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Creating new contractive mappings to obtain fixed points and data-dependence results under auxiliary functions

Hasanen A. Hammad[1,2,3*] and Doha A. Kattan[4]

*Correspondence:
h.abdelwareth@qu.edu.sa
[1]Department of Mathematics, College of Science, Qassim University, Buraydah, 51452, Saudi Arabia
[2]Department of Mathematics, Saveetha School of Engineering, SIMATS, Saveetha University, Chennai, 602105, India
Full list of author information is available at the end of the article

Abstract

This manuscript is concerned with obtaining results for fixed points that arise from new contractive mappings on controlled metric spaces. These mappings are a mixture of Wardowski's contractions with both multivalued, nonlinear mappings and auxiliary functions. It is also proved that the obtained fixed-point outcomes are well-posed. Additionally, a data-dependence result for fixed points is given. To aid with understanding, several illustrative examples are also provided. Numerous findings that are currently in the literature are specific instances of the findings that were made.

Mathematics Subject Classification: 47H10; 47H20

Keywords: Fixed-point techniques; Hybrid contraction mappings; Wardowski's contractions; Evaluation metrics; Data-dependence results; Sequence analysis

1 Introduction

Fixed-point (FP) theory is a mathematical discipline that studies the existence, uniqueness, and properties of solutions to equations of the form ($\Upsilon\upsilon = \upsilon$), where ($\Upsilon$) is a given function. Although this equation appears simple, it has profound implications and finds applications across various domains, from pure mathematics to real-world problem solving in economics, physics, computer science, and beyond. On metric spaces (MSs), Stephan Banach established the renowned Banach contraction principle (BCP) [1] in 1922.

There are two ways to build new FP results: either with the contraction inequality or utilize a more generalized space. BCP is expanded upon and altered in a variety of ways. For instance, the authors of [2, 3] and [4, 5] altered the underlined space and examined Kannan-type contractions to support specific fixed-point conclusions.

In another direction, many authors have extended numerous previous findings using F-contractions. In 2015, Klim and Wardowski [6] extended an F-contraction in terms of nonlinear F-contractions. By using the dynamic processes, the same authors demonstrated a FP theorem and expanded the concept of F-contractive mappings to the situation of nonlinear F-contractions. Subsequently, Wardowski [7] eliminated one of the criteria on the

F-mappings to create nonlinear F-contractions. Some theorems on the presence of fixed points of nonlinear F-contractions and the sum of mappings of this kind with a compact operator can be found in another paper by Wardowski [8].

FP theory has found significant applications in the realm of fractional calculus, particularly in the analysis of fractional differential equations. By establishing the existence and uniqueness of solutions to these equations, fixed-point theorems provide a powerful framework for investigating various phenomena in fields such as physics, engineering, and biology. The nonlocal nature of fractional derivatives, combined with the ability of fixed-point theorems to handle nonlinear operators, makes them well-suited for addressing complex problems involving fractional dynamics, see [9–14]

The concept of b-metric space (bMS) was first introduced by Bakhtin in 1989 [4]. Czerwick [15] added to it in order to provide certain FP outcomes made possible by this environment. In order to reduce the triangle inequality of a bMS, Kamran et al. [5] established a new route using a function $z : \eta \times \eta \to [1, \infty)$. By defining an extended bMS as a controlled metric space (CMS) and expanding upon its concept, Mlaiki et al. [16] achieved another breakthrough in this regard.

Estimating the separation between the sets of FPs of two mappings is a data-dependence problem. This notion becomes important only if we are certain that these two operators have nonempty FP sets. Since multivalued mappings (MVMs) frequently have larger FP sets than single-valued mappings, the data-dependence problem primarily affects set-valued mappings. Iqbal et al. [17] addressed data dependence, strict FPs, and the well-posedness of certain multivalued generalized contractions in the context of complete MSs in 2021. They also covered the existence of FPs. In the setting of CMSs, we generalize and unify the findings of Iqbal et al. [17] in this study under new contractive mappings.

2 Basic facts

This section is devoted to recalling some basic facts, which are needed to understand the manuscript. We shall consider (η, ϖ), $Q(\eta)$, η^c, η^{cb}, and η^{cp} to denote, respectively, a MS, containing all subsets of η, the sets of nonempty, closed subsets of η, nonempty, closed, and bounded subsets of η, and nonempty, compact subsets of η.

Assume that $\mho : \eta \to Q(\eta)$ is a MVM, the point $\vartheta \in \eta$ is called a FP of $\mho$ if $\vartheta \in \mho\vartheta$. The point $\widehat{\vartheta} \in \eta$ is said to be a strict FP if $\{\widehat{\vartheta}\} = \mho\vartheta$. The set of all (s.o.a.) FPs, and the s.o.a. strict FPs of $\mho$ are denoted by $F_{ix}(\mho)$ and $SF_{ix}(\mho)$, respectively.

Definition 2.1 [16] Assume that $\eta \neq \emptyset$ and $\gamma : \eta \times \eta \to [1, \infty)$ is a given function. The distance mapping $\varpi : \eta \times \eta \to [0, \infty)$ is called a CMS if the assertions below hold, for all $\vartheta_1, \vartheta_2, \vartheta_3 \in \eta$,

(ϖ_1) $\varpi(\vartheta_1, \vartheta_2) = 0$ if and only if $\vartheta_1 = \vartheta_2$;

(ϖ_2) $\varpi(\vartheta_1, \vartheta_2) = \varpi(\vartheta_2, \vartheta_1)$;

(ϖ_3) $\varpi(\vartheta_1, \vartheta_2) \leq \gamma(\vartheta_1, \vartheta_3)\varpi(\vartheta_1, \vartheta_3) + \gamma(\vartheta_3, \vartheta_2)\varpi(\vartheta_3, \vartheta_2)$.

Then, the trio (η, ϖ, γ) is called a CMS.

The definition of a Pompei–Hausdorff (PH) MS is defined by the authors in [18], where they considered that $\Re, \Theta \in NCB(\eta)$ and defined the mapping $\Upsilon : \eta^{cb} \times \eta^{cb} \to [0,\infty)$ by

$$\Upsilon(\Re,\Theta) = \max\left\{\sup_{\vartheta\in\Re} B(\vartheta,\Theta), \sup_{\widetilde{\vartheta}\in\Theta} B\left(\widetilde{\vartheta},\Re\right)\right\},$$

where $B(\vartheta,\Theta) = \left\{\inf \varpi\left(\vartheta,\widetilde{\vartheta}\right) : \widetilde{\vartheta}\in\Theta\right\}$ and $\Upsilon(\Re,\Theta)$ is called a Hausdorff distance.

In [7], Wardowski presented a wonderful definition (F-contraction) by selecting a strictly increasing function to the Banach contraction mapping: the function $F : (0,\infty) \to \mathbb{R}$ fulfills the axioms below:

($\heartsuit_1$) for each $\vartheta, \widetilde{\vartheta} \in (0,\infty)$, if $\vartheta < \widetilde{\vartheta}$, then $F(\vartheta) < F(\widetilde{\vartheta})$, that is, F is strictly increasing;

($\heartsuit_2$) $\lim_{u\to\infty}\Phi_u = 0 \Leftrightarrow \lim_{u\to\infty} F(\Phi_u) = -\infty$, for all sequences $\Phi_u \subseteq (0,\infty)$;

($\heartsuit_3$) there is $l \in (0,1)$ such that $\lim_{u\to 0^+}\Phi^l F(\Phi) = 0$.

Assume that $\triangledown(\Xi)$ is the s.o.a. functions F, which satisfy ($\heartsuit_1$), ($\heartsuit_2$), and ($\heartsuit_3$). Further, let

$$\triangledown(Z) = \{F \in \triangledown(\Xi) : (\heartsuit_3) \text{ is true for } F\},$$

where

($\heartsuit_4$) for all $\Re \in (0,\infty)$ with $\inf(\Re) > 0$, $F(\inf\Re) = \inf F(\Re)$.

The results of Turinici [19] can be obtained if we change the axiom ($\heartsuit_2$) to

($\heartsuit_2'$) $\lim_{u\to\infty} F(s) = -\infty$.

Let us consider $\triangledown(\widetilde{Z})$ to denote the s.o.a. functions F that fulfill ($\heartsuit_1$), ($\heartsuit_2'$), ($\heartsuit_3$), and ($\heartsuit_4$).

Now, for all $\vartheta, \widetilde{\vartheta} \in \eta$, if there are $\nu > 0$ and $F \in \triangledown(Z)$ such that

$$\Upsilon\left(\mho\vartheta, \mho\widetilde{\vartheta}\right) > 0 \Rightarrow \nu + F\left(\Upsilon\left(\mho\vartheta, \mho\widetilde{\vartheta}\right)\right) \le F\left(\varpi\left(\vartheta,\widetilde{\vartheta}\right)\right),$$

then the mapping $\mho : \eta \to \eta^{cb}$ is said to be a multivalued F-contraction [20].

Definition 2.2 [21] Assume that there exist $F \in \triangledown(Z)$ and a function $\sigma : (0,\infty) \to (0,\infty)$ such that the assumptions below hold:

(A_1) for all $\Phi > 0$, $\liminf_{\kappa\to\Phi^+}\sigma(\kappa) > 0$;

(A_2) for all $\vartheta, \widetilde{\vartheta} \in \eta$ with $\mho\vartheta \neq \mho\widetilde{\vartheta}$,

$$\sigma\left(\varpi\left(\vartheta,\widetilde{\vartheta}\right)\right) + F\left(\Upsilon\left(\mho\vartheta,\mho\widetilde{\vartheta}\right)\right) \le F\left(\varpi\left(\vartheta,\widetilde{\vartheta}\right)\right).$$

Then, the mapping $\mho : \eta \to \eta$ is called a (σ, F)-contraction.

Definition 2.3 [17] Assume that Ψ represents the s.o.a functions $\psi : (0,\infty) \to (0,\infty)$ such that, for each $\Phi \ge 0$, we have

$$\lim\inf_{\kappa\to\Phi^+}\psi(\kappa) > 0.$$

3 Auxiliary functions

The following new definitions are very important in the following.

Definition 3.1 Suppose that $\aleph$ refers to the s.o.a. continuous mappings $\ell : [0,\infty)^7 \to [0,\infty)$ such that the hypotheses below hold:

(ℓ_1) for $\lambda, \upsilon \geq 1$, $\ell\left(1,1,1,\lambda+\upsilon,0,1,\frac{\lambda+\upsilon}{2}\right) \in [0,1)$;

(ℓ_2) for all $(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5,\vartheta_6,\vartheta_7) \in [0,\infty)^7$ and $m \geq 0$, we obtain

$$\ell\left(m\vartheta_1,m\vartheta_2,m\vartheta_3,m\vartheta_4,m\vartheta_5,m\vartheta_6,m\vartheta_7\right) \leq m\ell\left(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5,\vartheta_6,\vartheta_7\right),$$

that is, ℓ is subhomogeneous;

(ℓ_3) for $\vartheta_i, \widetilde{\vartheta}_i \in [0,\infty)$ with $\vartheta_i \leq \widetilde{\vartheta}_i$ $(i = 1,2,\ldots,7)$, we obtain

$$\ell\left(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5,\vartheta_6,\vartheta_7\right) \leq \ell\left(\widetilde{\vartheta}_1,\widetilde{\vartheta}_2,\widetilde{\vartheta}_3,\widetilde{\vartheta}_4,\widetilde{\vartheta}_5,\widetilde{\vartheta}_6,\widetilde{\vartheta}_7\right),$$

that is, ℓ is nondecreasing. Moreover, if $\vartheta_i < \widetilde{\vartheta}_i$ $(i = 1,2,3,4,6,7)$, we have

$$\ell\left(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,0,\vartheta_6,\vartheta_7\right) < \ell\left(\widetilde{\vartheta}_1,\widetilde{\vartheta}_2,\widetilde{\vartheta}_3,\widetilde{\vartheta}_4,0,\widetilde{\vartheta}_6,\widetilde{\vartheta}_7\right)$$

and

$$\ell\left(\vartheta_1,\vartheta_2,\vartheta_3,0,\vartheta_4,\vartheta_6,\vartheta_7\right) < \ell\left(\widetilde{\vartheta}_1,\widetilde{\vartheta}_2,\widetilde{\vartheta}_3,0,\widetilde{\vartheta}_4,\widetilde{\vartheta}_6,\widetilde{\vartheta}_7\right).$$

Further, let $\widetilde{\aleph} = \left\{\ell \in \aleph : \ell\left(1,0,0,\lambda,\upsilon,0,\frac{\lambda}{2}\right) \in (0,1]\right\}$, where $\widetilde{\aleph} \subseteq \aleph$.

Example 3.2 The following functions support the above definition:

(1) Describe $\ell_1 : [0,\infty)^7 \to [0,\infty)$ as

$$\ell_1\left(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5,\vartheta_6,\vartheta_7\right) = \hbar \min\left\{\vartheta_1, \frac{\vartheta_2+\vartheta_3}{2}, \frac{\vartheta_4+\vartheta_5}{2}, \frac{\vartheta_6+\vartheta_7}{2}\right\},$$

where $\hbar \in (0,1)$, then, $\ell_1 \in \aleph$. Since $\ell_1\left(1,0,0,\lambda,\upsilon,0,\frac{\lambda}{2}\right) = 0 \notin (0,1]$. Hence, $\ell_1 \notin \widetilde{\aleph}$. This proves that $\widetilde{\aleph} \subseteq \aleph$, but the converse is not true.

(2) Describe $\ell_2 : [0,\infty)^7 \to [0,\infty)$ as

$$\ell_2\left(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5,\vartheta_6,\vartheta_7\right) = \frac{\vartheta_1}{2} + \frac{\vartheta_2+\vartheta_3}{4} + \frac{\vartheta_6}{8},$$

then $\ell_2\left(1,0,0,\lambda,\upsilon,0,\frac{\lambda}{2}\right) = \frac{1}{2} \in (0,1]$. Thus, $\ell_2 \in \widetilde{\aleph}$.

(3) Describe $\ell_3 : [0,\infty)^7 \to [0,\infty)$ as

$$\ell_3\left(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5,\vartheta_6,\vartheta_7\right) = \hbar \min\left\{\frac{\vartheta_1+\vartheta_3}{2}, \frac{\vartheta_4+\vartheta_5}{2}, \frac{\vartheta_6+\vartheta_7}{2}\right\},$$

where $\hbar \in (0,1)$, then $\ell_3\left(1,0,0,\lambda,\upsilon,0,\frac{\lambda}{2}\right) = \frac{1}{2} \in (0,1]$. Thus, $\ell_3 \in \widetilde{\aleph}$.

The Lemma below is very important in the following:

Lemma 3.3 *Assume that* $\ell \in \aleph$, $\beta,\theta \in [0,\infty)$, $\lambda,\upsilon \geq 1$, *and*

$$\beta \leq \max\left\{\ell\left(\theta,\theta,\beta,\lambda\theta+\upsilon\beta,0,\frac{\theta+\beta}{2},\frac{\lambda\theta+\upsilon\beta}{2}\right),\right.$$

$$\ell\left(\theta,\theta,\beta,0,\lambda\theta+\upsilon\beta,\frac{\theta+\beta}{2},\frac{\lambda\theta+\upsilon\beta}{2}\right),$$
$$\ell\left(\theta,\beta,\theta,\lambda\theta+\upsilon\beta,0,\frac{\theta+\beta}{2},\frac{\lambda\theta+\upsilon\beta}{2}\right),$$
$$\ell\left(\theta,\beta,\theta,0,\lambda\theta+\upsilon\beta,\frac{\theta+\beta}{2},\frac{\lambda\theta+\upsilon\beta}{2}\right)\Bigg\}.$$

Then, $\beta\le\theta$.

Proof Keeping the generalization intact, we can suppose that

$$\beta\le\ell\left(\theta,\theta,\beta,\lambda\theta+\upsilon\beta,0,\frac{\theta+\beta}{2},\frac{\lambda\theta+\upsilon\beta}{2}\right). \tag{3.1}$$

Conversely, let us assume that $\theta<\beta$. Now, we examine

$$\begin{aligned}\ell\left(\theta,\theta,\beta,\lambda\theta+\upsilon\beta,0,\frac{\theta+\beta}{2},\frac{\lambda\theta+\upsilon\beta}{2}\right)&<\ell\left(\beta,\beta,\beta,\lambda\beta+\upsilon\beta,0,\beta,\frac{\lambda\beta+\upsilon\beta}{2}\right)\\&\le\beta\ell\left(1,1,1,\lambda+\upsilon,0,1,\frac{\lambda+\upsilon}{2}\right)\\&\le\beta\,(1)\,.\end{aligned}$$

Hence,

$$\ell\left(\theta,\theta,\beta,\lambda\theta+\upsilon\beta,0,\frac{\theta+\beta}{2},\frac{\lambda\theta+\upsilon\beta}{2}\right)<\beta,$$

which contradicts (3.1). Therefore, $\beta\le\theta$. □

4 Existence of fixed points

According to a new definition 3.1, we present our contraction mapping here as follows:

Definition 4.1 ($\widetilde{\psi}F$-contraction) We say that the mapping $\mho:\eta\to\eta^{cb}$ is an $\widetilde{\psi}F$-contraction if

$(\widetilde{\psi}F)_i$ for all $q>0$, $F_1(q)\le F_2(q)$;
$(\widetilde{\psi}F)_{ii}$ $\Upsilon\left(\mho\vartheta,\mho\widetilde{\vartheta}\right)>0$, implies

$$\begin{aligned}&\widetilde{\psi}\left(\varpi\left(\vartheta,\widetilde{\vartheta}\right)\right)+F_2\left(\Upsilon\left(\mho\vartheta,\mho\widetilde{\vartheta}\right)\right)\\&\le F_1\left\{\ell\begin{pmatrix}\varpi\left(\vartheta,\widetilde{\vartheta}\right),B\left(\vartheta,\mho\vartheta\right),B\left(\widetilde{\vartheta},\mho\widetilde{\vartheta}\right),B\left(\vartheta,\mho\widetilde{\vartheta}\right),\\ B\left(\widetilde{\vartheta},\mho\vartheta\right),\frac{B(\vartheta,\mho\vartheta)+B(\widetilde{\vartheta},\mho\widetilde{\vartheta})}{2},\frac{B(\vartheta,\mho\widetilde{\vartheta})+B(\widetilde{\vartheta},\mho\vartheta)}{2}\end{pmatrix}\right\},\end{aligned}$$

for all $\vartheta,\widetilde{\vartheta}\in\eta$, where ϖ is described in Definition 2.1, F_1,F_2 are real-valued functions on $(0,\infty)$, $\ell\in\aleph$, and $\widetilde{\boldsymbol{\psi}}\in\Psi$.

Theorem 4.2 *Let* $\mho:\eta\to\eta^{cp}$ *be an* $\widetilde{\psi}F$*-contraction defined on a complete CMS* (η,ϖ,γ). *If the conditions below hold*:

(C_1) F_1 *is a nondecreasing function*;
(C_2) F_2 *fulfills axioms* $(\heartsuit_2')$, *and* $(\heartsuit_3)$;

(C_3) *for $\vartheta_0 \in \eta$, define the Picard sequence $\{\vartheta_u = \mho^u \vartheta_0\}$ such that*

$$\sup_{n\geq 1} \lim_{j\to\infty} \frac{\gamma\left(\vartheta_{j+1}, \vartheta_{j+2}\right)\gamma\left(\vartheta_{j+1}, \vartheta_n\right)}{\gamma\left(\vartheta_j, \vartheta_{j+1}\right)} < 1;$$

(C_4) *for $\vartheta \in \eta$, $\lim_{u\to\infty} \gamma(\vartheta_u, \vartheta) \leq 1$.*
Then, $\mho$ has at least one FP, that is, $F_{ix}(\mho) \neq \emptyset$.

Proof Assume that $\vartheta_0 \in \eta$ and $\vartheta_1 \in \mho\vartheta_0$. Clearly, if $\vartheta_1 \in \mho\vartheta_1$, $\vartheta_1 \in F_{ix}(\mho)$ and the proof is completed. Hence, let $\vartheta_1 \notin \mho\vartheta_1$, which means $B(\vartheta_1, \mho\vartheta_1) > 0$. Thus, $\Upsilon(\mho\vartheta_0, \mho\vartheta_1) > 0$. As $\mho\vartheta_1$ is compact, there is $\vartheta_2 \in \mho\vartheta_1$ such that $\varpi(\vartheta_1, \vartheta_2) = B(\vartheta_1, \mho\vartheta_1)$. Consider

$$\begin{aligned}
F_1(\varpi(\vartheta_1, \vartheta_2)) &= F_1(B(\vartheta_1, \mho\vartheta_1)) \\
&\leq F_1(\Upsilon(\mho\vartheta_0, \mho\vartheta_1)) \\
&\leq F_2(\Upsilon(\mho\vartheta_0, \mho\vartheta_1)) \\
&\leq F_1\left\{\ell\begin{pmatrix} \varpi(\vartheta_0, \vartheta_1), B(\vartheta_0, \mho\vartheta_0), B(\vartheta_1, \mho\vartheta_1), B(\vartheta_0, \mho\vartheta_1), \\ B(\vartheta_1, \mho\vartheta_0), \frac{B(\vartheta_0,\mho\vartheta_0)+B(\vartheta_1,\mho\vartheta_1)}{2}, \frac{B(\vartheta_0,\mho\vartheta_1)+B(\vartheta_1,\mho\vartheta_0)}{2} \end{pmatrix}\right\} \\
&\quad -\widetilde{\psi}(\varpi(\vartheta_0, \vartheta_1)) \\
&< F_1\left\{\ell\begin{pmatrix} \varpi(\vartheta_0, \vartheta_1), \varpi(\vartheta_0, \vartheta_1), \varpi(\vartheta_1, \vartheta_2), \varpi(\vartheta_0, \vartheta_2), \\ \varpi(\vartheta_1, \vartheta_1), \frac{\varpi(\vartheta_0,\vartheta_1)+\varpi(\vartheta_1,\vartheta_2)}{2}, \frac{\varpi(\vartheta_0,\vartheta_2)+\varpi(\vartheta_1,\vartheta_1)}{2} \end{pmatrix}\right\}.
\end{aligned}$$

Since F_1 is nondecreasing, one has

$$\begin{aligned}
\varpi(\vartheta_1, \vartheta_2) &< \ell\begin{pmatrix} \varpi(\vartheta_0, \vartheta_1), \varpi(\vartheta_0, \vartheta_1), \varpi(\vartheta_1, \vartheta_2), \varpi(\vartheta_0, \vartheta_2), \\ 0, \frac{\varpi(\vartheta_0,\vartheta_1)+\varpi(\vartheta_1,\vartheta_2)}{2}, \frac{\varpi(\vartheta_0,\vartheta_2)}{2} \end{pmatrix} \\
&\leq \ell\begin{pmatrix} \varpi(\vartheta_0, \vartheta_1), \varpi(\vartheta_0, \vartheta_1), \varpi(\vartheta_1, \vartheta_2), \\ \gamma(\vartheta_0, \vartheta_1)\varpi(\vartheta_0, \vartheta_1) + \gamma(\vartheta_1, \vartheta_2)\varpi(\vartheta_1, \vartheta_2), \\ 0, \frac{\varpi(\vartheta_0,\vartheta_1)+\varpi(\vartheta_1,\vartheta_2)}{2}, \frac{\gamma(\vartheta_0,\vartheta_1)\varpi(\vartheta_0,\vartheta_1)+\gamma(\vartheta_1,\vartheta_2)\varpi(\vartheta_1,\vartheta_2)}{2} \end{pmatrix}.
\end{aligned}$$

Based on Lemma 3.3, we conclude that

$$\varpi(\vartheta_1, \vartheta_2) < \varpi(\vartheta_0, \vartheta_1).$$

In the same way, we have $\vartheta_3 \in \mho\vartheta_2$ such that $\varpi(\vartheta_2, \vartheta_3) = B(\vartheta_2, \mho\vartheta_2)$ with $B(\vartheta_2, \mho\vartheta_2) > 0$ and

$$\varpi(\vartheta_2, \vartheta_3) < \varpi(\vartheta_1, \vartheta_2).$$

Repeating this technique, we have a sequence $\{\vartheta_u\} \subset \eta$ in order that $\vartheta_{u+1} \in \mho\vartheta_u$ fulfills $\varpi(\vartheta_u, \vartheta_{u+1}) = B(\vartheta_u, \mho\vartheta_u)$ with $B(\vartheta_u, \mho\vartheta_u) > 0$ and

$$\varpi(\vartheta_u, \vartheta_{u+1}) < \varpi(\vartheta_{u-1}, \vartheta_u), \ \forall u \in \mathbb{N}.$$

It follows that $\{\varpi(\vartheta_u, \vartheta_{u+1})\}_{s\in\mathbb{N}}$ is a decreasing sequence. Next, we can write

$$\widetilde{\psi}(\varpi(\vartheta_u, \vartheta_{u+1})) + F_2(\Upsilon(\mho\vartheta_u, \mho\vartheta_{u+1}))$$

$$
\begin{aligned}
&\leq F_1\left\{\ell\begin{pmatrix}\varpi(\vartheta_u,\vartheta_{u+1}), B(\vartheta_u,\mho\vartheta_u), B(\vartheta_{u+1},\mho\vartheta_{u+1}), B(\vartheta_u,\mho\vartheta_{u+1}),\\ B(\vartheta_{u+1},\mho\vartheta_u), \frac{B(\vartheta_u,\mho\vartheta_u)+B(\vartheta_{u+1},\mho\vartheta_{u+1})}{2}, \frac{B(\vartheta_u,\mho\vartheta_{u+1})+B(\vartheta_{u+1},\mho\vartheta_u)}{2}\end{pmatrix}\right\}\\
&\leq F_1\left\{\ell\begin{pmatrix}\varpi(\vartheta_u,\vartheta_{u+1}), \varpi(\vartheta_u,\vartheta_{u+1}), \varpi(\vartheta_{u+1},\vartheta_{u+2}), \varpi(\vartheta_u,\vartheta_{u+2}),\\ \varpi(\vartheta_{u+1},\vartheta_{u+1}), \frac{\varpi(\vartheta_u,\vartheta_{u+1})+\varpi(\vartheta_{u+1},\vartheta_{u+2})}{2}, \frac{\varpi(\vartheta_u,\vartheta_{u+2})+\varpi(\vartheta_{u+1},\vartheta_{u+1})}{2}\end{pmatrix}\right\}\\
&\leq F_1\left\{\ell\begin{pmatrix}\varpi(\vartheta_u,\vartheta_{u+1}), \varpi(\vartheta_u,\vartheta_{u+1}), \varpi(\vartheta_{u+1},\vartheta_{u+2}),\\ \gamma(\vartheta_u,\vartheta_{u+1})\varpi(\vartheta_u,\vartheta_{u+1})+\gamma(\vartheta_{u+1},\vartheta_{u+2})\varpi(\vartheta_{u+1},\vartheta_{u+2}),\\ 0, \frac{\varpi(\vartheta_u,\vartheta_{u+1})+\varpi(\vartheta_{u+1},\vartheta_{u+2})}{2}, \frac{\gamma(\vartheta_u,\vartheta_{u+1})\varpi(\vartheta_u,\vartheta_{u+1})+\gamma(\vartheta_{u+1},\vartheta_{u+2})\varpi(\vartheta_{u+1},\vartheta_{u+2})}{2}\end{pmatrix}\right\}\\
&< F_1\left\{\ell\begin{pmatrix}\varpi(\vartheta_u,\vartheta_{u+1}), \varpi(\vartheta_u,\vartheta_{u+1}), \varpi(\vartheta_u,\vartheta_{u+1}),\\ \gamma(\vartheta_u,\vartheta_{u+1})\varpi(\vartheta_u,\vartheta_{u+1})+\gamma(\vartheta_{u+1},\vartheta_{u+2})\varpi(\vartheta_u,\vartheta_{u+1}),\\ 0, \frac{\varpi(\vartheta_u,\vartheta_{u+1})+\varpi(\vartheta_u,\vartheta_{u+1})}{2}, \frac{\gamma(\vartheta_u,\vartheta_{u+1})\varpi(\vartheta_u,\vartheta_{u+1})+\gamma(\vartheta_{u+1},\vartheta_{u+2})\varpi(\vartheta_u,\vartheta_{u+1})}{2}\end{pmatrix}\right\}\\
&\leq F_1\left\{\varpi(\vartheta_u,\vartheta_{u+1})\,\ell\begin{pmatrix}1,1,1,\gamma(\vartheta_u,\vartheta_{u+1})+\gamma(\vartheta_{u+1},\vartheta_{u+2}),\\ 0,1,\frac{\gamma(\vartheta_u,\vartheta_{u+1})+\gamma(\vartheta_{u+1},\vartheta_{u+2})}{2}\end{pmatrix}\right\}\\
&\leq F_1(\varpi(\vartheta_u,\vartheta_{u+1}))\\
&= F_1(B(\vartheta_u,\mho\vartheta_u))\\
&\leq F_1(\Upsilon(\mho\vartheta_{u-1},\mho\vartheta_u))\\
&\leq F_2(\Upsilon(\mho\vartheta_{u-1},\mho\vartheta_u)).
\end{aligned}
$$

Hence, for each $u\in\mathbb{N}$, we conclude that

$$
F_2(\Upsilon(\mho\vartheta_u,\mho\vartheta_{u+1}))\leq F_2(\Upsilon(\mho\vartheta_{u-1},\mho\vartheta_u))-\widetilde{\psi}(\varpi(\vartheta_u,\vartheta_{u+1})). \tag{4.1}
$$

Since $\widetilde{\boldsymbol{\psi}}\in\Psi$, there is $\zeta>0$ so that $\widetilde{\psi}(\varpi(\vartheta_u,\vartheta_{u+1}))>\zeta$, for all $u\geq u_0$. From (4.1), we obtain

$$
\begin{aligned}
F_2(\Upsilon(\mho\vartheta_u,\mho\vartheta_{u+1})) &\leq F_2(\Upsilon(\mho\vartheta_{u-1},\mho\vartheta_u))-\widetilde{\psi}(\varpi(\vartheta_u,\vartheta_{u+1}))\\
&< F_2(\Upsilon(\mho\vartheta_{u-2},\mho\vartheta_{u-1}))-\widetilde{\psi}(\varpi(\vartheta_{u-1},\vartheta_u))-\widetilde{\psi}(\varpi(\vartheta_u,\vartheta_{u+1}))\\
&\ \ \vdots\\
&< F_2(\Upsilon(\mho\vartheta_0,\mho\vartheta_1))-\sum_{j=1}^{u}\widetilde{\psi}\left(\varpi\left(\vartheta_j,\vartheta_{j+1}\right)\right)\\
&= F_2(\Upsilon(\mho\vartheta_0,\mho\vartheta_1))-\sum_{j=1}^{u_0-1}\widetilde{\psi}\left(\varpi\left(\vartheta_j,\vartheta_{j+1}\right)\right)-\sum_{j=u_0}^{u}\widetilde{\psi}\left(\varpi\left(\vartheta_j,\vartheta_{j+1}\right)\right)\\
&< F_2(\Upsilon(\mho\vartheta_0,\mho\vartheta_1))-(u-u_0)\zeta,\ u\geq u_0.
\end{aligned}
$$

Hence,

$$
F_2(\Upsilon(\mho\vartheta_u,\mho\vartheta_{u+1}))<F_2(\Upsilon(\mho\vartheta_0,\mho\vartheta_1))-(u-u_0)\zeta,\ u\geq u_0. \tag{4.2}
$$

Letting $u\to\infty$ in (4.2), we have

$$
\lim_{u\to\infty}F_2(\Upsilon(\mho\vartheta_u,\mho\vartheta_{u+1}))=-\infty.
$$

Applying ($\heartsuit_2'$), we obtain

$$\lim_{u\to\infty} \Upsilon(\mho\vartheta_u, \mho\vartheta_{u+1}) = 0,$$

which yields,

$$\lim_{u\to\infty} \varpi(\vartheta_u, \vartheta_{u+1}) = \lim_{u\to\infty} D(\vartheta_u, \mho\vartheta_u) \leq \lim_{u\to\infty} \Upsilon(\mho\vartheta_{u-1}, \mho\vartheta_u) = 0. \tag{4.3}$$

According to ($\heartsuit_3$), there is $l \in (0,1)$ so that

$$\lim_{u\to\infty} (\Upsilon(\mho\vartheta_u, \mho\vartheta_{u+1}))^l F_2(\Upsilon(\mho\vartheta_u, \mho\vartheta_{u+1})) = 0. \tag{4.4}$$

For all $u \geq u_0$, from (4.2), one has

$$\begin{aligned}
&(\Upsilon(\mho\vartheta_u, \mho\vartheta_{u+1}))^l F_2(\Upsilon(\mho\vartheta_u, \mho\vartheta_{u+1})) - (\Upsilon(\mho\vartheta_u, \mho\vartheta_{u+1}))^l F_2(\Upsilon(\mho\vartheta_0, \mho\vartheta_1))\\
&\leq (\Upsilon(\mho\vartheta_u, \mho\vartheta_{u+1}))^l [F_2(\Upsilon(\mho\vartheta_0, \mho\vartheta_1)) - (u-u_0)\zeta]\\
&\quad - (\Upsilon(\mho\vartheta_u, \mho\vartheta_{u+1}))^l F_2(\Upsilon(\mho\vartheta_0, \mho\vartheta_1))\\
&= -(\Upsilon(\mho\vartheta_u, \mho\vartheta_{u+1}))^l (u-u_0)\zeta\\
&\leq 0.
\end{aligned}$$

As $u \to \infty$ in (4.3) and (4.4), we can write

$$0 \leq \lim_{u\to\infty} u(\Upsilon(\mho\vartheta_u, \mho\vartheta_{u+1}))^l \leq 0$$

and it follows that

$$\lim_{u\to\infty} u(\Upsilon(\mho\vartheta_u, \mho\vartheta_{u+1}))^l = 0. \tag{4.5}$$

Based on (4.5), there is $u_1 \in \mathbb{N}$ so that $u(\Upsilon(\mho\vartheta_u, \mho\vartheta_{u+1}))^l \leq 1$, for all $u \geq u_1$. Thus, we have

$$\Upsilon(\mho\vartheta_u, \mho\vartheta_{u+1}) \leq \frac{1}{u^{\frac{1}{l}}} \text{ for all } u \geq u_1.$$

Therefore,

$$\varpi(\vartheta_u, \vartheta_{u+1}) = B(\vartheta_u, \mho\vartheta_u) \leq \Upsilon(\mho\vartheta_{u-1}, \mho\vartheta_u) \leq \frac{1}{u^{\frac{1}{l}}} \text{ for all } u \geq u_1.$$

Now, we prove that $\{\vartheta_u\}$ is a Cauchy sequence (CS). In this regard, let $v, u \in \mathbb{N}$ in order that $v > u > u_1$. Then,

$$\begin{aligned}
\varpi(\vartheta_u, \vartheta_v) &= \gamma(\vartheta_u, \vartheta_{u+1})\varpi(\vartheta_u, \vartheta_{u+1}) + \gamma(\vartheta_{u+1}, \vartheta_v)\varpi(\vartheta_{u+1}, \vartheta_v)\\
&\leq \gamma(\vartheta_u, \vartheta_{u+1})\varpi(\vartheta_u, \vartheta_{u+1}) + \gamma(\vartheta_{u+1}, \vartheta_v)\gamma(\vartheta_{u+1}, \vartheta_{u+2})\varpi(\vartheta_{u+1}, \vartheta_{u+2})\\
&\quad + \gamma(\vartheta_{u+1}, \vartheta_v)\gamma(\vartheta_{u+2}, \vartheta_v)\varpi(\vartheta_{u+2}, \vartheta_v)
\end{aligned}$$

$$
\begin{aligned}
&\leq \gamma(\vartheta_u,\vartheta_{u+1})\varpi(\vartheta_u,\vartheta_{u+1})+\gamma(\vartheta_{u+1},\vartheta_v)\gamma(\vartheta_{u+1},\vartheta_{u+2})\varpi(\vartheta_{u+1},\vartheta_{u+2})\\
&\quad+\gamma(\vartheta_{u+1},\vartheta_v)\gamma(\vartheta_{u+2},\vartheta_v)\gamma(\vartheta_{u+2},\vartheta_{u+3})\varpi(\vartheta_{u+2},\vartheta_{u+3})\\
&\quad+\gamma(\vartheta_{u+1},\vartheta_v)\gamma(\vartheta_{u+2},\vartheta_v)\gamma(\vartheta_{u+3},\vartheta_v)\varpi(\vartheta_{u+3},\vartheta_v)\\
&\leq\\
&\quad\vdots\\
&\leq \gamma(\vartheta_u,\vartheta_{u+1})\varpi(\vartheta_u,\vartheta_{u+1})+\sum_{j=u+1}^{v-2}\left(\prod_{b=u+1}^{j}\gamma(\vartheta_b,\vartheta_v)\right)\gamma\left(\vartheta_j,\vartheta_{j+1}\right)\varpi\left(\vartheta_j,\vartheta_{j+1}\right)\\
&\quad+\left(\prod_{b=u+1}^{v-1}\gamma\left(\vartheta_j,\vartheta_v\right)\right)\varpi(\vartheta_{v-1},\vartheta_v)\\
&\leq \gamma(\vartheta_u,\vartheta_{u+1})\varpi(\vartheta_u,\vartheta_{u+1})+\sum_{j=u+1}^{v-2}\left(\prod_{b=u+1}^{j}\gamma(\vartheta_b,\vartheta_v)\right)\gamma\left(\vartheta_j,\vartheta_{j+1}\right)\varpi\left(\vartheta_j,\vartheta_{j+1}\right)\\
&\quad+\left(\prod_{b=u+1}^{v-1}\gamma\left(\vartheta_j,\vartheta_v\right)\right)\gamma(\vartheta_{v-1},\vartheta_v)\varpi(\vartheta_{v-1},\vartheta_v)\\
&= \gamma(\vartheta_u,\vartheta_{u+1})\varpi(\vartheta_u,\vartheta_{u+1})+\sum_{j=u+1}^{v-2}\left(\prod_{b=u+1}^{j}\gamma(\vartheta_b,\vartheta_v)\right)\gamma\left(\vartheta_j,\vartheta_{j+1}\right)\varpi\left(\vartheta_j,\vartheta_{j+1}\right)\\
&\leq \gamma(\vartheta_u,\vartheta_{u+1})\varpi(\vartheta_u,\vartheta_{u+1})+\sum_{j=u+1}^{v-2}\left(\prod_{b=0}^{j}\gamma(\vartheta_b,\vartheta_v)\right)\gamma\left(\vartheta_j,\vartheta_{j+1}\right)\varpi\left(\vartheta_j,\vartheta_{j+1}\right).
\end{aligned}
$$

Therefore,

$$
\varpi(\vartheta_u,\vartheta_v)\leq \gamma(\vartheta_u,\vartheta_{u+1})\varpi(\vartheta_u,\vartheta_{u+1})+\sum_{j=u+1}^{v-2}\left(\prod_{b=0}^{j}\gamma(\vartheta_b,\vartheta_v)\right)\gamma\left(\vartheta_j,\vartheta_{j+1}\right)\times\frac{1}{j^{\frac{1}{l}}}. \tag{4.6}
$$

Consider

$$
\begin{aligned}
\sum_{j=u+1}^{v-2}\left(\prod_{b=0}^{j}\gamma(\vartheta_b,\vartheta_v)\right)\gamma\left(\vartheta_j,\vartheta_{j+1}\right)\times\frac{1}{j^{\frac{1}{l}}} &\leq \sum_{j=u+1}^{\infty}\frac{1}{j^{\frac{1}{l}}}\left(\prod_{b=0}^{j}\gamma(\vartheta_b,\vartheta_v)\right)\gamma\left(\vartheta_j,\vartheta_{j+1}\right)\\
&= \sum_{j=u+1}^{\infty}M_jN_j,
\end{aligned}
$$

where $M_j=\frac{1}{j^{\frac{1}{l}}}$ and $N_j=\left(\prod_{b=0}^{j}\gamma(\vartheta_b,\vartheta_v)\right)\gamma\left(\vartheta_j,\vartheta_{j+1}\right)$. As $\frac{1}{l}>0$, the series $\sum_{j=u+1}^{\infty}\left(\frac{1}{j^{\frac{1}{l}}}\right)$ converges. Since $\{N_j\}_j$ is bounded above and increasing, the nonzero $\lim_{j\to\infty}\{N_j\}$ exists. Hence, $\lim_{j\to\infty}\{M_jN_j\}$ converges.

Take the partial sums $\wp=\sum_{j=0}^{\infty}\left(\prod_{b=0}^{j}\gamma(\vartheta_b,\vartheta_v)\right)\gamma\left(\vartheta_j,\vartheta_{j+1}\right)\times\frac{1}{j^{\frac{1}{l}}}$. From (4.6), we can write

$$
\varpi(\vartheta_u,\vartheta_v)\leq \gamma(\vartheta_u,\vartheta_{u+1})\varpi(\vartheta_u,\vartheta_{u+1})+(\wp_{v-1}-\wp_u). \tag{4.7}
$$

Utilizing the ratio test and Condition (C_3), we have that $\lim_{u\to\infty}\{\wp_u\}$ exists. Letting $u\to\infty$ in (4.7), we conclude that

$$\lim_{u\to\infty}\varpi\left(\vartheta_u,\vartheta_v\right)=0.$$

This proves that $\{\vartheta_u\}$ is a CS. Since η is complete, there is $\vartheta^*\in\eta$ so that

$$\lim_{u\to\infty}\vartheta_u=\vartheta^*. \tag{4.8}$$

Consider

$$\begin{aligned} F_1\left(\Upsilon\left(\mho\vartheta,\mho\widetilde{\vartheta}\right)\right) &\le F_2\left(\Upsilon\left(\mho\vartheta,\mho\widetilde{\vartheta}\right)\right)\le\widetilde{\psi}\left(\varpi\left(\vartheta,\widetilde{\vartheta}\right)\right)+F_2\left(\Upsilon\left(\mho\vartheta,\mho\widetilde{\vartheta}\right)\right)\\ &\le F_1\left\{\ell\begin{pmatrix}\varpi\left(\vartheta,\widetilde{\vartheta}\right),B\left(\vartheta,\mho\vartheta\right),B\left(\widetilde{\vartheta},\mho\widetilde{\vartheta}\right),B\left(\vartheta,\mho\widetilde{\vartheta}\right),\\ B\left(\widetilde{\vartheta},\mho\vartheta\right),\frac{B(\vartheta,\mho\vartheta)+B(\widetilde{\vartheta},\mho\widetilde{\vartheta})}{2},\frac{B(\vartheta,\mho\widetilde{\vartheta})+B(\widetilde{\vartheta},\mho\vartheta)}{2}\end{pmatrix}\right\}.\end{aligned}$$

Since F_1 is a nondecreasing function, then for $\vartheta,\widetilde{\vartheta}\in\xi$, one can write

$$\Upsilon\left(\mho\vartheta,\mho\widetilde{\vartheta}\right)\le\ell\begin{pmatrix}\varpi\left(\vartheta,\widetilde{\vartheta}\right),B\left(\vartheta,\mho\vartheta\right),B\left(\widetilde{\vartheta},\mho\widetilde{\vartheta}\right),B\left(\vartheta,\mho\widetilde{\vartheta}\right),B\left(\widetilde{\vartheta},\mho\vartheta\right),\\ \frac{B(\vartheta,\mho\vartheta)+B(\widetilde{\vartheta},\mho\widetilde{\vartheta})}{2},\frac{B(\vartheta,\mho\widetilde{\vartheta})+B(\widetilde{\vartheta},\mho\vartheta)}{2}\end{pmatrix}. \tag{4.9}$$

Then, to illustrate the existence of the FP of $\mho$, assume the contrary, that is, $B\left(\vartheta^*,\mho\vartheta^*\right)>0$. Using (4.9) and the compactness of $\mho\vartheta^*$ implies that there is $\vartheta\in\mho\vartheta^*$ such that

$$\begin{aligned} &B\left(\vartheta^*,\mho\vartheta^*\right)\\ &=\varpi\left(\vartheta^*,\vartheta\right)\\ &\le\gamma\left(\vartheta^*,\vartheta_{u+1}\right)\varpi\left(\vartheta^*,\vartheta_{u+1}\right)+\gamma\left(\vartheta_{u+1},\vartheta\right)\varpi\left(\vartheta_{u+1},\vartheta\right)\\ &=\gamma\left(\vartheta^*,\vartheta_{u+1}\right)\varpi\left(\vartheta^*,\vartheta_{u+1}\right)+\gamma\left(\vartheta_{u+1},\vartheta\right)B\left(\vartheta_{u+1},\mho\vartheta^*\right)\\ &\le\gamma\left(\vartheta^*,\vartheta_{u+1}\right)\varpi\left(\vartheta^*,\vartheta_{u+1}\right)+\gamma\left(\vartheta_{u+1},\vartheta\right)\Upsilon\left(\mho\vartheta_u,\mho\vartheta^*\right)\\ &\le\gamma\left(\vartheta^*,\vartheta_{u+1}\right)\varpi\left(\vartheta^*,\vartheta_{u+1}\right)+\gamma\left(\vartheta_{u+1},\vartheta\right)\\ &\quad\times\ell\begin{pmatrix}\varpi\left(\vartheta_u,\vartheta^*\right),B\left(\vartheta_u,\mho\vartheta_u\right),B\left(\vartheta^*,\mho\vartheta^*\right),\\ B\left(\vartheta_u,\mho\vartheta^*\right),B\left(\vartheta^*,\mho\vartheta_u\right),\\ \frac{B(\vartheta_u,\mho\vartheta_u)+B(\vartheta^*,\mho\vartheta^*)}{2},\frac{B(\vartheta_u,\mho\vartheta^*)+B(\vartheta^*,\mho\vartheta_u)}{2}\end{pmatrix}\\ &\le\gamma\left(\vartheta^*,\vartheta_{u+1}\right)\varpi\left(\vartheta^*,\vartheta_{u+1}\right)+\gamma\left(\vartheta_{u+1},\vartheta\right)\\ &\quad\times\ell\begin{pmatrix}\varpi\left(\vartheta_u,\vartheta^*\right),\varpi\left(\vartheta_u,\vartheta_{u+1}\right),B\left(\vartheta^*,\mho\vartheta^*\right),\\ \gamma\left(\vartheta_u,\vartheta^*\right)\varpi\left(\vartheta_u,\vartheta^*\right)+\gamma\left(\vartheta^*,\mho\vartheta^*\right)\varpi\left(\vartheta^*,\mho\vartheta^*\right),\\ \varpi\left(\vartheta^*,\vartheta_{u+1}\right),\frac{\varpi(\vartheta_u,\vartheta_{u+1})+B(\vartheta^*,\mho\vartheta^*)}{2},\\ \frac{\gamma(\vartheta_u,\vartheta^*)\varpi(\vartheta_u,\vartheta^*)+\gamma(\vartheta^*,\mho\vartheta^*)\varpi(\vartheta^*,\mho\vartheta^*)+\varpi(\vartheta^*,\vartheta_{u+1})}{2}\end{pmatrix}.\end{aligned}$$

In the above inequality, letting $u\to\infty$, using Condition (C_4), and (4.8), we have

$$B\left(\vartheta^*,\mho\vartheta^*\right)\le(1)\ell\begin{pmatrix}0,0,B\left(\vartheta^*,\mho\vartheta^*\right),0+\gamma\left(\vartheta^*,\mho\vartheta^*\right)\varpi\left(\vartheta^*,\mho\vartheta^*\right),\\ 0,\frac{0+B(\vartheta^*,\mho\vartheta^*)}{2},\frac{0+\gamma(\vartheta^*,\mho\vartheta^*)\varpi(\vartheta^*,\mho\vartheta^*)}{2}\end{pmatrix}$$

$$\leq \ell\left(0,0,B\left(\vartheta^*,\mho\vartheta^*\right),\varpi\left(\vartheta^*,\mho\vartheta^*\right),0,\frac{B\left(\vartheta^*,\mho\vartheta^*\right)}{2},\frac{\varpi\left(\vartheta^*,\mho\vartheta^*\right)}{2}\right).$$

Applying Lemma 3.3, we obtain that $B(\vartheta^*,\mho\vartheta^*)\leq 0$. Hence, $B(\vartheta^*,\mho\vartheta^*)=0$. As $\mho\vartheta^*$ is closed, we have $\vartheta^*\in\mho\vartheta^*$, and this completes the proof. □

Theorem 4.2 can be supported by the following example:

Example 4.3 Consider $\eta=\left\{0,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5}\right\}$. Describe $\varpi:\eta\times\eta\to[0,\infty)$ and $\gamma:\eta\times\eta\to[1,\infty)$ as mapping $\varpi\left(\vartheta,\widetilde{\vartheta}\right)=\left|\vartheta-\widetilde{\vartheta}\right|^2$ and

$$\gamma\left(\vartheta,\widetilde{\vartheta}\right)=\begin{cases}1, & \text{if } \vartheta=\widetilde{\vartheta}=0,\\ \frac{1}{\left(\vartheta+\widetilde{\vartheta}\right)^2}, & \text{if } \vartheta\neq 0 \text{ or } \widetilde{\vartheta}\neq 0,\end{cases}$$

respectively. Clearly, (η,ϖ,γ) is a complete CMS. Further, define $F_1,F_2:\mathbb{R}^+\to\mathbb{R}$ by

$$F_1(h)=\begin{cases}\frac{-1}{h}, & \text{if } h\in(0,1),\\ h^2, & \text{if } h\in[1,\infty)\end{cases}$$

and $F_2(h)=\ln(h)+h^2$, for $h\in\mathbb{R}^+$. From the definition of F_1 and F_2, we find that F_1 is nondecreasing, F_2 fulfills the conditions $(\heartsuit_2')$ and $(\heartsuit_3)$, and for all $h\in\mathbb{R}^+$, $F_1(h)\leq F_2(h)$. Let us define $\mho:\eta\to\eta^{cp}$, $\ell:[0,\infty)^7\to[0,\infty)$, and $\widetilde{\psi}:\mathbb{R}^+\to\mathbb{R}^+$ by

$$\mho\vartheta=\begin{cases}\{0\} & \text{if } \vartheta=0,\\ \{0,\frac{1}{2},\frac{1}{3}\} & \text{if } \vartheta\neq 0,\end{cases}$$

$\ell(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5,\vartheta_6,\vartheta_7)=\frac{\vartheta_1}{2}+30\vartheta_5$, and $\widetilde{\psi}(s)=\frac{1}{s^2}$, $s\in\mathbb{R}^+$, respectively. It is clear that $\ell\in\aleph$, $\widetilde{\psi}\in\Psi$. As $\Upsilon\left(\mho\vartheta,\mho\widetilde{\vartheta}\right)>0$, it follows that

$$\begin{aligned}&\widetilde{\psi}\left(\varpi\left(\vartheta,\widetilde{\vartheta}\right)\right)+F_2\left(\Upsilon\left(\mho\vartheta,\mho\widetilde{\vartheta}\right)\right)\\&\quad\leq F_1\left\{\ell\begin{pmatrix}\varpi\left(\vartheta,\widetilde{\vartheta}\right),B\left(\vartheta,\mho\vartheta\right),B\left(\widetilde{\vartheta},\mho\widetilde{\vartheta}\right),B\left(\vartheta,\mho\widetilde{\vartheta}\right),\\ B\left(\widetilde{\vartheta},\mho\vartheta\right),\frac{B(\vartheta,\mho\vartheta)+B\left(\widetilde{\vartheta},\mho\widetilde{\vartheta}\right)}{2},\frac{B\left(\vartheta,\mho\widetilde{\vartheta}\right)+B\left(\widetilde{\vartheta},\mho\vartheta\right)}{2}\end{pmatrix}\right\}.\end{aligned}$$

Moreover, $\lim_{u\to\infty}\gamma(\vartheta_u,\vartheta)\leq 1$. Therefore, all the requirements of Theorem 4.2 are fulfilled and $F_{ix}(\mho)=\{0,\frac{1}{2},\frac{1}{3}\}$.

We can relax the conditions of Theorem 4.2, by neglecting conditions $(\heartsuit_3)$ and (C_3) as follows:

Theorem 4.4 *Let $\mho:\eta\to\eta^{cp}$ be an MVM described on a complete CMS (η,ϖ,γ). Assume that F_1 and F_2 are functions verifying $\widetilde{\psi}F$-contraction. Also, suppose that the assertions below are true*:

(i) F_1 is nondecreasing;

(ii) F_2 fulfills $(\heartsuit_2')$;

(iii) for $\vartheta\in\eta$, $\lim_{l\to\infty}\gamma\left(\vartheta_{v_l},\vartheta_{u_l}\right)\leq 1$.

Then, $F_{ix}(\mho)\neq\emptyset$.

Proof Assume that $\vartheta_0 \in \eta$ and $\vartheta_1 \in \mho\vartheta_0$. Similar to the proof of Theorem 4.2, consider that $\{\vartheta_u\} \subset \eta$ is a sequence such that $\vartheta_{u+1} \in \mho\vartheta_u$. It fulfills $\varpi(\vartheta_u, \vartheta_{u+1}) = B(\vartheta_u, \mho\vartheta_u)$ with $B(\vartheta_u, \mho\vartheta_u) > 0$ and

$$\varpi(\vartheta_u, \vartheta_{u+1}) < \varpi(\vartheta_{u-1}, \vartheta_u), \ \forall u \in \mathbb{N}.$$

Also, we have

$$F_2(\Upsilon(\mho\vartheta_u, \mho\vartheta_{u+1})) < F_2(\Upsilon(\mho\vartheta_0, \mho\vartheta_1)) - (u - u_0)\zeta, \ u \geq u_0. \tag{4.10}$$

Passing $u \to \infty$ in (4.10), we obtain

$$\lim_{u\to\infty} F_2(\Upsilon(\mho\vartheta_u, \mho\vartheta_{u+1})) = -\infty.$$

From $(\heartsuit_2')$, we have

$$\lim_{u\to\infty} \Upsilon(\mho\vartheta_u, \mho\vartheta_{u+1}) = 0, \tag{4.11}$$

which implies that

$$\lim_{u\to\infty} \varpi(\vartheta_u, \vartheta_{u+1}) = \lim_{u\to\infty} D(\vartheta_u, \mho\vartheta_u) \leq \lim_{u\to\infty} \Upsilon(\mho\vartheta_{u-1}, \mho\vartheta_u) = 0.$$

Now, we claim that

$$\lim_{u,v\to\infty} \varpi(\vartheta_u, \vartheta_v) = 0. \tag{4.12}$$

Assume the converse, i.e., there is $\theta > 0$ so that for each $\widehat{r} \geq 0$, there exists $v_l > u_l > \widehat{r}$ such that

$$\varpi\left(\vartheta_{u_l}, \vartheta_{v_l}\right) > \theta.$$

Further, there is $\widehat{r}_0 \in \mathbb{N}$ in order that

$$m_{\widehat{r}_0} = \varpi(\vartheta_{u-1}, \vartheta_u) < \theta, \ \forall u \geq \widehat{r}_0.$$

Also, there are two subsequences $\{\vartheta_{u_l}\}$ and $\{\vartheta_{v_l}\}$ of $\{\vartheta_u\}$ in order that

$$\widehat{r}_0 \leq u_l \leq v_l + 1 \text{ and } \varpi\left(\vartheta_{u_l}, \vartheta_{v_l}\right) > \theta, \ \forall l \geq 0. \tag{4.13}$$

It should be noted that

$$\varpi\left(\vartheta_{v_l-1}, \vartheta_{u_l}\right) < \theta, \ \forall l \tag{4.14}$$

and v_l is the minimal index in order that (4.14) is satisfied. From (4.13) and (4.14), it is impossible to verify that $\vartheta_u + 1 \leq \vartheta_u$, then, $\vartheta_u + 2 \leq v_l$. This proves that

$$\vartheta_u + 1 < v_l < v_l + 1, \ \forall l.$$

Again, using (4.13), (4.14), and (ϖ_3), one can write

$$\begin{aligned}
\theta &< \varpi\left(\vartheta_{v_l},\vartheta_{u_l}\right)\\
&\leq \gamma\left(\vartheta_{v_l},\vartheta_{v_l-1}\right)\varpi\left(\vartheta_{v_l},\vartheta_{v_l-1}\right)+\gamma\left(\vartheta_{v_l-1},\vartheta_{u_l}\right)\varpi\left(\vartheta_{v_l-1},\vartheta_{u_l}\right)\\
&\leq \gamma\left(\vartheta_{v_l},\vartheta_{v_l-1}\right)\varpi\left(\vartheta_{v_l},\vartheta_{v_l-1}\right)+\theta\gamma\left(\vartheta_{v_l-1},\vartheta_{u_l}\right).
\end{aligned}$$

Passing $l\to\infty$ in the above inequality and using the condition (iii) of Theorem 4.4, one has

$$\begin{aligned}
\theta &< \lim_{l\to\infty}\varpi\left(\vartheta_{v_l},\vartheta_{u_l}\right)\\
&\leq 0+\theta\lim_{l\to\infty}\gamma\left(\vartheta_{v_l-1},\vartheta_{u_l}\right)\\
&= \theta\lim_{l\to\infty}\gamma\left(\vartheta_{v_l-1},\vartheta_{u_l}\right)\\
&\leq \theta.
\end{aligned}$$

This proves that

$$\lim_{l\to\infty}\varpi\left(\vartheta_{v_l},\vartheta_{u_l}\right)=\theta. \tag{4.15}$$

From (4.11) and (4.15), we deduce that

$$\lim_{l\to\infty}\varpi\left(\vartheta_{v_l+1},\vartheta_{u_l+1}\right)=\theta. \tag{4.16}$$

Let

$$\begin{aligned}
&\widetilde{\psi}\left(\varpi\left(\vartheta_{v_l},\vartheta_{u_l}\right)\right)+F_1\left(\varpi\left(\vartheta_{v_l+1},\vartheta_{u_l+1}\right)\right)\\
&=\widetilde{\psi}\left(\varpi\left(\vartheta_{v_l},\vartheta_{u_l}\right)\right)+F_1\left(B\left(\vartheta_{v_l+1},\mho\vartheta_{u_l}\right)\right)\\
&\leq\widetilde{\psi}\left(\varpi\left(\vartheta_{v_l},\vartheta_{u_l}\right)\right)+F_1\left(\Upsilon\left(\mho\vartheta_{v_l},\mho\vartheta_{u_l}\right)\right)\\
&\leq\widetilde{\psi}\left(\varpi\left(\vartheta_{v_l},\vartheta_{u_l}\right)\right)+F_2\left(\Upsilon\left(\mho\vartheta_{v_l},\mho\vartheta_{u_l}\right)\right)\\
&\leq F_1\left\{\ell\left(\begin{array}{c}\varpi\left(\vartheta_{v_l},\vartheta_{u_l}\right),B\left(\vartheta_{v_l},\mho\vartheta_{v_l}\right),B\left(\vartheta_{u_l},\mho\vartheta_{u_l}\right),B\left(\vartheta_{v_l},\mho\vartheta_{u_l}\right),\\ B\left(\vartheta_{u_l},\mho\vartheta_{v_l}\right),\frac{B\left(\vartheta_{v_l},\mho\vartheta_{v_l}\right)+B\left(\vartheta_{u_l},\mho\vartheta_{u_l}\right)}{2},\frac{B\left(\vartheta_{v_l},\mho\vartheta_{u_l}\right)+B\left(\vartheta_{u_l},\mho\vartheta_{v_l}\right)}{2}\end{array}\right)\right\}\\
&=F_1\left\{\ell\left(\begin{array}{c}\varpi\left(\vartheta_{v_l},\vartheta_{u_l}\right),\varpi\left(\vartheta_{v_l},\vartheta_{v_l+1}\right),\varpi\left(\vartheta_{u_l},\vartheta_{u_l+1}\right),\varpi\left(\vartheta_{v_l},\vartheta_{u_l+1}\right),\\ \varpi\left(\vartheta_{u_l},\vartheta_{v_l+1}\right),\frac{\varpi\left(\vartheta_{v_l},\vartheta_{v_l+1}\right)+\varpi\left(\vartheta_{u_l},\vartheta_{u_l+1}\right)}{2},\frac{\varpi\left(\vartheta_{v_l},\vartheta_{u_l+1}\right)+\varpi\left(\vartheta_{u_l},\vartheta_{v_l+1}\right)}{2}\end{array}\right)\right\}\\
&\leq F_1\left\{\ell\left(\begin{array}{c}\varpi\left(\vartheta_{v_l},\vartheta_{u_l}\right),\varpi\left(\vartheta_{v_l},\vartheta_{v_l+1}\right),\varpi\left(\vartheta_{u_l},\vartheta_{u_l+1}\right),\\ \gamma\left(\vartheta_{u_l},\vartheta_{u_l+1}\right)\varpi\left(\vartheta_{u_l},\vartheta_{u_l+1}\right)+\gamma\left(\vartheta_{u_l},\vartheta_{v_l}\right)\varpi\left(\vartheta_{u_l},\vartheta_{v_l}\right),\\ \gamma\left(\vartheta_{u_l},\vartheta_{u_l+1}\right)\varpi\left(\vartheta_{u_l},\vartheta_{u_l+1}\right)+\gamma\left(\vartheta_{u_l+1},\vartheta_{v_l+1}\right)\varpi\left(\vartheta_{u_l+1},\vartheta_{v_l+1}\right),\\ \frac{\varpi\left(\vartheta_{v_l},\vartheta_{v_l+1}\right)+\varpi\left(\vartheta_{u_l},\vartheta_{u_l+1}\right)}{2},\frac{\gamma\left(\vartheta_{u_l},\vartheta_{u_l+1}\right)\varpi\left(\vartheta_{u_l},\vartheta_{u_l+1}\right)+\gamma\left(\vartheta_{u_l},\vartheta_{v_l}\right)\varpi\left(\vartheta_{u_l},\vartheta_{v_l}\right)}{2}\\ +\frac{\gamma\left(\vartheta_{u_l},\vartheta_{u_l+1}\right)\varpi\left(\vartheta_{u_l},\vartheta_{u_l+1}\right)+\gamma\left(\vartheta_{u_l+1},\vartheta_{v_l+1}\right)\varpi\left(\vartheta_{u_l+1},\vartheta_{v_l+1}\right)}{2}\end{array}\right)\right\}.
\end{aligned}$$

Since F_1 is continuous, letting $l \to \infty$, and using (4.15) and (4.16), we have

$$\begin{aligned}
&\lim_{l\to\infty} \widetilde{\psi}\left(\varpi\left(\vartheta_{v_l},\vartheta_{u_l}\right)\right)+F_1(\theta)\\
&\leq F_1\left\{\ell\begin{pmatrix}\theta,0,0,0+\theta\gamma\left(\vartheta_{u_l},\vartheta_{v_l}\right),0+\theta\gamma\left(\vartheta_{u_l+1},\vartheta_{v_l+1}\right),\\ 0,\frac{0+\theta\gamma\left(\vartheta_{u_l},\vartheta_{v_l}\right)+\theta\gamma\left(\vartheta_{u_l+1},\vartheta_{v_l+1}\right)}{2}\end{pmatrix}\right\}\\
&=F_1\left\{\ell\left(\theta,0,0,\theta\gamma\left(\vartheta_{u_l},\vartheta_{v_l}\right),\theta\gamma\left(\vartheta_{u_l+1},\vartheta_{v_l+1}\right),0,\frac{\theta\gamma\left(\vartheta_{u_l},\vartheta_{v_l}\right)+\theta\gamma\left(\vartheta_{u_l+1},\vartheta_{v_l+1}\right)}{2}\right)\right\}\\
&\leq F_1\left\{\theta\ell\left(1,0,0,\gamma\left(\vartheta_{u_l},\vartheta_{v_l}\right),\gamma\left(\vartheta_{u_l+1},\vartheta_{v_l+1}\right),0,\frac{\gamma\left(\vartheta_{u_l},\vartheta_{v_l}\right)+\gamma\left(\vartheta_{u_l+1},\vartheta_{v_l+1}\right)}{2}\right)\right\}\\
&\leq F_1(\theta),
\end{aligned}$$

since $\ell \in \widetilde{\aleph}$, thus, $\ell\left(1,0,0,\gamma\left(\vartheta_{u_l},\vartheta_{v_l}\right),\gamma\left(\vartheta_{u_l+1},\vartheta_{v_l+1}\right),0,\frac{\gamma\left(\vartheta_{u_l},\vartheta_{v_l}\right)+\gamma\left(\vartheta_{u_l+1},\vartheta_{v_l+1}\right)}{2}\right) \in (0,1]$. Hence,

$$\lim_{l\to\infty} \widetilde{\psi}\left(\varpi\left(\vartheta_{v_l},\vartheta_{u_l}\right)\right)+F_1(\theta) \leq F_1(\theta),$$

which implies that

$$\lim_{l\to\infty} \widetilde{\psi}\left(\varpi\left(\vartheta_{v_l},\vartheta_{u_l}\right)\right) \leq 0.$$

Therefore,

$$\lim_{\wp\to\theta^+} \inf \widetilde{\psi}(\wp) \leq 0,$$

which is a contradiction. Hence, (4.13) is true. Thus, $\{\vartheta_u\}$ is a CS and the completeness of η implies that there is $\vartheta^* \in \eta$ in order that $\vartheta_u \to \vartheta^*$ as $u \to \infty$. Theorem 4.2 provides the remainder of the proof, which leads to $\vartheta^* \in \mho\vartheta^*$. □

If we take $F \in \nabla(\widetilde{Z})$, we can present the following theorem:

Theorem 4.5 *Let $\mho : \eta \to \eta^c$ be an MVM defined on a complete CMS (η,ϖ,γ). Assume that the following conditions are true*:

(c_i) $\widetilde{\psi} \in \Psi$ *and* $F \in \nabla(\widetilde{Z})$;

(c_{ii}) *for all* $\vartheta > 0$, $F(\vartheta) \leq \eth(\vartheta)$, *where* $\eth$ *is a real-valued function on* $\mathbb{R}^+$;

(c_{iii}) $\Upsilon\left(\mho\vartheta,\mho\widetilde{\vartheta}\right) > 0$, *implies*

$$\begin{aligned}
&\widetilde{\psi}\left(\varpi\left(\vartheta,\widetilde{\vartheta}\right)\right)+\eth\left(\Upsilon\left(\mho\vartheta,\mho\widetilde{\vartheta}\right)\right)\\
&\leq F\left\{\ell\begin{pmatrix}\varpi\left(\vartheta,\widetilde{\vartheta}\right),B\left(\vartheta,\mho\vartheta\right),B\left(\widetilde{\vartheta},\mho\widetilde{\vartheta}\right),B\left(\vartheta,\mho\widetilde{\vartheta}\right),\\ B\left(\widetilde{\vartheta},\mho\vartheta\right),\frac{B(\vartheta,\mho\vartheta)+B\left(\widetilde{\vartheta},\mho\widetilde{\vartheta}\right)}{2},\frac{B\left(\vartheta,\mho\widetilde{\vartheta}\right)+B\left(\widetilde{\vartheta},\mho\vartheta\right)}{2}\end{pmatrix}\right\},
\end{aligned}$$

for all $\vartheta,\widetilde{\vartheta} \in \eta$ *and* $\ell \in \aleph$;

(c_{iv}) *for* $\vartheta_0 \in \eta$, *define the Picard sequence* $\{\vartheta_u = \mho^u \vartheta_0\}$ *such that*

$$\sup_{n\geq 1} \lim_{j\to\infty} \frac{\gamma\left(\vartheta_{j+1},\vartheta_{j+2}\right)\gamma\left(\vartheta_{j+1},\vartheta_{n}\right)}{\gamma\left(\vartheta_{j},\vartheta_{j+1}\right)} < 1;$$

(c_v) *for all* $\vartheta \in \eta$, $\lim_{u\to\infty} \gamma(\vartheta_u,\vartheta) \leq 1$.

Then, $F_{ix}(\mho) \neq \emptyset$.

Proof Suppose that $\vartheta_0 \in \eta$ and $\vartheta_1 \in \mho\vartheta_0$. If $\vartheta_1 \in \mho\vartheta_1$, $\vartheta_1 \in F_{ix}(\mho)$ and this completes the proof. Hence, consider that $\vartheta_1 \notin \mho\vartheta_1$, that is, $B(\vartheta_1,\mho\vartheta_1) > 0$. Thus, $\Upsilon(\mho\vartheta_0,\mho\vartheta_1) > 0$. From ($\heartsuit_4$), one has

$$F(B(\vartheta_1,\mho\vartheta_1)) = \inf_{a\in\mho\vartheta_1} F(\varpi(\vartheta_1,a)). \tag{4.17}$$

It follows from (4.17), (c_{ii}), and (c_{iii}) that

$$\begin{aligned}
\inf_{a\in\mho\vartheta_1} F(\varpi(\vartheta_1,a)) &= F(B(\vartheta_1,\mho\vartheta_1)) \\
&\leq F(\Upsilon(\mho\vartheta_0,\mho\vartheta_1)) \\
&\leq \eth(\Upsilon(\mho\vartheta_0,\mho\vartheta_1)) \\
&\leq F\left\{\ell\begin{pmatrix} \varpi(\vartheta_0,\vartheta_1), B(\vartheta_0,\mho\vartheta_0), B(\vartheta_1,\mho\vartheta_1), B(\vartheta_0,\mho\vartheta_1), \\ B(\vartheta_1,\mho\vartheta_0), \frac{B(\vartheta_0,\mho\vartheta_0)+B(\vartheta_1,\mho\vartheta_1)}{2}, \frac{B(\vartheta_0,\mho\vartheta_1)+B(\vartheta_1,\mho\vartheta_0)}{2} \end{pmatrix}\right\} \\
&\quad -\widetilde{\psi}(\varpi(\vartheta_0,\vartheta_1)) \\
&< F\left\{\ell\begin{pmatrix} \varpi(\vartheta_0,\vartheta_1), \varpi(\vartheta_0,\vartheta_1), \varpi(\vartheta_1,\vartheta_2), \varpi(\vartheta_0,\vartheta_2), 0, \\ \frac{\varpi(\vartheta_0,\vartheta_1)+\varpi(\vartheta_1,\vartheta_2)}{2}, \frac{\varpi(\vartheta_0,\vartheta_2)+\varpi(\vartheta_1,\vartheta_1)}{2} \end{pmatrix}\right\}.
\end{aligned}$$

Hence, there is $\vartheta_2 \in \mho\vartheta_1$ in order that

$$F(\varpi(\vartheta_1,\vartheta_2)) < F\left\{\ell\begin{pmatrix} \varpi(\vartheta_0,\vartheta_1), \varpi(\vartheta_0,\vartheta_1), \varpi(\vartheta_1,\vartheta_2), \varpi(\vartheta_0,\vartheta_2), 0, \\ \frac{\varpi(\vartheta_0,\vartheta_1)+\varpi(\vartheta_1,\vartheta_2)}{2}, \frac{\varpi(\vartheta_0,\vartheta_2)+\varpi(\vartheta_1,\vartheta_1)}{2} \end{pmatrix}\right\}. \tag{4.18}$$

Since F is nondecreasing, it follows from (4.18) and (ϖ_3) that

$$\begin{aligned}
\varpi(\vartheta_1,\vartheta_2) &< \ell\begin{pmatrix} \varpi(\vartheta_0,\vartheta_1), \varpi(\vartheta_0,\vartheta_1), \varpi(\vartheta_1,\vartheta_2), \varpi(\vartheta_0,\vartheta_2), \\ 0, \frac{\varpi(\vartheta_0,\vartheta_1)+\varpi(\vartheta_1,\vartheta_2)}{2}, \frac{\varpi(\vartheta_0,\vartheta_2)}{2} \end{pmatrix} \\
&\leq \ell\begin{pmatrix} \varpi(\vartheta_0,\vartheta_1), \varpi(\vartheta_0,\vartheta_1), \varpi(\vartheta_1,\vartheta_2), \\ \gamma(\vartheta_0,\vartheta_1)\varpi(\vartheta_0,\vartheta_1) + \gamma(\vartheta_1,\vartheta_2)\varpi(\vartheta_1,\vartheta_2), 0, \\ \frac{\varpi(\vartheta_0,\vartheta_1)+\varpi(\vartheta_1,\vartheta_2)}{2}, \frac{\gamma(\vartheta_0,\vartheta_1)\varpi(\vartheta_0,\vartheta_1)+\gamma(\vartheta_1,\vartheta_2)\varpi(\vartheta_1,\vartheta_2)}{2} \end{pmatrix}.
\end{aligned}$$

From Lemma 3.3, we obtain

$$\varpi(\vartheta_1,\vartheta_2) < \varpi(\vartheta_0,\vartheta_1).$$

Similarly, we have $\vartheta_3 \in \mho\vartheta_2$ with $B(\vartheta_2,\mho\vartheta_2) > 0$. Using Lemma 3.3, ($c_{ii}$), and ($c_{iii}$), we have

$$\varpi(\vartheta_2,\vartheta_3) < \varpi(\vartheta_1,\vartheta_2).$$

As we stated before, we have a sequence $\{\vartheta_u\} \subset \eta$ in order that $\vartheta_{u+1} \in \mho\vartheta_u$ with $B(\vartheta_u, \mho\vartheta_u) > 0$ and

$$\varpi(\vartheta_u, \vartheta_{u+1}) < \varpi(\vartheta_{u-1}, \vartheta_u), \ \forall u \in \mathbb{N}. \tag{4.19}$$

Inequality ((4.19) proves that $\{\varpi(\vartheta_u, \vartheta_{u+1})\}_{s\in\mathbb{N}}$ is a decreasing sequence. From $(\heartsuit_4)$, one can write

$$\begin{aligned}
&\inf_{a\in\mho\vartheta_u} F(\varpi(\vartheta_u, a)) \\
&= F(B(\vartheta_u, \mho\vartheta_u)) \\
&\leq F(\Upsilon(\mho\vartheta_{u-1}, \mho\vartheta_u)) \\
&\leq \backepsilon(\Upsilon(\mho\vartheta_{u-1}, \mho\vartheta_u)) \\
&\leq F\left\{\ell\begin{pmatrix} \varpi(\vartheta_{u-1}, \vartheta_u), B(\vartheta_{u-1}, \mho\vartheta_{u-1}), B(\vartheta_u, \mho\vartheta_u), B(\vartheta_{u-1}, \mho\vartheta_u), \\ B(\vartheta_u, \mho\vartheta_{u-1}), \frac{B(\vartheta_{u-1}, \mho\vartheta_{u-1}) + B(\vartheta_u, \mho\vartheta_u)}{2}, \frac{B(\vartheta_{u-1}, \mho\vartheta_u) + B(\vartheta_u, \mho\vartheta_{u-1})}{2} \end{pmatrix}\right\} \\
&\quad - \widetilde{\psi}(\varpi(\vartheta_{u-1}, \vartheta_u)) \\
&\leq F\left\{\ell\begin{pmatrix} \varpi(\vartheta_{u-1}, \vartheta_u), \varpi(\vartheta_{u-1}, \vartheta_u), \varpi(\vartheta_u, \vartheta_{u+1}), \varpi(\vartheta_{u-1}, \vartheta_{u+1}), \\ \varpi(\vartheta_u, \vartheta_u), \frac{\varpi(\vartheta_{u-1}, \vartheta_u) + \varpi(\vartheta_u, \vartheta_{u+1})}{2}, \frac{\varpi(\vartheta_{u-1}, \vartheta_{u+1}) + \varpi(\vartheta_u, \vartheta_u)}{2} \end{pmatrix}\right\} \\
&\quad - \widetilde{\psi}(\varpi(\vartheta_{u-1}, \vartheta_u)) \\
&\leq F\left\{\ell\begin{pmatrix} \varpi(\vartheta_{u-1}, \vartheta_u), \varpi(\vartheta_{u-1}, \vartheta_u), \varpi(\vartheta_u, \vartheta_{u+1}), \\ \gamma(\vartheta_{u-1}, \vartheta_u)\varpi(\vartheta_{u-1}, \vartheta_u) + \gamma(\vartheta_u, \vartheta_{u+1})\varpi(\vartheta_u, \vartheta_{u+1}), 0, \\ \frac{\varpi(\vartheta_{u-1}, \vartheta_u) + \varpi(\vartheta_u, \vartheta_{u+1})}{2}, \frac{\gamma(\vartheta_{u-1}, \vartheta_u)\varpi(\vartheta_{u-1}, \vartheta_u) + \gamma(\vartheta_u, \vartheta_{u+1})\varpi(\vartheta_u, \vartheta_{u+1}) + 0}{2} \end{pmatrix}\right\} \\
&\quad - \widetilde{\psi}(\varpi(\vartheta_{u-1}, \vartheta_u)) \\
&< F\left\{\ell\begin{pmatrix} \varpi(\vartheta_{u-1}, \vartheta_u), \varpi(\vartheta_{u-1}, \vartheta_u), \varpi(\vartheta_{u-1}, \vartheta_u), \\ \gamma(\vartheta_{u-1}, \vartheta_u)\varpi(\vartheta_{u-1}, \vartheta_u) + \gamma(\vartheta_u, \vartheta_{u+1})\varpi(\vartheta_{u-1}, \vartheta_u), 0, \\ \frac{\varpi(\vartheta_{u-1}, \vartheta_u) + \varpi(\vartheta_{u-1}, \vartheta_u)}{2}, \frac{\gamma(\vartheta_{u-1}, \vartheta_u)\varpi(\vartheta_{u-1}, \vartheta_u) + \gamma(\vartheta_u, \vartheta_{u+1})\varpi(\vartheta_{u-1}, \vartheta_u)}{2} \end{pmatrix}\right\} \\
&\quad - \widetilde{\psi}(\varpi(\vartheta_{u-1}, \vartheta_u)) \\
&\leq F\left\{\varpi(\vartheta_{u-1}, \vartheta_u)\,\ell\begin{pmatrix} 1, 1, 1, \gamma(\vartheta_{u-1}, \vartheta_u) + \gamma(\vartheta_u, \vartheta_{u+1}), 0, \\ 1, \frac{\gamma(\vartheta_{u-1}, \vartheta_u) + \gamma(\vartheta_u, \vartheta_{u+1})}{2} \end{pmatrix}\right\} \\
&\quad - \widetilde{\psi}(\varpi(\vartheta_{u-1}, \vartheta_u)) \\
&\leq F(\varpi(\vartheta_{u-1}, \vartheta_u)) - \widetilde{\psi}(\varpi(\vartheta_{u-1}, \vartheta_u)).
\end{aligned}$$

Hence, for each $u \in \mathbb{N}$, we have

$$\inf_{a\in\mho\vartheta_u} F(\varpi(\vartheta_u, a)) \leq F(\varpi(\vartheta_{u-1}, \vartheta_u)) - \widetilde{\psi}(\varpi(\vartheta_{u-1}, \vartheta_u)), \ \forall u \in \mathbb{N}. \tag{4.20}$$

Since $\widetilde{\boldsymbol{\psi}} \in \Psi$, there is $\zeta > 0$ and $u_0 \in \mathbb{N}$ so that $\widetilde{\psi}(\varpi(\vartheta_u, \vartheta_{u+1})) > \zeta$, for all $u \geq u_0$. From (4.22), we obtain

$$F(\varpi(\vartheta_u, \vartheta_{u+1})) \leq F(\varpi(\vartheta_{u-1}, \vartheta_u)) - \widetilde{\psi}(\varpi(\vartheta_{u-1}, \vartheta_u))$$

$$
\begin{aligned}
&\leq F\left(\varpi\left(\vartheta_{u-2},\vartheta_{u-1}\right)\right)-\widetilde{\psi}\left(\varpi\left(\vartheta_{u-2},\vartheta_{u-1}\right)\right)-\widetilde{\psi}\left(\varpi\left(\vartheta_{u-1},\vartheta_{u}\right)\right)\\
&\ \ \vdots\\
&\leq F\left(\varpi\left(\vartheta_{0},\vartheta_{1}\right)\right)-\sum_{j=1}^{u-1}\widetilde{\psi}\left(\varpi\left(\vartheta_{j-1},\vartheta_{j}\right)\right)\\
&=F\left(\varpi\left(\vartheta_{0},\vartheta_{1}\right)\right)-\sum_{j=1}^{u_0-1}\widetilde{\psi}\left(\varpi\left(\vartheta_{j-1},\vartheta_{j}\right)\right)-\sum_{j=u_0}^{u}\widetilde{\psi}\left(\varpi\left(\vartheta_{j-1},\vartheta_{j}\right)\right)\\
&=F\left(\varpi\left(\vartheta_{0},\vartheta_{1}\right)\right)-(u-u_0)\zeta,\ u\geq u_0.
\end{aligned}
\tag{4.21}
$$

In (4.21), take $u \to \infty$, we have

$$\lim_{u\to\infty} F\left(\varpi\left(\vartheta_{u-1},\vartheta_{u}\right)\right)=-\infty.$$

From ($\heartsuit_2'$), we obtain

$$\lim_{u\to\infty} \varpi\left(\vartheta_{u-1},\vartheta_{u}\right)=0. \tag{4.22}$$

Based on ($\heartsuit_3$), there is $l \in (0,1)$ such that that

$$\lim_{u\to\infty} \left(\varpi\left(\vartheta_{u-1},\vartheta_{u}\right)\right)^l F_2\left(\varpi\left(\vartheta_{u-1},\vartheta_{u}\right)\right)=0. \tag{4.23}$$

For all $u \geq u_0$, by (4.22), one can write

$$
\begin{aligned}
&\left(\varpi\left(\vartheta_{u-1},\vartheta_{u}\right)\right)^l F_2\left(\varpi\left(\vartheta_{u-1},\vartheta_{u}\right)\right)-\left(\varpi\left(\vartheta_{u-1},\vartheta_{u}\right)\right)^l F\left(\varpi\left(\vartheta_{0},\vartheta_{1}\right)\right)\\
&\leq\left(\varpi\left(\vartheta_{u-1},\vartheta_{u}\right)\right)^l\left[F\left(\varpi\left(\vartheta_{0},\vartheta_{1}\right)\right)-(u-u_0)\zeta\right]\\
&\quad-\left(\varpi\left(\vartheta_{u-1},\vartheta_{u}\right)\right)^l F\left(\varpi\left(\vartheta_{0},\vartheta_{1}\right)\right)\\
&=-\left(\varpi\left(\vartheta_{0},\vartheta_{1}\right)\right)^l(u-u_0)\zeta\leq 0.
\end{aligned}
\tag{4.24}
$$

Letting $u \to \infty$ in (4.24) and utilizing (4.22) and (4.23), we obtain that

$$0\leq-\lim_{u\to\infty} u\left(\varpi\left(\vartheta_{u-1},\vartheta_{u}\right)\right)^l\leq 0,$$

which yields

$$\lim_{u\to\infty} u\left(\varpi\left(\vartheta_{u-1},\vartheta_{u}\right)\right)^l=0. \tag{4.25}$$

By (4.5), there is $u_1 \in \mathbb{N}$ so that $u\left(\varpi\left(\vartheta_{u-1},\vartheta_{u}\right)\right)^l \leq 1$, for all $u \geq u_1$. Thus, we have

$$\varpi\left(\vartheta_{u-1},\vartheta_{u}\right)\leq\frac{1}{u^{\frac{1}{l}}}\ \text{for all } u\geq u_1.$$

In order to demonstrate that $\{\vartheta_u\}_{u\in\mathbb{N}}$ is a CS, let us look at $\nu, u \in \mathbb{N}$ such that $\nu > u > u_1$. The remainder of the proof proceeds from Theorem 4.2. Using (c_{iv}) and the ratio test, we determine that $\{\vartheta_u\}_{u\in\mathbb{N}}$ is a CS and thus, there is $\vartheta^* \in \eta$ so that

$$\lim_{u\to\infty} \vartheta_u = \vartheta^*.$$

Consider

$$\begin{aligned} F\left(\Upsilon\left(\mho\vartheta,\mho\widetilde{\vartheta}\right)\right) &\le L\left(\Upsilon\left(\mho\vartheta,\mho\widetilde{\vartheta}\right)\right) \le \widetilde{\psi}\left(\varpi\left(\vartheta,\widetilde{\vartheta}\right)\right) + L\left(\Upsilon\left(\mho\vartheta,\mho\widetilde{\vartheta}\right)\right) \\ &\le F\left\{\ell\begin{pmatrix} \varpi\left(\vartheta,\widetilde{\vartheta}\right), B\left(\vartheta,\mho\vartheta\right), B\left(\widetilde{\vartheta},\mho\widetilde{\vartheta}\right), B\left(\vartheta,\mho\widetilde{\vartheta}\right), \\ B\left(\widetilde{\vartheta},\mho\vartheta\right), \frac{B(\vartheta,\mho\vartheta)+B(\widetilde{\vartheta},\mho\widetilde{\vartheta})}{2}, \frac{B(\vartheta,\mho\widetilde{\vartheta})+B(\widetilde{\vartheta},\mho\vartheta)}{2}\end{pmatrix}\right\}. \end{aligned}$$

Since F_1 is nondecreasing, one can write for all $\vartheta,\widetilde{\vartheta} \in \eta$,

$$\Upsilon\left(\mho\vartheta,\mho\widetilde{\vartheta}\right) \le \ell\begin{pmatrix} \varpi\left(\vartheta,\widetilde{\vartheta}\right), B\left(\vartheta,\mho\vartheta\right), B\left(\widetilde{\vartheta},\mho\widetilde{\vartheta}\right), B\left(\vartheta,\mho\widetilde{\vartheta}\right), B\left(\widetilde{\vartheta},\mho\vartheta\right), \\ \frac{B(\vartheta,\mho\vartheta)+B(\widetilde{\vartheta},\mho\widetilde{\vartheta})}{2}, \frac{B(\vartheta,\mho\widetilde{\vartheta})+B(\widetilde{\vartheta},\mho\vartheta)}{2}\end{pmatrix}.$$

Finally, to find the FP of $\mho$, assume the contrary, that is, $B(\vartheta^*,\mho\vartheta^*) > 0$. Along the same lines as Theorem 4.2, we have $B(\vartheta^*,\mho\vartheta^*)$. Since $\mho\vartheta^*$ is closed, $\vartheta^* \in \mho\vartheta^*$. This completes the proof. □

Now, if we take $\ell \in \widetilde{\aleph}$, we have the following theorem:

Theorem 4.6 *Let $\mho : \eta \to \eta^c$ be an MVM defined on a complete CMS (η,ϖ,γ). Assume that the following conditions are satisfied*:

($\bigstar_i$) *$\widetilde{\psi} \in \Psi$, $\ell \in \widetilde{\aleph}$, and F satisfy condition ($\heartsuit_2'$), where $F : \mathbb{R}^+ \to \mathbb{R}$ is a nondecreasing, continuous, and real-valued function*;

($\bigstar_{ii}$) *for all $\vartheta > 0$, $F(\vartheta) \le \eth(\vartheta)$, where $\eth$ is a real-valued function on $\mathbb{R}^+$*;

($\bigstar_{iii}$) *$\Upsilon\left(\mho\vartheta,\mho\widetilde{\vartheta}\right) > 0$, implies*

$$\begin{aligned} &\widetilde{\psi}\left(\varpi\left(\vartheta,\widetilde{\vartheta}\right)\right) + \eth\left(\Upsilon\left(\mho\vartheta,\mho\widetilde{\vartheta}\right)\right) \\ &\le F\left\{\ell\begin{pmatrix} \varpi\left(\vartheta,\widetilde{\vartheta}\right), B\left(\vartheta,\mho\vartheta\right), B\left(\widetilde{\vartheta},\mho\widetilde{\vartheta}\right), B\left(\vartheta,\mho\widetilde{\vartheta}\right), \\ B\left(\widetilde{\vartheta},\mho\vartheta\right), \frac{B(\vartheta,\mho\vartheta)+B(\widetilde{\vartheta},\mho\widetilde{\vartheta})}{2}, \frac{B(\vartheta,\mho\widetilde{\vartheta})+B(\widetilde{\vartheta},\mho\vartheta)}{2}\end{pmatrix}\right\}, \end{aligned}$$

for all $\vartheta,\widetilde{\vartheta} \in \eta$;

($\bigstar_{iv}$) *for all $\vartheta \in \eta$, $\lim_{u\to\infty} \gamma(\vartheta_u,\vartheta) \le 1$.*

Then, $F_{ix}(\mho) \ne \emptyset$.

Proof Assume that $\vartheta_0 \in \eta$ and $\vartheta_1 \in \mho\vartheta_0$. Similar to the proof of Theorem 4.2, we have a sequence $\{\vartheta_u\} \subset \eta$ such that $\vartheta_{u+1} \in \mho\vartheta_u$ with $B(\vartheta_u,\mho\vartheta_{u+1}) > 0$, and

$$\varpi(\vartheta_u,\vartheta_{u+1}) < \varpi(\vartheta_{u-1},\vartheta_u)$$

and

$$F(\varpi(\vartheta_{u-1},\vartheta_u)) < F(\varpi(\vartheta_0,\vartheta_1)) - (u-u_0)\zeta,\ \forall u \ge u_0. \tag{4.26}$$

In (4.26), letting $u \to \infty$, we have

$$\lim_{u\to\infty} F\left(\varpi\left(\vartheta_{u-1},\vartheta_u\right)\right) = -\infty.$$

By ($\heartsuit_2'$), we obtain

$$\lim_{u\to\infty} \varpi\left(\vartheta_{u-1},\vartheta_u\right) = 0.$$

Now, we show that

$$\lim_{u,v\to\infty} \varpi\left(\vartheta_u,\vartheta_v\right) = 0. \tag{4.27}$$

Assume that (4.27) is not true, there is $\theta > 0$ so that for each $\widehat{r} \geq 0$, and we have $v_l > u_l > \widehat{r}$ and

$$\varpi\left(\vartheta_u,\vartheta_v\right) < \theta.$$

In addition, there is $\widehat{r}_0 \in \mathbb{N}$ in order that

$$m_{\widehat{r}_0} = \varpi\left(\vartheta_{u-1},\vartheta_u\right) < \theta,\ \forall u \geq \widehat{r}_0.$$

There are two subsequences $\{\vartheta_{u_l}\}$ and $\{\vartheta_{v_l}\}$ of $\{\vartheta_u\}$, and following the same steps as Theorem 4.4, we obtain that

$$\lim_{l\to\infty} \varpi\left(\vartheta_{v_l},\vartheta_{u_l}\right) = \theta$$

and

$$\lim_{l\to\infty} \varpi\left(\vartheta_{v_l+1},\vartheta_{u_l+1}\right) = \theta. \tag{4.28}$$

The monotonicity of F and the conditions ($\bigstar_{ii}$) and ($\bigstar_{iii}$) imply that

$$\begin{aligned}
&\widetilde{\psi}\left(\varpi\left(\vartheta_{v_l},\vartheta_{u_l}\right)\right) + F\left(\varpi\left(\vartheta_{v_l+1},\vartheta_{u_l+1}\right)\right)\\
&= \widetilde{\psi}\left(\varpi\left(\vartheta_{v_l},\vartheta_{u_l}\right)\right) + F\left(B\left(\vartheta_{v_l+1},\mho\vartheta_{u_l}\right)\right)\\
&\leq \widetilde{\psi}\left(\varpi\left(\vartheta_{v_l},\vartheta_{u_l}\right)\right) + F\left(\Upsilon\left(\mho\vartheta_{v_l},\mho\vartheta_{u_l}\right)\right)\\
&\leq \widetilde{\psi}\left(\varpi\left(\vartheta_{v_l},\vartheta_{u_l}\right)\right) + \eth\left(\Upsilon\left(\mho\vartheta_{v_l},\mho\vartheta_{u_l}\right)\right)\\
&\leq F\left\{\ell\left(\begin{array}{c}
\varpi\left(\vartheta_{v_l},\vartheta_{u_l}\right), \varpi\left(\vartheta_{v_l},\vartheta_{v_l+1}\right), \varpi\left(\vartheta_{u_l},\vartheta_{u_l+1}\right),\\
\gamma\left(\vartheta_{u_l},\vartheta_{u_l+1}\right)\varpi\left(\vartheta_{u_l},\vartheta_{u_l+1}\right) + \gamma\left(\vartheta_{u_l},\vartheta_{v_l}\right)\varpi\left(\vartheta_{u_l},\vartheta_{v_l}\right),\\
\gamma\left(\vartheta_{u_l},\vartheta_{u_l+1}\right)\varpi\left(\vartheta_{u_l},\vartheta_{u_l+1}\right) + \gamma\left(\vartheta_{u_l+1},\vartheta_{v_l+1}\right)\varpi\left(\vartheta_{u_l+1},\vartheta_{v_l+1}\right),\\
\frac{\varpi\left(\vartheta_{v_l},\vartheta_{v_l+1}\right)+\varpi\left(\vartheta_{u_l},\vartheta_{u_l+1}\right)}{2}, \frac{\gamma\left(\vartheta_{u_l},\vartheta_{u_l+1}\right)\varpi\left(\vartheta_{u_l},\vartheta_{u_l+1}\right)+\gamma\left(\vartheta_{u_l},\vartheta_{v_l}\right)\varpi\left(\vartheta_{u_l},\vartheta_{v_l}\right)}{2}\\
+\frac{\gamma\left(\vartheta_{u_l},\vartheta_{u_l+1}\right)\varpi\left(\vartheta_{u_l},\vartheta_{u_l+1}\right)+\gamma\left(\vartheta_{u_l+1},\vartheta_{v_l+1}\right)\varpi\left(\vartheta_{u_l+1},\vartheta_{v_l+1}\right)}{2}
\end{array}\right)\right\}.
\end{aligned}$$

In the above inequality, letting $l \to \infty$, and using the continuity of F and (4.28), we have

$$\lim_{l\to\infty} \widetilde{\psi}\left(\varpi\left(\vartheta_{v_l},\vartheta_{u_l}\right)\right) + F(\theta)$$

$$
\begin{aligned}
&\leq F\left\{\ell\begin{pmatrix}\theta,0,0,0+\theta\gamma\left(\vartheta_{u_l},\vartheta_{v_l}\right),0+\theta\gamma\left(\vartheta_{u_l+1},\vartheta_{v_l+1}\right),\\ 0,\frac{0+\theta\gamma\left(\vartheta_{u_l},\vartheta_{v_l}\right)+\theta\gamma\left(\vartheta_{u_l+1},\vartheta_{v_l+1}\right)}{2}\end{pmatrix}\right\}\\
&=F\left\{\ell\left(\theta,0,0,\theta\gamma\left(\vartheta_{u_l},\vartheta_{v_l}\right),\theta\gamma\left(\vartheta_{u_l+1},\vartheta_{v_l+1}\right),0,\frac{\theta\gamma\left(\vartheta_{u_l},\vartheta_{v_l}\right)+\theta\gamma\left(\vartheta_{u_l+1},\vartheta_{v_l+1}\right)}{2}\right)\right\}\\
&\leq F\left\{\theta\ell\left(1,0,0,\gamma\left(\vartheta_{u_l},\vartheta_{v_l}\right),\gamma\left(\vartheta_{u_l+1},\vartheta_{v_l+1}\right),0,\frac{\gamma\left(\vartheta_{u_l},\vartheta_{v_l}\right)+\gamma\left(\vartheta_{u_l+1},\vartheta_{v_l+1}\right)}{2}\right)\right\}\\
&\leq F(\theta),
\end{aligned}
$$

since $\ell\in\widetilde{\aleph}$, and thus $\ell\left(1,0,0,\gamma\left(\vartheta_{u_l},\vartheta_{v_l}\right),\gamma\left(\vartheta_{u_l+1},\vartheta_{v_l+1}\right),0,\frac{\gamma\left(\vartheta_{u_l},\vartheta_{v_l}\right)+\gamma\left(\vartheta_{u_l+1},\vartheta_{v_l+1}\right)}{2}\right)\in(0,1]$. Hence,

$$
\lim_{l\to\infty}\widetilde{\psi}\left(\varpi\left(\vartheta_{v_l},\vartheta_{u_l}\right)\right)+F\left(\theta\right)\leq F(\theta),
$$

which implies that

$$
\lim_{l\to\infty}\widetilde{\psi}\left(\varpi\left(\vartheta_{v_l},\vartheta_{u_l}\right)\right)\leq 0.
$$

Therefore,

$$
\lim_{\wp\to\theta^+}\inf\widetilde{\psi}(\wp)\leq 0,
$$

which is a contradiction of the definition of Ψ. Hence, (4.26) is true. Thus, $\{\vartheta_u\}$ is a CS and the completeness of η implies that there is $\vartheta^*\in\eta$ in order that $\vartheta_u\to\vartheta^*$ as $u\to\infty$. Theorem 4.5 provides the remainder of the proof, which leads to $\vartheta^*\in\mho\vartheta^*$. □

5 Data-dependence result

The FP sets $F_{ix}(\mho_1)$ and $F_{ix}(\mho_2)$ are nonempty for a MS (η,ϖ) and mappings $\mho_1,\mho_2:\eta\to Q(\eta)$. Numerous authors have tackled the topic of determining the PH distance Υ between $F_{ix}(\mho_1)$ and $F_{ix}(\mho_2)$, provided that for $k>0$, $\Upsilon(\mho_1\vartheta,\mho_2\vartheta)<k$ for all $\vartheta\in\eta$. For instance, see [22–24].

We provide a data-dependence result for the established result in this section.

Definition 5.1 Suppose that (η,ϖ) is a MS and $\mho:\eta\to\eta^c$ is a MVM such that for all $\vartheta\in\eta$ and $\widetilde{\vartheta}\in\mho\vartheta$, there is a sequence $\{\vartheta_u\}_{u\in\mathbb{N}}$ satisfying

(1) $\vartheta_0=\vartheta$ and $\vartheta_1=\widetilde{\vartheta}$;

(2) $\vartheta_{u+1}\in\mho\vartheta_u$, for all $u\in\mathbb{N}$;

(3) $\{\vartheta_u\}_{u\in\mathbb{N}}$ is convergent to a FP of $\mho$.

Then, $\mho$ is called a multivalued, weakly Picard operator (MWPO, for short). A sequence $\{\vartheta_u\}_{u\in\mathbb{N}}$ that satisfies conditions (2) and (3) of Definition 5.1 is described as a sequence of successive approximations (SAM).

Our main theorem in this section is as follows:

Theorem 5.2 *Assume that $\mho_1, \mho_2 : \eta \to \eta^{cp}$ are MVMs on a complete CMS (η, ϖ, γ) such that an $\widetilde{\psi}F$-contraction is true for $\mho_j$, where $j = 1, 2$. Also, assume that the hypotheses below hold*:

(D_1) *F_1 is a real-valued, nondecreasing function on $\mathbb{R}^+$;*
(D_2) *F_2 is a real-valued function on $\mathbb{R}^+$ verifying $(\heartsuit_2')$ and $(\heartsuit_3)$;*
(D_4) *for all $\vartheta \in \eta$, there exists $\zeta > 0$ so that $\Upsilon(\mho_1\vartheta, \mho_2\vartheta) \le \zeta$;*
(D_5) *for $\vartheta_0 \in \eta$, define the Picard sequence $\{\vartheta_u = \mho^u \vartheta_0\}$ such that*

$$\sup_{n\ge 1} \lim_{j\to\infty} \frac{\gamma\left(\vartheta_{j+1}, \vartheta_{j+2}\right)\gamma\left(\vartheta_{j+1}, \vartheta_n\right)}{\gamma\left(\vartheta_j, \vartheta_{j+1}\right)} < 1;$$

(D_6) *for all $\vartheta \in \eta$, $\lim_{u\to\infty} \gamma(\vartheta_u, \vartheta) \le 1$.*
Then, the following results are obtained:
(R_1) *for $j \in \{1, 2\}$, $F_{ix}\left(\mho_j\right) \in \eta^c$;*
(R_2) *$\mho_1$ are $\mho_2$ are MWPOs, and*

$$\begin{aligned}&\Upsilon\left(F_{ix}(\mho_1), F_{ix}(\mho_2)\right)\\ &\quad\le \frac{\zeta}{1 - \max\left\{\ell_1\left(1,1,1,\lambda+\upsilon,0,1,\frac{\lambda+\upsilon}{2}\right), \ell_2\left(1,1,1,\lambda+\upsilon,0,1,\frac{\lambda+\upsilon}{2}\right)\right\}},\end{aligned}$$

where $\lambda, \upsilon \ge 1$.

Proof (R_1) Thanks to Theorem 4.2, $F_{ix}\left(\mho_j\right) \ne \emptyset$ for $j \in \{1, 2\}$. We claim that $F_{ix}\left(\mho_j\right)$ is closed for $j \in \{1, 2\}$. Assume that a there is a sequence $\{\vartheta_u\}$ in $F_{ix}\left(\mho_j\right)$ such that $\lim_{u\to\infty} \vartheta_u = \vartheta$. Now,

$$\begin{aligned}&F_1\left(\Upsilon\left(\mho\vartheta, \mho\widetilde{\vartheta}\right)\right)\\ &\le F_2\left(\Upsilon\left(\mho\vartheta, \mho\widetilde{\vartheta}\right)\right)\\ &\le \widetilde{\psi}\left(\varpi\left(\vartheta, \widetilde{\vartheta}\right)\right) + F_2\left(\Upsilon\left(\mho\vartheta, \mho\widetilde{\vartheta}\right)\right)\\ &\le F_1\left\{\ell\begin{pmatrix}\varpi\left(\vartheta, \widetilde{\vartheta}\right), B\left(\vartheta, \mho\vartheta\right), B\left(\widetilde{\vartheta}, \mho\widetilde{\vartheta}\right), B\left(\vartheta, \mho\widetilde{\vartheta}\right),\\ B\left(\widetilde{\vartheta}, \mho\vartheta\right), \frac{B(\vartheta,\mho\vartheta)+B\left(\widetilde{\vartheta},\mho\widetilde{\vartheta}\right)}{2}, \frac{B\left(\vartheta,\mho\widetilde{\vartheta}\right)+B\left(\widetilde{\vartheta},\mho\vartheta\right)}{2}\end{pmatrix}\right\}.\end{aligned}$$

The monotonicity of F_1, implies that

$$\Upsilon\left(\mho\vartheta, \mho\widetilde{\vartheta}\right) \le \ell\begin{pmatrix}\varpi\left(\vartheta, \widetilde{\vartheta}\right), B\left(\vartheta, \mho\vartheta\right), B\left(\widetilde{\vartheta}, \mho\widetilde{\vartheta}\right), B\left(\vartheta, \mho\widetilde{\vartheta}\right),\\ B\left(\widetilde{\vartheta}, \mho\vartheta\right), \frac{B(\vartheta,\mho\vartheta)+B\left(\widetilde{\vartheta},\mho\widetilde{\vartheta}\right)}{2}, \frac{B\left(\vartheta,\mho\widetilde{\vartheta}\right)+B\left(\widetilde{\vartheta},\mho\vartheta\right)}{2}\end{pmatrix}, \tag{5.1}$$

for all $\vartheta, \widetilde{\vartheta} \in \eta$. Let $B\left(\widetilde{\vartheta}, \mho\widetilde{\vartheta}\right) > 0$. Then, by (5.1), there is $\vartheta \in \mho\widetilde{\vartheta}$ such that

$$\begin{aligned}B\left(\widetilde{\vartheta}, \mho\widetilde{\vartheta}\right) &= \varpi\left(\widetilde{\vartheta}, \vartheta\right)\\ &\le \gamma\left(\widetilde{\vartheta}, \vartheta_{u+1}\right)\varpi\left(\widetilde{\vartheta}, \vartheta_{u+1}\right) + \gamma\left(\vartheta_{u+1}, \vartheta\right)\varpi\left(\vartheta_{u+1}, \vartheta\right)\\ &= \gamma\left(\widetilde{\vartheta}, \vartheta_{u+1}\right)\varpi\left(\widetilde{\vartheta}, \vartheta_{u+1}\right) + \gamma\left(\vartheta_{u+1}, \vartheta\right)B\left(\vartheta_{u+1}, \mho\widetilde{\vartheta}\right)\\ &\le \gamma\left(\widetilde{\vartheta}, \vartheta_{u+1}\right)\varpi\left(\widetilde{\vartheta}, \vartheta_{u+1}\right) + \gamma\left(\vartheta_{u+1}, \vartheta\right)\Upsilon\left(\mho\vartheta_u, \mho\widetilde{\vartheta}\right)\\ &\le \gamma\left(\widetilde{\vartheta}, \vartheta_{u+1}\right)\varpi\left(\widetilde{\vartheta}, \vartheta_{u+1}\right)\end{aligned}$$

$$
\begin{aligned}
&+\gamma(\vartheta_{u+1},\vartheta)\,\ell\begin{pmatrix}\varpi(\vartheta_u,\widetilde{\vartheta}),B(\vartheta_u,\mho\vartheta_u),B(\widetilde{\vartheta},\mho\widetilde{\vartheta}),B(\vartheta_u,\mho\widetilde{\vartheta}),\\ B(\widetilde{\vartheta},\mho\vartheta_u),\frac{B(\vartheta_u,\mho\vartheta_u)+B(\widetilde{\vartheta},\mho\widetilde{\vartheta})}{2},\frac{B(\vartheta_u,\mho\widetilde{\vartheta})+B(\widetilde{\vartheta},\mho\vartheta_u)}{2}\end{pmatrix}\\
&\le\gamma(\widetilde{\vartheta},\vartheta_{u+1})\,\varpi(\widetilde{\vartheta},\vartheta_{u+1})\\
&+\gamma(\vartheta_{u+1},\vartheta)\,\ell\begin{pmatrix}\varpi(\vartheta_u,\widetilde{\vartheta}),B(\vartheta_u,\mho\vartheta_u),B(\widetilde{\vartheta},\mho\widetilde{\vartheta}),\\ \gamma(\vartheta_u,\widetilde{\vartheta})B(\vartheta_u,\widetilde{\vartheta})+\gamma(\widetilde{\vartheta},\mho\widetilde{\vartheta})B(\widetilde{\vartheta},\mho\widetilde{\vartheta}),\\ B(\widetilde{\vartheta},\mho\vartheta_u),\frac{B(\vartheta_u,\mho\vartheta_u)+B(\widetilde{\vartheta},\mho\widetilde{\vartheta})}{2},\\ \frac{\gamma(\vartheta_u,\widetilde{\vartheta})B(\vartheta_u,\widetilde{\vartheta})+\gamma(\widetilde{\vartheta},\mho\widetilde{\vartheta})B(\widetilde{\vartheta},\mho\widetilde{\vartheta})+B(\vartheta_u,\mho\widetilde{\vartheta})}{2}\end{pmatrix}\\
&\le\gamma(\widetilde{\vartheta},\vartheta_{u+1})\,\varpi(\widetilde{\vartheta},\vartheta_{u+1})\\
&+\gamma(\vartheta_{u+1},\vartheta)\,\ell\begin{pmatrix}\varpi(\vartheta_u,\widetilde{\vartheta}),\varpi(\vartheta_u,\vartheta_{u+1}),B(\widetilde{\vartheta},\mho\widetilde{\vartheta}),\\ \gamma(\vartheta_u,\widetilde{\vartheta})\,\varpi(\vartheta_u,\widetilde{\vartheta})+\gamma(\widetilde{\vartheta},\vartheta)B(\widetilde{\vartheta},\mho\widetilde{\vartheta}),\\ \varpi(\widetilde{\vartheta},\vartheta_{u+1}),\frac{\varpi(\vartheta_u,\vartheta_{u+1})+B(\widetilde{\vartheta},\mho\widetilde{\vartheta})}{2},\\ \frac{\gamma(\vartheta_u,\widetilde{\vartheta})\varpi(\vartheta_u,\widetilde{\vartheta})+\gamma(\widetilde{\vartheta},\vartheta)B(\widetilde{\vartheta},\mho\widetilde{\vartheta})+\varpi(\widetilde{\vartheta},\vartheta_{u+1})}{2}\end{pmatrix}.
\end{aligned}
$$

Letting $u\to\infty$ in the above inequality and using the definition of ℓ and (D_6), we have

$$
\begin{aligned}
&B(\widetilde{\vartheta},\mho\widetilde{\vartheta})\\
&\le(1)\ell\left(0,0,B(\widetilde{\vartheta},\mho\widetilde{\vartheta}),0+\gamma(\widetilde{\vartheta},\vartheta)B(\widetilde{\vartheta},\mho\widetilde{\vartheta}),0,\frac{B(\widetilde{\vartheta},\mho\widetilde{\vartheta})}{2},\frac{0+\gamma(\widetilde{\vartheta},\vartheta)B(\widetilde{\vartheta},\mho\widetilde{\vartheta})}{2}\right)\\
&=\ell\left(0,0,B(\widetilde{\vartheta},\mho\widetilde{\vartheta}),\gamma(\widetilde{\vartheta},\vartheta)B(\widetilde{\vartheta},\mho\widetilde{\vartheta}),0,\frac{B(\widetilde{\vartheta},\mho\widetilde{\vartheta})}{2},\frac{\gamma(\widetilde{\vartheta},\vartheta)B(\widetilde{\vartheta},\mho\widetilde{\vartheta})}{2}\right).
\end{aligned}
$$

Using Lemma 3.3, we observe that $B(\widetilde{\vartheta},\mho\widetilde{\vartheta})\le 0$, hence $B(\widetilde{\vartheta},\mho\widetilde{\vartheta})=0$. Since $\mho\widetilde{\vartheta}$ is closed, $\widetilde{\vartheta}\in\mho\widetilde{\vartheta}$. Therefore, $F_{ix}(\mho_j)$ is closed for $j\in\{1,2\}$.

(R_2) According to Theorem 4.2, we conclude that $\mho_1$ are $\mho_2$ are MWPOs. It remains to prove that

$$
\begin{aligned}
&\Upsilon(F_{ix}(\mho_1),F_{ix}(\mho_2))\\
&\quad\le\frac{\zeta}{1-\max\left\{\ell_1\left(1,1,1,\lambda+\upsilon,0,1,\frac{\lambda+\upsilon}{2}\right),\ell_2\left(1,1,1,\lambda+\upsilon,0,1,\frac{\lambda+\upsilon}{2}\right)\right\}}.
\end{aligned}
$$

Let us consider $p>0$ and $\vartheta_0\in F_{ix}(\mho_2)$. Then, there is $\vartheta_1\in\mho_2(\vartheta_0)$ in order that

$$
\varpi(\vartheta_0,\vartheta_1)=B(\vartheta_0,\mho_2(\vartheta_0))\ \text{and}\ \varpi(\vartheta_1,\vartheta_2)\le p\Upsilon(\mho_1(\vartheta_0),\mho_2(\vartheta_0)).
$$

Now, there is $\vartheta_2\in\mho_2(\vartheta_1)$ so that

$$
\varpi(\vartheta_0,\vartheta_1)=B(\vartheta_0,\mho_2(\vartheta_0))\ \text{and}\ \varpi(\vartheta_1,\vartheta_2)\le p\Upsilon(\mho_2(\vartheta_0),\mho_2(\vartheta_1)).
$$

Further, we obtain $\varpi(\vartheta_1,\vartheta_2)<\varpi(\vartheta_0,\vartheta_1)$ and

$$
\begin{aligned}
&\varpi(\vartheta_1,\vartheta_2)\\
&\quad\le p\Upsilon(\mho_2(\vartheta_0),\mho(\vartheta_1))
\end{aligned}
$$

$$\leq p\ell_1 \begin{pmatrix} \varpi(\vartheta_0,\vartheta_1), B(\vartheta_0,\mho_2\vartheta_0), B(\vartheta_1,\mho_2\vartheta_1), B(\vartheta_0,\mho_2\vartheta_1), \\ B(\vartheta_1,\mho_2\vartheta_0), \frac{\varpi(\vartheta_0,\vartheta_0)+B(\vartheta_1,\mho_2\vartheta_1)}{2}, \frac{B(\vartheta_0,\mho_2\vartheta_1)+B(\vartheta_1,\mho_2\vartheta_0)}{2} \end{pmatrix}$$
$$\leq p\ell_1 \begin{pmatrix} \varpi(\vartheta_0,\vartheta_1), \varpi(\vartheta_0,\vartheta_1), \varpi(\vartheta_1,\vartheta_2), \varpi(\vartheta_0,\vartheta_2), \\ \varpi(\vartheta_1,\vartheta_1), \frac{\varpi(\vartheta_0,\vartheta_1)+\varpi(\vartheta_1,\vartheta_2)}{2}, \frac{\varpi(\vartheta_0,\vartheta_2)+\varpi(\vartheta_1,\vartheta_1)}{2} \end{pmatrix}$$
$$\leq p\ell_1 \begin{pmatrix} \varpi(\vartheta_0,\vartheta_1), \varpi(\vartheta_0,\vartheta_1), \varpi(\vartheta_1,\vartheta_2), \\ \gamma(\vartheta_0,\vartheta_1)\varpi(\vartheta_0,\vartheta_1) + \gamma(\vartheta_1,\vartheta_2)\varpi(\vartheta_1,\vartheta_2), \\ 0, \frac{\varpi(\vartheta_0,\vartheta_1)+\varpi(\vartheta_1,\vartheta_2)}{2}, \frac{\gamma(\vartheta_0,\vartheta_1)\varpi(\vartheta_0,\vartheta_1)+\gamma(\vartheta_1,\vartheta_2)\varpi(\vartheta_1,\vartheta_2)+0}{2} \end{pmatrix}$$
$$< p\ell_1 \begin{pmatrix} \varpi(\vartheta_0,\vartheta_1), \varpi(\vartheta_0,\vartheta_1), \varpi(\vartheta_0,\vartheta_1), \\ \gamma(\vartheta_0,\vartheta_1)\varpi(\vartheta_0,\vartheta_1) + \gamma(\vartheta_1,\vartheta_2)\varpi(\vartheta_0,\vartheta_1), \\ 0, \frac{\varpi(\vartheta_0,\vartheta_1)+\varpi(\vartheta_0,\vartheta_1)}{2}, \frac{\gamma(\vartheta_0,\vartheta_1)\varpi(\vartheta_0,\vartheta_1)+\gamma(\vartheta_1,\vartheta_2)\varpi(\vartheta_0,\vartheta_1)}{2} \end{pmatrix}$$
$$\leq p\varpi(\vartheta_0,\vartheta_1)\,\ell_1\left(1,1,1,\gamma(\vartheta_0,\vartheta_1)+\gamma(\vartheta_1,\vartheta_2),0,1,\frac{\gamma(\vartheta_0,\vartheta_1)+\gamma(\vartheta_1,\vartheta_2)}{2}\right),$$

where $\ell_1 \in \ell \in \aleph$. Therefore, we obtain a sequence of SAM of $\mho$ at starting point ϑ_0, which fulfills

$$\varpi(\vartheta_u,\vartheta_{u+1}) \leq \left(p\ell_1\left(1,1,1,\lambda+\upsilon,0,1,\frac{\lambda+\upsilon}{2}\right)^u \varpi(\vartheta_0,\vartheta_1)\right),$$

for all $\lambda,\upsilon \geq 1$ and all $u \in \mathbb{N}$.

In another form, we can write

$$\varpi(\vartheta_u,\vartheta_{u+n}) \leq \frac{\left(p\ell_1\left(1,1,1,\lambda+\upsilon,0,1,\frac{\lambda+\upsilon}{2}\right)\right)^u}{1-p\ell_1\left(1,1,1,\lambda+\upsilon,0,1,\frac{\lambda+\upsilon}{2}\right)}\varpi(\vartheta_0,\vartheta_1), \text{ for all } u \in \mathbb{N}. \tag{5.2}$$

In (5.2), letting $u \to \infty$, we find that $\{\vartheta_u\}$ is a CS in η, and thus, converges to some $\sigma \in \eta$. From the proof of Theorem 4.2, we obtain that $\sigma \in F_{ix}(\mho_2)$. Again, passing $n \to \infty$ in (5.2), one has

$$\varpi(\vartheta_u,\sigma) \leq \frac{\left(p\ell_1\left(1,1,1,\lambda+\upsilon,0,1,\frac{\lambda+\upsilon}{2}\right)\right)^u}{1-p\ell_1\left(1,1,1,\lambda+\upsilon,0,1,\frac{\lambda+\upsilon}{2}\right)}\varpi(\vartheta_0,\vartheta_1), \text{ for all } u \in \mathbb{N}.$$

Setting $u = 0$, and using (D_4), we obtain

$$\varpi(\vartheta_0,\sigma) \leq \frac{1}{1-p\ell_1\left(1,1,1,\lambda+\upsilon,0,1,\frac{\lambda+\upsilon}{2}\right)}\varpi(\vartheta_0,\vartheta_1)$$
$$\leq \frac{p\zeta}{1-p\ell_1\left(1,1,1,\lambda+\upsilon,0,1,\frac{\lambda+\upsilon}{2}\right)}.$$

Switching the roles of $\mho_1$ and $\mho_2$, for every $\sigma_0 \in F_{ix}(\mho_1)$, one can write

$$\varpi(\vartheta_0,\sigma_0) \leq \frac{1}{1-p\ell_2\left(1,1,1,\lambda+\upsilon,0,1,\frac{\lambda+\upsilon}{2}\right)}\varpi(\vartheta_0,\vartheta_1)$$
$$\leq \frac{p\zeta}{1-p\ell_2\left(1,1,1,\lambda+\upsilon,0,1,\frac{\lambda+\upsilon}{2}\right)}.$$

Hence,

$$\begin{aligned}&\Upsilon\left(F_{ix}\left(\mho_1\right),F_{ix}\left(\mho_2\right)\right)\\&\quad\leq\frac{P\zeta}{1-\max\left\{p\ell_1\left(1,1,1,\lambda+\upsilon,0,1,\frac{\lambda+\upsilon}{2}\right),p\ell_2\left(1,1,1,\lambda+\upsilon,0,1,\frac{\lambda+\upsilon}{2}\right)\right\}},\end{aligned}$$

where $\lambda,\upsilon\geq 1$. Letting $p\to 1$ in the above inequality, we have the result. □

6 Well-posednees and strict FPs

The definition of the well-posedness for the FP problem is presented in [25] as follows:

Definition 6.1 Assume that (η,ϖ) is an MS, $\Lambda\in Q(\eta)$, and $\mho:\Lambda\to\eta^c$ is a MVM. The FP issue is called well-posed for $\mho$ with respect to (w.r.t.) B if

(w_1) $F_{ix}(\mho)=\{\widehat{\vartheta}\}$;

(w_2) if $\vartheta_u\in\Lambda$, for all $u\in\mathbb{N}$, $\lim_{u\to\infty}B(\vartheta_u,\mho\vartheta_u)=0$.

Then, $\lim_{u\to\infty}\vartheta_u=\widehat{\vartheta}\in F_{ix}(\mho)$.

Definition 6.2 Assume that (η,ϖ) is an MS, $\Lambda\in Q(\eta)$, and $\mho:\Lambda\to\eta^c$ is a MVM. The FP issue is called well-posed for $\mho$ w.r.t. Υ if

(w_1) $SF_{ix}(\mho)=\{\widehat{\vartheta}\}$;

(w_2) if $\vartheta_u\in\Lambda$, for all $u\in\mathbb{N}$, $\lim_{u\to\infty}\Upsilon(\vartheta_u,\mho\vartheta_u)=0$.

Then, $\lim_{u\to\infty}\vartheta_u=\widehat{\vartheta}\in SF_{ix}(\mho)$.

The main theorem in this part is as follows:

Theorem 6.3 *Suppose that (η,ϖ,γ) is a complete CMS, $\mho:\eta\to\eta^{cp}$ is an MVM and F_1,F_2 are functions verifying an $\widetilde{\psi}F$-contraction. Assume that the following presumptions hold*:

(P_1) *F_1 is nondecreasing*;

(P_2) *F_2 verifies ($\heartsuit_2'$) with $\ell(1,0,0,1,1,0,1)\in(0,1)$*;

(P_3) *$SF_{ix}(\mho)$ is nonempty*;

(P_4) *for all $\vartheta\in\eta$, $\lim_{u\to\infty}\gamma(\vartheta_u,\vartheta)\leq 1$.*

Then,

(I) *$F_{ix}(\mho)=SF_{ix}(\mho)=\left\{\widehat{\vartheta}\right\}$*;

(II) *The FP problem is well-posed for the MVM $\mho$ w.r.t. Υ.*

Proof (I) According to Theorem 4.4, we have $F_{ix}(\mho)\neq\emptyset$. Next, we will show that $F_{ix}(\mho)=\left\{\widehat{\vartheta}\right\}$. Utilizing $\left(\widetilde{\psi}F\right)_i$ and $\left(\widetilde{\psi}F\right)_{ii}$, we can write

$$\begin{aligned}&F_1\left(\Upsilon\left(\mho\vartheta,\mho\widetilde{\vartheta}\right)\right)\\&\leq F_2\left(\Upsilon\left(\mho\vartheta,\mho\widetilde{\vartheta}\right)\right)\\&\leq\widetilde{\psi}\left(\varpi\left(\vartheta,\widetilde{\vartheta}\right)\right)+F_2\left(\Upsilon\left(\mho\vartheta,\mho\widetilde{\vartheta}\right)\right)\\&\leq F_1\left\{\ell\begin{pmatrix}\varpi\left(\vartheta,\widetilde{\vartheta}\right),B\left(\vartheta,\mho\vartheta\right),B\left(\widetilde{\vartheta},\mho\widetilde{\vartheta}\right),B\left(\vartheta,\mho\widetilde{\vartheta}\right),\\B\left(\widetilde{\vartheta},\mho\vartheta\right),\frac{B(\vartheta,\mho\vartheta)+B\left(\widetilde{\vartheta},\mho\widetilde{\vartheta}\right)}{2},\frac{B\left(\vartheta,\mho\widetilde{\vartheta}\right)+B\left(\widetilde{\vartheta},\mho\vartheta\right)}{2}\end{pmatrix}\right\}.\end{aligned}$$

The monotonicity of F_1, implies that

$$\Upsilon\left(\mho\vartheta,\mho\widetilde{\vartheta}\right)\leq\ell\begin{pmatrix}\varpi\left(\vartheta,\widetilde{\vartheta}\right),B\left(\vartheta,\mho\vartheta\right),B\left(\widetilde{\vartheta},\mho\widetilde{\vartheta}\right),B\left(\vartheta,\mho\widetilde{\vartheta}\right),\\ B\left(\widetilde{\vartheta},\mho\vartheta\right),\frac{B(\vartheta,\mho\vartheta)+B\left(\widetilde{\vartheta},\mho\widetilde{\vartheta}\right)}{2},\frac{B\left(\vartheta,\mho\widetilde{\vartheta}\right)+B\left(\widetilde{\vartheta},\mho\vartheta\right)}{2}\end{pmatrix},$$

for all $\vartheta,\widetilde{\vartheta}\in\eta$. Consider $\sigma\in F_{ix}(\mho)$ with $\sigma\neq\widehat{\vartheta}$. Then, $B\left(\widehat{\vartheta},\mho\sigma\right)>0$. Now, we obtain

$$\begin{aligned}B\left(\widehat{\vartheta},\mho\sigma\right) &= \Upsilon\left(\mho\widehat{\vartheta},\mho\sigma\right)\\ &\leq\ell\begin{pmatrix}\varpi\left(\widehat{\vartheta},\sigma\right),B\left(\widehat{\vartheta},\mho\widehat{\vartheta}\right),B\left(\sigma,\mho\sigma\right),B\left(\widehat{\vartheta},\mho\sigma\right),\\ B\left(\sigma,\mho\widehat{\vartheta}\right),\frac{B\left(\widehat{\vartheta},\mho\widehat{\vartheta}\right)+B(\sigma,\mho\sigma)}{2},\frac{B\left(\widehat{\vartheta},\mho\sigma\right)+B\left(\sigma,\mho\widehat{\vartheta}\right)}{2}\end{pmatrix}\\ &\leq\ell\left(\varpi\left(\widehat{\vartheta},\sigma\right),0,0,B\left(\widehat{\vartheta},\sigma\right),B\left(\sigma,\widehat{\vartheta}\right),0,\frac{B\left(\widehat{\vartheta},\sigma\right)+B\left(\sigma,\widehat{\vartheta}\right)}{2}\right)\\ &\leq\varpi\left(\widehat{\vartheta},\sigma\right)\ell\left(1,0,0,1,1,0,1\right).\end{aligned}$$

Applying the condition (P_2), we obtain

$$\varpi\left(\widehat{\vartheta},\sigma\right)=B\left(\widehat{\vartheta},\mho\sigma\right)\leq\varpi\left(\widehat{\vartheta},\sigma\right),$$

which is a contradiction. Hence, $\varpi\left(\widehat{\vartheta},\sigma\right)=0$, that is, $\widehat{\vartheta}=\sigma$.

(II) Assume that $\vartheta_u\in\Lambda$ and $u\in\mathbb{N}$ in order that

$$\lim_{u\to\infty}B\left(\vartheta_u,\mho\vartheta_u\right)=0. \tag{6.1}$$

We prove that

$$\lim_{u\to\infty}\varpi\left(\vartheta_u,\widehat{\vartheta}\right)=0,$$

where $\widehat{\vartheta}\in F_{ix}(\mho)$. Assume the contrary, then for each $u\in\mathbb{N}$, there is $\varepsilon>0$ so that

$$\varpi\left(\vartheta_u,\widehat{\vartheta}\right)>\varepsilon.$$

Equation (6.1) leads to the fact that there is $u_\varepsilon\in\mathbb{N}-\{0\}$ so that

$$\lim_{u\to\infty}B\left(\vartheta_u,\mho\vartheta_u\right)<\varepsilon,\ \text{for each } u>u_\varepsilon.$$

It follows that

$$\varpi\left(\vartheta_u,\widehat{\vartheta}\right)=B\left(\vartheta_u,\mho\widehat{\vartheta}\right),\ \text{for each } u>u_\varepsilon.$$

Since $\mho\widehat{\vartheta}$ is compact, there is $\vartheta\in\mho\widehat{\vartheta}$ so that

$$\begin{aligned}\varpi\left(\vartheta_u,\widehat{\vartheta}\right) &= B\left(\vartheta_u,\mho\widehat{\vartheta}\right)=\varpi\left(\vartheta_u,\vartheta\right)\\ &\leq\gamma\left(\vartheta_u,\vartheta_{u+1}\right)\varpi\left(\vartheta_u,\vartheta_{u+1}\right)+\gamma\left(\vartheta_{u+1},\vartheta\right)\varpi\left(\vartheta_{u+1},\vartheta\right)\\ &= \gamma\left(\vartheta_u,\vartheta_{u+1}\right)B\left(\vartheta_u,\mho\vartheta_u\right)+\gamma\left(\vartheta_{u+1},\vartheta\right)B\left(\vartheta_{u+1},\mho\widehat{\vartheta}\right)\end{aligned}$$

$$\begin{aligned}
&\leq \gamma\left(\vartheta_u, \vartheta_{u+1}\right) B\left(\vartheta_u, \mho\vartheta_u\right) + \gamma\left(\vartheta_{u+1}, \vartheta\right) \Upsilon\left(\mho\vartheta_u, \mho\widehat{\vartheta}\right) \\
&\leq \gamma\left(\vartheta_u, \vartheta_{u+1}\right) B\left(\vartheta_u, \mho\vartheta_u\right) \\
&\quad + \gamma\left(\vartheta_{u+1}, \vartheta\right) \ell \left(\begin{array}{c} \varpi\left(\vartheta_u, \widehat{\vartheta}\right), B\left(\vartheta_u, \mho\vartheta_u\right), B\left(\widehat{\vartheta}, \mho\widehat{\vartheta}\right), B\left(\vartheta_u, \mho\widehat{\vartheta}\right), \\ B\left(\widehat{\vartheta}, \mho\vartheta_u\right), \frac{B(\vartheta_u, \mho\vartheta_u) + B\left(\widehat{\vartheta}, \mho\widehat{\vartheta}\right)}{2}, \frac{B\left(\vartheta_u, \mho\widehat{\vartheta}\right) + B\left(\widehat{\vartheta}, \mho\vartheta_u\right)}{2} \end{array} \right) \\
&\leq \gamma\left(\vartheta_u, \vartheta_{u+1}\right) B\left(\vartheta_u, \mho\vartheta_u\right) \\
&\quad + \gamma\left(\vartheta_{u+1}, \vartheta\right) \ell \left(\begin{array}{c} \varpi\left(\vartheta_u, \widehat{\vartheta}\right), B\left(\vartheta_u, \mho\vartheta_u\right), \varpi\left(\widehat{\vartheta}, \widehat{\vartheta}\right), B\left(\vartheta_u, \widehat{\vartheta}\right), \\ \gamma\left(\widehat{\vartheta}, \vartheta_u\right) B\left(\widehat{\vartheta}, \vartheta_u\right) + \gamma\left(\vartheta_u, \mho\vartheta_u\right) \varpi\left(\vartheta_u, \vartheta_{u+1}\right), \\ \frac{B(\vartheta_u, \mho\vartheta_u), \varpi\left(\widehat{\vartheta}, \widehat{\vartheta}\right)}{2}, \frac{B\left(\vartheta_u, \widehat{\vartheta}\right) + \gamma\left(\widehat{\vartheta}, \vartheta_u\right) B\left(\widehat{\vartheta}, \vartheta_u\right) + \gamma\left(\vartheta_u, \mho\vartheta_u\right) \varpi\left(\vartheta_u, \vartheta_{u+1}\right)}{2} \end{array} \right).
\end{aligned}$$

From conditions (P_2) and (P_4), letting $u \to \infty$ in the above inequality and using (6.1), we have $\lim_{u\to\infty} \varpi\left(\vartheta_u, \widehat{\vartheta}\right) = 0$, which is a contradiction. Therefore, the FP issue is well-posed for the MVM $\mho$ w.r.t. B. Additionally, $F_{ix}(\mho) = SF_{ix}(\mho)$ and the FP issue is well-posed for the MVM $\mho$ w.r.t. Υ. □

7 Conclusion

Several strict and FP results on CMSs have been established in this study. As we utilized the controlled metric setting platform and adhered to the plan of Iqbal et al. [17], the results presented in [17] are specific instances of those presented in this study. We have also given the theorems' well-posedness. Additionally, the FP data-dependence issue of the considered mappings is established. For the sake of authenticity, numerous nontrivial examples are included.

Abbreviations
FP, Fixed point; MS, Metric space; bMS, b-metric space; CMS, controlled metric space; MVM, multivalued mapping; s.o.a., set of all; PH, Pompei–Hausdorff; CS, Cauchy sequence; MWPO, multivalued weakly Picard operator; SAM, successive approximations; w.r.t., with respect to.

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

Author details
[1]Department of Mathematics, College of Science, Qassim University, Buraydah, 51452, Saudi Arabia. [2]Department of Mathematics, Saveetha School of Engineering, SIMATS, Saveetha University, Chennai, 602105, India. [3]Department of Mathematics, Faculty of Science, Sohag University, Sohag, 82524, Egypt. [4]Department of Mathematics, College of Sciences and Art, King Abdulaziz University, Rabigh, Saudi Arabia.

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