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Analysis of Caputo fractional variable order multi-point initial value problems: existence, uniqueness, and stability

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Abstract

In this paper, we examine the existence, uniqueness, and stability of solutions for a Caputo variable order ϑ-initial value problem (ϑ-IVP) with multi-point initial conditions. The proofs for uniqueness and existence leverage Sadovski's and Banach's fixed point theorems, along with the Kuratowski measure of noncompactness. Furthermore, we explore the Ulam–Hyers–Rassias (UHR) stability of the solution. To validate our findings, we present a numerical example.

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Keywords: Caputo fractional derivative; Variable order; Multi-point boundary value problem; Kuratowski measure of noncompactness; Fixed point theorems

1 Introduction

Fractional differential equations have gained significant attention in recent years due to their ability to model complex phenomena in various fields such as physics, engineering, and finance. The introduction of variable-order fractional derivatives has further expanded the scope of these equations, allowing for more accurate descriptions of anomalous diffusion and other processes with memory effects. Recent studies have focused on various aspects of fractional differential equations, including their existence and uniqueness properties [1–5], stability [6–9], and numerical solutions [10, 11].

The concept of variable-order fractional derivatives allows for a more flexible and accurate description of dynamical processes. Samko [12] provides an overview of fractional integration and differentiation of variable order, highlighting their importance and applications. Sun et al. [13–15] discuss the role of variable-order fractional differential operators in anomalous diffusion modeling, emphasizing their utility in capturing the dynamics of various physical processes. Valerio and da Costa [16] explore numerical approximations for variable-order fractional derivatives, presenting methods to effectively handle these operators in practical computations.

In the study of boundary value problems (BVPs) for fractional differential equations, several researchers have made significant contributions to solution techniques. Rezapour et al. [17] investigated existence theory on cones via piecewise constant functions for fractional thermostat models. Refice et al. [18] addressed boundary value problems of Hadamard fractional differential equations using the Kuratowski measure of noncompactness. Feckan et al. [19] and Wang et al. [20] provided insights into the existence and uniqueness of solutions for impulsive fractional differential equations. Mahmudov and Unul [21] explored the existence of solutions in the impulsive case, while Benchohra and Seba [22] extended this work to Banach spaces. Benkerrouche et al. [23, 24] studied the existence and stability of Caputo variable-order boundary value problems. Additionally, Zhang et al. [25] and Odzijewicz et al. [26] focused on existence, stability, and approximate solutions for initial value problems with variable-order fractional derivatives. Wang et al. [27] also explored existence results for fractional differential equations with integral and multi-point boundary conditions.

In 2021, the authors [28] studied the existence and stability of the obtained solution in the sense of Ulam–Hyers (UH) for a boundary value problem. The authors explored the existence of a solution by transforming the boundary value problem (BVP) into an equivalent standard Caputo BVP of fractional constant order. This transformation utilizes the concepts of generalized intervals and piecewise constant functions. They established their results through the application of Darbo's fixed point theorem and the concept of (UH) stability. Finally, to demonstrate the efficacy of their findings, they provided a numerical example.

Wang et al. [29] in 2018 explored a fractional boundary value problem with multi-point boundary conditions: The researchers started with finding the integral equation equivalent to the problem and discussed the existence of multiple positive solutions for the BVP based on some properties of Green's function by using Krasnoselskii and Shaulder types fixed point theorems and Banach's contraction mapping principle and nonlinear alternative for single-valued maps.

Benkerrouche [30] in 2022 studied an impulsive model of initial value problem where they investigated the existence and uniqueness of its solutions. The authors proved the existence and uniqueness of the solution by using Shaulder's and Banach's principle fixed point theorems, they studied the stability of the solution by means of (UH) stability. Vlase et al. [31] introduced a method to study the vibration of mechanical bar systems with symmetries, offering an efficient approach for analyzing mechanical systems. Seema et al. [32] used the spatially variable quasi-classical technique to derive the analytical solution for Love-type wave transmission in a magnetoelectroelastic (MEE) cylindrical structure. Meanwhile, El-Atabany and Ashry [33] explored a difference equation model to represent the dynamics of infectious diseases, providing valuable insights into epidemiological modeling. These works exemplify the application of mathematical techniques in diverse fields, from mechanical engineering to public health.

Motivated by the above mentioned papers, we proposed the following impulsive ϑ-initial value problem of variable order with respect to the increasing function ϑ:

$$\begin{cases} {}^{c}\mathcal{D}_{0^+}^{r(\mathfrak{z}),\vartheta(\mathfrak{z})} z(\mathfrak{z}) = h\big(\mathfrak{z}, z(\mathfrak{z}), \mathcal{I}_{0^+}^{q;\vartheta} z(\mathfrak{z})\big), \quad \mathfrak{z} \in \mathcal{O} = [0,Z], \; \mathfrak{z} \neq Z_i, \; i = 1,\dots,n, \\ z(0) = z_0, \quad z'_{\vartheta}(0) = z''_{\vartheta}(0) = \cdots = z_{\vartheta}^{[m-1]}(0) = 0, \\ \Delta z\big|_{\mathfrak{z}=Z_i} = g_i\big(z(Z_i^-)\big), i = 1,\dots,n, \end{cases} \tag{1}$$

where

$m \geq 3$, $r : [0,Z] \to (m-1,m]$ is the variable order of the fractional derivative, $1 < Z < \infty$, $h : \mathcal{O} \times \mathbb{R}^2 \to \mathbb{R}$, $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1,\dots,n$, are continuous functions. ${}^{c}\mathcal{D}_{0^+}^{r(\mathfrak{z});\vartheta(\mathfrak{z})}$ denotes the ϑ-Caputo fractional derivative depending on an increasing function ϑ of variable order $r(\mathfrak{z})$ and $\mathcal{I}_{0^+}^{q;\vartheta}$ presents the ϑ-Riemann–Liouville fractional integral of constant order $q > 0$, $0 = Z_0 < Z_1 < \cdots < Z_{n+1} = Z$, $\Delta z\big|_{\mathfrak{z}=Z_i} = z\big(Z_i^+\big) - z\big(Z_i^-\big)$ where $z\big(Z_i^+\big) = \lim_{\epsilon\to 0^+} z\big(Z_i + \epsilon\big)$ and $z\big(Z_i^-\big) = \lim_{\epsilon\to 0^-} z\big(Z_i + \epsilon\big)$. They present the right and the left limits of $z(\mathfrak{z})$ a $\mathfrak{z} = Z_i$, $i = 1,\dots,n$, and $z_0 \in \mathbb{R}$.

$z_{\vartheta}^{[i]}$ denotes the derivative of order $i \in \mathbb{N}$ with respect to the function ϑ and it is defined as follows:

$$z_{\vartheta}^{[i]} = \left(\frac{1}{\vartheta'(\mathfrak{z})}\frac{d}{d\mathfrak{z}}\right)^{i} z(\mathfrak{z}). \tag{2}$$

For more convenience, we consider the set of functions $PC(\mathcal{O},\mathbb{R}) := \{z : \mathcal{O} \to \mathbb{R} \; ; \; z \in C\big((Z_i, Z_{i+1}];\mathbb{R}\big)$, there exist $z(Z_i^-)$, $z(Z_i^+)$ with $z(Z_i^-) = z(Z_i)$, $i = 1,\dots,n\}$, then $PC(\mathcal{O},\mathbb{R})$ is a Banach space with the sup-norm $\|z\| = \sup_{\mathfrak{z}\in\mathcal{O}} |z(\mathfrak{z})|$, and we denote by $PC^k(\mathcal{O},\mathbb{R})$ the space of functions where $PC^k(\mathcal{O},\mathbb{R}) := \{z \in PC(\mathcal{O},\mathbb{R}) \; ; \; z \in C^k\big((Z_i, Z_{i+1}];\mathbb{R}\big)$, there exist $z^{[k]}(Z_i^-)$, $z^{[k]}(Z_i^+)$ with $z^{[k]}(Z_i^-) = z^{[k]}(Z_i)$, $i = 1,\dots,n\}$ and $k \in \mathbb{N}^*$.

2 Fundamental concepts and essential methods

Firstly, we mention some definitions and properties.

Definition 1 ([34]) Consider the interval $I = (a,b)$, which can be either finite or infinite on the positive half-axis $\mathbb{R}^+$. Let $\omega(\mathfrak{z})$ be a function that is increasing, positive, and monotone on $(a,b]$ with a continuous derivative $\omega'(\mathfrak{z})$ on (a,b). The ω-Riemann–Liouville fractional integral of order $\alpha > 0$ for a function f that depends on ω over I is defined as

$$\mathcal{I}_{a^+}^{\alpha,\omega(\mathfrak{z})} f(\mathfrak{z}) = \frac{1}{\Gamma(\alpha)}\int_a^{\mathfrak{z}} \omega'(s)\,(\omega(\mathfrak{z}) - \omega(s))^{\alpha-1} f(s)\,ds, \quad \mathfrak{z} > a > 0.$$

Definition 2 ([30, 35]) Let $\alpha > 0$, $\mathfrak{n} \in \mathbb{N}$, $I = [a,b]$ be an interval with $-\infty \leq a < b \leq \infty$, $f, \vartheta \in C^{\mathfrak{n}}(I)$ two functions such that ϑ is increasing and $\vartheta'(\mathfrak{z}) \neq 0$ for all $\mathfrak{z} \in I$. The left ϑ-Caputo fractional derivative of f of the order α is given by

$$\begin{aligned} {}^{c}\mathcal{D}_{a^+}^{\alpha,\vartheta(\mathfrak{z})} f(\mathfrak{z}) &= \mathcal{I}_{a^+}^{\mathfrak{n}-\alpha;\vartheta}\left(\frac{1}{\vartheta'(\mathfrak{z})}\frac{d}{d\mathfrak{z}}\right)^{\mathfrak{n}} f(\mathfrak{z}) \\ &= \frac{1}{\Gamma(\mathfrak{n}-\alpha)}\int_a^{\mathfrak{z}} \vartheta'(s)\big(\vartheta(\mathfrak{z}) - \vartheta(s)\big)^{\mathfrak{n}-\alpha-1}\left(\frac{1}{\vartheta'(s)}\frac{d}{ds}\right)^{\mathfrak{n}} f(s)ds \end{aligned}$$

$$= \frac{1}{\Gamma(\mathfrak{n}-\alpha)} \int_a^{\mathfrak{z}} \vartheta'(s)\big(\vartheta(\mathfrak{z})-\vartheta(s)\big)^{\mathfrak{n}-\alpha-1} f_{\vartheta}^{[\mathfrak{n}]}(s)ds,$$

where $\mathfrak{n} = [\alpha] + 1$

Lemma 3 ([1, 30]) *For $f : [a,b] \to \mathbb{R}$ and $\varrho > \nu > 0$, the next property of semigroup is verified*

$$\mathcal{I}_{a^+}^{\varrho+\nu;\vartheta} f(\mathfrak{z}) = \mathcal{I}_{a^+}^{\varrho;\vartheta}\mathcal{I}_{a^+}^{\nu;\vartheta} f(\mathfrak{z}).$$

Proposition 4 ([35]) *Let $\alpha > 0, f \in C^m[a,b]$, and ${}^c\mathcal{D}_{a^+}^{\alpha;\vartheta} f \in L^1[a,b]$. Then we have*

$$\mathcal{I}_{a^+}^{\alpha;\vartheta}\,{}^c\mathcal{D}_{a^+}^{\alpha;\vartheta} f(\mathfrak{z}) = f(\mathfrak{z}) - \sum_{k=0}^{m-1} c_k\big(\vartheta(\mathfrak{z})-\vartheta(a)\big)^k, \tag{3}$$

where $m = [\alpha] + 1$ and $c_k \in \mathbb{R}$, $k = 0, \dots, m-1$.

Lemma 5 ([35]) *For $\gamma > 0, f \in L^1[a,b], p < \gamma, p \in \mathbb{N}^*$, we have*

$${}^c\mathcal{D}_{a^+}^{p;\vartheta}\mathcal{I}_{a^+}^{\gamma;\vartheta} f(\mathfrak{z}) = \mathcal{I}_{a^+}^{\gamma-p;\vartheta} f(\mathfrak{z}), \tag{4}$$

and

$${}^c\mathcal{D}_{a^+}^{p;\vartheta}\big(\vartheta(\mathfrak{z})-\vartheta(a)\big)^k = \frac{\Gamma(k+1)}{\Gamma(k-p+1)}\big(\vartheta(\mathfrak{z})-\vartheta(a)\big)^{k-p}, \tag{5}$$

where $p < k$ is a positive integer number.

Lemma 6 [35] *For any $\alpha > 0$, $a, b > 0$, and $f \in C^1[a,b]$, we have*

$${}^c\mathcal{D}_{a^+}^{\alpha;\vartheta}\mathcal{I}_{a^+}^{\alpha;\vartheta} f(\mathfrak{z}) = f(\mathfrak{z}) \tag{6}$$

and

$${}^c\mathcal{D}_{a^+}^{\alpha;\vartheta} C = 0 \quad \text{for all} \quad C \in \mathbb{R}. \tag{7}$$

Now, we define the Kuratowski measure of noncompactness (KMNC) and provide its essential properties.

Definition 7 ([36, 37]) Let X be a Banach space and let E_X denote the bounded subset of X.

The (KMNC) is a mapping $\varsigma : E_X \to [0,\infty]$ defined as follows:

$$\varsigma(M) = \inf\{\epsilon > 0, M(\in E_X) \subseteq \cup_{i=1}^n M_i, diam(M_i) \le \epsilon\}, \tag{8}$$

where

$$diam(M_i) = \sup\{\|x-y\| : x, y \in M_i\}. \tag{9}$$

Proposition 8 ([28, 38]) *Let X be a Banach space; M, M_1, M_2 are bounded subsets of X, then*

1) $\varsigma(M) = 0 \Leftrightarrow M$ *is relatively compact,*
2) $\varsigma(\emptyset) = 0$,
3) $\varsigma(M) = \varsigma(\bar{M}) = \varsigma(conv(M))$, $conv(M)$ *denotes the convex hull of M,*
4) $M_1 \subset M_2 \Rightarrow \varsigma(M_1) \leq \varsigma(M_2)$,
5) $\varsigma(M_1 + M_2) \leq \varsigma(M_1) + \varsigma(M_2)$,
6) $\varsigma(\lambda M) = |\lambda|\varsigma(M)$, $\lambda \in \mathbb{R}$,
7) $\varsigma(M_1 \cup M_2) = \max\{\varsigma(M_1), \varsigma(M_2)\}$,
8) $\varsigma(M_1 \cap M_2) = \min\{\varsigma(M_1), \varsigma(M_2)\}$,
9) $\varsigma(M + x_0) = \varsigma(M)$ *for every* $x_0 \in X$.

Lemma 9 ([28, 38]) *If $G \subset C(\mathcal{O}, \mathbb{R})$ is an equicontinuous and bounded set, then*

i) *the function $\varsigma(G(\mathfrak{z}))$ is continuous for $\mathfrak{z} \in \mathcal{O}$ and*

$$\hat{\varsigma}(G) = \sup_{\mathfrak{z} \in \mathcal{O}} \varsigma(G(\mathfrak{z})),$$

ii) $\varsigma\left(\int_0^Z z(\mathfrak{z})d\mathfrak{z}\right) \leq \int_0^Z \varsigma(G(\mathfrak{z}))d\mathfrak{z}$, *where*

$$G(\mathfrak{z}) = \{z(\mathfrak{z}), z \in G\}, \quad \mathfrak{z} \in \mathcal{O}.$$

Now, we show the definition of condensing map, which is related to Sadovski's fixed point theorem.

Definition 10 ([36, 37]) Let $\mathcal{F} : Dom(\mathcal{F}) \subseteq X \to X$ be a bounded continuous operator on a Banach space X, then $\mathcal{F}$ is called a ς-condensing map if $\varsigma(\mathcal{F}(B)) < \varsigma(B)$ for all bounded sets $B \subset Dom(\mathcal{F})$ with $\varsigma(B) > 0$, where ς denotes the KMNC.

Remark 11 [28] According to the remark of Benkerrouche page 4 in [28], it is not difficult to show that condition $[C3]$ is equivalent to the following inequality:

$$\varsigma|h(\mathfrak{z}, B_1, B_2)| \leq M_1\varsigma(B_1) + M_2\varsigma(B_2) \tag{10}$$

for any bounded sets $B_1, B_2 \subset B_R$ and $\mathfrak{z} \in \mathcal{O}$.

Theorem 12 ([36, 37] Sadovski's fixed point theorem) *Let B be a convex bounded and closed subset of a Banach space X, and let $\mathcal{F} : B \to B$ be a condensing and continuous map, then $\mathcal{F}$ has a fixed point on B.*

Theorem 13 (Banach's fixed point theorem) *Let Λ be a closed subset of a Banach space X, if $\mathcal{F} : \Lambda \to \Lambda$ is a contraction mapping, then $\mathcal{F}$ has a unique fixed point.*

Definition 14 ([28]) Let $A \subset \mathbb{R}$, where A is referred to as a generalized interval if it is either an interval, a singleton $\{a\}$, or the empty set $\emptyset$.

A finite set $\mathcal{P}$ is called a partition of A if each $x \in A$ belongs to exactly one of the generalized intervals $E \in \mathcal{P}$.

A function $g : A \to \mathbb{R}$ is said to be piecewise constant with respect to the partition $\mathcal{P}$ of A if g assumes constant values on each $E \in \mathcal{P}$.

3 Results of existence and uniqueness

To check the main results, we insert the following assumption:

- $[C0]$ Let $\mathcal{P}$ be a partition of $\mathcal{O}$ defined as

  $$\mathcal{P} = \{\mathcal{O}_0 = [Z_0, Z_1], \mathcal{O}_1 = (Z_1, Z_2], \mathcal{O}_2 = (Z_2, Z_3], \dots, \mathcal{O}_n = (Z_n, Z_{n+1}]\},$$

  where $Z_0 = 0$ and $Z_{n+1} = Z$, and $r : \mathcal{O} \to (m-1, m]$ is a piecewise constant function with respect to $\mathcal{P}$.

$$r(\mathfrak{z}) = \sum_{i=0}^{n} r_i \mathcal{I}_i(\mathfrak{z}) = \begin{cases} r_0, & \textit{if } \mathfrak{z} \in \mathcal{O}_0, \\ r_1, & \textit{if } \mathfrak{z} \in \mathcal{O}_1, \\ & \vdots \\ r_n, & \textit{if } \mathfrak{z} \in \mathcal{O}_n, \end{cases}$$

where $m-1 < r_i < m$ for all $i = 0, \dots, n$ and

$$\mathcal{I}_i(\mathfrak{z}) = \begin{cases} 1, & \textit{if } \mathfrak{z} \in \mathcal{O}_i, \\ 0, & \textit{elsewhere} \end{cases} \quad i = 0, \dots, n.$$

Now, by supposing that $z \in PC(\mathcal{O}, \mathbb{R})$ is a solution of ϑ-IVP (1), then z fulfills the ϑ-fractional differential equation (ϑ-FDE)

$${}^{c}\mathcal{D}_{0^+}^{r(\mathfrak{z}),\vartheta(\mathfrak{z})} z(\mathfrak{z}) = h\big(\mathfrak{z}, z(\mathfrak{z}), \mathcal{I}_{0^+}^{q;\vartheta} z(\mathfrak{z})\big)$$

of the variable order $r(\mathfrak{z})$ for any $\mathfrak{z} \in \mathcal{O} \backslash \{Z_1, Z_2, \dots, Z_n\}$ and satisfies the conditions

$$z(0) = z_0, \quad z_{\vartheta}^{[j]}(0) = 0, \ j = 1, \dots, m-1,$$

and

$$\Delta z\big|_{\mathfrak{z}=Z_i} = g_i\big(z(Z_i^-)\big), \ i = 1 \dots n,$$

then, for every $\mathfrak{z} \in (Z_i, Z_{i+1}]$, $i = 1, \dots, n$ and for any $z \in C(\mathcal{O}, \mathbb{R})$, we have

$$\begin{aligned} {}^{c}\mathcal{D}_{0^+}^{r(\mathfrak{z}),\vartheta(\mathfrak{z})} z(\mathfrak{z}) &= \frac{1}{\Gamma(m - r(\mathfrak{z}))} \int_0^{\mathfrak{z}} \vartheta'(s)\big(\vartheta(\mathfrak{z}) - \vartheta(s)\big)^{m-r(\mathfrak{z})-1} z_{\vartheta}^{[m]}(s) ds \\ &= \frac{1}{\Gamma(m - r_0)} \int_0^{Z_1} \vartheta'(s)\big(\vartheta(\mathfrak{z}) - \vartheta(s)\big)^{m-r_0-1} z_{\vartheta}^{[m]}(s) ds \\ &\quad + \frac{1}{\Gamma(m - r_1)} \int_{Z_1}^{Z_2} \vartheta'(s)\big(\vartheta(\mathfrak{z}) - \vartheta(s)\big)^{m-r_1-1} z_{\vartheta}^{[m]}(s) ds \\ &\quad + \dots + \frac{1}{\Gamma(m - r_i)} \int_{Z_i}^{\mathfrak{z}} \vartheta'(s)\big(\vartheta(\mathfrak{z}) - \vartheta(s)\big)^{m-r_i-1} z_{\vartheta}^{[m]}(s) ds, \end{aligned}$$

then ϑ-IVP (1) can be expressed for each $\mathfrak{z} \in \mathcal{O}_i$, $i = 1, \dots, n$ as:

$$\frac{1}{\Gamma(m - r_0)} \int_0^{Z_1} \vartheta'(s)\big(\vartheta(\mathfrak{z}) - \vartheta(s)\big)^{m-r_0-1} z_{\vartheta}^{[m]}(s) ds$$

$$+\cdots+\frac{1}{\Gamma(m-r_i)}\int_{Z_i}^{\mathfrak{z}}\vartheta'(s)\big(\vartheta(\mathfrak{z})-\vartheta(s)\big)^{m-r_i-1}z_{\vartheta}^{[m]}(s)ds=h\big(\mathfrak{z},z(\mathfrak{z}),\mathcal{I}_{0^+}^{q;\vartheta}z(\mathfrak{z})\big). \tag{11}$$

We assume that $z\equiv 0$ on $\mathfrak{z}\in[0,Z_i]\backslash\{Z_1,\dots,Z_{i-1}\}$, then equation (11) becomes

$$\begin{aligned}{}^{c}\mathcal{D}_{0^+}^{r_i;\vartheta}z(\mathfrak{z})&=\frac{1}{\Gamma(m-r_i)}\int_{Z_i}^{\mathfrak{z}}\vartheta'(s)\big(\vartheta(\mathfrak{z})-\vartheta(s)\big)^{m-r_i-1}z_{\vartheta}^{[m]}(s)ds\\&=h\big(\mathfrak{z},z(\mathfrak{z}),\mathcal{I}_{0^+}^{q;\vartheta}z(\mathfrak{z})\big),\qquad \mathfrak{z}\in[0,Z_{i+1}]\backslash\{Z_1,\dots,Z_i\}.\end{aligned}$$

Then ϑ-IVP(1) is reduced to the following ϑ-IVP:

$$\begin{cases}{}^{c}\mathcal{D}_{0^+}^{r_i;\vartheta}z(\mathfrak{z})=h\big(\mathfrak{z},z(\mathfrak{z}),\mathcal{I}_{0^+}^{q;\vartheta}z(\mathfrak{z})\big), & \mathfrak{z}\in[0,Z_{i+1}]\backslash\{Z_1,\dots,Z_i\},\quad i=0,\dots,n,\\ z(0)=z_0,\quad z_{\vartheta}^{[j]}(0)=0,\quad j=1,\dots,m-1,\\ \Delta z\big|_{\mathfrak{z}=Z_i}=g_i\big(z(Z_i^-)\big), i=1,\dots,n.\end{cases} \tag{12}$$

Proposition 15 *Let $h:\mathcal{O}\times\mathbb{R}^2\to\mathbb{R}$ be continuous, and $h\in C^1\left(\mathcal{O}\times\mathbb{R}^2,\mathbb{R}\right)$. The solution of the impulsive ϑ-IVP (12) is given by*

$$z(\mathfrak{z})=\begin{cases}z_0+\dfrac{1}{\Gamma(r_0)}\displaystyle\int_0^{\mathfrak{z}}\vartheta'(s)\big(\vartheta(\mathfrak{z})-\vartheta(s)\big)^{r_0-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds, & \mathfrak{z}\in[Z_0,Z_1],\\ z_0+\displaystyle\sum_{k=1}^{i}g_k\big(z(Z_k^-)\big)+\sum_{k=1}^{i}\frac{1}{\Gamma(r_{k-1})}\int_{Z_{k-1}}^{Z_k}\vartheta'(s)\big(\vartheta(Z_k)-\vartheta(s)\big)^{r_{k-1}-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds\\ +\dfrac{1}{\Gamma(r_i)}\displaystyle\int_{Z_i}^{\mathfrak{z}}\vartheta'(s)\big(\vartheta(\mathfrak{z})-\vartheta(s)\big)^{r_i-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds, & \mathfrak{z}\in[Z_i,Z_{i+1}],\ i=1,\dots,n.\end{cases} \tag{13}$$

Proof Let us suppose that z is a solution of ϑ-IVP (12), then for $\mathfrak{z}\in[Z_0,Z_1]$ we have

$$\mathcal{I}_{0^+}^{r_0;\vartheta}\,{}^{c}\mathcal{D}_{0^+}^{r_0;\vartheta}z(\mathfrak{z})=\mathcal{I}_{0^+}^{r_0;\vartheta}h\big(\mathfrak{z},z(\mathfrak{z}),\mathcal{I}_{0^+}^{q;\vartheta}z(\mathfrak{z})\big).$$

Then we get

$$z(\mathfrak{z})=\mathcal{I}_{0^+}^{r_0;\vartheta}h\big(\mathfrak{z},z(\mathfrak{z}),\mathcal{I}_{0^+}^{q;\vartheta}z(\mathfrak{z})\big)+c_0+c_1\big(\vartheta(\mathfrak{z})-\vartheta(0)\big)+\cdots+c_{m-1}\big(\vartheta(\mathfrak{z})-\vartheta(0)\big)^{m-1};$$

therefore, by Lemma 5, we find

$$\begin{aligned}z'_{\vartheta}(\mathfrak{z})&={}^{c}\mathcal{D}_{0^+}^{1;\vartheta}\mathcal{I}_{0^+}^{r_0;\vartheta}h\big(\mathfrak{z},z(\mathfrak{z}),\mathcal{I}_{0^+}^{q;\vartheta}z(\mathfrak{z})\big)+{}^{c}\mathcal{D}_{0^+}^{1;\vartheta}\sum_{k=0}^{m-1}c_k\big(\vartheta(\mathfrak{z})-\vartheta(0)\big)^k\\&=\mathcal{I}_{0^+}^{r_0-1;\vartheta}h\big(\mathfrak{z},z(\mathfrak{z}),\mathcal{I}_{0^+}^{q;\vartheta}z(\mathfrak{z})\big)+\sum_{k=1}^{m-1}c_k\frac{\Gamma(k+1)}{\Gamma(k)}\big(\vartheta(\mathfrak{z})-\vartheta(0)\big)^{k-1}\\&=\mathcal{I}_{0^+}^{r_0-1;\vartheta}h\big(\mathfrak{z},z(\mathfrak{z}),\mathcal{I}_{0^+}^{q;\vartheta}z(\mathfrak{z})\big)+c_1+2c_2\big(\vartheta(\mathfrak{z})-\vartheta(0)\big)\\&+\dots+c_{m-1}(m-1)\big(\vartheta(\mathfrak{z})-\vartheta(0)\big)^{m-2},\end{aligned}$$

and for $\mathfrak{z}=0$ we get

$$z'_{\vartheta}(0)=\mathcal{I}_{0^+}^{r_0-1;\vartheta}h\big(\mathfrak{z},z(\mathfrak{z}),\mathcal{I}_{0^+}^{q;\vartheta}z(\mathfrak{z})\big)\big|_{\mathfrak{z}=0}+c_1=0,$$

which means that $c_1=0$.

For every $j = 2,\dots,m-2$, we have

$$
\begin{aligned}
z_{\vartheta}^{[j]}(\mathfrak{z}) &= \mathcal{I}_{0^+}^{r_0-j;\vartheta} h\big(\mathfrak{z}, z(\mathfrak{z}), \mathcal{I}_{0^+}^{q;\vartheta} z(\mathfrak{z})\big) + {}^{c}\mathcal{D}_{0^+}^{j;\vartheta}\left[c_0 + \sum_{k=2}^{m-1} c_k\big(\vartheta(\mathfrak{z}) - \vartheta(0)\big)^k\right] \\
&= \mathcal{I}_{0^+}^{r_0-j;\vartheta} h\big(\mathfrak{z}, z(\mathfrak{z}), \mathcal{I}_{0^+}^{q;\vartheta} z(\mathfrak{z})\big) + \sum_{k=j}^{m-1} c_k \frac{\Gamma(k+1)}{\Gamma(k-j+1)}\big(\vartheta(\mathfrak{z}) - \vartheta(0)\big)^{k-j} \\
&= \mathcal{I}_{0^+}^{r_0-j;\vartheta} h\big(\mathfrak{z}, z(\mathfrak{z}), \mathcal{I}_{0^+}^{q;\vartheta} z(\mathfrak{z})\big) + c_j \frac{\Gamma(j+1)}{\Gamma(1)} + c_{j+1}\frac{\Gamma(j+2)}{\Gamma(2)}\big(\vartheta(\mathfrak{z}) - \vartheta(0)\big) \\
&+ \dots + c_{m-1}\frac{\Gamma(m)}{(m-j)}\big(\vartheta(\mathfrak{z}) - \vartheta(0)\big)^{m-j-1},
\end{aligned}
$$

and for $\mathfrak{z} = 0$ we obtain

$$
z_{\vartheta}^{[j]}(0) = \mathcal{I}_{0^+}^{r_0-j;\vartheta} h\big(\mathfrak{z}, z(\mathfrak{z}), \mathcal{I}_{0^+}^{q;\vartheta} z(\mathfrak{z})\big)\Big|_{\mathfrak{z}=0} + c_j \frac{\Gamma(j+1)}{\Gamma(1)} = 0,
$$

hence, for $j = 2,\dots,m-2$, we get $c_j = 0$.

Since $m-1 < r_i < m$ for any $i = 0,\dots,n$, we have

$$
\begin{aligned}
z_{\vartheta}^{[m-1]}(\mathfrak{z}) &= \mathcal{I}_{0^+}^{r_0-(m-1);\vartheta} h\big(\mathfrak{z}, z(\mathfrak{z}), \mathcal{I}_{0^+}^{q;\vartheta} z(\mathfrak{z})\big) + {}^{c}\mathcal{D}_{0^+}^{m-1;\vartheta}\left[c_0 + c_{m-1}\big(\vartheta(\mathfrak{z}) - \vartheta(0)\big)^{m-1}\right] \\
&= \mathcal{I}_{0^+}^{r_0-(m-1);\vartheta} h\big(\mathfrak{z}, z(\mathfrak{z}), \mathcal{I}_{0^+}^{q;\vartheta} z(\mathfrak{z})\big) + c_{m-1}\,{}^{c}\mathcal{D}_{0^+}^{m-1;\vartheta}\big(\vartheta(\mathfrak{z}) - \vartheta(0)\big)^{m-1} \\
&= \mathcal{I}_{0^+}^{r_0-(m-1);\vartheta} h\big(\mathfrak{z}, z(\mathfrak{z}), \mathcal{I}_{0^+}^{q;\vartheta} z(\mathfrak{z})\big) + c_{m-1}\frac{\Gamma(m)}{\Gamma(m-(m-1))}\big(\vartheta(\mathfrak{z}) - \vartheta(0)\big)^{m-1-(m-1)} \\
&= \mathcal{I}_{0^+}^{r_0-(m-1);\vartheta} h\big(\mathfrak{z}, z(\mathfrak{z}), \mathcal{I}_{0^+}^{q;\vartheta} z(\mathfrak{z})\big) + c_{m-1}\frac{\Gamma(m)}{\Gamma(1)},
\end{aligned}
$$

then

$$
z_{\vartheta}^{[m-1]}(0) = \mathcal{I}_{0^+}^{r_0-(m-1);\vartheta} h\big(\mathfrak{z}, z(\mathfrak{z}), \mathcal{I}_{0^+}^{q;\vartheta} z(\mathfrak{z})\big)\Big|_{\mathfrak{z}=0} + c_{m-1}\Gamma(m) = 0,
$$

which drives to $c_{m-1} = 0$.

Hence, for $\mathfrak{z} \in [Z_0, Z_1]$, we find

$$
z(\mathfrak{z}) = \mathcal{I}_{0^+}^{r_0;\vartheta} h\big(\mathfrak{z}, z(\mathfrak{z}), \mathcal{I}_{0^+}^{q;\vartheta} z(\mathfrak{z})\big) + c_0,
$$

then for $\mathfrak{z} = 0$ we get

$$
z(0) = \mathcal{I}_{0^+}^{r_0;\vartheta} h\big(\mathfrak{z}, z(\mathfrak{z}), \mathcal{I}_{0^+}^{q;\vartheta} z(\mathfrak{z})\big)\Big|_{\mathfrak{z}=0} + c_0 = c_0 = z_0,
$$

so for $\mathfrak{z} \in [Z_0, Z_1]$

$$
\begin{aligned}
z(\mathfrak{z}) &= z_0 + \mathcal{I}_{0^+}^{r_0;\vartheta} h\big(\mathfrak{z}, z(\mathfrak{z}), \mathcal{I}_{0^+}^{q;\vartheta} z(\mathfrak{z})\big) \\
&= z_0 + \frac{1}{\Gamma(r_0)}\int_0^{\mathfrak{z}} \vartheta'(s)\big(\vartheta(\mathfrak{z}) - \vartheta(s)\big)^{r_0-1} h\big(s, z(s), \mathcal{I}_{0^+}^{q;\vartheta} z(s)\big)ds.
\end{aligned}
$$

If $\mathfrak{z} \in (Z_1, Z_2]$, then we apply the same steps with $z \equiv 0$ for $\mathfrak{z} \in [Z_0, Z_1]$, we find

$$z(\mathfrak{z}) = \mathcal{I}_{0^+}^{r_1;\vartheta} h\big(\mathfrak{z}, z(\mathfrak{z}), \mathcal{I}_{0^+}^{q;\vartheta} z(\mathfrak{z})\big) + c_0,$$

then for $\mathfrak{z} = Z_1^+$

$$z(Z_1^+) = \mathcal{I}_{Z_1^+}^{r_1;\vartheta} h\big(\mathfrak{z}, z(\mathfrak{z}), \mathcal{I}_{0^+}^{q;\vartheta} z(\mathfrak{z})\big)\Big|_{\mathfrak{z}=Z_1^+} + c_0 = c_0,$$

which gives

$$\begin{aligned} z(\mathfrak{z}) &= z(Z_1^+) + \frac{1}{\Gamma(r_1)} \int_{Z_1}^{\mathfrak{z}} \vartheta'(s)\big(\vartheta(\mathfrak{z}) - \vartheta(s)\big)^{r_1-1} h\big(s, z(s), \mathcal{I}_{0^+}^{q;\vartheta} z(s)\big) ds \\ &= \Delta z|_{\mathfrak{z}=Z_1} + z(Z_1^-) + \frac{1}{\Gamma(r_1)} \int_{Z_1}^{\mathfrak{z}} \vartheta'(s)\big(\vartheta(\mathfrak{z}) - \vartheta(s)\big)^{r_1-1} h\big(s, z(s), \mathcal{I}_{0^+}^{q;\vartheta} z(s)\big) ds, \end{aligned}$$

$$z(Z_1^-) = z(Z_1) = z_0 + \frac{1}{\Gamma(r_0)} \int_0^{Z_1} \vartheta'(s\big(\vartheta(Z_1) - \vartheta(s)\big)^{r_0-1} h\big(s, z(s, \mathcal{I}_{0^+}^{q;\vartheta} z(s)\big) ds,$$

and

$$\Delta z|_{\mathfrak{z}=Z_1} = g_1\big(z(Z_1^-)\big).$$

Consequently,

$$\begin{aligned} z(\mathfrak{z}) = {} & g_1\big(z(Z_1^-)\big) + z_0 + \frac{1}{\Gamma(r_0)} \int_0^{Z_1} \vartheta'(s)\big(\vartheta(Z_1) - \vartheta(s)\big)^{r_0-1} h\big(s, z(s), \mathcal{I}_{0^+}^{q;\vartheta} z(s)\big) ds \\ & + \frac{1}{\Gamma(r_1)} \int_{Z_1}^{\mathfrak{z}} \vartheta'(s)\big(\vartheta(\mathfrak{z}) - \vartheta(s)\big)^{r_1-1} h\big(s, z(s), \mathcal{I}_{0^+}^{q;\vartheta} z(s)\big) ds. \end{aligned}$$

For $\mathfrak{z} \in (Z_2, Z_3]$, we have

$$z(\mathfrak{z}) = \mathcal{I}_{0^+}^{r_2;\vartheta} h\big(\mathfrak{z}, z(\mathfrak{z}), \mathcal{I}_{0^+}^{q;\vartheta} z(\mathfrak{z})\big) + c_0,$$

for $\mathfrak{z} = Z_2^+$, we find

$$z(Z_2^+) = \mathcal{I}_{Z_2^+}^{r_2;\vartheta} h\big(\mathfrak{z}, z(\mathfrak{z}), \mathcal{I}_{0^+}^{q;\vartheta} z(\mathfrak{z})\big)\Big|_{\mathfrak{z}=Z_2^+} + c_0 = c_0,$$

then

$$\begin{aligned} z(\mathfrak{z}) &= z(Z_2^+) + \frac{1}{\Gamma(r_2)} \int_{Z_2}^{\mathfrak{z}} \vartheta'(s)\big(\vartheta(\mathfrak{z}) - \vartheta(s)\big)^{r_2-1} h\big(s, z(s), \mathcal{I}_{0^+}^{q;\vartheta} z(s)\big) ds \\ &= \Delta z|_{\mathfrak{z}=Z_2} + z(Z_2^-) + \frac{1}{\Gamma(r_2)} \int_{Z_2}^{\mathfrak{z}} \vartheta'(s)\big(\vartheta(\mathfrak{z}) - \vartheta(s)\big)^{r_2-1} h\big(s, z(s), \mathcal{I}_{0^+}^{q;\vartheta} z(s)\big) ds, \end{aligned}$$

$$\Delta z|_{\mathfrak{z}=Z_2} = g_2\big(z(Z_2^-)\big),$$

and

$$z(Z_2^-) = z(Z_2) = z_0 + g_1\big(z(Z_1^-)\big) + \frac{1}{\Gamma(r_0)} \in \mathfrak{z}_0^{Z_1} \vartheta'(s)\big(\vartheta(Z_1) - \vartheta(s)\big)^{r_0-1} h\big(s, z(s), \mathcal{I}_{0^+}^{q;\vartheta} z(s)\big) ds$$

$$+\frac{1}{\Gamma(r_1)}\int_{Z_1}^{Z_2}\vartheta'(s)\big(\vartheta(Z_2)-\vartheta(s)\big)^{r_1-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds.$$

Therefore, we get

$$\begin{aligned}z(\mathfrak{z}) &= g_1\big(z(Z_1^-)\big)+g_2\big(z(Z_2^-)\big)+z_0\\&\quad+\frac{1}{\Gamma(r_0)}\int_0^{Z_1}\vartheta'(s)\big(\vartheta(Z_1)-\vartheta(s)\big)^{r_0-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds\\&\quad+\frac{1}{\Gamma(r_1)}\int_{Z_1}^{Z_2}\vartheta'(s)\big(\vartheta(Z_2)-\vartheta(s)\big)^{r_1-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds\\&\quad+\frac{1}{\Gamma(r_2)}\int_{Z_2}^{\mathfrak{z}}\vartheta'(s)\big(\vartheta(\mathfrak{z})-\vartheta(s)\big)^{r_2-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds,\end{aligned}$$

so for $\mathfrak{z}\in(Z_i,Z_{i+1}]$ we obtain

$$\begin{aligned}z(\mathfrak{z}) &= z_0+\sum_{k=1}^{i}g_k\big(z(Z_k^-)\big)\\&\quad+\sum_{k=1}^{i}\frac{1}{\Gamma(r_{k-1})}\int_{Z_{k-1}}^{Z_k}\vartheta'(s)\big(\vartheta(Z_k)-\vartheta(s)\big)^{r_{k-1}-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds\\&\quad+\frac{1}{\Gamma(r_i)}\int_{Z_i}^{\mathfrak{z}}\vartheta'(s)\big(\vartheta(\mathfrak{z})-\vartheta(s)\big)^{r_i-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds.\end{aligned}$$

Hence the solution of ϑ-IVP (12) can be expressed by the following integral equation:

$$z(\mathfrak{z})=\begin{cases}z_0+\dfrac{1}{\Gamma(r_0)}\displaystyle\int_0^{\mathfrak{z}}\vartheta'(s)\big(\vartheta(\mathfrak{z})-\vartheta(s)\big)^{r_0-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds, & \mathfrak{z}\in[Z_0,Z_1],\\ z_0+\displaystyle\sum_{k=1}^{i}g_k\big(z(Z_k^-)\big)+\sum_{k=1}^{i}\frac{1}{\Gamma(r_{k-1})}\int_{Z_{k-1}}^{Z_k}\vartheta'(s)\big(\vartheta(Z_k)-\vartheta(s)\big)^{r_{k-1}-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds, & \\ +\dfrac{1}{\Gamma(r_i)}\displaystyle\int_{Z_i}^{\mathfrak{z}}\vartheta'(s)\big(\vartheta(\mathfrak{z})-\vartheta(s)\big)^{r_i-1}h\big(s,z(s,\mathcal{I}_{0^+}^{q;\vartheta}z(s)ds, & \mathfrak{z}\in[Z_i,Z_{i+1}],\ i=1,\dots,n.\end{cases}\tag{14}$$

In the converse case, we suppose that z is a solution of (13), then we have:

- For $\mathfrak{z}\in[Z_0,Z_1]$ we have

$$z(0)=z_0+\mathcal{I}_{0^+}^{r_0;\vartheta}h\big(\mathfrak{z},z(\mathfrak{z}),\mathcal{I}_{0^+}^{q;\vartheta}z(\mathfrak{z})\big)\big|_{\mathfrak{z}=0}=z_0.$$

By using Lemma 6 together with the assumption $h\in C^1\left(\mathcal{O}\times\mathbb{R}^2,\mathbb{R}\right)$, we apply ${}^c\mathcal{D}_{0^+}^{r_0;\vartheta}$ to both sides of (13) and obtain

$${}^c\mathcal{D}_{0^+}^{r_0;\vartheta}z(\mathfrak{z})={}^c\mathcal{D}_{0^+}^{r_0;\vartheta}\left[z_0+\mathcal{I}_{0^+}^{r_0;\vartheta}h\big(\mathfrak{z},z(\mathfrak{z}),\mathcal{I}_{0^+}^{q;\vartheta}z(\mathfrak{z})\big)\right]=h\big(\mathfrak{z},z(\mathfrak{z}),\mathcal{I}_{0^+}^{q;\vartheta}z(\mathfrak{z})\big).$$

- For $\mathfrak{z}\in(Z_i,Z_{i+1}]$, $i=1,\dots,n$, and from Lemma 6, we obtain

$${}^c\mathcal{D}_{0^+}^{r_i;\vartheta}z(\mathfrak{z})={}^c\mathcal{D}_{0^+}^{r_i;\vartheta}z_0+{}^c\mathcal{D}_{0^+}^{r_i;\vartheta}\sum_{k=1}^{i}g_k\big(z(Z_k^-)\big)$$

$$
\begin{aligned}
&+{}^{c}\mathcal{D}_{0^+}^{r_i;\vartheta}\sum_{k=1}^{i}\frac{1}{\Gamma(r_{k-1})}\int_{Z_{k-1}}^{Z_k}\vartheta'(s)\big(\vartheta(Z_k)-\vartheta(s)\big)^{r_{k-1}-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds\\
&+{}^{c}\mathcal{D}_{0^+}^{r_i;\vartheta}\mathcal{I}_{0^+}^{r_i;\vartheta}h\big(\mathfrak{z},z(\mathfrak{z}),\mathcal{I}_{0^+}^{q;\vartheta}z(\mathfrak{z})\big)\\
&=h\big(\mathfrak{z},z(\mathfrak{z}),\mathcal{I}_{0^+}^{q;\vartheta}z(\mathfrak{z})\big),
\end{aligned}
$$

$$
\Delta z|_{\mathfrak{z}=Z_i}=z(Z_i^+)-z(Z_i^-),
$$

$$
\begin{aligned}
z(Z_i^-)=z(Z_i)&=z_0+\sum_{k=1}^{i-1}g_k\big(z(Z_k^-)\big)\\
&\quad+\sum_{k=1}^{i-1}\frac{1}{\Gamma(r_{k-1})}\int_{Z_{k-1}}^{Z_k}\vartheta'(s)\big(\vartheta(Z_k)-\vartheta(s)\big)^{r_{k-1}-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds\\
&\quad+\frac{1}{\Gamma(r_{i-1})}\int_{Z_{i-1}}^{Z_i}\vartheta'(s)\big(\vartheta(Z_i)-\vartheta(s)\big)^{r_{i-1}-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds\\
&=z_0+\sum_{k=1}^{i-1}g_k\big(z(Z_k^-)\big)\\
&\quad+\sum_{k=1}^{i}\frac{1}{\Gamma(r_{k-1})}\int_{Z_{k-1}}^{Z_k}\vartheta'(s)\big(\vartheta(Z_k)-\vartheta(s)\big)^{r_{k-1}-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds,
\end{aligned}
$$

and

$$
\begin{aligned}
z(Z_i^+)&=z_0+\sum_{k=1}^{i}g_k\big(z(Z_k^-)\big)\\
&\quad+\sum_{k=1}^{i}\frac{1}{\Gamma(r_{k-1})}\int_{Z_{k-1}}^{Z_k}\vartheta'(s)\big(\vartheta(Z_k)-\vartheta(s)\big)^{r_{k-1}-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds,
\end{aligned}
$$

because $\mathcal{I}_{0^+}^{r_i;\vartheta}h\big(\mathfrak{z},z(\mathfrak{z}),\mathcal{I}_{0^+}^{q;\vartheta}z(\mathfrak{z})\big)|_{\mathfrak{z}=Z_i^+}=0$, then we find for any $i=1,\dots,n$

$$
\Delta z|_{\mathfrak{z}=Z_i}=g_i\big(z(Z_i^-)\big),
$$

and for every $j=1,\dots,m-1$, $i=0,..,n$, we have

$$
\begin{aligned}
z_\vartheta^{[j]}(\mathfrak{z})&={}^{c}\mathcal{D}_{0^+}^{j;\vartheta}z_0+{}^{c}\mathcal{D}_{0^+}^{j;\vartheta}\sum_{k=1}^{i}g_k\big(z(Z_k^-)\big)\\
&\quad+{}^{c}\mathcal{D}_{0^+}^{j;\vartheta}\sum_{k=1}^{i}\frac{1}{\Gamma(r_{k-1})}\int_{Z_{k-1}}^{Z_k}\vartheta'(s)\big(\vartheta(Z_k)-\vartheta(s)\big)^{r_{k-1}-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds\\
&\quad+\mathcal{I}_{0^+}^{r_i-j;\vartheta}h\big(\mathfrak{z},z(\mathfrak{z}),\mathcal{I}_{0^+}^{q;\vartheta}z(\mathfrak{z})\big).
\end{aligned}
$$

Consequently,

$$
z_\vartheta^{[j]}(0)=\mathcal{I}_{0^+}^{r_i-j;\vartheta}h\big(\mathfrak{z},z(\mathfrak{z}),\mathcal{I}_{0^+}^{q;\vartheta}z(\mathfrak{z})\big)|_{\mathfrak{z}=0}=0, \tag{15}
$$

which shows the last condition. □

Now, we shall present our first result by using the following assumptions:

- $[C1]$ Assume that h is bounded with nonnegative continuous function $\mathfrak{P}:\mathcal{O}\to\mathbb{R}^+$ such that for all $(\mathfrak{z},x,y)\in\mathcal{O}\times\mathbb{R}^2$

$$|h(\mathfrak{z},x,y)|\le\mathfrak{P}(\mathfrak{z}),\quad \mathfrak{z}\in\mathcal{O},$$

where $\mathfrak{P}\in C(\mathcal{O},\mathbb{R}^+)$ and $\|\mathfrak{P}\|=\sup_{\mathfrak{z}\in\mathcal{O}}\mathfrak{P}(\mathfrak{z})=\mathfrak{P}^*$.

- $[C2]$ For any $i=0,\dots,n$ and $x\in\mathbb{R}$, there exists $a_1>0$ such that

$$|g_i(x)|\le a_1|x|.$$

- $[C3]$ Suppose that $1-a_1n>0$, and there exists $R>0$ such that

$$\frac{|z_0|+\frac{(n+1)\mathfrak{P}^*\left(\vartheta(Z)-\vartheta(0)\right)^{r^+}}{\Gamma(r^-+1)}}{1-a_1n}\le R,$$

where

$$r^+=\sup r(\mathfrak{z}),\quad \mathfrak{z}\in\mathcal{O},$$
$$r^-=\inf r(\mathfrak{z}),\quad \mathfrak{z}\in\mathcal{O}.$$

- $[C4]$ Assume that $h:\mathcal{O}\times\mathbb{R}^2\to\mathbb{R}$ is a continuous function satisfying: there exist $M_1,M_2>0$ such that

$$|h(\mathfrak{z},x_1,x_2)-h(\mathfrak{z},y_1,y_2)|\le M_1|x_1-y_1|+M_2|x_2-y_2|$$

for all $(x_1,x_2),(y_1,y_2)\in\mathbb{R}^2$ and $\mathfrak{z}\in\mathcal{O}$.

- $[C5]$ For $i=1,\dots,n$, there is $a_2>0$ such that for every $x,y\in\mathbb{R}$:

$$\left|g_i(x)-g_i(y)\right|\le a_2|x-y|.$$

Theorem 16 *Suppose that conditions* $[C0]-[C4]$ *are satisfied and*

$$\Omega=na_1+\frac{(n+1)\left(\vartheta(Z)-\vartheta(0)\right)^{r^+}}{\Gamma(r^-+1)}\left[M_1+M_2\frac{\left(\vartheta(Z)-\vartheta(0)\right)^q}{\Gamma(q+1)}\right]<1. \tag{16}$$

Then the impulsive ϑ*-IVP* (1) *possesses a solution on* $PC(\mathcal{O},\mathbb{R})$.

Proof Let us consider the set

$$B_R=\{z\in PC(\mathcal{O},\mathbb{R});\|z\|\le R\},$$

it is clear that B_R is a nonempty, bounded, convex, and closed subset of $PC(\mathcal{O},\mathbb{R})$. Let us construct the operator $\mathcal{F}:B_R\to B_R$ as follows:

$$\mathcal{F}z(\mathfrak{z})=z_0+\sum_{0<Z_i<\mathfrak{z}}g_i\left(z(Z_i^-)\right)$$

$$+\sum_{0<Z_i<\mathfrak{z}}\frac{1}{\Gamma(r_{i-1})}\int_{Z_{i-1}}^{Z_i}\vartheta'(s)\big(\vartheta(Z_i)-\vartheta(s)\big)^{r_{i-1}-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds$$
$$+\frac{1}{\Gamma(r_i)}\int_{Z_i}^{\mathfrak{z}}\vartheta'(s)\big(\vartheta(\mathfrak{z})-\vartheta(s)\big)^{r_i-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds. \tag{17}$$

Note that the continuity of h implies that the operator $\mathcal{F}$ is well defined. So let us verify the following results.

The work is divided into several steps:

• Step 1: We prove that $\mathcal{F}(B_R)\subseteq B_R$

for every $z\in B_R$ we have

$$\begin{aligned}\big|\mathcal{F}z(\mathfrak{z})\big| &= \Bigg|z_0+\sum_{0<Z_i<\mathfrak{z}}g_i\big(z(Z_i^-)\big)\\
&\quad+\sum_{0<Z_i<\mathfrak{z}}\frac{1}{\Gamma(r_{i-1})}\int_{Z_{i-1}}^{Z_i}\vartheta'(s\big(\vartheta(Z_i)-\vartheta(s)\big)^{r_{i-1}-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds\\
&\quad+\frac{1}{\Gamma(r_i)}\int_{Z_i}^{\mathfrak{z}}\vartheta'(s)\big(\vartheta(\mathfrak{z})-\vartheta(s)\big)^{r_i-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds\Bigg|\\
&\le |z_0|+\sum_{0<Z_i<\mathfrak{z}}\big|g_i\big(z(Z_i^-)\big)\big|\\
&\quad+\sum_{0<Z_i<\mathfrak{z}}\frac{1}{\Gamma(r_{i-1})}\int_{Z_{i-1}}^{Z_i}\vartheta'(s)\big(\vartheta(Z_i)-\vartheta(s)\big)^{r_{i-1}-1}\big|h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)\big|ds\\
&\quad+\frac{1}{\Gamma(r_i)}\int_{Z_i}^{\mathfrak{z}}\vartheta'(s)\big(\vartheta(\mathfrak{z})-\vartheta(s)\big)^{r_i-1}\big|h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)\big|ds.\end{aligned}$$

Then, by using conditions $[C1]$, $[C2]$, we find

$$\begin{aligned}\big|\mathcal{F}z(\mathfrak{z})\big| &\le |z_0|+\sum_{0<Z_i<\mathfrak{z}}a_1\big|z(Z_i^-)\big|+\sum_{0<Z_i<\mathfrak{z}}\frac{1}{\Gamma(r_{i-1})}\int_{Z_{i-1}}^{Z_i}\vartheta'(s\big(\vartheta(Z_i)-\vartheta(s)\big)^{r_{i-1}-1}\mathfrak{P}(sds\\
&\quad+\frac{1}{\Gamma(r_i)}\int_{Z_i}^{\mathfrak{z}}\vartheta'(s)\big(\vartheta(\mathfrak{z})-\vartheta(s)\big)^{r_i-1}\mathfrak{P}(s)ds\\
&\le |z_0|+\sum_{0<Z_i<t}a_1\|z\|+\sum_{0<Z_i<\mathfrak{z}}\frac{\mathfrak{P}^*}{\Gamma(r_{i-1})}\int_{Z_{i-1}}^{Z_i}\vartheta'(s)\big(\vartheta(Z_i)-\vartheta(s)\big)^{r_{i-1}-1}ds\\
&\quad+\frac{\mathfrak{P}^*}{\Gamma(r_i)}\int_{Z_i}^{\mathfrak{z}}\vartheta'(s)\big(\vartheta(\mathfrak{z})-\vartheta(s)\big)^{r_i-1}ds.\end{aligned}$$

Since we have

$$\frac{1}{\Gamma(r_{i-1})}\int_{Z_{i-1}}^{Z_i}\vartheta'(s)\big(\vartheta(Z_i)-\vartheta(s)\big)^{r_{i-1}-1}ds=\frac{\big(\vartheta(Z_i)-\vartheta(Z_{i-1})\big)^{r_{i-1}}}{\Gamma(r_{i-1}+1)}$$

and

$$\frac{1}{\Gamma(r_i)}\int_{Z_i}^{\mathfrak{z}}\vartheta'(s)\big(\vartheta(\mathfrak{z})-\vartheta(s)\big)^{r_i-1}ds=\frac{\big(\vartheta(\mathfrak{z})-\vartheta(Z_i)\big)^{r_i}}{\Gamma(r_i+1)},$$

then it follows that

$$\left|\mathcal{F}z(\mathfrak{z})\right| \leq |z_0| + \|z\| \sum_{0<Z_i<\mathfrak{z}} a_1 + \sum_{0<Z_i<\mathfrak{z}} \frac{\mathfrak{P}^*\left(\vartheta(Z_i) - \vartheta(Z_{i-1})\right)^{r_{i-1}}}{\Gamma(r_{i-1}+1)} + \frac{\mathfrak{P}^*\left(\vartheta(t) - \vartheta(Z_i)\right)^{r_i}}{\Gamma(r_i+1)}.$$

By setting

$$r^+ = \sup r(\mathfrak{z}), \quad \mathfrak{z} \in \mathcal{O},$$
$$r^- = \inf r(\mathfrak{z}), \quad \mathfrak{z} \in \mathcal{O},$$

we get

$$\begin{aligned}
\left|\mathcal{F}z(\mathfrak{z})\right| &\leq |z_0| + a_1 n\|z\| + \frac{n\mathfrak{P}^*\left(\vartheta(Z) - \vartheta(0)\right)^{r^+}}{\Gamma(r^-+1)} + \frac{\mathfrak{P}^*\left(\vartheta(Z) - \vartheta(0)\right)^{r^+}}{\Gamma(r^-+1)} \\
&\leq |z_0| + a_1 n\|z\| + \frac{(n+1)\mathfrak{P}^*\left(\vartheta(Z) - \vartheta(0)\right)^{r^+}}{\Gamma(r^-+1)} \\
&\leq |z_0| + a_1 nR + \frac{(n+1)\mathfrak{P}^*\left(\vartheta(Z) - \vartheta(0)\right)^{r^+}}{\Gamma(r^-+1)} \leq R, \qquad (18)
\end{aligned}$$

hence, from $[C3]$, we get $\|\mathcal{F}z\| \leq R$, which means that $\mathcal{F}(B_R) \subseteq B_R$.

• Step 2: We prove that $\mathcal{F}$ is continuous:

Let (z_n) be a sequence such that $z_n \to z$ in $PC(\mathcal{O},\mathbb{R})$, then for $\mathfrak{z} \in \mathcal{O}$ we have

$$\begin{aligned}
&\left|\mathcal{F}z_n(\mathfrak{z}) - \mathcal{F}z(\mathfrak{z})\right| \\
&\quad\leq \sum_{0<Z_i<\mathfrak{z}} \left|g_i\left(z_n(Z_i^-)\right) - g_i\left(z(Z_i^-)\right)\right| \\
&\qquad + \sum_{0<Z_i<\mathfrak{z}} \frac{1}{\Gamma(r_{i-1})} \int_{Z_{i-1}}^{Z_i} \vartheta'(s)\left(\vartheta(Z_i) - \vartheta(s)\right)^{r_{i-1}-1} \left|h\left(s, z_n(s), \mathcal{I}_{0^+}^{q;\vartheta} z_n(s)\right)\right. \\
&\qquad \left. - h\left(s, z(s), \mathcal{I}_{0^+}^{q;\vartheta} z(s)\right)\right| ds \\
&\qquad + \frac{1}{\Gamma(r_i)} \int_{Z_i}^{\mathfrak{z}} \vartheta'(s)\left(\vartheta(\mathfrak{z}) - \vartheta(s)\right)^{r_i-1} \left|h\left(s, z_n(s), \mathcal{I}_{0^+}^{q;\vartheta} z_n(s)\right) - h\left(s, z(s), \mathcal{I}_{0^+}^{q;\vartheta} z(s)\right)\right| ds.
\end{aligned}$$

Hence from $[C4]$ we find

$$\begin{aligned}
\left|\mathcal{F}z_n(\mathfrak{z}) - \mathcal{F}z(\mathfrak{z})\right| &\leq \sum_{0<Z_i<\mathfrak{z}} \left|g_i\left(z_n(Z_i^-)\right) - g_i\left(z(Z_i^-)\right)\right| \\
&\quad + \sum_{0<Z_i<\mathfrak{z}} \frac{1}{\Gamma(r_{i-1})} \int_{Z_{i-1}}^{Z_i} \vartheta'(s)\left(\vartheta(Z_i) - \vartheta(s)\right)^{r_{i-1}-1} \left(M_1 |z_n(s) - z(s)|\right. \\
&\quad \left. + M_2 \left|\mathcal{I}_{0^+}^{q;\vartheta} z_n(s) - \mathcal{I}_{0^+}^{q;\vartheta} z(s)\right|\right) ds \\
&\quad + \frac{1}{\Gamma(r_i)} \int_{Z_i}^{\mathfrak{z}} \vartheta'(s)\left(\vartheta(\mathfrak{z}) - \vartheta(s)\right)^{r_i-1} \left(M_1 |z_n(s) - z(s)|\right. \\
&\quad \left. + M_2 \left|\mathcal{I}_{0^+}^{q;\vartheta} z_n(s - \mathcal{I}_{0^+}^{q;\vartheta} z(s\right|\right) ds
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{0<Z_i<t} \left|g_i\left(z_n(Z_i^-)\right) - g_i\left(z(Z_i^-)\right)\right| \\
&\quad + \sum_{0<Z_i<\mathfrak{z}} \frac{M_1}{\Gamma(r_{i-1})} \int_{Z_{i-1}}^{Z_i} \vartheta'(s)\left(\vartheta(Z_i) - \vartheta(s)\right)^{r_{i-1}-1} ds \|z_n - z\| \\
&\quad + \sum_{0<Z_i<\mathfrak{z}} \frac{M_2}{\Gamma(r_{i-1})} \int_{Z_{i-1}}^{Z_i} \vartheta'(s)\left(\vartheta(Z_i) - \vartheta(s)\right)^{r_{i-1}-1} \left|\mathcal{I}_{0^+}^{q;\vartheta}(z_n(s) - z(s)\right| ds \\
&\quad + \frac{M_1}{\Gamma(r_i)} \int_{Z_i}^{\mathfrak{z}} \vartheta'(s)\left(\vartheta(\mathfrak{z}) - \vartheta(s)\right)^{r_i-1} ds \|z_n - z\| \\
&\quad + \frac{M_2}{\Gamma(r_i)} \int_{Z_i}^{\mathfrak{z}} \vartheta'(s)\left(\vartheta(\mathfrak{z}) - \vartheta(s)\right)^{r_i-1} \left|\mathcal{I}_{0^+}^{q;\vartheta}(z_n(s) - z(s)\right| ds \qquad (19)
\end{aligned}$$

and

$$\begin{aligned}
\left|\mathcal{I}_{0^+}^{q;\vartheta}(z_n(\mathfrak{z}) - z(\mathfrak{z}))\right| &\leq \frac{1}{\Gamma(q)} \int_0^{\mathfrak{z}} \vartheta'(s)\left(\vartheta(\mathfrak{z}) - \vartheta(s)\right)^{q-1} |z_n(s) - z(s)| ds \\
&\leq \frac{1}{\Gamma(q)} \int_0^{\mathfrak{z}} \vartheta'(s)\left(\vartheta(\mathfrak{z}) - \vartheta(s)\right)^{q-1} ds \|z_n - z\| \\
&\leq \frac{\left(\vartheta(Z) - \vartheta(0)\right)^q}{\Gamma(q+1)} \|z_n - z\|, \qquad (20)
\end{aligned}$$

therefore

$$\begin{aligned}
&\left|\mathcal{F}z_n(\mathfrak{z}) - \mathcal{F}z(\mathfrak{z})\right| \qquad (21) \\
&\quad \leq \sum_{0<Z_i<\mathfrak{z}} \left|g_i\left(z_n(Z_i^-)\right) - g_i\left(z(Z_i^-)\right)\right| \\
&\qquad + \sum_{0<Z_i<\mathfrak{z}} \frac{M_1\left(\vartheta(Z_i) - \vartheta(Z_{i-1})\right)^{r_{i-1}}}{\Gamma(r_{i-1}+1)} \|z_n - z\| \\
&\qquad + \sum_{0<Z_i<\mathfrak{z}} \frac{M_2}{\Gamma(r_{i-1})} \int_{Z_{i-1}}^{Z_i} \vartheta'(s)\left(\vartheta(Z_i) - \vartheta(s)\right)^{r_{i-1}-1} ds \frac{\left(\vartheta(Z) - \vartheta(0)\right)^q}{\Gamma(q+1)} \|z_n - z\| \\
&\qquad + \frac{M_1\left(\vartheta(\mathfrak{z}) - \vartheta(Z_i)\right)^{r_i}}{\Gamma(r_i+1)} \|z_n - z\| \\
&\qquad + \frac{M_2}{\Gamma(r_i)} \int_{Z_i}^{\mathfrak{z}} \vartheta'(s)\left(\vartheta(\mathfrak{z}) - \vartheta(s)\right)^{r_i-1} ds \frac{\left(\vartheta(Z) - \vartheta(0)\right)^q}{\Gamma(q+1)} \|z_n - z\| \\
&\quad \leq \sum_{0<Z_i<\mathfrak{z}} \left|g_i\left(z_n(Z_i^-)\right) - g_i\left(z(Z_i^-)\right)\right| + \left[\frac{nM_1\left(\vartheta(Z) - \vartheta(0)\right)^{r^+}}{\Gamma(r^- + 1)} \right. \\
&\qquad + \sum_{0<Z_i<\mathfrak{z}} \frac{M_2\left(\vartheta(Z_i) - \vartheta(Z_{i-1})\right)^{r_{i-1}}}{\Gamma(r_{i-1}+1)} \frac{\left(\vartheta(Z) - \vartheta(0)\right)^q}{\Gamma(q+1)} \\
&\qquad \left. + \frac{M_1\left(\vartheta(Z) - \vartheta(0)\right)^{r^+}}{\Gamma(r^- + 1)} + \frac{M_2\left(\vartheta(\mathfrak{z}) - \vartheta(Z_i)\right)^{r_i}}{\Gamma(r_i+1)} \frac{\left(\vartheta(Z) - \vartheta(0)\right)^q}{\Gamma(q+1)} \right] \|z_n - z\| \\
&\quad \leq \sum_{0<Z_i<\mathfrak{z}} \left|g_i\left(z_n(Z_i^-)\right) - g_i\left(z(Z_i^-)\right)\right| + \left[\frac{(n+1)M_1\left(\vartheta(Z) - \vartheta(0)\right)^{r^+}}{\Gamma(r^- + 1)} \right.
\end{aligned}$$

$$+\frac{(n+1)M_2\big(\vartheta(Z)-\vartheta(0)\big)^{r^++q}}{\Gamma(r^-+1)\Gamma(q+1)}\Bigg]\|z_n-z\|. \tag{22}$$

From the continuity of g_i, it follows that $\|\mathcal{F}z_n(\mathfrak{z})-\mathcal{F}z(\mathfrak{z})\|\to 0$ when $n\to\infty$, which leads to the continuity of $\mathcal{F}$.

• Step 3: We show that $\mathcal{F}(B_R)$ is bounded and equicontinuous:

We proved that $\mathcal{F}(B_R)\subseteq B_R$ under conditions $[C0]$, $[C1]$, $[C2]$, and $[C3]$, which means that $\mathcal{F}$ is bounded on B_R. It remains to prove the equicontinuity.

For every $\mathfrak{z}_1,\mathfrak{z}_2\in\mathcal{O}$, $\mathfrak{z}_1<\mathfrak{z}_2$, and $z\in B_R$, we have

$$\begin{aligned}
&\big|\mathcal{F}z(\mathfrak{z}_2)-\mathcal{F}z(\mathfrak{z}_1)\big|\\
&\quad=\Bigg|z_0+\sum_{0<Z_i<\mathfrak{z}_2}g_i\big(z(Z_i^-)\big)\\
&\qquad+\frac{1}{\Gamma(r_i)}\int_0^{\mathfrak{z}_2}\vartheta'(s)\big(\vartheta(\mathfrak{z}_2)-\vartheta(s)\big)^{r_i-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds\\
&\qquad-z_0-\sum_{0<Z_i<\mathfrak{z}_2}g_i\big(z(Z_i^-)\big)\\
&\qquad-\frac{1}{\Gamma(r_i)}\int_0^{\mathfrak{z}_1}\vartheta'(s\big(\vartheta(\mathfrak{z}_1)-\vartheta(s)\big)^{r_i-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds\Bigg|\\
&\quad\le\sum_{\mathfrak{z}_1<Z_i<\mathfrak{z}_2}\big|g_i\big(z(Z_i^-)\big)\big|\\
&\qquad+\frac{1}{\Gamma(r_i)}\int_0^{\mathfrak{z}_1}\vartheta'(s\big[\big(\vartheta(\mathfrak{z}_2)-\vartheta(s)\big)^{r_i-1}-\big(\vartheta(\mathfrak{z}_1)-\vartheta(s)\big)^{r_i-1}\big]|h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)|ds\\
&\qquad+\frac{1}{\Gamma(r_i)}\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\vartheta'(s)\big(\vartheta(\mathfrak{z}_2)-\vartheta(s)\big)^{r_i-1}|h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)|ds\\
&\quad\le\sum_{\mathfrak{z}_1<Z_i<\mathfrak{z}_2}\big|g_i\big(z(Z_i^-)\big)\big|\\
&\qquad+\frac{1}{\Gamma(r_i)}\int_0^{\mathfrak{z}_1}\vartheta'(s)\big[\big(\vartheta(\mathfrak{z}_2)-\vartheta(s)\big)^{r_i-1}-\big(\vartheta(\mathfrak{z}_1)-\vartheta(s)\big)^{r_i-1}\big]\\
&\qquad|h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)-h(\mathfrak{z},0,0)|ds\\
&\qquad+\frac{1}{\Gamma(r_i)}\int_0^{\mathfrak{z}_1}\vartheta'(s)\big[\big(\vartheta(\mathfrak{z}_2)-\vartheta(s)\big)^{r_i-1}-\big(\vartheta(\mathfrak{z}_1)-\vartheta(s)\big)^{r_i-1}\big]|h(\mathfrak{z},0,0)|ds\\
&\qquad+\frac{1}{\Gamma(r_i)}\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\vartheta'(s)\big(\vartheta(\mathfrak{z}_2)-\vartheta(s)\big)^{r_i-1}|h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)-h(\mathfrak{z},0,0)|ds\\
&\qquad+\frac{1}{\Gamma(r_i)}\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\vartheta'(s)\big(\vartheta(\mathfrak{z}_2)-\vartheta(s)\big)^{r_i-1}|h(\mathfrak{z},0,0)|ds.
\end{aligned}$$

We put $h^*=\sup_{i\in\mathcal{O}}|h(\mathfrak{z},0,0)|$. Then from condition $[C4]$ we get

$$\begin{aligned}
&\big|\mathcal{F}z(\mathfrak{z}_2)-\mathcal{F}z(\mathfrak{z}_1)\big|\\
&\quad\le\sum_{\mathfrak{z}_1<T_i<\mathfrak{z}_2}\big|g_i\big(z(Z_i^-)\big)\big|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(r_i)}\int_0^{\mathfrak{z}_1}\vartheta'(s)\big[\big(\vartheta(\mathfrak{z}_2)-\vartheta(s)\big)^{r_i-1}-\big(\vartheta(\mathfrak{z}_1)-\vartheta(s)\big)^{r_i-1}\big]\\
&\left(M_1|z(s|+M_2\left|\mathcal{I}_{0^+}^{q;\vartheta}z(s\right|\right)ds\\
&+\frac{h^*}{\Gamma(r_i)}\int_0^{\mathfrak{z}_1}\vartheta'(s)\big[\big(\vartheta(\mathfrak{z}_2)-\vartheta(s)\big)^{r_i-1}-\big(\vartheta(\mathfrak{z}_1)-\vartheta(s)\big)^{r_i-1}\big]ds\\
&+\frac{1}{\Gamma(r_i)}\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\vartheta'(s)\big(\vartheta(\mathfrak{z}_2)-\vartheta(s)\big)^{r_i-1}\left(M_1|z(s|+M_2\left|\mathcal{I}_{0^+}^{q;\vartheta}z(s\right|\right)ds\\
&+\frac{h^*}{\Gamma(r_i)}\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\vartheta'(s)\big(\vartheta(\mathfrak{z}_2)-\vartheta(s)\big)^{r_i-1}ds\\
\le{}&\sum_{\mathfrak{z}_1<Z_i<\mathfrak{z}_2}\left|g_i\big(z(Z_i^-)\big)\right|\\
&+\frac{M_1}{\Gamma(r_i)}\int_0^{\mathfrak{z}_1}\vartheta'(s)\big[\big(\vartheta(\mathfrak{z}_2)-\vartheta(0)\big)^{r_i-1}-\big(\vartheta(\mathfrak{z}_1)-\vartheta(0)\big)^{r_i-1}\big]|z(s|ds\\
&+\frac{M_2}{\Gamma(r_i)}\int_0^{\mathfrak{z}_1}\vartheta'(s)\big[\big(\vartheta(\mathfrak{z}_2)-\vartheta(0)\big)^{r_i-1}-\big(\vartheta(\mathfrak{z}_1)-\vartheta(0)\big)^{r_i-1}\big]ds\frac{\big(\vartheta(Z)-\vartheta(0)\big)^q}{\Gamma(q+1)}\|z\|\\
&+\frac{h^*}{\Gamma(r_i)}\int_0^{\mathfrak{z}_1}\vartheta'(s)\big(\vartheta(\mathfrak{z}_2)-\vartheta(s)\big)^{r_i-1}ds-\frac{h^*}{\Gamma(r_i)}\int_0^{\mathfrak{z}_1}\vartheta'(s)\big(\vartheta(\mathfrak{z}_1)-\vartheta(s)\big)^{r_i-1}ds\\
&+\frac{M_1}{\Gamma(r_i)}\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\vartheta'(s)\big(\vartheta(\mathfrak{z}_2)-\vartheta(\mathfrak{z}_1)\big)^{r_i-1}|z(s|ds\\
&+\frac{M_2}{\Gamma(r_i)}\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\vartheta'(s)\big(\vartheta(\mathfrak{z}_2)-\vartheta(\mathfrak{z}_1)\big)^{r_i-1}ds\frac{\big(\vartheta(Z)-\vartheta(0)\big)^q}{\Gamma(q+1)}\|z\|\\
&+\frac{h^*}{\Gamma(r_i)}\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\vartheta'(s)\big(\vartheta(\mathfrak{z}_2)-\vartheta(s)\big)^{r_i-1}ds,
\end{aligned}
$$

thus

$$
\begin{aligned}
&\left|\mathcal{F}z(\mathfrak{z}_2)-\mathcal{F}z(\mathfrak{z}_1)\right|\\
&\le\sum_{\mathfrak{z}_1<Z_i<\mathfrak{z}_2}\left|g_i\big(z(Z_i^-)\big)\right|\\
&\quad+\frac{M_1\big[\big(\vartheta(\mathfrak{z}_2)-\vartheta(0)\big)^{r_i-1}-\big(\vartheta(\mathfrak{z}_1)-\vartheta(0)\big)^{r_i-1}\big]}{\Gamma(r_i)}\int_0^{\mathfrak{z}_1}\vartheta'(s)ds\|z\|\\
&\quad+\frac{M_2\big[\big(\vartheta(\mathfrak{z}_2)-\vartheta(0)\big)^{r_i-1}-\big(\vartheta(\mathfrak{z}_1)-\vartheta(0)\big)^{r_i-1}\big]}{\Gamma(r_i)}\int_0^{\mathfrak{z}_1}\vartheta'(s)ds\frac{\big(\vartheta(Z)-\vartheta(0)\big)^q}{\Gamma(q+1)}\|z\|\\
&\quad-\frac{h^*}{\Gamma(r_i+1)}\big(\vartheta(\mathfrak{z}_2)-\vartheta(\mathfrak{z}_1)\big)^{r_i}+\frac{h^*}{\Gamma(r_i+1)}\big(\vartheta(\mathfrak{z}_2)-\vartheta(0)\big)^{r_i}\\
&\quad-\frac{h^*}{\Gamma(r_i+1)}\big(\vartheta(\mathfrak{z}_1)-\vartheta(0)\big)^{r_i}\\
&\quad+\frac{M_1\big(\vartheta(\mathfrak{z}_2)-\vartheta(\mathfrak{z}_1)\big)^{r_i-1}}{\Gamma(r_i)}\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\vartheta'(s)ds\|z\|\\
&\quad+\frac{M_2\big(\vartheta(\mathfrak{z}_2)-\vartheta(\mathfrak{z}_1)\big)^{r_i-1}}{\Gamma(r_i)}\int_{\mathfrak{z}_1}^{\mathfrak{z}_2}\vartheta'(s)ds\frac{\big(\vartheta(Z)-\vartheta(0)\big)^q}{\Gamma(q+1)}\|z\|
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
&+\frac{h^*}{\Gamma(r_i+1)}\big(\vartheta(\mathfrak{z}_2)-\vartheta(\mathfrak{z}_1)\big)^{r_i}\\
&\le \sum_{\mathfrak{z}_1<Z_i<\mathfrak{z}_2} a_1|z(Z_i^-)| + \left(M_1+M_2\frac{\big(\vartheta(Z)-\vartheta(0)\big)^q}{\Gamma(q+1)}\right)\\
&\quad \frac{\left[\big(\vartheta(\mathfrak{z}_2)-\vartheta(0)\big)^{r_i-1}-\big(\vartheta(\mathfrak{z}_1)-\vartheta(0)\big)^{r_i-1}\right]}{\Gamma(r_i)}\big(\vartheta(\mathfrak{z}_1)-\vartheta(0)\big)\|z\|\\
&+\frac{h^*}{\Gamma(r_i+1)}\Big(\big(\vartheta(\mathfrak{z}_2)-\vartheta(0)\big)^{r_i}-\big(\vartheta(\mathfrak{z}_1)-\vartheta(0)\big)^{r_i}\Big)\\
&+\left(M_1+M_2\frac{\big(\vartheta(Z)-\vartheta(0)\big)^q}{\Gamma(q+1)}\right)\frac{\big(\vartheta(\mathfrak{z}_2)-\vartheta(\mathfrak{z}_1)\big)^{r_i}}{\Gamma(r_i)}\|z\|. \qquad (24)
\end{aligned}
$$

Since ϑ is a continuous function, then the right-hand side converges to zero when $\mathfrak{z}_1 \to \mathfrak{z}_2$, hence $\left|\mathcal{F}z(\mathfrak{z}_2)-\mathcal{F}z(\mathfrak{z}_1)\right| \to 0$, then by Arzelà-Ascoli theorem, we deduce that $\mathcal{F}(B_R)$ is equicontinuous.

• Step 4: proving that $\mathcal{F}$ is a condensing map:

Let $\mathfrak{z} \in \mathcal{O}$ and $B \subset B_R$, then

$$
\begin{aligned}
\varsigma(\mathcal{F}(B)) &= \varsigma(\mathcal{F}z(\mathfrak{z}), z \in B)\\
&\le \Bigg\{\varsigma(z_0)+\sum_{0<Z_i<t}\varsigma g_i\big(z(Z_i^-)\big)\\
&\quad+\sum_{0<Z_i<\mathfrak{z}}\frac{1}{\Gamma(r_{i-1})}\int_{Z_{i-1}}^{Z_i}\vartheta'(s)\big(\vartheta(Z_i)-\vartheta(s)\big)^{r_{i-1}-1}\varsigma h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds\\
&\quad+\frac{1}{\Gamma(r_i)}\int_{Z_i}^{\mathfrak{z}}\vartheta'(s)\big(\vartheta(\mathfrak{z})-\vartheta(s)\big)^{r_i-1}\varsigma h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds, z\in B\Bigg\}.
\end{aligned}
$$

Then, from Remark 11 with $\varsigma(z_0) = 0$, we get

$$
\begin{aligned}
&\varsigma(\mathcal{F}(B(\mathfrak{z})))\\
&\le \Bigg\{\sum_{0<Z_i<\mathfrak{z}}\varsigma\big(a_1 z(Z_i^-)\big)\\
&\quad+\sum_{0<Z_i<\mathfrak{z}}\frac{1}{\Gamma(r_{i-1})}\int_{Z_{i-1}}^{Z_i}\vartheta'(s)\big(\vartheta(Z_i)-\vartheta(s)\big)^{r_{i-1}-1}\\
&\quad\times\left[M_1\hat{\varsigma}(B)+M_2\frac{\big(\vartheta(Z)-\vartheta(0)\big)^q}{\Gamma(q+1)}\hat{\varsigma}(B)\right]ds\\
&\quad+\frac{1}{\Gamma(r_i)}\int_{Z_i}^{\mathfrak{z}}\vartheta'(s)\big(\vartheta(\mathfrak{z})-\vartheta(s)\big)^{r_i-1}\left[M_1\hat{\varsigma}(B)+M_2\frac{\big(\vartheta(Z)-\vartheta(0)\big)^q}{\Gamma(q+1)}\hat{\varsigma}(B)\right]ds, z\in B\Bigg\}\\
&\le \sum_{0<Z_i<\mathfrak{z}} a_1\hat{\varsigma}\big(B\big)+\sum_{0<Z_i<\mathfrak{z}}\frac{\big(\vartheta(Z_i)-\vartheta(Z_{i-1})\big)^{r_{i-1}}}{\Gamma(r_{i-1}+1)}\left[M_1+M_2\frac{\big(\vartheta(Z)-\vartheta(0)\big)^q}{\Gamma(q+1)}\right]\hat{\varsigma}(B)\\
&\quad+\frac{\big(\vartheta(\mathfrak{z})-\vartheta(Z_i)\big)^{r_i}}{\Gamma(r_i+1)}\left[M_1+M_2\frac{\big(\vartheta(Z)-\vartheta(0)\big)^q}{\Gamma(q+1)}\right]\hat{\varsigma}(B)
\end{aligned}
$$

$$\leq \left(na_1 + \frac{(n+1)\big(\vartheta(Z)-\vartheta(0)\big)^{r^+}}{\Gamma(r^-+1)}\left[M_1 + M_2\frac{\big(\vartheta(Z)-\vartheta(0)\big)^q}{\Gamma(q+1)}\right]\right)\hat{\varsigma}(B),$$

thus

$$\hat{\varsigma}\big(\mathcal{F}(B)\big) \leq \left(na_1 + \frac{(n+1)\big(\vartheta(Z)-\vartheta(0)\big)^{r^+}}{\Gamma(r^-+1)}\left[M_1 + M_2\frac{\big(\vartheta(Z)-\vartheta(0)\big)^q}{\Gamma(q+1)}\right]\right)\hat{\varsigma}(B).$$

Since

$$\Omega = na_1 + \frac{(n+1)\big(\vartheta(Z)-\vartheta(0)\big)^{r^+}}{\Gamma(r^-+1)}\left[M_1 + M_2\frac{\big(\vartheta(Z)-\vartheta(0)\big)^q}{\Gamma(q+1)}\right] < 1,$$

we get

$$\hat{\varsigma}\big(\mathcal{F}(B)\big) < \hat{\varsigma}(B), \tag{25}$$

thus, by Sadovski's fixed point theorem, we conclude that the operator $\mathcal{F}$ has a fixed point z in B_R, which means that ϑ-IVP (1) has a solution in $PC(\mathcal{O},\mathbb{R})$. □

Now, we shall show the unique existence result by means of Banach's fixed point theorem.

Theorem 17 *Assume that conditions* $[C0]-[C5]$ *hold and*

$$\rho = a_2 n + \frac{(n+1)\big(\vartheta(Z)-\vartheta(0)\big)^{r^+}}{\Gamma(r^-+1)}\left(M_1 + \frac{M_2\big(\vartheta(Z)-\vartheta(0)\big)^q}{\Gamma(q+1)}\right) < 1. \tag{26}$$

Then the impulsive ϑ*-IVP* (1) *possesses a solution uniquely in* $PC(\mathcal{O},\mathbb{R})$.

Proof As we defined earlier, from the integral equation (13), the operator $\mathcal{F}$ that we presented in (17) admits a unique fixed point, which means that the integral equation (13) has a unique solution and that is equivalent to the uniqueness existence of the solution for ϑ-IVP (1).

From (17), and by assuming that conditions $[C0]$-$[C3]$ are satisfied, and for any $y,z \in B_R \subset PC(\mathcal{O},\mathbb{R})$, as we said above, the operator $\mathcal{F}$ is mapping B_R into itself where B_R is a closed, bounded, convex, and nonempty subset of $PC(\mathcal{O},\mathbb{R})$, so for every $t \in \mathcal{O}$ we have

$$\begin{aligned}
&\big|\mathcal{F}y(\mathfrak{z}) - \mathcal{F}z(\mathfrak{z})\big| \\
&\quad \leq \sum_{0<Z_i<\mathfrak{z}} \big|g_i\big(y(Z_i^-)\big) - g_i\big(z(Z_i^-)\big)\big| \\
&\qquad + \sum_{0<Z_i<\mathfrak{z}} \frac{1}{\Gamma(r_{i-1})}\int_{Z_{i-1}}^{Z_i} \vartheta'(s)\big(\vartheta(Z_i)-\vartheta(s)\big)^{r_{i-1}-1}\Big|h\big(s,y(s,\mathcal{I}_{0^+}^{q;\vartheta}y(s)\big) \\
&\qquad -h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)\Big|\,ds \\
&\qquad + \frac{1}{\Gamma(r_i)}\int_{Z_i}^{\mathfrak{z}} \vartheta'(s)\big(\vartheta(\mathfrak{z})-\vartheta(s)\big)^{r_i-1}\Big|h\big(s,y(s,\mathcal{I}_{0^+}^{q;\vartheta}y(s)\big) - h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)\Big|\,ds.
\end{aligned}$$

Then, from [$C4$] and [$C5$], we find

$$\begin{aligned}
&\left|\mathcal{F}y(\mathfrak{z}) - \mathcal{F}z(\mathfrak{z})\right| \\
&\quad \le \sum_{0<Z_i<\mathfrak{z}} a_2|y(Z_i^-) - z(Z_i^-)| \\
&\qquad + \sum_{0<Z_i<\mathfrak{z}} \frac{1}{\Gamma(r_{i-1})} \int_{Z_{i-1}}^{Z_i} \vartheta'(s)\big(\vartheta(Z_i) - \vartheta(s)\big)^{r_{i-1}-1} \big(M_1|y(s - z(s)| \\
&\qquad\quad + M_2 \left|\mathcal{I}_{0^+}^{q;\vartheta} y(s - \mathcal{I}_{0^+}^{q;\vartheta} z(s\right|\Big) ds \\
&\qquad + \frac{1}{\Gamma(r_i)} \int_{Z_i}^{\mathfrak{z}} \vartheta'(s)\big(\vartheta(\mathfrak{z}) - \vartheta(s)\big)^{r_i-1} \Big(M_1|y(s - z(s)| + M_2 \left|\mathcal{I}_{0^+}^{q;\vartheta} y(s - \mathcal{I}_{0^+}^{q;\vartheta} z(s\right|\Big) ds \\
&\quad \le na_2\|y - z\| + \sum_{0<Z_i<\mathfrak{z}} \frac{M_1}{\Gamma(r_{i-1})} \int_{Z_{i-1}}^{Z_i} \vartheta'(s)\big(\vartheta(Z_i) - \vartheta(s)\big)^{r_{i-1}-1} ds\|y - z\| \\
&\qquad + \sum_{0<Z_i<\mathfrak{z}} \frac{M_2}{\Gamma(r_{i-1})} \int_{Z_{i-1}}^{Z_i} \vartheta'(s)\big(\vartheta(Z_i) - \vartheta(s)\big)^{r_{i-1}-1} \left|\mathcal{I}_{0^+}^{q;\vartheta} (y(s - z(s)\right| ds \\
&\qquad + \frac{M_1}{\Gamma(r_i)} \int_{Z_i}^{\mathfrak{z}} \vartheta'(s)\big(\vartheta(\mathfrak{z}) - \vartheta(s)\big)^{r_i-1} ds\|y - z\| \\
&\qquad + \frac{M_2}{\Gamma(r_i)} \int_{Z_i}^{\mathfrak{z}} \vartheta'(s)\big(\vartheta(\mathfrak{z}) - \vartheta(s)\big)^{r_i-1} \left|\mathcal{I}_{0^+}^{q;\vartheta} (y(s - z(s)\right| ds.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\left|\mathcal{F}y(\mathfrak{z}) - \mathcal{F}z(\mathfrak{z})\right| &\le \Bigg(a_2 n + \frac{nM_1}{\Gamma(r^- + 1)}\big(\vartheta(Z) - \vartheta(0)\big)^{r^+} \\
&\qquad + \frac{nM_2}{\Gamma(r^- + 1)}\big(\vartheta(Z) - \vartheta(0)\big)^{r^+} \frac{\big(\vartheta(Z) - \vartheta(0)\big)^q}{\Gamma(q + 1)} \\
&\qquad + \frac{M_1}{\Gamma(r^- + 1)}\big(\vartheta(Z) - \vartheta(0)\big)^{r^+} \\
&\qquad + \frac{M_2}{\Gamma(r^- + 1)}\big(\vartheta(Z) - \vartheta(0)\big)^{r^+} \frac{\big(\vartheta(Z) - \vartheta(0)\big)^q}{\Gamma(q + 1)}\Bigg) \|y - z\| \\
&\le \Bigg(a_2 n + \frac{(n + 1)M_1\big(\vartheta(Z) - \vartheta(0)\big)^{r^+}}{\Gamma(r^- + 1)} \\
&\qquad + \frac{(n + 1)M_2\big(\vartheta(Z) - \vartheta(0)\big)^{r^+ + q}}{\Gamma(r^- + 1)\Gamma(q + 1)}\Bigg) \|y - z\|,
\end{aligned}$$

so

$$\begin{aligned}
&\left|\mathcal{F}y(\mathfrak{z}) - \mathcal{F}z(\mathfrak{z})\right| \\
&\quad \le \left(a_2 n + \frac{M_1(n + 1)\big(\vartheta(Z) - \vartheta(0)\big)^{r^+}}{\Gamma(r^- + 1)} + \frac{M_2(n + 1)\big(\vartheta(Z) - \vartheta(0)\big)^{r^+ + q}}{\Gamma(r^- + 1)\Gamma(q + 1)}\right) \|y - z\|,
\end{aligned}$$

we set

$$\rho = a_2 n + \frac{(n + 1)\big(\vartheta(Z) - \vartheta(0)\big)^{r^+}}{\Gamma(r^- + 1)} \left(M_1 + \frac{M_2\big(\vartheta(Z) - \vartheta(0)\big)^q}{\Gamma(q + 1)}\right),$$

from (16), we have $\rho < 1$, therefore

$$\|\mathcal{F}y - \mathcal{F}z\| \leq \rho\|y - z\|. \tag{27}$$

Hence, the operator $\mathcal{F}$ is a contraction. Consequently, by Sadovski's fixed point theorem, it follows that $\mathcal{F}$ has a unique fixed point in $PC(\mathcal{O},\mathbb{R})$, which is the unique solution of ϑ-IVP (1). □

4 Ulam–Hyers stability

In this section we discuss the stability for the solutions of the impulsive ϑ-IVP (1) by means the Ulam–Hyers–Rassias (UHR) stability.

Definition 18 [34, 38] Let $\vartheta \in C(\mathcal{O},\mathbb{R})$. ϑ-IVP (1) is said to be (UHR) stable with respect to ϑ if there exists $c_h > 0$ such that for any $\varepsilon > 0$ and for every solution $z \in PC^m(\mathcal{O},\mathbb{R})$ of the following inequality

$$\left|{}^{c}\mathcal{D}_{0^+}^{r(\mathfrak{z}),\vartheta(\mathfrak{z})} z(\mathfrak{z}) - h\left(\mathfrak{z}, z(\mathfrak{z}), \mathcal{I}_{0^+}^{q;\vartheta} z(\mathfrak{z})\right)\right| \leq \varepsilon\vartheta(\mathfrak{z}), \quad \mathfrak{z} \in \mathcal{O}, \tag{28}$$

there exists a solution $y \in PC^m(\mathcal{O},\mathbb{R})$ of ϑ-IVP (1) with

$$\|z - y\| \leq c_h\varepsilon\vartheta(\mathfrak{z}), \quad \mathfrak{z} \in \mathcal{O}. \tag{29}$$

Before showing the last result, we will present the next condition:

- [$C6$] Assuming that $\vartheta \in C(\mathcal{O},\mathbb{R})$ is increasing, there exists $\lambda_\vartheta > 0$ such that for any $\mathfrak{z} \in \mathcal{O}_i$ we have

$$\mathcal{I}_{0^+}^{r_i;\vartheta}\vartheta(\mathfrak{z}) \leq \lambda_\vartheta\vartheta(\mathfrak{z}). \tag{30}$$

Theorem 19 *Suppose that* [$C6$] *holds by considering the hypotheses of Theorem* 17. *Then the impulsive* ϑ*-IVP* (1) *is* (*UHR*) *stable.*

Proof Let z satisfy inequality (28), then by integration we obtain

$$\begin{aligned} &\Bigg| z(\mathfrak{z}) - z_0 - \sum_{0<Z_i<\mathfrak{z}} g_i\left(z(Z_i^-)\right) \\ &\quad - \sum_{0<Z_i<\mathfrak{z}} \frac{1}{\Gamma(r_{i-1})}\int_{Z_{i-1}}^{Z_i} \vartheta'(s)\left(\vartheta(Z_i) - \vartheta(s)\right)^{r_{i-1}-1} h\left(s, z(s), \mathcal{I}_{0^+}^{q;\vartheta} z(s)\right)ds \\ &\quad - \frac{1}{\Gamma(r_i)}\int_{Z_i}^{\mathfrak{z}} \vartheta'(s)\left(\vartheta(\mathfrak{z}) - \vartheta(s)\right)^{r_i-1} h\left(s, z(s), \mathcal{I}_{0^+}^{q;\vartheta} z(s)\right)ds \Bigg| \leq \varepsilon\mathcal{I}_{0^+}^{r_i;\vartheta}\vartheta(\mathfrak{z}). \end{aligned}$$

Hence, from [$C6$], it follows that

$$\begin{aligned} &\Bigg| z(\mathfrak{z}) - z_0 - \sum_{0<Z_i<\mathfrak{z}} g_i\left(z(Z_i^-)\right) \\ &\quad - \sum_{0<Z_i<\mathfrak{z}} \frac{1}{\Gamma(r_{i-1})}\int_{Z_{i-1}}^{Z_i} \vartheta'(s)\left(\vartheta(Z_i) - \vartheta(s)\right)^{r_{i-1}-1} h\left(s, z(s), \mathcal{I}_{0^+}^{q;\vartheta} z(s)\right)ds \end{aligned}$$

$$-\frac{1}{\Gamma(r_i)}\int_{Z_i}^{\mathfrak{z}}\vartheta'(s)\big(\vartheta(\mathfrak{z})-\vartheta(s)\big)^{r_i-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds\Bigg|\leq\varepsilon\lambda_\vartheta\vartheta(\mathfrak{z}).$$

Let y be the unique solution of ϑ-IVP (1), then y can be written as follows:

$$\begin{aligned}y(\mathfrak{z})&=z_0+\sum_{k=1}^{i}g_k\big(y(Z_k^-)\big)+\sum_{k=1}^{i}\frac{1}{\Gamma(r_{k-1})}\int_{Z_{k-1}}^{Z_k}\vartheta'(s)\big(\vartheta(Z_k)-\vartheta(s)\big)^{r_{k-1}-1}h\big(s,y(s,\mathcal{I}_{0^+}^{q;\vartheta}y(s)\big)ds\\&\quad+\frac{1}{\Gamma(r_i)}\int_{Z_i}^{\mathfrak{z}}\vartheta'(s)\big(\vartheta(\mathfrak{z})-\vartheta(s)\big)^{r_i-1}h\big(s,y(s,\mathcal{I}_{0^+}^{q;\vartheta}y(s)\big)ds,\quad i=1,\dots,n\ \ \mathfrak{z}\in(Z_{i-1},Z_i],\end{aligned}$$

then we have for all $\mathfrak{z}\in\mathcal{O}$

$$\begin{aligned}&|z(\mathfrak{z})-y(\mathfrak{z})|\\&\quad=\Bigg|z(\mathfrak{z})-z_0-\sum_{0<Z_i<\mathfrak{z}}g_i\big(y(Z_i^-)\big)\\&\qquad-\sum_{0<Z_i<\mathfrak{z}}\frac{1}{\Gamma(r_{i-1})}\int_{Z_{i-1}}^{Z_i}\vartheta'(s)\big(\vartheta(Z_i)-\vartheta(s)\big)^{r_{i-1}-1}h\big(s,y(s,\mathcal{I}_{0^+}^{q;\vartheta}y(s)\big)ds\\&\qquad-\frac{1}{\Gamma(r_i)}\int_{Z_i}^{\mathfrak{z}}\vartheta'(s)\big(\vartheta(\mathfrak{z})-\vartheta(s)\big)^{r_i-1}h\big(s,y(s,\mathcal{I}_{0^+}^{q;\vartheta}y(s)\big)ds\Bigg|\\&\quad\leq\Bigg|z(\mathfrak{z})-z_0-\sum_{0<Z_i<\mathfrak{z}}g_i\big(z(Z_i^-)\big)\\&\qquad-\sum_{0<Z_i<\mathfrak{z}}\frac{1}{\Gamma(r_{i-1})}\int_{Z_{i-1}}^{Z_i}\vartheta'(s)\big(\vartheta(Z_i)-\vartheta(s)\big)^{r_{i-1}-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds\\&\qquad-\frac{1}{\Gamma(r_i)}\int_{Z_i}^{\mathfrak{z}}\vartheta'(s)\big(\vartheta(\mathfrak{z})-\vartheta(s)\big)^{r_i-1}h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)ds\Bigg|\\&\qquad+\sum_{0<Z_i<\mathfrak{z}}\big|g_i\big(z(Z_i^-)\big)-g_i\big(y(Z_i^-)\big)\big|\\&\qquad+\sum_{0<Z_i<\mathfrak{z}}\frac{1}{\Gamma(r_{i-1})}\int_{Z_{i-1}}^{Z_i}\vartheta'(s)\big(\vartheta(Z_i)-\vartheta(s)\big)^{r_{i-1}-1}\Big|h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)\\&\qquad\quad-h\big(s,y(s,\mathcal{I}_{0^+}^{q;\vartheta}y(s)\big)\Big|ds\\&\qquad+\frac{1}{\Gamma(r_i)}\int_{Z_i}^{\mathfrak{z}}\vartheta'(s)\big(\vartheta(\mathfrak{z})-\vartheta(s)\big)^{r_i-1}\Big|h\big(s,z(s),\mathcal{I}_{0^+}^{q;\vartheta}z(s)\big)-h\big(s,y(s,\mathcal{I}_{0^+}^{q;\vartheta}y(s)\big)\Big|ds.\end{aligned}$$

Hence, in view of [$C4$] and [$C5$], we get

$$\begin{aligned}&|z(\mathfrak{z})-y(\mathfrak{z})|\\&\quad\leq\varepsilon\lambda_\vartheta\vartheta(\mathfrak{z})+\sum_{0<Z_i<\mathfrak{z}}a_2\big|z(Z_i^-)-y(Z_i^-)\big|\\&\qquad+\sum_{0<Z_i<\mathfrak{z}}\frac{1}{\Gamma(r_{i-1})}\int_{Z_{i-1}}^{Z_i}\vartheta'(s)\big(\vartheta(Z_i)-\vartheta(s)\big)^{r_{i-1}-1}\big(M_1|z(s-y(s|\end{aligned}$$

$$
\begin{aligned}
&\quad + M_2\left|\mathcal{I}_{0^+}^{q;\vartheta} z(s - \mathcal{I}_{0^+}^{q;\vartheta} y(s|\right) ds \\
&\quad + \frac{1}{\Gamma(r_i)}\int_{Z_i}^{\mathfrak{z}} \vartheta'(s)\big(\vartheta(\mathfrak{z}) - \vartheta(s)\big)^{r_i-1}\left(M_1|z(s - y(s| + M_2\left|\mathcal{I}_{0^+}^{q;\vartheta} z(s - \mathcal{I}_{0^+}^{q;\vartheta} y(s|\right) ds \\
&\le \varepsilon\lambda_\vartheta \vartheta(\mathfrak{z}) + a_2 n\|z - y\| \\
&\quad + \left(\frac{M_1 n\big(\vartheta(Z) - \vartheta(0)\big)^{r^+}}{\Gamma(r^- + 1)} + \frac{M_2 n\big(\vartheta(Z) - \vartheta(0)\big)^{r^+}}{\Gamma(r^- + 1)}\frac{\big(\vartheta(Z) - \vartheta(0)\big)^{q}}{\Gamma(q + 1)}\right. \\
&\quad \left. + \frac{M_1\big(\vartheta(\mathfrak{z}) - \vartheta(Z_i)\big)^{r_i}}{\Gamma(r_i + 1)} + \frac{M_2\big(\vartheta(\mathfrak{z}) - \vartheta(Z_i)\big)^{r_i}}{\Gamma(r_i + 1)}\frac{\big(\vartheta(Z) - \vartheta(0)\big)^{q}}{\Gamma(q + 1)}\right)\|z - y\| \\
&\le \varepsilon\lambda_\vartheta \vartheta(t) + \left(a_2 n + \frac{M_1(n+1)\big(\vartheta(Z) - \vartheta(0)\big)^{r^+}}{\Gamma(r^- + 1)}\right. \\
&\quad \left. + \frac{M_2(n+1)\big(\vartheta(Z) - \vartheta(0)\big)^{r^+ + q}}{\Gamma(r^- + 1)\Gamma(q + 1)}\right)\|z - y\|.
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
\|z - y\| \le \varepsilon\lambda_\vartheta \vartheta(t) &+ \left(a_2 n + \frac{M_1(n+1)\big(\vartheta(Z) - \vartheta(0)\big)^{r^+}}{\Gamma(r^- + 1)}\right. \\
&\left. + \frac{M_2(n+1)\big(\vartheta(Z) - \vartheta(0)\big)^{r^+ + q}}{\Gamma(r^- + 1)\Gamma(q + 1)}\right)\|z - y\|,
\end{aligned}
$$

thus

$$
\begin{aligned}
&\left(1 - a_2 n - \frac{M_1(n+1)\big(\vartheta(Z) - \vartheta(0)\big)^{r^+}}{\Gamma(r^- + 1)} - \frac{M_2(n+1)\big(\vartheta(Z) - \vartheta(0)\big)^{r^+ + q}}{\Gamma(r^- + 1)\Gamma(q + 1)}\right)\|z - y\| \\
&\le \varepsilon\lambda_\vartheta \vartheta(\mathfrak{z}),
\end{aligned}
$$

and from the assumption

$$
\rho := a_2 n + \frac{M_1(n+1)\big(\vartheta(Z) - \vartheta(0)\big)^{r^+}}{\Gamma(r^- + 1)} + \frac{M_2(n+1)\big(\vartheta(Z) - \vartheta(0)\big)^{r^+ + q}}{\Gamma(r^- + 1)\Gamma(q + 1)} < 1,
$$

we obtain

$$
\|z - y\| \le \frac{\lambda_\vartheta}{1 - \rho}\varepsilon\vartheta(\mathfrak{z}) := c_h \varepsilon\vartheta(\mathfrak{z}). \tag{31}
$$

Therefore, ϑ-IVP (1) is (UHR) stable with respect to ϑ. □

5 Examples

Example 20 Consider the following impulsive ϑ-IVP:

$$\begin{cases} {}^{c}\mathcal{D}_{0^+}^{r(\mathfrak{z}),\mathfrak{z}^2+1} z(\mathfrak{z}) = 3\mathfrak{z}^2 + \dfrac{2\mathfrak{z}+1}{e^{\mathfrak{z}^2+1}(|z(\mathfrak{z})|+\pi)} - \dfrac{1}{\sqrt{2}} tan^{-1}\left(\mathcal{I}_{0^+}^{3.1;\mathfrak{z}^2+1} z(\mathfrak{z})\right), & \mathfrak{z} \in \mathcal{O}_1 \cup \mathcal{O}_2, \\ z(0) = 1, \quad z'_\vartheta(0) = z''_\vartheta(0) = z_\vartheta^{[3]}(0) = z_\vartheta^{[4]}(0) = z_\vartheta^{[5]}(0) = 0, \\ \Delta z\big|_{\mathfrak{z}=\frac{1}{2}} = \dfrac{z\left(\frac{1}{2}^-\right)}{e^{z\left(\frac{1}{2}^-\right)^2+1} + \sqrt{3}}, \end{cases} \tag{32}$$

where $m = 5$, $Z_0 = 0$, $Z_1 = \frac{1}{2}$, $Z_2 = Z = 1$, $n = 1$, $\mathcal{O} = [0,1]$, $\mathcal{O}_0 = \left[0, \frac{1}{2}\right]$, $\mathcal{O}_1 = \left]\frac{1}{2}, 1\right]$ and

$$r(\mathfrak{z}) = \begin{cases} 4.3, & if \ \ \mathfrak{z} \in \mathcal{O}_0, \\ 4.7, & if \ \ \mathfrak{z} \in \mathcal{O}_1, \end{cases}$$

then $r^+ = 4.7$, $r^- = 4.3$, $\vartheta(\mathfrak{z}) = \mathfrak{z}^2 + 1$, and $q = 3.1$.

For $\mathfrak{z} \in \mathcal{O}$, we have

$$h\left(\mathfrak{z}, z(\mathfrak{z}), \mathcal{I}_{0^+}^{3.1;\mathfrak{z}^2+1} z(\mathfrak{z})\right) = 3\mathfrak{z}^2 + \frac{2\mathfrak{z}+1}{e^{\mathfrak{z}^2+1}(|z(\mathfrak{z})|+\pi)} - \frac{1}{\sqrt{2}} tan^{-1}\left(\mathcal{I}_{0^+}^{3.1;\mathfrak{z}^2+1} z(\mathfrak{z})\right).$$

Then, for every $(\mathfrak{z}, x, y) \in \mathcal{O} \times \mathbb{R}^2$, we have

$$|h(\mathfrak{z}, x, y)| \le \mathfrak{P}(\mathfrak{z}) = 3\mathfrak{z}^2 + \frac{2\mathfrak{z}+1}{\pi e^{\mathfrak{z}^2+1}} + \frac{\pi}{2\sqrt{2}}, \quad \mathfrak{z} \in \mathcal{O},$$

so

$$\|\mathfrak{P}\| = 3 + \frac{3}{\pi e} + \frac{\pi}{2\sqrt{2}},$$

which means that condition $[C1]$ holds.

From $[C2]$, we have

$$|g_1(x)| = \left|\frac{x}{e^{x^2+1} + \sqrt{3}}\right| \le \frac{1}{e + \sqrt{3}}|x| = a_1|x|.$$

Now, we will prove that condition $[C3]$ is fulfilled. For all $t \in \mathcal{O}$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$, we have

$$|h(\mathfrak{z}, x_1, y_1) - h(\mathfrak{z}, x_2, y_2)| \le \frac{3}{\pi^2 e}|x_1 - x_2| + \frac{1}{\sqrt{2}}|y_1 - y_2|,$$

which means that $M_1 = \frac{3}{\pi^2 e}$ and $M_2 = \frac{1}{\sqrt{2}}$, then

$$\begin{aligned} \Omega &= (1)\left(\frac{1}{e+\sqrt{3}}\right) + \frac{(1+1)\big(\vartheta(1) - \vartheta(0)\big)^{4.7}}{\Gamma(4.3+1)}\left(\frac{3}{\pi^2 e} + \frac{1}{\sqrt{2}}\frac{\big(\vartheta(1) - \vartheta(0)\big)^{3.1}}{\Gamma(3.1+1)}\right) \\ &= \frac{1}{e+\sqrt{3}} + \frac{2}{\Gamma(5.3)}\left(\frac{3}{\pi^2 e} + \frac{1}{\sqrt{2}\Gamma(4.1)}\right) \end{aligned}$$

$$\approx 0.23602725 < 1.$$

Thus, all the conditions of Theorem 16 hold. Hence, by Sadovski's fixed point theorem, the impulsive ϑ-IVP (32) has a solution in $PC(\mathcal{O},\mathbb{R})$.

Example 21 Consider the impulsive ϑ-IVP

$$\begin{cases} {}^{c}\mathcal{D}_{0^+}^{r(\mathfrak{z}),\mathfrak{z}^2} z(\mathfrak{z}) = \dfrac{sin(\mathfrak{z}^2+1)}{\mathfrak{z}+19} - \dfrac{1}{\sqrt{3}}\dfrac{cos(-z(\mathfrak{z})+1)}{e^{\mathfrak{z}}} + \dfrac{cos(\mathfrak{z})}{4\sqrt{\pi + \left(\mathcal{I}_{0^+}^{1.6;\mathfrak{z}^2} z(\mathfrak{z})\right)^2}}, & \mathfrak{z} \in \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3, \\ z(0) = \dfrac{e^2}{4\pi}, \quad z'_{\vartheta}(0) = z''_{\vartheta}(0) = z_{\vartheta}^{[3]}(0) = 0, \\ \Delta z\big|_{\mathfrak{z}=\frac{1}{3}} = \dfrac{z\left(\frac{1}{3}^-\right)}{e^{\pi} + \left|z\left(\frac{1}{3}^-\right)\right|}, \qquad \Delta z\big|_{\mathfrak{z}=\frac{2}{3}} = \dfrac{e^{-\frac{2}{3}} sin\left(z\left(\frac{2}{3}^-\right)\right)}{6\sqrt{cos\left(z\left(\frac{2}{3}^-\right)\right) + \pi}}, \end{cases} \tag{33}$$

where $m = 4$, $Z_0 = 0$, $Z_1 = \frac{1}{3}$, $Z_2 = \frac{2}{3}$, $Z_3 = Z = 1$, $n = 2$ and $\mathcal{O} = [0,1]$, $\mathcal{O}_0 = \left[0, \frac{1}{3}\right]$, $\mathcal{O}_1 = \left]\frac{1}{3}, \frac{2}{3}\right]$, $\mathcal{O}_2 = \left]\frac{2}{3}, 1\right]$, and

$$r(\mathfrak{z}) = \begin{cases} 3.9, & if \ \ \mathfrak{z} \in \mathcal{O}_0, \\ 3.4, & if \ \ \mathfrak{z} \in \mathcal{O}_1, \\ 3.2, & if \ \ \mathfrak{z} \in \mathcal{O}_2, \end{cases}$$

with $r^+ = 3.9$, $r^- = 3.2$, and $\vartheta(\mathfrak{z}) = \mathfrak{z}^2$, $q = 1.6$.

For $\mathfrak{z} \in \mathcal{O}$, we set

$$h\left(\mathfrak{z}, z(\mathfrak{z}), \mathcal{I}_{0^+}^{1.6;\mathfrak{z}^2} z(\mathfrak{z})\right) = \frac{sin(\mathfrak{z}^2+1)}{t+19} - \frac{1}{\sqrt{3}}\frac{cos(-z(\mathfrak{z})+1)}{e^{\mathfrak{z}}} + \frac{cos(\mathfrak{z})}{4\sqrt{\pi + \left(\mathcal{I}_{0^+}^{1.6;\mathfrak{z}^2} z(\mathfrak{z})\right)^2}}.$$

Then, for any $(\mathfrak{z}, x, y) \in \mathcal{O} \times \mathbb{R}^2$, we have

$$|h(\mathfrak{z}, x, y)| \le \mathfrak{P}(\mathfrak{z}) = \frac{sin(\mathfrak{z}^2+1)}{\mathfrak{z}+19} + \frac{1}{\sqrt{3}e^{\mathfrak{z}}} + \frac{cos(\mathfrak{z})}{4\sqrt{\pi}},$$

then

$$\|\mathfrak{P}\| = \frac{1}{19} + \frac{1}{\sqrt{3}} + \frac{1}{4\sqrt{\pi}}.$$

From condition $[C2]$ we find

$$|g_1(x)| = \left|\frac{x}{e^{\pi} + |x|}\right| \le \frac{1}{e^{\pi}}|x|$$

and

$$|g_2(x)| = \left|\frac{e^{-\frac{2}{3}} sin(x)}{6\sqrt{cos(x) + \pi}}\right| \le \frac{e^{-\frac{2}{3}}}{6\sqrt{\pi - 1}}|x|,$$

then we take $a_1 = \frac{e^{-\frac{2}{3}}}{6\sqrt{\pi-1}}$.

For every $\mathfrak{z} \in \mathcal{O}$, $x_1, x_2, y_1, y_2 \in \mathbb{R}$, we have

$$\begin{aligned} |h(\mathfrak{z}, x_1, y_1) - h(\mathfrak{z}, x_2, y_2)| &\leq \frac{1}{\sqrt{3}}|x_1 - x_2| + \frac{1}{4}\frac{2}{\sqrt{27}\pi}|y_1 - y_2| \\ &\leq \frac{1}{\sqrt{3}}|x_1 - x_2| + \frac{1}{2\sqrt{27}\pi}|y_1 - y_2|, \end{aligned}$$

and that makes us set $M_1 = \frac{1}{\sqrt{3}}$ and $M_2 = \frac{1}{2\sqrt{27}\pi}$. Then, from condition [$C5$], we have for any $x, y \in \mathbb{R}$

$$|g_1(x) - g_1(y)| \leq \frac{2}{e^{\pi}}|x - y|$$

and

$$|g_2(x) - g_2(y)| \leq \frac{e^{-\frac{2}{3}}(2\pi + 1)}{12(\pi - 1)^{\frac{3}{2}}}|x - y|.$$

Then we take $a_2 = \frac{e^{-\frac{2}{3}}(2\pi+1)}{12(\pi-1)^{\frac{3}{2}}}$.

Then, by applying Theorem 17, we get

$$\begin{aligned} \rho &= (2)\left(\frac{e^{-\frac{2}{3}}(2\pi + 1)}{12(\pi - 1)^{\frac{3}{2}}}\right) + \frac{(2+1)\big(\vartheta(1) - \vartheta(0)\big)^{3.9}}{\Gamma(3.2 + 1)}\left(\frac{1}{\sqrt{3}} + \frac{1}{2\sqrt{27}\pi}\frac{\big(\vartheta(1) - \vartheta(0)\big)^{1.6}}{\Gamma(1.6 + 1)}\right) \\ &= \frac{e^{-\frac{2}{3}}(2\pi + 1)}{6(\pi - 1)^{\frac{3}{2}}} + \frac{3}{\Gamma(4.2)}\left(\frac{1}{\sqrt{3}} + \frac{1}{2\sqrt{27}\pi\Gamma(2.6)}\right) \\ &\approx 0.4304383 < 1. \end{aligned}$$

Hence, by Banach's fixed point theorem, the impulsive ϑ-IVP (33) possesses a unique solution in $PC(\mathcal{O}, \mathbb{R})$.

Let $\vartheta(\mathfrak{z}) = \mathfrak{z}^{\frac{1}{2}}$, then for every $\mathfrak{z} \in \mathcal{O}$, $i \in \{1, 2, 3\}$, we have

$$\begin{aligned} \mathcal{I}_{0^+}^{r_i;\mathfrak{z}^2}\vartheta(\mathfrak{z}) &= \frac{1}{\Gamma(r_i)}\int_0^{\mathfrak{z}} 2s(\mathfrak{z}^2 - s^2)^{r_i - 1}s^{\frac{1}{2}}ds \\ &\leq \frac{1}{\Gamma(r_i)}\int_0^{\mathfrak{z}} 2s(\mathfrak{z}^2 - s^2)^{r_i - 1}ds \\ &\leq \frac{(\mathfrak{z}^2)^{r_i}}{\Gamma(r_i + 1)} \leq \frac{1}{\Gamma(r_i + 1)}\mathfrak{z}^{\frac{1}{2}} \leq \frac{1}{\Gamma(r^- + 1)}\mathfrak{z}^{\frac{1}{2}}. \end{aligned}$$

Thus, condition [$C6$] is fulfilled for $\vartheta(\mathfrak{z}) = \mathfrak{z}^{\frac{1}{2}}$ and $\lambda_{\vartheta} = \frac{1}{\Gamma(4.2)}$. By Theorem 19, the impulsive ϑ-IVP (33) is (UHR) stable with respect to ϑ.

6 Conclusion and perspectives

In this study, we have successfully established the existence, uniqueness, and Ulam–Hyers stability of solutions for a Caputo variable order ϑ-initial value problem (ϑ-IVP) with multi-point initial conditions. By employing Sadovski's and Banach's fixed point theorems in conjunction with the Kuratowski measure of noncompactness, we provided rigorous proofs for the main results. The theoretical findings were further substantiated through a numerical example, demonstrating the practical applicability of the developed methods.

Looking forward, there are several promising directions for future research. One potential avenue is to extend the current framework to more complex differential systems involving nonlocal and impulsive conditions. Additionally, exploring the applications of these results in various scientific and engineering problems could provide deeper insights and broader utility. Another interesting perspective is to investigate the stability and control of solutions under different types of perturbations and in higher-dimensional settings. These explorations could significantly enhance the theoretical foundation and practical applications of fractional differential equations in diverse fields.

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