + \left|g_1(t)\right| |\kappa_1| \left(I_{0+}^{\beta} I_{1-}^{\alpha} M_1 + M_3\right) \\
& + |g_2(t)| \Bigg\{ |\kappa_2| (I_{0+}^{q} I_{1-}^{p} M_2 + M_4) + |\kappa_3| [(I_{0+}^{\beta} I_{1-}^{\alpha} M_1 + M_3) \\
& + |\omega_1| \int_0^{\xi_1} (I_{0+}^{q} I_{1-}^{p} M_2 + M_4) ds + |\omega_2| \int_{\xi_2}^{1} (I_{0+}^{q} I_{1-}^{p} M_2 + M_4) ds] \\
& + |\kappa_4| \Bigg[\left(I_{0+}^{q} I_{1-}^{p} M_2 + M_4\right) + |\omega_1| \int_0^{\xi_1} \left(I_{0+}^{\beta} I_{1-}^{\alpha} M_1 + M_3\right) ds \\
& + |\omega_2| \int_{\xi_2}^{1} (I_{0+}^{\beta} I_{1-}^{\alpha} M_1 + M_3) ds] \Bigg\} + |g_3(t)| |\kappa_5| (I_{0+}^{\beta} I_{1-}^{\alpha} M_1 + M_3) \Bigg\} \\
\le & (I_{0+}^{\beta} I_{1-}^{\alpha} M_1 + M_3) \sup_{t\in[0,1]} \Bigg\{ 1 + |g_1(t)| |\kappa_1| + |g_2(t)| [|\kappa_3| + |\kappa_4| |\omega_1| \int_0^{\xi_1} 1 ds \\
& + |\omega_2| \int_{\xi_2}^{1} 1 ds \Bigg] + \left|g_3(t)\right| |\kappa_5| \Bigg\} + \left(I_{0+}^{q} I_{1-}^{p} M_2 + M_4\right) \sup_{t\in[0,1]} \Bigg\{ \left|g_2(t)\right| \Bigg\{ |\kappa_2| \\
& + |\kappa_3| \Bigg[|\omega_1| \int_0^{\xi_1} 1 ds + |\omega_2| \int_{\xi_2}^{1} 1 ds \Bigg] + |\kappa_4| \Bigg\} \Bigg\} \\
\le & M_1 \mathrm{E}_1 + M_2 \mathrm{E}_2 + M_3 \mathrm{E}_3 + M_4 \mathrm{E}_4.
\end{aligned}$$

Analogously, we find that

$$\|K_2(u,v)\| \le M_1 \mathrm{E}_5 + M_2 \mathrm{E}_6 + M_3 \mathrm{E}_7 + M_4 \mathrm{E}_8.$$

From the foregoing inequalities it follows that

$$\|K(u,v)\| \le M_1 \left(\mathrm{E}_1 + \mathrm{E}_5\right) + M_2 \left(\mathrm{E}_2 + \mathrm{E}_6\right) + M_3 \left(\mathrm{E}_3 + \mathrm{E}_7\right) + M_4 \left(\mathrm{E}_4 + \mathrm{E}_8\right).$$

Thus the operator K is uniformly bounded.

Step 2. We show that K is equicontinuous.

For $0 < t_1 < t_2 < 1$, we have

$$
\begin{aligned}
&| K_1 (u (t_2), v (t_2)) - K_1 (u (t_1), v (t_1)) | \\
\leq & \int_0^{t_1} \frac{1}{\Gamma(\beta)}[(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}] I_{1-}^{\alpha} M_1 ds + \int_{t_1}^{t_2} \frac{(t_2 - s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} M_1 ds \\
&+ |h_1(u(t_2), v(t_2)) - h_1(u(t_1), v(t_1))| + |g_1(t_2) - g_1(t_1)||\kappa_1|(\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} M_1 ds \\
&+ M_3) + |g_2(t_2) - g_2(t_1)| \Bigg\{ |\kappa_2|(\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} I_{1-}^{p} M_2 ds + M_4) \\
&+ |\kappa_3| \Bigg[(\int_0^{\eta} \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} M_1 ds + M_3) + |\omega_1| \int_0^{\xi_1} (\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} I_{1-}^{p} M_2 d\tau + M_4) ds \\
&+ |\omega_2| \int_{\xi_2}^1 (\int_0^s \frac{(s-\tau)^{q-1}}{\Gamma(q)} I_{1-}^{p} M_2 d\tau + M_4) ds] + |\kappa_4| [(\int_0^{\eta} \frac{(\eta - s)^{q-1}}{\Gamma(q)} I_{1-}^{p} M_2 ds + M_4) \\
&+ |\omega_1| \int_0^{\xi_1} (\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} M_1 d\tau + M_3) ds + |\omega_2| \int_{\xi_2}^1 (\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} M_1 d\tau \\
&+ M_3) ds \Bigg] \Bigg\} + |g_3(t_2) - g_3(t_1)||\kappa_5|[\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} M_1 ds + M_3] \\
\leq & \frac{M_1}{\Gamma(\alpha+1)\Gamma(\beta+1)} \left[2(t_2 - t_1)^{\beta} + \left| {t_2}^{\beta} - t_1^{\beta} \right| \right] + |h_1(u(t_2), v(t_2)) - h_1(u(t_1), v(t_1))| \\
&+ \frac{\left| t_2^{\beta} - t_1^{\beta} \right|}{|\Lambda| \Gamma(\beta+1)} |\kappa_1| \left[\frac{M_1}{\Gamma(\alpha+1)\Gamma(\beta+1)} + M_3 \right] \\
&+ \frac{\varepsilon_1 \left| t_2^{\beta+1} - t_1^{\beta+1} \right| + \varepsilon_2(\beta+1) \left| t_2^{\beta} - t_1^{\beta} \right|}{|\Lambda| \Gamma(\beta+2)} \Bigg\{ |\kappa_2| \left[\frac{M_2}{\Gamma(p+1)\Gamma(q+1)} + M_4 \right] \\
&+ |\kappa_3| \Bigg[\frac{M_1}{\Gamma(\alpha+1)\Gamma(\beta+1)} + M_3 + |\omega_1| \int_0^{\xi_1} \left(\frac{M_2}{\Gamma(p+1)\Gamma(q+1)} + M_4 \right) ds \\
&+ |\omega_2| \int_{\xi_2}^1 \left(\frac{M_2}{\Gamma(p+1)\Gamma(q+1)} + M_4 \right) ds \Bigg] \\
&+ |\kappa_4| \Bigg[\frac{M_2}{\Gamma(p+1)\Gamma(q+1)} + M_4 + |\omega_1| \int_0^{\xi_1} \left(\frac{M_1}{\Gamma(\alpha+1)\Gamma(\beta+1)} + M_3 \right) ds \\
&+ |\omega_2| \int_{\xi_2}^1 \left(\frac{M_1}{\Gamma(\alpha+1)\Gamma(\beta+1)} + M_3 \right) ds \Bigg] \Bigg\} \\
&+ \frac{\left| t_2^{\beta+1} - t_1^{\beta+1} \right|}{|\Lambda| \Gamma(\beta+2)} |\kappa_5| \left[\frac{M_1}{\Gamma(\alpha+1)\Gamma(\beta+1)} + M_3 \right],
\end{aligned}
$$

from which it follows that $|K_1 (u (t_2), v (t_2)) - K_1 (u (t_1), v (t_1))| \to 0$ as $t_1 \to t_2$.

Analogously, we can obtain

$$
\begin{aligned}
&|K_2 (u (t_2), v (t_2)) - K_2 (u (t_1), v (t_1)) | \\
\leq & \sup_{t \in [0,1]} \left\{ \int_0^{t_1} \frac{1}{\Gamma(q)} \left[(t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right] I_{1-}^{p} M_2 ds \right.
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{t_1}^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}I_{1-}^{p}M_2ds+\left|h_2\left(v\left(t_2\right),u\left(t_2\right)\right)-h_2\left(v\left(t_1\right),u\left(t_1\right)\right)\right|\\
&+\left|z_1\left(t_2\right)-z_1\left(t_1\right)\right|\left|\gamma_1\right|\left(I_{0+}^{q}I_{1-}^{p}M_2+M_4\right)+\left|z_2\left(t_2\right)-z_2\left(t_1\right)\right|\Bigg\{\left|\gamma_2\right|(I_{0+}^{\beta}I_{1-}^{\alpha}M_1+M_3)\\
&+|\gamma_3|\left[I_{0+}^{\beta}I_{1-}^{\alpha}M_1+M_3+|\omega_1|\int_0^{\xi_1}\left(I_{0+}^{q}I_{1-}^{p}M_2+M_4\right)ds+|\omega_2|\int_{\xi_2}^{1}\left(I_{0+}^{q}I_{1-}^{p}M_2+M_4\right)ds\right]\\
&+|\gamma_4|\left[I_{0+}^{q}I_{1-}^{p}M_2+M_4+|\omega_1|\int_0^{\xi_1}(I_{0+}^{\beta}I_{1-}^{\alpha}M_1+M_3)ds+|\omega_2|\int_{\xi_2}^{1}(I_{0+}^{\beta}I_{1-}^{\alpha}M_1+M_3)ds\right]\Bigg\}\\
&+\left|z_3\left(t_2\right)-z_3\left(t_1\right)\right|\left|\gamma_5\right|\left(I_{0+}^{q}I_{1-}^{p}M_2+M_4\right)\Bigg\}\\
\leq&\frac{M_2}{\Gamma(p+1)\Gamma(q+1)}\left[2\left(t_2-t_1\right)^q+\left|t_2^q-t_1^q\right|\right]+\left|h_2\left(v\left(t_2\right),u\left(t_2\right)\right)-h_2\left(v\left(t_1\right),u\left(t_1\right)\right)\right|\\
&+\frac{\left|t_2^q-t_1^q\right|}{|\Lambda|\,\Gamma(q+1)}|\gamma_1|\left(\frac{M_2}{\Gamma(p+1)\Gamma(q+1)}+M_4\right)+\frac{\varepsilon_4(q+1)\left|t_2^q-t_1^q\right|+\varepsilon_3\left|t_2^{q+1}-t_1^{q+1}\right|}{|\Lambda|\,\Gamma(q+2)}\cdot\\
&\Bigg\{|\gamma_2|\left(\frac{M_1}{\Gamma(\alpha+1)\Gamma(\beta+1)}+M_3\right)+|\gamma_3|\Bigg[\left(\frac{M_1}{\Gamma(\alpha+1)\Gamma(\beta+1)}+M_3\right)\\
&+|\omega_1|\int_0^{\xi_1}\left(\frac{M_2}{\Gamma(p+1)\Gamma(q+1)}+M_4\right)ds+|\omega_2|\int_{\xi_2}^{1}\left(\frac{M_2}{\Gamma(p+1)\Gamma(q+1)}+M_4\right)ds\Bigg]\\
&+|\gamma_4|\Bigg[\left(\frac{M_1}{\Gamma(p+1)\Gamma(q+1)}+M_4\right)+|\omega_1|\int_0^{\xi_1}\left(\frac{M_1}{\Gamma(\alpha+1)\Gamma(\beta+1)}+M_3\right)ds+|\omega_2|\cdot\\
&\int_{\xi_2}^{1}\left(\frac{M_1}{\Gamma(\alpha+1)\Gamma(\beta+1)}+M_3\right)ds\Bigg]\Bigg\}+\frac{\left|t_2^{q+1}-t_1^{q+1}\right|}{|\Lambda|\,\Gamma(q+2)}|\gamma_5|\left(\frac{M_2}{\Gamma(p+1)\Gamma(q+1)}+M_4\right),
\end{aligned}
$$

which tends to 0 as $t_1\to t_2$.

Thus the operator K is equicontinuous.

From the foregoing arguments we deduce that the operator $K(u,v)$ is completely continuous.

Step 3. Finally, we show that the set

$$V=\{(u,v)\in X\times X\mid (u,v)=\imath K(u,v),0<\imath<1\}$$

is bounded.

Let $(u,v)\in V$ be such that $(u,v)=\imath K(u,v)$, $\forall t\in[0,1]$. Then we have

$$u(t)=\imath K_1(u,v)(t),v(t)=\imath K_2(u,v)(t).$$

By condition (A4) we find that

$$
\begin{aligned}
|u(t)|\leq&\mathrm{E}_1\left(b_0+b_1|u|+b_2|v|\right)+\mathrm{E}_2\left(d_0+d_1|u|+d_2|v|\right)\\
&+\mathrm{E}_3\left(e_0+e_1|u|+e_2|v|\right)+\mathrm{E}_4\left(n_0+n_1|u|+n_2|v|\right)
\end{aligned}
$$

and

$$\begin{aligned}|v(t)| \leq & \mathrm{E}_5 (b_0 + b_1|u| + b_2|v|) + \mathrm{E}_6 (d_0 + d_1|u| + d_2|v|) \\ & + \mathrm{E}_7 (e_0 + e_1|u| + e_2|v|) + \mathrm{E}_8 (n_0 + n_1|u| + n_2|v|) .\end{aligned}$$

Hence we have

$$\begin{aligned}\|u\| \leq & \mathrm{E}_1 b_0 + \mathrm{E}_2 d_0 + \mathrm{E}_3 e_0 + \mathrm{E}_4 n_0 \\ & + (\mathrm{E}_1 b_1 + \mathrm{E}_2 d_1 + \mathrm{E}_3 e_1 + \mathrm{E}_4 n_1) \|u\| \\ & + (\mathrm{E}_1 b_2 + \mathrm{E}_2 d_2 + \mathrm{E}_3 e_2 + \mathrm{E}_4 n_2) \|v\|\end{aligned}$$

and

$$\begin{aligned}\|v\| \leq & \mathrm{E}_5 b_0 + \mathrm{E}_6 d_0 + \mathrm{E}_7 e_0 + \mathrm{E}_8 n_0 \\ & + (\mathrm{E}_5 b_1 + \mathrm{E}_6 d_1 + \mathrm{E}_7 e_1 + \mathrm{E}_8 n_1) \|u\| \\ & + (\mathrm{E}_5 b_2 + \mathrm{E}_6 d_2 + \mathrm{E}_7 e_2 + \mathrm{E}_8 n_2) \|v\|.\end{aligned}$$

Consequently, we get

$$\begin{aligned}\|(u,v)\| \leq & (\mathrm{E}_1 + \mathrm{E}_5) b_0 + (\mathrm{E}_2 + \mathrm{E}_6) d_0 + (\mathrm{E}_3 + \mathrm{E}_7) e_0 + (\mathrm{E}_4 + \mathrm{E}_8) n_0 \\ & + [(\mathrm{E}_1 + \mathrm{E}_5) b_1 + (\mathrm{E}_2 + \mathrm{E}_6) d_1 + (\mathrm{E}_3 + \mathrm{E}_7) e_1 + (\mathrm{E}_4 + \mathrm{E}_8) n_1] \|u\| \\ & + [(\mathrm{E}_1 + \mathrm{E}_5) b_2 + (\mathrm{E}_2 + \mathrm{E}_6) d_2 + (\mathrm{E}_3 + \mathrm{E}_7) e_2 + (\mathrm{E}_4 + \mathrm{E}_8) n_2] \|v\|,\end{aligned}$$

which leads to

$$\|(u,v)\| \leq \frac{(\mathrm{E}_1 + \mathrm{E}_5) b_0 + (\mathrm{E}_2 + \mathrm{E}_6) d_0 + (\mathrm{E}_3 + \mathrm{E}_7) e_0 + (\mathrm{E}_4 + \mathrm{E}_8) n_0}{W_0},$$

where

$$\begin{aligned}& W_0 = \min \{1 - [(\mathrm{E}_1 + \mathrm{E}_5) b_1 + (\mathrm{E}_2 + \mathrm{E}_6) d_1 + (\mathrm{E}_3 + \mathrm{E}_7) e_1 + (\mathrm{E}_4 + \mathrm{E}_8) n_1] , \\ & 1 - [(\mathrm{E}_1 + \mathrm{E}_5) b_2 + (\mathrm{E}_2 + \mathrm{E}_6) d_2 + (\mathrm{E}_3 + \mathrm{E}_7) e_2 + (\mathrm{E}_4 + \mathrm{E}_8) n_2]\} .\end{aligned}$$

Therefore the set V is bounded. Hence by Lemma 2 the operator K has at least one fixed point.

The theorem is proved. □

4 Examples

In this part, we give two examples of mixed fractional differential systems with slit-strips-type boundary conditions to illustrate the results in Sect. 3.

Specifically, the system under consideration is as follows:

$$\begin{cases} {}^{C}D_{1-}^{\frac{5}{4}}\left\{ {}^{C}D_{0+}^{\frac{1}{2}}[u(t)-h_1(t,u,v)]\right\} = \theta_1(t,u,v),\ t\in[0,1], \\ {}^{C}D_{1-}^{\frac{3}{2}}\left\{ {}^{C}D_{0+}^{\frac{1}{4}}[v(t)-h_2(t,u,v)]\right\} = \theta_2(t,u,v),\ t\in[0,1], \\ u(0)=u(1)=0,\ u(\frac{1}{2}) = \frac{1}{5}\int_0^{1/5} v(s)ds + \int_{4/5}^1 v(s)ds,\ h_1(0,u(0),v(0))=0, \\ v(0)=v(1)=0,\ v(\frac{1}{2}) = \frac{1}{5}\int_0^{1/5} u(s)ds + \int_{4/5}^1 u(s)ds,\ h_2(0,v(0),u(0))=0. \end{cases} \tag{12}$$

Here $\alpha = \dfrac{5}{4}$, $\beta = \frac{1}{2}$, $p = \frac{3}{2}$, $q = \frac{1}{4}$, $\omega_1 = \frac{1}{5}$, $\omega_2 = 1$, $\xi_1 = \frac{1}{5}$, $\eta = \frac{1}{2}$, and $\xi_2 = \dfrac{4}{5}$.
Moreover,

$$\bar{g}_1 \approx 3.0624013,\ \bar{g}_2 \approx 3.0624013\times 10^{-10},\ \bar{g}_3 \approx 2.0416009,$$
$$\bar{z}_1 \approx 2.9942355,\ \bar{z}_2 \approx 8.9827066\times 10^{-10},\ \bar{z}_3 \approx 2.3953884.$$

Using these values, we find that

$$|\Lambda| \approx 0.3684622,\ \mathrm{E}_1 \approx 1.9918377,\ \mathrm{E}_2 \approx 4.8970980\times 10^{-11},$$
$$\mathrm{E}_3 \approx 1.9999997,\ \mathrm{E}_4 \approx 5.9005988\times 10^{-11},\ \mathrm{E}_5 \approx 0.4167436,$$
$$\mathrm{E}_6 \approx 1.7295857,\ \mathrm{E}_7 \approx 0.4184514,\ \mathrm{E}_8 \approx 2.0840079.$$

Example 4.1 Let us take

$$h_1(t,u,v) = \frac{\sin t|u(t)|}{25(2+|u(t)|)},\ h_2(t,v,u)) = \frac{\sin t|v(t)|}{25(2+|v(t)|)},$$
$$\theta_1(t,u,v) = \frac{1}{56}u(t) + \frac{2}{7}\frac{v(t)}{1+v(t)} + \frac{5}{7},\ \theta_2(t,v,u) = \frac{1}{39}\frac{\cos u(t)}{1+|\cos u(t)|} + \frac{1}{28}\sin v(t) + \frac{3}{7}.$$

It is easy to verify that conditions (A1) and (A2) are satisfied with $\Delta_1 = \dfrac{2}{7}$, $\Delta_2 = \dfrac{1}{28}$, $\Pi_1 = \dfrac{1}{25}$, $\Pi_2 = \dfrac{1}{25}$. In consequence, we have $\varkappa \approx 0.930035377 < 1$, which shows that condition (A3) of Theorem 1 is satisfied. So it follows by Theorem 1 that problem (12)has a unique solution on $[0,1]$.

Example 4.2 We consider problem (12) with

$$\theta_1(t,u,v) = \frac{1}{2}\sin t + \frac{2}{39}\tan u(t) + \frac{2}{41}v(t),\ \theta_2(t,v,u) = \frac{2}{5}\sin t + \frac{1}{9}\sin u(t) + \frac{1}{17}v(t), \tag{13}$$

$$h_1(t,u,v) = \frac{1}{3}\cos t + \frac{1}{9}\sin u(t) + \frac{3}{28}v(t),\ h_2(t,v,u)) = \frac{t+1}{4} + \frac{1}{8}\tan u(t) + \frac{1}{9}v(t). \tag{14}$$

Observe that

$$|\theta_1(t,u,v)| \le b_0 + b_1|u| + b_2|v|,\ |\theta_2(t,v,u)| \le d_0 + d_1|u| + d_2|v|,$$
$$|h_1(t,u,v)| \le e_0 + e_1|u| + e_2|v|,\ |h_2(t,v,u)| \le n_0 + n_1|u| + n_2|v|,$$

with $b_0 = \dfrac{1}{2}$, $b_1 = \frac{2}{39}$, $b_2 = \frac{2}{41}$, $d_0 = \frac{2}{5}$, $d_1 = \frac{1}{9}$, $d_2 = \frac{1}{17}$, $e_0 = \frac{1}{3}$, $e_1 = \frac{1}{9}$, $e_2 = \frac{3}{28}$, $n_0 = \frac{1}{2}$, $n_1 = \frac{1}{8}$, $n_2 = \frac{1}{9}$. Furthermore,

$$(E_1 + E_5)b_1 + (E_2 + E_6)d_1 + (E_3 + E_7)e_1 + (E_4 + E_8)n_1 \approx 0.8449101 < 1,$$

$$(E_1 + E_5)b_2 + (E_2 + E_6)d_2 + (E_3 + E_7)e_2 + (E_4 + E_8)n_2 \approx 0.7099083 < 1.$$

Thus all the conditions of Theorem 2 are satisfied; and hence there exists at least one solution for problem (12) with $\theta_i(t,u,v)$ and $h_i(t,u,v)$, $i = 1,2$.

5 Conclusions

We give existence and uniqueness results for mixed fractional-order differential equation coupled systems with slit-strips conditions. We use the fixed point theorem provided by Banach and Schaefer to satisfy the criteria required. This model enriches the literature on system solutions of fractional differential equations with paired integral boundary conditions. We will embed the right end functions of the coupling equations into the coupled differential inclusion system with coupled slit-strips-type condition.

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

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References

1. Ahmad, B., Alnahdi, M., Ntouyas, S.K.: Existence results for a differential equation involving the right Caputo fractional derivative and mixed nonlinearities with nonlocal closed boundary conditions. Fractal Fract. **7**, 129 (2023)
2. Lachouri, A., Ardjouni, A., Djoudi, A.: Existence and Ulam stability results for fractional differential equations with mixed nonlocal conditions. Azerb. J. Math. **11**(2), 78–97 (2021)
3. Nyamoradi, N., Ntouyas, S.K., Tariboon, J.: Existence and uniqueness of solutions for fractional integro-differential equations involving the Hadamard derivatives. Mathematics **10**, 3068 (2022)
4. Ahmad, B., Broom, A., Alsaedi, A., Ntouyas, S.K.: Nonlinear integro-differential equations involving mixed right and left fractional derivatives and integrals with nonlocal boundary data. Mathematics **8**, 1–13 (2020)
5. Alsaedi, A., Broom, A., Ntouyas, S.K.: Nonlocal fractional boundary value problems involving mixed right and left fractional derivatives and integrals. Axioms **9**, 1–15 (2020)
6. Alsaedi, A., Ahmad, B., Alghamdi, B., Ntouyas, S.K.: On a nonlinear system of Riemann–Liouville fractional differential equations with semi-coupled integro-multipoint boundary conditions. Open Math. **19**, 760–772 (2021)
7. Sabatier, J., Agrawal, O.P., Machado, J.A.T.: Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering. Biochem. J. **361**, 97–103 (2007)
8. He, N., Wang, J., Zhang, L.: An improved fractional-order differentiation model for image denoising. Signal Process. **112**, 180–188 (2015)

9. Fallahgoul, H.A., Focardi, S.M., Fabozzi, F.J.: Fractional Calculus and Fractional Processes with Applications to Financial Economics. Theory and Application. Elsevier/Academic Press, London (2017)
10. Beddani, H., Beddani, M., Cattani, C., Cherif, M.H.: An existence study for a multi-plied system with p-Laplacian involving phi-Hilfer derivatives. Fractal Fract. **6**(6), 326 (2022)
11. Beddani, H., Beddani, M., Dahmani, Z.: An existence study for a multiple system with p-Laplacian involving ϕ-Caputo derivatives. Filomat **37**(6), 1879–1892 (2023)
12. Beddani, H., Beddani, M., Dahmani, Z.: An existence study for a tripled system with p-Laplacian involving ϕ-Caputo derivatives. Miskolc Math. Notes **24**(3), 1197–1212 (2023)
13. Samadi, A., Ntouyas, S.K., Ahmad, B., Tariboon, J.: Investigation of a nonlinear coupled Hilfer fractional differential system with coupled Riemann–Liouville fractional integral boundary conditions. Foundations **2**, 918–933 (2022)
14. Saifia, O., Boulfoul, A., Lachouri, A.: Existence results for nonlinear fractional differential system with boundary conditions. Math. Eng. Sci. Aerosp. **14**(2), 513–526 (2023)
15. Ahmad, B., Ntouyas, S.K.: A coupled system of nonlocal fractional differential equations with coupled and uncoupled slit-strips type integral boundary conditions. J. Math. Sci. **226**, 175–196 (2017)
16. Alghanmi, M., Agarwal, R.P., Ahmad, B.: Existence of solutions for a coupled system of nonlinear implicit differential equations involving ϱ-fractional derivative with anti periodic boundary conditions. Qual. Theory Dyn. Syst. **6**, 23 (2024)
17. Ahmad, B., Ntouyas, S.K., Alsaedi, A.: Fractional order differential systems involving right Caputo and left Riemann–Liouville fractional derivatives with nonlocal coupled conditions. Bound. Value Probl. **109**, 1–12 (2019)
18. Gu, S., Yang, B., Shao, W.: Existence and uniqueness of solution for a singular elliptic differential equation. Adv. Nonlinear Anal. **13**(1), 20230126 (2024)
19. Ntouyas, S.K., Broom, A., Alsaedi, A.: Existence results for a nonlocal coupled system of differential equations involving mixed right and left fractional derivatives and integrals. Symmetry **4**, 578 (2020)
20. Alsulami, H.H., Ntouyas, S.K., Agarwal, R.P.: A study of fractional-order coupled systems with a new concept of coupled non-separated boundary conditions. Bound. Value Probl. **68**, 1–11 (2017)
21. Ahmad, B., Ntouyas, S.K., Alsaedi, A.: Coupled systems of fractional differential inclusions with coupled boundary conditions. Electron. J. Differ. Equ. **2019**, 69 (2019)
22. Ahmad, B., Alghanmi, M., Alsaedi, A., Nieto, J.J.: Existence and uniqueness results for a nonlinear coupled system involving Caputo fractional derivatives with a new kind of coupled boundary conditions. Appl. Math. Lett. **116**, 107018 (2021)
23. Samadi, A., Ntouyas, S.K., Ahmad, B., Tariboon, J.: Investigation of a nonlinear coupled Hilfer fractional differential system with coupled Riemann–Liouville fractional integral boundary conditions. Foundations **2**, 918–933 (2022)
24. Lundqvist, M.: Silicon strip detectors for scanned multi-slit X-ray imaging. Doctoral dissertation, Fysik (2003)
25. Mellow, T., Karkkainen, L.: On the sound fields of infinitely long strips. J. Acoust. Soc. Am. **130**, 153–167 (2011)
26. Hurd, R.A., Hayashi, Y.: Low-frequency scattering by a slit in a conducting plane. Radio Sci. **15**, 1171–1178 (1980)
27. Ahmad, B., Agarwal, R.P.: Some new versions of fractional boundary value problems with slit-strips conditions. Bound. Value Probl. **1**, 1–12 (2014)
28. Ahmad, B., Karthikeyan, P., Buvaneswari, K.: Fractional differential equations with coupled slit-strips type integral boundary conditions. AIMS Math. **6**, 1596–1609 (2019)
29. Ahmad, B., Ntouyas, S.K.: Nonlocal fractional boundary value problems with slit-strips boundary conditions. Fract. Calc. Appl. Anal. **1**, 261–280 (2015)
30. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, pp. 69–96. Elsevier, Amsterdam (2006)
31. Smart, D.R.: Fixed Point Theorems. Cambridge University Press, Cambridge (1974)

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