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Research Article

Eigenvalue Problems and Bifurcation of Nonhomogeneous Semilinear Elliptic Equations in Exterior Strip Domains

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We consider the following eigenvalue problems: $-\Delta u + u = \lambda(f(u) + h(x))$ in Ω, $u > 0$ in Ω, $u \in H_0^1(\Omega)$, where $\lambda > 0$, $N = m + n \geq 2$, $n \geq 1$, $0 \in \omega \subseteq \mathbb{R}^m$ is a smooth bounded domain, $\mathbb{S} = \omega \times \mathbb{R}^n$, D is a smooth bounded domain in $\mathbb{R}^N$ such that $D \subset\subset \mathbb{S}$, $\Omega = \mathbb{S} \setminus \overline{D}$. Under some suitable conditions on f and h, we show that there exists a positive constant λ^* such that the above-mentioned problems have at least two solutions if $\lambda \in (0, \lambda^*)$, a unique positive solution if $\lambda = \lambda^*$, and no solution if $\lambda > \lambda^*$. We also obtain some bifurcation results of the solutions at $\lambda = \lambda^*$.

1. Introduction

Throughout this article, let $N = m + n \geq 2$, $n \geq 1$, $2^* = 2N/(N-2)$ for $N \geq 3$, $2^* = \infty$ for $N = 2$, $x = (y, z)$ be the generic point of $\mathbb{R}^N$ with $y \in \mathbb{R}^m$, $z \in \mathbb{R}^n$.

In this article, we are concerned with the following eigenvalue problems:

$$-\Delta u + u = \lambda(f(u) + h(x)) \text{ in } \Omega, \quad u \text{ in } H_0^1(\Omega), \quad u > 0 \text{ in } \Omega, \quad N \geq 2, \tag{1.1$_\lambda$}$$

where $\lambda > 0$, $0 \in \omega \subseteq \mathbb{R}^m$ is a smooth bounded domain, $\mathbb{S} = \omega \times \mathbb{R}^n$, D is a smooth bounded domain in $\mathbb{R}^N$ such that $D \subset\subset \mathbb{S}$, $\Omega = \mathbb{S} \setminus \overline{D}$ is an exterior strip domain in $\mathbb{R}^N$, $h(x) \in L^2(\Omega) \cap L^{q_0}(\Omega)$ for some $q_0 > N/2$ if $N \geq 4$, $q_0 = 2$ if $N = 2, 3$, $h(x) \geq 0$, $h(x) \not\equiv 0$ and f satisfies the following conditions:

(f1) $f \in C^1([0, +\infty), \mathbb{R}^+)$, $f(0) = 0$, and $f(t) \equiv 0$ if $t < 0$;

(f2) there is a positive constant C such that

$$|f(t)| \leq C(|t| + |t|^p) \quad \text{for some } 1 < p < 2^* - 1; \tag{1.1}$$

(f3) $\lim_{t\to 0} t^{-1}f(t) = 0$;

(f4) there is a number $\theta \in (0,1)$ such that

$$\theta t f'(t) \geq f(t) > 0 \quad \text{for } t > 0; \tag{1.2}$$

(f5) $f \in C^2(0,+\infty)$ and $f''(t) \geq 0$ for $t > 0$;

(f5)* $f \in C^2(0,+\infty)$ and $f''(t) > 0$ for $t > 0$;

(f6) $\lim_{t\to 0^+} t^{1-q_1} f''(t) \leq C$ where C is some constant, $0 < q_1 < 4/(N-2)$ if $N \geq 3$, $q_1 > 0$ if $N = 2$.

If $\Omega = \mathbb{R}^N$ or $\Omega = \mathbb{R}^N \setminus \overline{D}$ ($m = 0$ in our case), then the homogeneous case of problem $(1.1)_\lambda$ (i.e., the case $h(x) \equiv 0$) has been studied by many authors (see Cao [4] and the references therein). For the nonhomogeneous case ($h(x) \not\equiv 0$), Zhu [18] has studied the special problem

$$\begin{gathered} -\Delta u + u = u^p + h(x) \text{ in } \mathbb{R}^N, \\ u \text{ in } H^1(\mathbb{R}^N), \quad u > 0 \text{ in } \mathbb{R}^N, \quad N \geq 2. \end{gathered} \tag{1.3}$$

They have proved that (1.3) has at least two positive solutions for $\|h\|_{L^2}$ sufficiently small and h exponentially decaying.

Cao and Zhou [5] have considered the following general problems:

$$\begin{gathered} -\Delta u + u = f(x,u) + h(x) \text{ in } \mathbb{R}^N, \\ u \text{ in } H^1(\mathbb{R}^N), \quad u > 0 \text{ in } \mathbb{R}^N, \quad N \geq 2, \end{gathered} \tag{1.4}$$

where $h \in H^{-1}(\mathbb{R}^N)$, $0 \leq f(x,u) \leq c_1 u^p + c_2 u$ with $c_1 > 0$, $c_2 \in [0,1)$ being some constants. They also have shown that (1.4) has at least two positive solutions for $\|h\|_{H^{-1}} < C_p S^{(p+1)/2(p-1)}$ and $h \geq 0$, $h \not\equiv 0$ in $\mathbb{R}^N$, where S is the best Sobolev constant and $C_p = c_1^{-1/(p-1)}(p-1)[(1-c_2)/p]^{p/(p-1)}$.

Zhu and Zhou [19] have investigated the existence and multiplicity of positive solutions of $(1.1)_\lambda$ in $\mathbb{R}^N \setminus \overline{D}$ for $N \geq 3$. They have shown that there exists $\lambda^* > 0$ such that $(1.1)_\lambda$ admits at least two positive solutions if $\lambda \in (0,\lambda^*)$ and $(1.1)_\lambda$ has no positive solutions if $\lambda > \lambda^*$ under the conditions that $h(x) \geq 0$, $h(x) \not\equiv 0$, $h(x) \in L^2(\Omega) \cap L^{(N+\gamma)/2}(\Omega)$ ($\gamma > 0$ if $N \geq 4$ and $\gamma = 0$ if $N = 3$), and f satisfies conditions (f1)–(f5). However, their method cannot know whether λ^* is bounded or infinite.

In the present paper, motivated by [19], we extend and improve the paper by Zhu and Zhou [19]. First, we deal with the more general domains instead of the exterior domains, and second, we prove that λ^* is finite, and third, we also obtain the behavior of the two solutions on $(0,\lambda^*)$ and some bifurcation results of the solutions at $\lambda = \lambda^*$. Now, we state our main results.

Theorem 1.1. *Let $\Omega = \mathbb{S} \setminus \overline{D}$ or $\Omega = \mathbb{R}^N \setminus \overline{D}$. Suppose $h(x) \geq 0$, $h(x) \not\equiv 0$, $h(x) \in L^2(\Omega) \cap L^{q_0}(\Omega)$ for some $q_0 > N/2$ if $N \geq 4$, $q_0 = 2$ if $N = 2,3$, and $f(t)$ satisfies (f1)–(f5). Then there exists $\lambda^* > 0$, $0 < \lambda^* < \infty$ such that*

(i) *equation $(1.1)_\lambda$ has at least two positive solutions u_λ, U_λ, and $u_\lambda < U_\lambda$ if $\lambda \in (0,\lambda^*)$, where u_λ is the minimal solution of $(1.1)_\lambda$ and U_λ is the second solution of $(1.1)_\lambda$ constructed in Section 5;*

(ii) *equation* $(1.1)_\lambda$ *has at least one minimal positive solution* u_{λ^*}*;*
(iii) *equation* $(1.1)_\lambda$ *has no positive solutions if* $\lambda > \lambda^*$.
Moreover, assume that condition (f5) holds, then* $(1.1)_{\lambda^*}$ *has a unique positive solution* u_{λ^*}.

Theorem 1.2. *Suppose the assumptions of Theorem 1.1 and condition (f5)* hold, then*
(i) u_λ *is strictly increasing with respect to* λ, u_λ *is uniformly bounded in* $L^\infty(\Omega)\cap H_0^1(\Omega)$ *for all* $\lambda \in (0,\lambda^*]$, *and*

$$u_\lambda \longrightarrow 0 \quad \text{in } L^\infty(\Omega)\cap H_0^1(\Omega) \text{ as } \lambda \longrightarrow 0^+, \tag{1.5}$$

(ii) U_λ *is unbounded in* $L^\infty(\Omega)\cap H_0^1(\Omega)$ *for* $\lambda \in (0,\lambda^*)$, *that is,*

$$\lim_{\lambda\to 0^+} \|U_\lambda\| = \lim_{\lambda\to 0^+} \|U_\lambda\|_\infty = \infty, \tag{1.6}$$

(iii) *moreover, assume that condition (f6) holds and* $h(x)$ *is in* $C^\alpha(\Omega)\cap L^2(\Omega)$, *then all solutions of* $(1.1)_\lambda$ *are in* $C^{2,\alpha}(\Omega)\cap H^2(\Omega)$, *and* $(\lambda^*, u_{\lambda^*})$ *is a bifurcation point for* $(1.1)_\lambda$ *and*

$$\begin{aligned} u_\lambda &\longrightarrow u_{\lambda^*} \quad \text{in } C^{2,\alpha}(\Omega)\cap H^2(\Omega) \text{ as } \lambda \longrightarrow \lambda^*, \\ U_\lambda &\longrightarrow u_{\lambda^*} \quad \text{in } C^{2,\alpha}(\Omega)\cap H^2(\Omega) \text{ as } \lambda \longrightarrow \lambda^*. \end{aligned} \tag{1.7}$$

2. Preliminaries

In this paper, we denote by C and C_i $(i = 1,2,\dots)$ the universal constants, unless otherwise specified. Now, we will establish some analytic tools and auxiliary results which will be used later. We set

$$\begin{gathered} F(u) = \int_0^u f(s)ds, \\ \|u\| = \left(\int_\Omega (|\nabla u|^2 + u^2)dx\right)^{1/2}, \\ \|u\|_p = \left(\int_\Omega |u|^q dx\right)^{1/q}, \quad 1 \le q < \infty, \\ \|u\|_\infty = \sup_{x\in\Omega} |u(x)|. \end{gathered} \tag{2.1}$$

First, we give some properties of $f(t)$. The proof can be found in Zhu and Zhou [19].

Lemma 2.1. *Under conditions (f1), (f4), and (f5),*
(i) *let* $\nu = 1+\theta^{-1} > 2$, *one has that* $tf(t) \ge \nu F(t)$ *for* $t > 0$;
(ii) $t^{-1/\theta} f(t)$ *is monotone nondecreasing for* $t > 0$ *and* $t^{-1}f(t)$ *is strictly monotone increasing if* $t > 0$;
(iii) *for any* $t_1, t_2 \in (0,+\infty)$, *one has*

$$f(t_1+t_2) \ge f(t_1) + f(t_2), \qquad f(t_1+t_2) \not\equiv f(t_1) + f(t_2). \tag{2.2}$$

In order to get the existence of positive solutions of (1.1)$_\lambda$, consider the energy functional $I : H_0^1(\Omega) \to \mathbb{R}$ *defined by*

$$I(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + u^2)\,dx - \lambda\int_\Omega F(u^+)\,dx - \lambda\int_\Omega hu\,dx. \tag{2.3}$$

By the strong maximum principle, it is easy to show that the critical points of I are the positive solutions of (1.1)$_\lambda$.

Now, introduce the following elliptic equation on $\mathbb{S}$:

$$-\Delta u + u = \lambda f(u) \text{ in } \mathbb{S}, \quad u \in H_0^1(\mathbb{S}), \qquad N \geq 2, \tag*{(2.4)$_\lambda$}$$

and its associated energy functional I^∞ defined by

$$I^\infty(u) = \frac{1}{2}\int_{\mathbb{S}} (|\nabla u|^2 + u^2)\,dx - \lambda\int_{\mathbb{S}} F(u^+)\,dx, \quad u \in H_0^1(\mathbb{S}). \tag{2.4}$$

If (f1)–(f4) hold, using results of Esteban [8] and Lions [15, 16], one knows that (2.4)$_\lambda$ has a ground state $w(x) > 0$ in $\mathbb{S}$ such that

$$S^\infty = I^\infty(w) = \sup_{t\geq 0} I^\infty(tw). \tag{2.5}$$

Now, establish the following decomposition lemma for later use.

Proposition 2.2. *Let conditions (f1), (f2), and (f4) be satisfied and suppose that $\{u_k\}$ is a $(\mathrm{PS})_\alpha$-sequence of I in $H_0^1(\Omega)$, that is, $I(u_k) = \alpha + o(1)$ and $I'(u_k) = o(1)$ strong in $H^{-1}(\Omega)$. Then there exist an integer $l \geq 0$, sequence $\{x_k^i\} \subseteq \mathbb{R}^N$ of the form $(0, z_k^i) \in \mathbb{S}$, a solution $\overline{u}$ of (1.1)$_\lambda$, and solutions u^i of (2.4)$_\lambda$, $1 \leq i \leq l$, such that for some subsequence $\{u_k\}$, one has*

$$\begin{gathered} u_k \rightharpoonup \overline{u} \quad \text{weakly in } H_0^1(\Omega), \\ I(u_k) \longrightarrow I(\overline{u}) + \sum_{i=1}^{l} I^\infty(u^i), \\ u_k - \left(\overline{u} + \sum_{i=1}^{m} u^i(x - x_k^i)\right) \longrightarrow 0 \quad \text{strong in } H_0^1(\Omega), \\ |x_k^i| \longrightarrow \infty, \quad |x_k^i - x_k^j| \longrightarrow \infty, \quad 1 \leq i \neq j \leq l, \end{gathered} \tag{2.6}$$

where one agrees that in the case $l = 0$, the above hold without u^i, x_k^i.

Proof. This result can be derived from the arguments in [3] (see also [15–17]). Here we omit it. □

3. Asymptotic behavior of solutions

In this section, we establish the decay estimate for solutions of (1.1)$_\lambda$ and (2.4)$_\lambda$. In order to get the asymptotic behavior of solutions of (1.1)$_\lambda$, we need the following lemmas. First, we quote regularity Lemma 1 (see Hsu [12] for the proof). Now, let $\mathbb{X}$ be a $C^{1,1}$ domain in $\mathbb{R}^N$.

Lemma 3.1 (regularity Lemma 1). *Let $g : \mathbb{X} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that for almost every $x \in \mathbb{X}$, there holds*

$$|f(x,u)| \leq C(|u| + |u|^p) \quad \textit{uniformly in } x \in \mathbb{X}, \tag{3.1}$$

where $1 < p < 2^* - 1$.

Also, let $u \in H_0^1(\mathbb{X})$ be a weak solution of equation $-\Delta u = f(x,u) + h(x)$ in $\mathbb{X}$, where $h \in L^{N/2}(\mathbb{X}) \cap L^2(\mathbb{X})$. Then $u \in L^q(\mathbb{X})$ for $q \in [2,\infty)$.

Now, we quote Regularity Lemmas 2–4, (see Gilbarg and Trudinger [9, Theorems 8.8, 9.11, and 9.16] for the proof).

Lemma 3.2 (regularity Lemma 2). *Let $\mathbb{X} \subset \mathbb{R}^N$ be a domain, $g \in L^2(\mathbb{X})$, and $u \in H^1(\mathbb{X})$ a weak solution of the equation $-\Delta u + u = g$ in $\mathbb{X}$. Then for any subdomain $\mathbb{X}' \subset\subset \mathbb{X}$ with $d' = \operatorname{dist}(\mathbb{X}', \partial\mathbb{X}) > 0$, $u \in H^2(\mathbb{X}')$ and*

$$\|u\|_{H^2(\mathbb{X}')} \leq C(\|u\|_{H^1(\mathbb{X})} + \|g\|_{L^2(\mathbb{X})}) \tag{3.2}$$

for some $C = C(N,d')$. Furthermore, u satisfies the equation $-\Delta u + u = g$ almost everywhere in $\mathbb{X}$.

Lemma 3.3 (regularity Lemma 3). *Let $g \in L^2(\mathbb{X})$ and let $u \in H_0^1(\mathbb{X})$ be a weak solution of the equation $-\Delta u + u = g$. Then $u \in H_0^2(\mathbb{X})$ satisfies*

$$\|u\|_{H^2(\mathbb{X})} \leq C\|g\|_{L^2(\mathbb{X})}, \tag{3.3}$$

where $C = C(N,\partial\mathbb{X})$.

Lemma 3.4 (regularity Lemma 4). *Let $g \in L^2(\mathbb{X}) \cap L^q(\mathbb{X})$ for some $q \in [2,\infty)$ and let $u \in H_0^1(\mathbb{X})$ be a weak solution of the equation $-\Delta u + u = g$ in $\mathbb{X}$. Then $u \in W^{2,q}(\mathbb{X})$ satisfies*

$$\|u\|_{W^{2,q}(\mathbb{X})} \leq C(\|u\|_{L^q(\mathbb{X})} + \|g\|_{L^q(\mathbb{X})}), \tag{3.4}$$

where $C = C(N,q,\partial\mathbb{X})$.

By Lemmas 3.1 and 3.4, we obtain the first asymptotic behavior of solution of $(1.1)_\lambda$.

Lemma 3.5 (asymptotic Lemma 1). *Let condition (f2) hold and let u be a weak solution of $(1.1)_\lambda$, then $u(y,z) \to 0$ as $|z| \to \infty$ uniformly for $y \in \omega$. Moreover, if $h(x)$ is bounded, then $u \in C^{1,\alpha}(\overline{\Omega})$ for any $0 < \alpha < 1$.*

Proof. Suppose that u is a solution of $(1.1)_\lambda$, then $-\Delta u + u = \lambda(f(u) + h(x))$ in Ω. Since f satisfies condition (f2) and $h \in L^2(\Omega) \cap L^{q_0}(\Omega)$ for some $q_0 > N/2$ if $N \geq 4$, $q_0 = 2$ if $N = 2,3$, this implies that $h \in L^2(\Omega) \cap L^{N/2}(\Omega)$ for $N \geq 4$ and $h \in L^2(\Omega)$ for $N = 2,3$. By Lemma 3.1, we conclude that

$$u \in L^q(\Omega) \quad \text{for } q \in [2,\infty). \tag{3.5}$$

Hence, $\lambda(f(u) + h(x)) \in L^2(\Omega) \cap L^{q_0}(\Omega)$ and by Lemma 3.4, we have

$$u \in W^{2,2}(\Omega) \cap W^{2,q_0}(\Omega), \quad q_0 > \frac{N}{2} \text{ if } N \geq 4,\ q_0 = 2 \text{ if } N = 2,3. \tag{3.6}$$

Now, by the Sobolev embedding theorem, we obtain that $u \in C_b(\overline{\Omega})$. It is well known that the Sobolev embedding constants are independent of domains (see [1]). Thus there exists a constant C such that, for $R > 0$,

$$\|u\|_{L^\infty(\Omega\setminus B_R)} \le C\|u\|_{W^{2,q_0}(\Omega\setminus B_R)} \quad \text{for } N \ge 2, \tag{3.7}$$

where $B_R = \{x = (y,z) \in \Omega \mid |z| \le R\}$. From this, we conclude that $u(y,z) \to 0$ as $|z| \to \infty$ uniformly for $y \in \omega$. By Lemma 3.4 and condition (f2), we also have that

$$\|u\|_\infty \le \|u\|_{W^{2,q_0}(\Omega)} \le C\Big(\|u\|_{q_0} + \big\|\lambda f(u) + \lambda h(x)\big\|_{q_0}\Big) \le C_1\|u\|_{q_0} + \lambda C_2\big(\|u\|_{pq_0}^p + \|h\|_{q_0}\big), \tag{3.8}$$

where C_1, C_2 are constants independent of λ.

Moreover, if $h(x)$ is bounded, then we have $u \in W^{2,q}(\Omega)$ for $q \in [2,\infty)$. Hence, by the Sobolev embedding theorem, we obtain that $u \in C^{1,\alpha}(\overline{\Omega})$ for $\alpha \in (0,1)$. □

We use Lemma 3.5, and modify the proof in Hsu [11]. We obtain the following precise asymptotic behavior of solutions of $(1.1)_\lambda$ and $(2.4)_\lambda$ at infinity.

Lemma 3.6 (asymptotic Lemma 2). *Let w be a positive solution of $(2.4)_\lambda$, let u be a positive solution of $(1.1)_\lambda$, and let φ be the first positive eigenfunction of the Dirichlet problem $-\Delta\varphi = \lambda_1\varphi$ in ω, then for any $\varepsilon > 0$ with $0 < \varepsilon < 1+\lambda_1$, there exist constants $C, C_\varepsilon > 0$ such that*

$$\begin{gathered} w(y,z) \le C_\varepsilon\varphi(y)\exp\Big(-\sqrt{1+\lambda_1-\varepsilon}\,|z|\Big), \\ w(y,z) \ge C\varphi(y)\exp\Big(-\sqrt{1+\lambda_1}\,|z|\Big)|z|^{-(n-1)/2} \quad \text{as } |z| \longrightarrow \infty,\ y \in \varpi, \\ u(y,z) \ge C\varphi(y)\exp\Big(-\sqrt{1+\lambda_1}\,|z|\Big)|z|^{-(n-1)/2}. \end{gathered} \tag{3.9}$$

Proof. (i) First, we claim that for any $\varepsilon > 0$ with $0 < \varepsilon < 1+\lambda_1$, there exists $C_\varepsilon > 0$ such that

$$w(y,z) \le C_\varepsilon\varphi(y)\exp\Big(-\sqrt{1+\lambda_1-\varepsilon}\,|z|\Big) \quad \text{as } |z| \longrightarrow \infty,\ y \in \varpi. \tag{3.10}$$

Without loss of generality, we may assume $\varepsilon < 1$. Now given $\varepsilon > 0$, by condition (f3) and Lemma 3.5, we may choose R_0 large enough such that

$$\lambda f(w(y,z)) \le \varepsilon w(y,z) \quad \text{for } |z| \ge R_0. \tag{3.11}$$

Let $q = (q_y, q_z)$, $q_y \in \partial\omega$, $|q_z| = R_0$, and B a small ball in Ω such that $q \in \partial B$. Since $\varphi(y) > 0$ for $x = (y,z) \in B$, $\varphi(q_y) = 0$, $w(x) > 0$ for $x \in B$, $w(q) = 0$, by the strong maximum principle $(\partial\varphi/\partial y)(q_y) < 0$, $(\partial w/\partial x)(q) < 0$. Thus

$$\lim_{\substack{x\to q\\ |z|=R_0}} \frac{w(x)}{\varphi(y)} = \frac{(\partial w/\partial x)(q)}{(\partial\varphi/\partial y)(q_y)} > 0. \tag{3.12}$$

Note that $w(x)\varphi^{-1}(y) > 0$ for $x = (y,z)$, $y \in \omega$, $|z| = R_0$. Thus $w(x)\varphi^{-1}(y) > 0$ for $x = (y,z)$, $y \in \varpi$, $|z| = R_0$. Since $\varphi(y)\exp(-\sqrt{1+\lambda_1-\varepsilon}\,|z|)$ and $w(x)$ are $C^1(\overline{\omega \times \partial B_{R_0}(0)})$, if set

$$C_\varepsilon = \sup_{y\in\varpi,\, |z|=R_0} \left(w(x)\varphi^{-1}(y)\exp\left(\sqrt{1+\lambda_1-\varepsilon}R_0\right)\right), \tag{3.13}$$

then $0 < C_\varepsilon < +\infty$ and

$$C_\varepsilon\varphi(y)\exp\left(-\sqrt{1+\lambda_1-\varepsilon}R_0\right) \geq w(x) \quad \text{for } y \in \varpi,\ |z| = R_0. \tag{3.14}$$

Let $\Phi_1(x) = C_\varepsilon\varphi(y)\exp(-\sqrt{1+\lambda_1-\varepsilon}|z|)$, for $x \in \overline{\Omega}$. Then, for $|z| \geq R_0$, we have

$$\begin{aligned}\Delta(w-\Phi_1)(x) - (w-\Phi_1)(x) &= -\lambda f(w(x)) + \left(\varepsilon + \frac{\sqrt{1+\lambda_1-\varepsilon}(n-1)}{|z|}\right)\Phi_1(x) \\ &\geq -\varepsilon w(x) + \varepsilon\Phi_1(x) = \varepsilon(\Phi_1 - w)(x).\end{aligned} \tag{3.15}$$

Hence $\Delta(w-\Phi_1)(x) - (1-\varepsilon)(w-\Phi_1)(x) \geq 0$, for $|z| \geq R_0$.

The strong maximum principle implies that $w(x) - \Phi_1(x) \leq 0$ for $x = (y,z)$, $y \in \varpi$, $|z| \geq R_0$, and therefore we get this claim.

(ii) Let

$$\Psi(y,z) = \left(1 + \frac{1}{\sqrt{|z|}}\right)\varphi(y)\exp\left(-\sqrt{1+\lambda_1}|z|\right)|z|^{-(n-1)/2} \quad \text{for } (y,z) \in \Omega. \tag{3.16}$$

It is very easy to show that

$$-\Delta\Psi + \Psi \leq 0 \quad \text{for } y \in \varpi,\ |z| \text{ large}. \tag{3.17}$$

Therefore, by means of the maximum principle, there exists a constant $C > 0$ such that

$$\left.\begin{aligned} w(y,z) &\geq C\varphi(y)\exp\left(-\sqrt{1+\lambda_1}|z|\right)|z|^{-(n-1)/2} \\ u(y,z) &\geq C\varphi(y)\exp\left(-\sqrt{1+\lambda_1}|z|\right)|z|^{-(n-1)/2}\end{aligned}\right. \quad \text{as } |z| \longrightarrow \infty,\ y \in \varpi. \tag{3.18}$$

This completes the proof of Lemma 3.6. □

4. Existence of minimal solution

In this section, by the barrier method, we prove that there exists some $\lambda^* > 0$ such that for $\lambda \in (0,\lambda^*)$, $(1.1)_\lambda$ has a minimal positive solution u_λ (i.e., for any positive solution u of $(1.1)_\lambda$, then $u \geq u_\lambda$).

Lemma 4.1. *If conditions (f1) and (f2) hold, then for any given $\rho > 0$, there exists $\lambda_0 > 0$ such that for $\lambda \in (0,\lambda_0)$, one has $I(u) > 0$ for all $u \in S_\rho = \{u \in H_0^1(\Omega) \mid \|u\| = \rho\}$.*

For the proof, see Zhu and Zhou [19].

Remark 4.2. For any $\varepsilon > 0$, there exists $\delta > 0$ ($\delta \leq \rho$) such that $I(u) \geq -\varepsilon$ for all $u \in \{u \in H_0^1(\Omega) \mid \rho - \delta \leq \|u\| \leq \rho\}$ and for $\lambda \in (0,\lambda_0)$ if λ_0 is small enough (see Zhu and Zhou [19]).

For the number $\rho > 0$ given in Lemma 4.1, we denote

$$B_\rho = \{u \in H_0^1(\Omega) \mid \|u\| < \rho\}. \tag{4.1}$$

Thus we have the following local minimum result.

Lemma 4.3. *Under conditions (f1), (f2), and (f4), if λ_0 is chosen as in Remark 4.2 and $\lambda \in (0,\lambda_0)$, then there is a $u_0 \in B_\rho$ such that $I(u_0) = \min\{I(u) \mid u \in \overline{B_\rho}\} < 0$ and u_0 is a positive solution of (1.1)$_\lambda$.*

Proof. Since $h \not\equiv 0$ and $h \geq 0$, we can choose a function $\varphi \in H_0^1(\Omega)$ such that $\int_\Omega h\varphi > 0$. For $t \in (0,+\infty)$, then

$$\begin{aligned} I(t\varphi) &= \frac{t^2}{2}\int_\Omega (|\nabla\varphi|^2 + \varphi^2) - \lambda\int_{\mathbb{R}_+^N} F(t\varphi^+) - \lambda t\int_\Omega h\varphi \\ &\leq \frac{t^2}{2}\|\varphi\|^2 + \lambda C t^2 \int_\Omega (|\varphi|^2 + t^{p-1}|\varphi|^{p+1}) - \lambda t \int_\Omega h\varphi. \end{aligned} \tag{4.2}$$

Then for t small enough, $I(t\varphi) < 0$. So $\alpha = \inf\{I(u) \mid u \in \overline{B_\rho}\}$. Clearly, $\alpha > -\infty$. By Remark 4.2, there is ρ' such that $0 < \rho' < \rho$ and $\alpha = \inf\{I(u) \mid u \in \overline{B_{\rho'}}\}$. By Ekeland variational principle [7], there exists a $(\mathrm{PS})_\alpha$-sequence $\{u_k\} \subset \overline{B_{\rho'}}$. By Proposition 2.2, there exists a subsequence $\{u_k\}$, an integer $l \geq 0$, a solution u^i of $(2.4)_\lambda$, $1 \leq i \leq l$, and a solution u_0 in $\overline{B_{\rho'}}$ of $(1.1)_\lambda$ such that $u_k \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$ and $\alpha = I(u_0) + \sum_{i=1}^{l} I^\infty(u^i)$. Note that $I^\infty(u^i) \geq S^\infty > 0$ for $i = 1,2,\ldots,m$. Since $u_0 \in \overline{B_\rho}$, we have $I(u_0) \geq \alpha$. We conclude that $l = 0$, $I(u_0) = \alpha$, and $I'(u_0) = 0$. □

By the standard barrier method, we prove the following lemma.

Lemma 4.4. *Let conditions (f1), (f2), and (f4) be satisfied, then there exists $\lambda^* > 0$ such that*

(i) *for any $\lambda \in (0,\lambda^*)$, (1.1)$_\lambda$ has a minimal positive solution u_λ and u_λ is strictly increasing in λ;*

(ii) *if $\lambda > \lambda^*$, (1.1)$_\lambda$ has no positive solution.*

Proof. Set $Q_\lambda = \{0 < \lambda < +\infty \mid (1.1)_\lambda \text{ is solvable}\}$, by Lemma 4.3, we have Q_λ is nonempty. Denoting $\lambda^* = \sup Q_\lambda > 0$, we claim that $(1.1)_\lambda$ has at least one solution for all $\lambda \in (0,\lambda^*)$. In fact, for any $\lambda \in (0,\lambda^*)$, by the definition of λ^*, we know that there exists $\lambda' > 0$ and $0 < \lambda < \lambda' < \lambda^*$ such that $(1.3)_{\lambda'}$ has a solution $u_{\lambda'} > 0$, that is,

$$-\Delta u_{\lambda'} + u_{\lambda'} = \lambda'(f(u_{\lambda'}) + h) \geq \lambda(f(u_{\lambda'}) + h). \tag{4.3}$$

Then $u_{\lambda'}$ is a supersolution of $(1.1)_\lambda$. From $h \geq 0$ and $h \not\equiv 0$, it is easy to see that 0 is a subsolution of $(1.1)_\lambda$. By the standard barrier method, there exists a solution $u_\lambda > 0$ of $(1.1)_\lambda$ such that $0 \leq u_\lambda \leq u_{\lambda'}$. Since 0 is not a solution of $(1.1)_\lambda$ and $\lambda' > \lambda$, the maximum principle implies that $0 < u_\lambda < u_{\lambda'}$. Using the result of Graham-Eagle [10], we can choose a minimal positive solution u_λ of $(1.1)_\lambda$. □

Let u_λ be the minimal positive solution of $(1.1)_\lambda$ for $\lambda \in (0,\lambda^*)$, we study the following eigenvalue problem

$$
\begin{gathered}
-\Delta v + v = \mu_\lambda f'(u_\lambda) v \text{ in } \Omega, \\
v \in H_0^1(\Omega), \quad v > 0 \text{ in } \Omega,
\end{gathered}
\tag{4.4}
$$

then we have the following lemma.

Lemma 4.5. *Under conditions (f1)–(f5), the first eigenvalue μ_λ of (4.4) is defined by*

$$
\mu_\lambda = \inf\left\{ \int_\Omega (|\nabla v|^2 + v^2)\,dx \mid v \in H_0^1(\Omega),\ \int_\Omega f'(u_\lambda) v^2 dx = 1 \right\}.
\tag{4.5}
$$

Then
(i) *μ_λ is achieved;*
(ii) *$\mu_\lambda > \lambda$ and is strictly decreasing in λ, $\lambda \in (0,\lambda^*)$;*
(iii) *$\lambda^* < +\infty$ and $(1.1)_{\lambda^*}$ has a minimal positive solution u_{λ^*}.*

Proof. (i) Indeed, by the definition of μ_λ, we know that $0 < \mu_\lambda < +\infty$. Let $\{v_k\} \subset H_0^1(\Omega)$ be a minimizing sequence of μ_λ, that is,

$$
\int_\Omega f'(u_\lambda) v_k^2 dx = 1, \quad \int_\Omega \left(|\nabla v_k|^2 + v_k^2 \right) dx \longrightarrow \mu_\lambda \quad \text{as } k \longrightarrow \infty.
\tag{4.6}
$$

This implies that $\{v_k\}$ is bounded in $H_0^1(\Omega)$, then there is a subsequence, still denoted by $\{v_k\}$ and some $v_0 \in H_0^1(\Omega)$ such that

$$
\begin{gathered}
v_k \rightharpoonup v_0 \quad \text{weakly in } H_0^1(\Omega), \\
v_k \longrightarrow v_0 \quad \text{a.e. in } \Omega.
\end{gathered}
\tag{4.7}
$$

Thus,

$$
\int_\Omega \left(|\nabla v_0|^2 + v_0^2 \right) dx \le \liminf \int_\Omega \left(|\nabla v_k|^2 + v_k^2 \right) dx = \mu_\lambda.
\tag{4.8}
$$

By Lemma 3.5 and the conditions (f1), (f3), we have $f'(u_\lambda) \to 0$ as $|x| \to \infty$, it follows that there exists a constant $C > 0$ such that

$$
|f'(u_\lambda)| \le C \quad \forall x \in \Omega.
\tag{4.9}
$$

Furthermore, for any $\varepsilon > 0$, there exists $R > 0$ such that for $x \in \Omega$ and $|x| \ge R$, $f'(u_\lambda) < \varepsilon$. Then

$$
\begin{aligned}
\left| \int_\Omega f'(u_\lambda) |v_k - v_0|^2 dx \right| &\le \int_{B_R \cap \Omega} f'(u_\lambda) |v_k - v_0|^2 dx + \int_{\Omega \setminus B_R} f'(u_\lambda) |v_k - v_0|^2 dx \\
&\le C \int_{B_R \cap \Omega} |v_k - v_0|^2 dx + \varepsilon \int_{\Omega \setminus B_R} |v_k - v_0|^2 dx.
\end{aligned}
\tag{4.10}
$$

It follows from the Sobolev embedding theorem that there exists k_1, such that for $k \geq k_1$,

$$\int_{B_R \cap \Omega} |v_k - v_0|^2 dx < \varepsilon. \tag{4.11}$$

Since $\{v_k\}$ is bounded in $H_0^1(\Omega)$, this implies that there exists a constant $C_1 > 0$ such that

$$\int_{\Omega \setminus B_R} |v_k - v_0|^2 dx \leq C_1. \tag{4.12}$$

Therefore, we conclude that for $k \geq k_1$,

$$\left| \int_\Omega f'(u_\lambda) |v_k - v_0|^2 dx \right| \leq C\varepsilon + C_1 \varepsilon. \tag{4.13}$$

Takeing $\varepsilon \to 0$, we obtain that

$$\int_\Omega f'(u_\lambda) v_0^2 dx = 1. \tag{4.14}$$

Hence

$$\int_\Omega \left(|\nabla v_0|^2 + v_0^2 \right) dx \geq \mu_\lambda. \tag{4.15}$$

This implies that v_0 achieves μ. Clearly, $|v_0|$ also achieves μ_λ. By (4.17) and the maximum principle, we may assume $v_0 > 0$ in Ω.

(ii) We now prove $\mu_\lambda > \lambda$. Setting $\lambda' > \lambda > 0$ and $\lambda' \in (0, \lambda^*)$, by Lemma 4.4, $(1.1)_{\lambda'}$ has a positive solution $u_{\lambda'}$. Since u_λ is the minimal positive solution of $(1.1)_\lambda$, then $u_{\lambda'} > u_\lambda$ as $\lambda' > \lambda$. By virtue of $(1.1)_{\lambda'}$ and $(1.1)_\lambda$, we see that

$$-\Delta(u_{\lambda'} - u_\lambda) + (u_{\lambda'} - u_\lambda) = \lambda' f(u_{\lambda'}) - \lambda f(u_\lambda) + (\lambda' - \lambda) h. \tag{4.16}$$

Applying the Taylor expansion and noting that $\lambda' > \lambda$, $h(x) \geq 0$ and $f''(t) \geq 0$, $f(t) > 0$ for all $t > 0$, we get

$$-\Delta(u_{\lambda'} - u_\lambda) + (u_{\lambda'} - u_\lambda) \geq (\lambda' - \lambda) f(u_\lambda) + \lambda' f'(u_\lambda)(u_{\lambda'} - u_\lambda) > \lambda f'(u_\lambda)(u_{\lambda'} - u_\lambda). \tag{4.17}$$

Let $v_0 \in H_0^1(\Omega)$ and $v_0 > 0$ solve (4.4). Multiplying (4.17) by v_0 and noting (4.4), then we get

$$\mu_\lambda \int_\Omega f'(u_\lambda)(u_{\lambda'} - u_\lambda) v_0 dx > \lambda \int_\Omega f'(u_\lambda)(u_{\lambda'} - u_\lambda) v_0 dx, \tag{4.18}$$

hence $\mu_\lambda > \lambda$. Now let v_λ be a minimizer of μ_λ, then

$$\int_\Omega f'(u_{\lambda'}) v_\lambda^2 dx > \int_\Omega f'(u_\lambda) v_\lambda^2 dx = 1, \tag{4.19}$$

and there is t, with $0 < t < 1$ such that

$$\int_\Omega f'(u_{\lambda'})(tv_\lambda)^2 dx = 1. \tag{4.20}$$

Therefore,

$$\mu_{\lambda'} \le t^2\|v_\lambda\|^2 < \|v_\lambda\|^2 = \mu_\lambda, \tag{4.21}$$

showing that μ_λ is strictly decreasing in λ, for $\lambda \in (0,\lambda^*)$.

(iii) We show next that $\lambda^* < +\infty$. Let $\lambda_0 \in (0,\lambda^*)$ be fixed. For any $\lambda \ge \lambda_0$, we have $\mu_\lambda > \lambda$ and by (4.21), then

$$\mu_{\lambda_0} \ge \mu_\lambda > \lambda \tag{4.22}$$

for all $\lambda \in [\lambda_0,\lambda^*)$. Thus $\lambda^* < +\infty$.

By (4.4) and $\mu_\lambda > \lambda$, we have

$$\int_\Omega \left(|\nabla u_\lambda|^2 + |u_\lambda|^2\right) dx > \int_\Omega \lambda f'(u_\lambda) u_\lambda^2 dx, \tag{4.23}$$

and also we have

$$\int_\Omega \left(|\nabla u_\lambda|^2 + |u_\lambda|^2\right) dx - \int_\Omega \lambda f(u_\lambda) u_\lambda dx - \int_\Omega \lambda h(x) u_\lambda dx = 0. \tag{4.24}$$

By condition (f4) and (4.23), we have that

$$\begin{aligned}\int_\Omega \left(|\nabla u_\lambda|^2 + |u_\lambda|^2\right) dx &= \int_\Omega \lambda f(u_\lambda) u_\lambda dx + \int_\Omega \lambda h(x) u_\lambda dx \le \theta \int_\Omega \lambda f'(u_\lambda) u_\lambda^2 dx \\ &\quad + \lambda\|h\|_2\|u_\lambda\| \le \theta\|u_\lambda\|^2 + \lambda\|h\|_2\|u_\lambda\|.\end{aligned} \tag{4.25}$$

This implies that

$$\|u_\lambda\| \le \frac{\lambda}{1-\theta}\|h\|_2 \tag{4.26}$$

for all $\lambda \in (0,\lambda^*)$. Since $\lambda^* < +\infty$, by (4.26) we can obtain that $\|u_\lambda\| \le C < +\infty$ for all $\lambda \in (0,\lambda^*)$. Thus, there exists $u_{\lambda^*} \in H_0^1(\Omega)$ such that

$$\begin{gathered} u_\lambda \rightharpoonup u_{\lambda^*} \quad \text{weakly in } H_0^1(\Omega), \\ u_\lambda \longrightarrow u_{\lambda^*} \quad \text{strongly in } L^q_{\text{loc}}(\Omega) \text{ for } 2 \le q < \frac{2N}{N-2}, \text{ as } \lambda \longrightarrow \lambda^*, \\ u_\lambda \longrightarrow u_{\lambda^*} \quad \text{almost everywhere in } \Omega. \end{gathered} \tag{4.27}$$

For $\varphi \in H_0^1(\Omega)$, by condition (f2), we obtain that

$$\left.\begin{aligned} \int_\Omega (\nabla u_\lambda \cdot \nabla\varphi + u_\lambda\varphi) dx &\longrightarrow \int_\Omega (\nabla u_{\lambda^*} \cdot \nabla\varphi + u_{\lambda^*}\varphi) dx \\ \lambda \int_\Omega (f(u_\lambda) + h)\varphi\, dx &\longrightarrow \lambda^* \int_\Omega (f(u_{\lambda^*}) + h)\varphi\, dx \end{aligned}\right. \quad \text{as } \lambda \longrightarrow \lambda^*. \tag{4.28}$$

From $\langle I'_\lambda(u_\lambda),\varphi\rangle = 0$ and let $\lambda \to \lambda^*$, we deduce $I'_{\lambda^*}(u_{\lambda^*}) = 0$ in $H^{-1}(\Omega)$. Hence, u_{λ^*} is a positive solution of $(1.1)_{\lambda^*}$.

Let u be any positive solution of $(1.1)_{\lambda^*}$. By adopting the argument as in Lemma 4.4, we have $u \geq u_\lambda$ in Ω for $\lambda \in (0,\lambda^*)$. Let $\lambda \to \lambda^*$, we deduce that $u \geq u_{\lambda^*}$ in Ω. This implies that u_{λ^*} is a minimal solution of $(1.1)_{\lambda^*}$. □

5. Existence of second solution

When $\lambda \in (0,\lambda^*)$, we have known that $(1.1)_\lambda$ has a minimal positive solution u_λ by Lemma 4.4, then we need only to prove that $(1.1)_\lambda$ has another positive solution in the form of $U_\lambda = u_\lambda + \overline{v}$, where $\overline{v}$ is a solution of the following equation:

$$\begin{gathered} -\Delta v + v = \lambda\big(f(u_\lambda + v) - f(u_\lambda)\big) \text{ in } \Omega, \\ v > 0 \text{ in } \Omega, \quad v \in H_0^1(\Omega). \end{gathered} \tag{5.1}$$

For (5.1), we define the energy functional $J : H_0^1(\Omega) \to \mathbb{R}$ as follows:

$$J(v) = \frac{1}{2}\int_\Omega (|\nabla v|^2 + v^2)\,dx - \lambda\int_\Omega \big(F(u_\lambda + v^+) - F(u_\lambda) - f(u_\lambda)v^+\big)\,dx. \tag{5.2}$$

Using the monotonicity of f and the maximum principle, we know that the nontrivial critical points of energy functional J are the positive solutions of (5.1).

First, we give an inequality about f and u_λ.

Lemma 5.1. *Under conditions (f1), (f2), and (f5), then for any* $\varepsilon > 0$, *there exists* $C_\varepsilon > 0$ *such that*

$$f(u_\lambda + s) - f(u_\lambda) - f'(u_\lambda)s \leq \varepsilon s + C_\varepsilon s^p, \quad s \geq 0, \tag{5.3}$$

where $1 < p < 2^* - 1$ *and* u_λ *is the minimal solution of* $(1.1)_\lambda$.

For the proof, see Zhu and Zhou [19].

Lemma 5.2. *Under conditions (f1), (f2), (f4), and (f5), there exist* $\rho > 0$ *and* $\alpha > 0$ *such that*

$$J(v)\,|_{S_\rho} \geq \alpha > 0, \tag{5.4}$$

where $S_\rho = \{u \in H_0^1(\Omega) \mid \|u\| = \rho\}$.

Proof. By Lemma 4.5, it is easy to see that, for all $v \in H_0^1(\Omega)$,

$$\int_\Omega (|\nabla v|^2 + v^2)\,dx \geq \mu_\lambda \int_\Omega f'(u_\lambda)v^2\,dx. \tag{5.5}$$

Again, by Lemma 5.1 and Sobolev embedding, we obtain that

$$
\begin{aligned}
J(v) &= \frac{1}{2}\int_{\Omega}(|\nabla v|^2+v^2)dx-\lambda\int_{\Omega}(F(u_\lambda+v^+)-F(u_\lambda)-f(u_\lambda)v^+)dx\\
&= \frac{1}{2}\|v\|^2-\frac{\lambda}{2}\int_{\Omega}f'(u_\lambda)\left|v^+\right|^2dx-\lambda\int_{\Omega}\int_0^{v^+}(f(u_\lambda+s)-f(u_\lambda)-f'(u_\lambda)s)\,ds\,dx\\
&\geq \frac{1}{2}\|v\|^2-\frac{\lambda}{2}\int_{\Omega}f'(u_\lambda)\left|v^+\right|^2dx-\frac{1}{2}\lambda\varepsilon\int_{\Omega}\left|v^+\right|^2dx-\frac{\lambda}{p+1}\int_{\Omega}C_\varepsilon\left|v^+\right|^{p+1}dx\\
&\geq \frac{1}{2}\|v\|^2-\frac{\lambda}{2}\mu^{-1}\|v\|^2-\frac{1}{2}\lambda\varepsilon\|v\|^2-\overline{C_\varepsilon}\|v\|^{p+1}\\
&= \frac{1}{2}\mu_\lambda^{-1}(\mu_\lambda-\lambda-\lambda\mu_\lambda\varepsilon)\|v\|^2-\overline{C_\varepsilon}\|v\|^{p+1}.
\end{aligned}
\tag{5.6}
$$

Since $\mu_\lambda>\lambda$, we may choose $\varepsilon>0$ small enough such that $\mu_\lambda-\lambda-\lambda\mu_\lambda\varepsilon>0$. If we take $\varepsilon=(\mu_\lambda-\lambda)/2\lambda\mu_\lambda$, then

$$
J(v)\geq\frac{1}{4}\mu_\lambda^{-1}(\mu_\lambda-\lambda)\|v\|^2-C\|v\|^{p+1}. \tag{5.7}
$$

Hence, there exist $\rho>0$ and $\alpha>0$ such that $J(v)\,|_{S_\rho}\geq\alpha>0$. □

Similar to Proposition 2.2, for the energy functional J, we also have the following result.

Proposition 5.3. *Under conditions (f1), (f2), and (f4), let $\{v_k\}$ be a $(\mathrm{PS})_c$-sequence of J. Then there exists a subsequence (still denoted by $\{v_k\}$) for which the following holds: there exist an integer $l\geq 0$, a sequence $\{x_k^i\}\subseteq\mathbb{R}^N$ of the form $(0,z_k^i)\in\mathbb{S}$, a solution $\overline{v}$ of (5.1), and solutions u^i of $(2.4)_\lambda$, $1\leq i\leq l$, such that for some subsequence $\{v_k\}$, as $k\to\infty$, one has*

$$
\begin{gathered}
v_k \rightharpoonup \overline{v}\quad\text{weakly in } H_0^1(\Omega),\\
J(v_k)\longrightarrow J(\overline{v})+\sum_{i=1}^{l}I^\infty(u^i),\\
v_k-\left(\overline{v}+\sum_{i=1}^{l}u^i(x-x_k^i)\right)\longrightarrow 0\quad\text{strongly in } H_0^1(\Omega),\\
\left|x_k^i\right|\longrightarrow\infty,\qquad\left|x_k^i-x_k^j\right|\longrightarrow\infty,\qquad 1\leq i\neq j\leq l,
\end{gathered}
\tag{5.8}
$$

where one agrees that in the case $l=0$, the above hold without u^i, x_k^i.

Now, let δ be small enough, D^δ a δ-tubular neighborhood of D such that $D^\delta\subset\subset\mathbb{S}$. Let $\eta(x):\mathbb{S}\to[0,1]$ be a C^∞ cut-off function such that $0\leq\eta\leq 1$ and

$$
\eta(x)=\begin{cases}0 & \text{if } x\in D;\\ 1 & \text{if } x\in\mathbb{S}\setminus\overline{D}^\delta.\end{cases} \tag{5.9}
$$

Let $e_N = (0,0,\dots,0,1) \in \mathbb{R}^N$, denote

$$\tau_0 = 2\sup_{x\in D^\delta}|x| + 1,$$
$$w_\tau(x) = w(x-\tau e_N), \quad \tau \in [0,\infty), \tag{5.10}$$

where w is a ground state solution of $(2.4)_\lambda$.

Lemma 5.4. *Let conditions (f1)–(f5) be satisfied. Then*
(i) *there exists $t_0 > 0$ such that $J(t\eta w_\tau) < 0$ for $t \geq t_0$, $\tau \geq \tau_0$,*
(ii) *there exists $\tau_* > 0$ such that the following inequality holds for $\tau \geq \tau_*$:*

$$0 < \sup_{t\geq 0} J(t\eta w_\tau) < I^\infty(w) = S^\infty. \tag{5.11}$$

Proof. (i) By the definition of η and Lemma 2.1(iii), we have

$$\begin{aligned} J(t\eta w_\tau) &= \frac{1}{2}\int_\Omega \left(|\nabla(t\eta w_\tau)|^2 + (t\eta w_\tau)^2\right)dx - \lambda\int_\Omega\int_0^{t\eta w_\tau}(f(u_\lambda+s)-f(u_\lambda))\,ds\,dx \\ &\leq \frac{t^2}{2}\int_\Omega\left(|\nabla(\eta w_\tau)|^2 + (\eta w_\tau)^2\right)dx - \lambda\int_{\mathbb{S}\setminus\overline{D}^\delta} F(tw_\tau)\,dx. \end{aligned} \tag{5.12}$$

Noting part (ii) of Lemma 2.1, we see that $F(u)/(\nu^{-1}u^\nu)$ is monotone nondecreasing for $u > 0$, where $\nu = 1+\theta^{-1} > 2$. Thus, for any given constant $C > 0$, there is $u_0 \geq 0$ such that

$$F(u) \geq Cu^\nu \quad \forall u \geq u_0. \tag{5.13}$$

Let r_0 be a positive constant such that $B^m(0;r_0) = \{y \mid |y| \leq r_0\} \subset\subset \omega$, $B^n(0;1) = \{z \mid |z| \leq 1\}$, $\Omega_1 = B^m(0;r_0)\times B^n(0;1)$, and $\Omega_{1\tau} = B^m(0;r_0)\times\{z+\tau e_N \mid |z|\leq 1\}$. By the definition of τ_0, we have that $\Omega_{1\tau} \subset\subset \Omega\setminus\overline{D}^\delta$ for all $\tau \geq \tau_0$. This also implies that there exists $t_0 \geq 0$, as $t \geq t_0$, we have

$$F(tw_\tau) \geq Ct^\nu w_\tau^\nu \quad \forall \tau \geq \tau_0,\ \forall x \in \Omega_{1\tau}. \tag{5.14}$$

Therefore, as $t > t_0$ and $\tau \geq \tau_0$,

$$\begin{aligned} J(t\eta w_\tau) &\leq \frac{t^2}{2}\int_\Omega\left(|\nabla(\eta w_\tau)|^2 + (\eta w_\tau)^2\right)dx - \lambda Ct^\nu\int_{\Omega_{1\tau}} w_\tau^\nu\,dx \\ &\leq \frac{t^2}{2}\|\eta w_\tau\|^2 - \lambda Ct^\nu\int_{\Omega_1} w^\nu\,dx. \end{aligned} \tag{5.15}$$

Since $\nu > 2$, we can choose $t_0 > 0$ large enough such that (i) holds.

(ii) By (i), J is continuous on $H_0^1(\Omega)$, $J(0) = 0$, and Lemma 5.2, we know that there exists t_1 with $0 < t_1 < t_0$ such that

$$\sup_{t\geq 0} J(t\eta w_\tau) = \sup_{t_1\leq t\leq t_0} J(t\eta w_\tau) \quad \forall \tau \geq \tau_0. \tag{5.16}$$

For $\tau \geq \tau_0$, $t_1 \leq t \leq t_0$, by condition (f2), (2.5), Lemmas 2.1 and 3.6, we have

$$\begin{aligned}
J(t\eta w_\tau) &= \frac{t^2}{2}\int_\Omega \left(|\nabla(\eta w_\tau)|^2 + (\eta w_\tau)^2\right)dx - \lambda\int_\Omega F(t\eta w_\tau)\,dx \\
&\quad - \lambda\int_\Omega\int_0^{t\eta w_\tau}\left(f(u_\lambda+s) - f(u_\lambda) - f(s)\right)ds\,dx \\
&\leq \frac{t^2}{2}\int_{\mathbb{S}}(-\Delta w + w)(\eta_\tau^2 w)\,dx + \frac{t^2}{2}\int_{\mathbb{S}}|\nabla\eta_\tau|^2|w|^2dx - \lambda\int_{\mathbb{S}}F(tw_\tau)\,dx \\
&\quad + \lambda\int_{\mathbb{S}}\int_{t\eta w_\tau}^{tw_\tau} f(s)\,ds\,dx - \lambda\int_\Omega\int_0^{t\eta w_\tau}\left(f(u_\lambda+s) - f(u_\lambda) - f(s)\right)ds\,dx \\
&\leq S^\infty + \frac{t_0^2}{2}\int_{D^\delta\setminus D}|\nabla\eta|^2|\varpi_\tau|^2dx + \lambda\int_{D^\delta}\int_0^{tw_\tau} f(s)\,ds\,dx \\
&\quad - \lambda\int_\Omega\int_0^{t\eta w_\tau}\left(f(u_\lambda+s) - f(u_\lambda) - f(s)\right)ds\,dx \\
&\leq S^\infty + C_\varepsilon\exp\left(-2\sqrt{1+\lambda_1-\varepsilon}\,\tau\right) + \lambda C\int_{D^\delta}\left[\frac{(tw_\tau)^2}{2} + \frac{(tw_\tau)^{p+1}}{p+1}\right]dx \\
&\quad - \lambda\int_\Omega\int_0^{t\eta w_\tau}\left(f(u_\lambda+s) - f(u_\lambda) - f(s)\right)ds\,dx \\
&\leq S^\infty + C_\varepsilon\exp\left(-2\sqrt{1+\lambda_1-\varepsilon}\,\tau\right) - \lambda\int_\Omega\int_0^{t\eta w_\tau}\left(f(u_\lambda+s) - f(u_\lambda) - f(s)\right)ds\,dx,
\end{aligned} \tag{5.17}$$

where $0 < \varepsilon < 1 + \lambda_1$ and C_ε is independent of τ.

It follows from the Taylor's expansion that

$$f(u_\lambda + s) = f(s) + f'(s)u_\lambda + \frac{1}{2}f''(\xi)u_\lambda^2, \quad \xi \in (s, u_\lambda + s). \tag{5.18}$$

From (f5) and the above formula, for $t_1 \leq t \leq t_0$, we obtain that

$$\begin{aligned}
&\int_0^{t\eta w_\tau}\left(f(u_\lambda+s) - f(u_\lambda) - f(s)\right)ds \\
&\quad \geq \int_0^{t_1\eta w_\tau}\left(f'(s)u_\lambda - f(u_\lambda)\right)ds = \left[(t_1 w_\tau)^{-1} f(t_1\eta w_\tau) - \eta u_\lambda^{-1} f(u_\lambda)\right]t_1 w_\tau u_\lambda.
\end{aligned} \tag{5.19}$$

Since $w_\tau > 0$ in $\mathbb{S}$, there exists $\gamma_1 > 0$ such that

$$w_\tau \geq \gamma_1 \text{ in } \Omega_{1\tau}. \tag{5.20}$$

By the definition of w_τ and $u_\lambda(x) \to 0$ as $|x| \to \infty$, we see that for τ large enough,

$$t_1 w_\tau \geq u_\lambda \text{ in } \Omega_{1\tau}, \tag{5.21}$$

then part (ii) of Lemma 2.1 implies that there exist $\gamma_2 > 0$ and $\tau_1 > 0$ such that, for $\tau \geq \tau_1$,

$$(t_1 w_\tau)^{-1} f(t_1 w_\tau) - u_\lambda^{-1} f(u_\lambda) > \gamma_2 \text{ in } \Omega_{1\tau}. \tag{5.22}$$

Now by Lemma 3.6, for $\tau \geq \max(\tau_0, \tau_1)$ and $t_1 \leq t \leq t_0$, we obtain that

$$\begin{aligned}
&\int_{\Omega_{1\tau}} \int_0^{t\eta w_\tau} \left(f(u_\lambda + s) - f(u_\lambda) - f(s)\right) ds\, dx \\
&\quad \geq \int_{\Omega_{1\tau}} \left[(t_1 w_\tau)^{-1} f(t_1 w_\tau) - u_\lambda^{-1} f(u_\lambda)\right] t_1 w_\tau u_\lambda dx \\
&\quad \geq \gamma_1 \gamma_2 \int_{\Omega_{1\tau}} t_1 u_\lambda dx \geq C_2 \exp\left(-\sqrt{1+\lambda_1}\tau\right),
\end{aligned} \tag{5.23}$$

where C_2 is independent of τ.

Therefore, we obtain that

$$J(t\eta w_\tau) \leq S^\infty + \lambda C_\varepsilon \exp\left(-2\sqrt{1+\lambda_1 - \varepsilon}\tau\right) - \lambda C_2 \exp\left(-\sqrt{1+\lambda_1}\tau\right), \tag{5.24}$$

for $t \in [t_1, t_0]$ and $\tau \geq \max(\tau_0, \tau_1)$.

Now, let $\varepsilon = (1+\lambda_1)/2$, then we can find some τ_* large enough such that

$$\lambda C_\varepsilon \exp\left(-\sqrt{2(1+\lambda_1)}\tau\right) - \lambda C_2 \exp\left(-\sqrt{1+\lambda_1}\tau\right) < 0, \tag{5.25}$$

for all $\tau \geq \tau_*$ and we complete the proof. □

Theorem 5.5. *Let conditions (f1)–(f5) be satisfied. Then (5.1) has a positive solution $\overline{v}$ if $\lambda \in (0, \lambda^*)$.*

Proof. Now, set

$$\begin{gathered}
\Gamma = \left\{p \in C([0,1], H_0^1(\Omega)) \mid p(0) = 0,\ p(1) = t_0 \eta w_{\tau_*}\right\}, \\
c = \inf_{p \in \Gamma} \max_{s \in [0,1]} J(p(s)).
\end{gathered} \tag{5.26}$$

By Lemmas 5.2 and 5.4, we have

$$0 < \alpha \leq c < S^\infty. \tag{5.27}$$

Applying the mountain pass theorem of Ambrosetti and Rabinowitz [2], there exists a $(\mathrm{PS})_c$-sequence $\{v_k\}$ such that

$$\begin{gathered}
J(v_k) \longrightarrow c, \\
J'(v_k) \longrightarrow 0 \quad \text{strongly in } H^{-1}(\Omega).
\end{gathered} \tag{5.28}$$

By Proposition 5.3, there exists a sequence (still denoted by $\{v_k\}$), an integer $l \geq 0$, a sequence $\{x_k^i\}$ in Ω, $1 \leq i \leq l$, a solution $\overline{v}$ of (5.1), and solutions u^i of $(2.4)_\lambda$ such that

$$c = J(\overline{v}) + \sum_{i=0}^{l} I^\infty(u^i). \tag{5.29}$$

By the strong maximum principle, to complete the proof, we only need to prove $\overline{v} \not\equiv 0$ in Ω. In fact, we have

$$c = J(\overline{v}) \geq \alpha > 0 \quad \text{if } l = 0, \qquad S^\infty > c \geq J(\overline{v}) + S^\infty \quad \text{if } l \geq 1. \tag{5.30}$$

This implies $\overline{v} \not\equiv 0$ in Ω. □

6. Properties and bifurcation of solutions

Denote by $A = \{(\lambda, u) \mid u \text{ solves problem } (1.1)_\lambda\}$ the set of solutions of $(1.1)_\lambda$, $\lambda \in (0, \lambda^*]$. For each $(\lambda, u) \in A$, let $\mu_\lambda(u)$ denote the number defined by

$$\mu_\lambda(u) = \inf\left\{\int_\Omega (|\nabla v|^2 + v^2)\,dx \mid v \in H_0^1(\Omega), \int_\Omega f'(u) v^2 dx = 1\right\}, \tag{6.1}$$

which is the smallest eigenvalue of the following problem:

$$\begin{gathered} -\Delta v + v = \mu_\lambda(u) f'(u) v \text{ in } \Omega, \\ v > 0, \quad v \in H_0^1(\Omega). \end{gathered} \tag{6.2}$$

In this section, we always assume that conditions (f1)–(f4), (f5)*, and (f6) hold. With the same arguments used in the proof of part (i) of Lemma 4.5, we can show that $\mu_\lambda(u)$ is achieved for all $(\lambda, u) \in A$. By Lemma 3.5, we have $A \subset \mathbb{R} \times L^\infty(\mathbb{R}^N) \cap H_0^1(\Omega)$. Moreover, if we assume that $h(x) \in C^\alpha(\Omega) \cap L^2(\Omega)$, then by elliptic regular theory (see [9]), we can deduce that $A \subset \mathbb{R} \times C^{2,\alpha}(\Omega) \cap H^2(\Omega)$.

Lemma 6.1. *Let u be a solution and let u_λ be the minimal solution of $(1.1)_\lambda$ for $\lambda \in (0, \lambda^*)$. Then*

(i) *$\mu_\lambda(u) > \lambda$ if and only if $u = u_\lambda$;*

(ii) *$\mu_\lambda(U_\lambda) < \lambda$, where U_λ is the second solution of $(1.1)_\lambda$ constructed in Section 5.*

Proof. Now, let $\psi \geq 0$ and $\psi \in H_0^1(\Omega)$. Since u and u_λ are the solution of $(1.1)_\lambda$, then

$$\begin{aligned} &\int_\Omega \nabla\psi \cdot \nabla(u_\lambda - u)\,dx + \int_\Omega \psi(u_\lambda - u)\,dx \\ &\quad = \lambda \int_\Omega (f(u_\lambda) - f(u))\psi\,dx = \lambda \int_\Omega \left(\int_u^{u_\lambda} f'(t)dt\right)\psi\,dx \geq \lambda \int_\Omega f'(u)(u_\lambda - u)\psi\,dx. \end{aligned} \tag{6.3}$$

Let $\psi = (u - u_\lambda)^+ \geq 0$ and $\psi \in H_0^1(\Omega)$. If $\psi \not\equiv 0$, then (6.3) implies

$$-\int_\Omega (|\nabla\psi|^2 + \psi^2)\,dx \geq -\lambda \int_\Omega f'(u)\psi^2 dx \tag{6.4}$$

and, therefore, the definition of $\mu_\lambda(u)$ implies

$$\int_\Omega (|\nabla\psi|^2 + \psi^2)\,dx \leq \lambda \int_\Omega f'(u)\psi^2 dx < \mu_\lambda(u) \int_\Omega f'(u)\psi^2 dx \leq \int_\Omega (|\nabla\psi|^2 + \psi^2)\,dx, \tag{6.5}$$

which is impossible. Hence $\psi \equiv 0$, and $u = u_\lambda$ in Ω. On the other hand, by Lemma 4.5, we also have that $\mu_\lambda(u_\lambda) > \lambda$. This completes the proof of (i).

By (i), we get that $\mu_\lambda(U_\lambda) \le \lambda$ for $\lambda \in (0,\lambda^*)$. We claim that $\mu_\lambda(U_\lambda) = \lambda$ cannot occur. We proceed by contradiction. Set $w = U_\lambda - u_\lambda$; we have

$$-\Delta w + w = \lambda[f(U_\lambda) - f(U_\lambda - w)], \quad w > 0 \text{ in } \Omega. \tag{6.6}$$

By $\mu_\lambda(U_\lambda) = \lambda$, we have that the problem

$$-\Delta\phi + \phi = \lambda f'(U_\lambda)\phi, \quad \phi \in H_0^1(\Omega) \tag{6.7}$$

possesses a positive solution ϕ_1.

Multiplying (6.6) by ϕ_1 and (6.7) by w, integrating and subtracting, we deduce that

$$0 = \int_\Omega \lambda[f(U_\lambda) - f(U_\lambda - w) - f'(U_\lambda)w]\phi_1 dx = -\frac{1}{2}\int_\Omega \lambda f''(\xi_\lambda)w^2\phi_1 dx, \tag{6.8}$$

where $\xi_\lambda \in (u_\lambda, U_\lambda)$. By condition (f5)*, we obtain that $w \equiv 0$, that is, $U_\lambda = u_\lambda$ for $\lambda \in (0,\lambda^*)$. This is a contradiction. Hence, we have that $\mu_\lambda(U_\lambda) < \lambda$ for $\lambda \in (0,\lambda^*)$. □

Lemma 6.2. *Let u_λ be the minimal solution of $(1.1)_\lambda$ for $\lambda \in [0,\lambda^*]$ and $\mu_\lambda(u_\lambda) > \lambda$. Then for any $g(x) \in H^{-1}(\Omega)$, problem*

$$-\Delta w + w = \lambda f'(u_\lambda)w + g(x), \quad w \in H_0^1(\Omega), \tag{6.9$_\lambda$}$$

has a solution.

Proof. Consider the functional

$$\Phi(w) = \frac{1}{2}\int_\Omega (|\nabla w|^2 + w^2)dx - \frac{1}{2}\lambda\int_\Omega f'(u_\lambda)w^2 dx - \int_\Omega g(x)w\,dx, \tag{6.9}$$

where $w \in H_0^1(\Omega)$. From Hölder inequality and Young's inequality, we have, for any $\epsilon > 0$, that

$$\Phi(w) \ge \frac{1}{2}(1 - \lambda\mu_\lambda(u_\lambda)^{-1})\|w\|^2 - \frac{1}{2}\epsilon\|w\|^2 - \frac{C_\epsilon}{2}\|g\|^2_{H^{-1}(\Omega)} \ge -C\|g\|^2_{H^{-1}(\Omega)} \tag{6.10}$$

if we choose ϵ small.

Now, let $\{w_k\} \subset H_0^1(\Omega)$ be the minimizing sequence of variational problem

$$d = \inf\{\Phi(w) \mid w \in H_0^1(\Omega)\}. \tag{6.11}$$

From (6.10) and $\mu_\lambda(u_\lambda) > \lambda$, we can also deduce that $\{w_k\}$ is bounded in $H_0^1(\Omega)$, if we choose ϵ small. So we may suppose that

$$\begin{aligned} w_k &\rightharpoonup w \quad \text{weakly in } H_0^1(\Omega) \text{ as } k \longrightarrow \infty, \\ w_k &\longrightarrow w \quad \text{a.e. in } \Omega \text{ as } k \longrightarrow \infty. \end{aligned} \tag{6.12}$$

By Fatou's lemma,

$$\|w\|^2 \leq \liminf \|w_k\|^2. \tag{6.13}$$

By Lemma 3.5, we have that $u_\lambda(x) \to 0$ as $|x| \to \infty$, conditions (f1)–(f5) and the weak convergence imply

$$\int_\Omega g w_k dx \longrightarrow \int_\Omega g w\, dx, \quad \int_\Omega f'(u_\lambda) w_k^2 dx \longrightarrow \int_\Omega f'(u_\lambda) w^2 dx \quad \text{as } k \longrightarrow \infty. \tag{6.14}$$

Therefore

$$\Phi(w) \leq \lim_{k\to\infty} \Phi(w_k) = d, \tag{6.15}$$

and hence $\Phi(w) = d$ which gives that w is a solution of $(6.9)_\lambda$. □

Remark 6.3. From Lemma 6.2, we know that $(6.9)_\lambda$ has a solution $w \in H_0^1(\Omega)$. Now, we also assume that $h(x)$ and $g(x)$ are in $C^\alpha(\Omega) \cap L^2(\Omega)$, then by Lemmas 3.1, 3.3, conditions (f1)–(f5), and the elliptic regular theory (see [9]), we can deduce that $w \in C^{2,\alpha}(\Omega) \cap H^2(\Omega)$.

LEMMA 6.4. *Suppose u_{λ^*} is a solution of $(1.1)_{\lambda^*}$, then $\mu_{\lambda^*}(u_{\lambda^*}) = \lambda^*$ and the solution u_{λ^*} is unique.*

Proof. Define $F : \mathbb{R} \times H_0^1(\Omega) \to H^{-1}(\Omega)$ by

$$F(\lambda, u) = \Delta u - u + \lambda(f(u) + h(x)). \tag{6.16}$$

Let $g(\lambda) = \mu_\lambda(u_\lambda) = \inf_{\int_\Omega f'(u_\lambda)v^2 dx = 1} \|v\|^2$ for $\lambda \in (0, \lambda^*]$, then it is easy to see that g is continuous on $(0, \lambda^*]$. Since $\mu_\lambda(u_\lambda) > \lambda$ for $\lambda \in (0, \lambda^*)$, so $\mu_{\lambda^*}(u_{\lambda^*}) \geq \lambda^*$. If $\mu_{\lambda^*}(u_{\lambda^*}) > \lambda^*$, the equation $F_u(\lambda^*, u_{\lambda^*})\phi = 0$ has no nontrivial solution. From Lemma 6.2, F_u maps $\mathbb{R} \times H_0^1(\Omega)$ onto $H^{-1}(\Omega)$. Applying the implicit function theorem to F, we can find a neighborhood $(\lambda^* - \delta,\ \lambda^* + \delta)$ of λ^* such that $(1.1)_\lambda$ possesses a solution u_λ if $\lambda \in (\lambda^* - \delta,\ \lambda^* + \delta)$. This is contradictory to the definition of λ^*. Hence, we obtain that $\mu_{\lambda^*}(u_{\lambda^*}) = \lambda^*$.

Next, we are going to prove that u_{λ^*} is unique. In fact, suppose $(1.1)_{\lambda^*}$ has another solution $U_{\lambda^*} \geq u_{\lambda^*}$. Set $w = U_{\lambda^*} - u_{\lambda^*}$; we have

$$-\Delta w + w = \lambda^*[f(w + u_{\lambda^*}) - f(u_{\lambda^*})], \quad w > 0 \text{ in } \Omega. \tag{6.17}$$

By $\mu_{\lambda^*}(u_{\lambda^*}) = \lambda^*$, we have that the problem

$$-\Delta\phi + \phi = \lambda^* f'(u_{\lambda^*})\phi, \quad \phi \in H_0^1(\Omega) \tag{6.18}$$

possesses a positive solution ϕ_1.

Multiplying (6.17) by ϕ_1 and (6.18) by w, integrating and subtracting, we deduce that

$$0 = \int_\Omega \lambda^*[f(w + u_{\lambda^*}) - f(u_{\lambda^*}) - f'(u_{\lambda^*})w]\phi_1 dx = \frac{1}{2}\int_\Omega \lambda^* f''(\xi_{\lambda^*}) w^2 \phi_1 dx, \tag{6.19}$$

where $\xi_{\lambda^*} \in (u_{\lambda^*}, u_{\lambda^*} + w)$. By condition (f5)*, we obtain that $w \equiv 0$. Thus, u_{λ^*} is unique. □

Proposition 6.5. *Let u_λ be the minimal solution of (1.1)$_\lambda$. Then u_λ is uniformly bounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$ for all $\lambda \in (0, \lambda^*]$, and*

$$u_\lambda \longrightarrow 0 \quad \text{in } L^\infty(\Omega) \cap H_0^1(\Omega) \text{ as } \lambda \longrightarrow 0^+. \tag{6.20}$$

Proof. By (4.26), we have that

$$||u_\lambda|| \le \frac{\lambda}{1-\theta} \|h\|_2 \tag{6.21}$$

for $\lambda \in (0, \lambda^*)$, and u_λ is strictly increasing with respect to λ, we can easily deduce that u_λ is uniformly bounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$ for $\lambda \in (0, \lambda^*]$ and $u_\lambda \to 0$ in $H_0^1(\Omega)$ as $\lambda \to 0^+$.

By (3.8), (4.26), and u_λ is uniformly bounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$, we have that

$$||u_\lambda||_\infty \le C_1 ||u_\lambda||_{q_0} + \lambda C_2 \left(||u_\lambda||_{pq_0}^p + \|h\|_{q_0}\right) \le C_1 ||u_\lambda||_\infty^{(q_0-2)/q_0} ||u_\lambda||_2^{2/q_0} + C_3 \lambda \le C\left(\lambda^{2/q_0} + \lambda\right), \tag{6.22}$$

where C is independent of λ, and $\lambda \in (0, \lambda^*]$. Hence, we obtain that $u_\lambda \to 0$ in $L^\infty(\Omega)$ as $\lambda \to 0^+$. □

Proposition 6.6. *For $\lambda \in (0, \lambda^*)$, let U_λ be the second solution of (1.1)$_\lambda$ constructed in Section 5. Then U_λ is unbounded in $L^\infty(\Omega) \cap H_0^1(\Omega)$, and*

$$\lim_{\lambda \to 0^+} ||U_\lambda|| = \lim_{\lambda \to 0^+} ||U_\lambda||_\infty = \infty. \tag{6.23}$$

Proof. First, we show that $\{U_\lambda : \lambda \in (0, \lambda_0)\}$ is unbounded in $L^\infty(\Omega)$ for any $\lambda_0 \in (0, \lambda^*)$. We proceed by contradiction. Assume to the contrary that there exists $c_0 > 0$ such that

$$||U_\lambda||_\infty \le c_0 < \infty \quad \forall \lambda \in (0, \lambda_0). \tag{6.24}$$

Now, let φ_λ be a minimizer of $\mu_\lambda(U_\lambda)$ for $\lambda \in (0, \lambda_0)$, that is,

$$\int_\Omega f'(U_\lambda) \varphi_\lambda^2 = 1, \qquad ||\varphi_\lambda||^2 = \mu_\lambda(U_\lambda). \tag{6.25}$$

By condition (f1) and (6.24), there exists a constant M independent of λ, such that $f'(U_\lambda(x)) \le M$ for all $\lambda \in (0, \lambda_0)$ and $x \in \Omega$. Hence, by (6.25) and $\mu_\lambda(U_\lambda) < \lambda$ for all $\lambda \in (0, \lambda_0)$, we obtain that

$$1 = \int_\Omega f'(U_\lambda) \varphi_\lambda^2 \le M ||\varphi_\lambda||^2 = M \mu_\lambda(U_\lambda) < M\lambda. \tag{6.26}$$

This is a contradiction for all $\lambda < 1/M$. Hence, for any $\lambda_0 \in (0, \lambda^*)$, we have that $\{U_\lambda : \lambda \in (0, \lambda^*)\}$ is unbounded in $L^\infty(\Omega)$. From this result, it is to be seen that $\lim_{\lambda \to 0^+} \|U_\lambda\|_\infty = \infty$.

Now, we show that $\{U_\lambda : \lambda \in (0, \lambda_0)\}$ is unbounded in $H_0^1(\Omega)$ for any $\lambda_0 \in (0, \lambda^*)$. If not, then there exists a constant M independent of λ such that

$$||U_\lambda|| \le M \quad \forall \lambda \in (0, \lambda_0). \tag{6.27}$$

Since U_λ is a solution of $(1.1)_\lambda$, and by condition (f2) and (6.27), we have that

$$\begin{aligned}||U_\lambda||^2 &= \int_\Omega \lambda f(U_\lambda)U_\lambda dx + \int_\Omega \lambda h U_\lambda dx \leq \lambda C\Big(\int_\Omega U_\lambda^2 dx + \int_\Omega U_\lambda^{p+1} dx\Big) + \lambda\|h\|_2||U_\lambda||_2 \\ &\leq \lambda C(||U_\lambda||^2 + ||U_\lambda||^{p+1}) + \lambda\|h\|_2||U_\lambda||_2 \leq \lambda C_1,\end{aligned} \tag{6.28}$$

where C_1 is independent of λ. Without loss of generality, we may assume that $q_0 = 2$ if $N = 2,3$ and $N/2 < q_0 < 2^*/(p-1)$ if $N \geq 4$. By (3.8), (6.27), and the Sobolev embedding theorem, we obtain that

$$\begin{aligned}||U_\lambda||_\infty &\leq C_1||U_\lambda||_{q_0} + \lambda C_2(||U_\lambda||_{pq_0}^p + \|h\|_{q_0}) \\ &\leq C_1||U_\lambda||_\infty^{1-2/q_0}||U_\lambda||_2^{2/q_0} + \lambda C_2||U_\lambda||_\infty^{p-2^*/q_0}||U_\lambda||_{2^*}^{2^*/q_0} + \lambda C_2\|h\|_{q_0} \\ &\leq C_3||U_\lambda||_\infty^{1-2/q_0} + \lambda C_4||U_\lambda||_\infty^{1-(2^*-q_0(p-1))/q_0} + \lambda C_2\|h\|_{q_0}.\end{aligned} \tag{6.29}$$

This implies that

$$1 \leq C_3||U_\lambda||_\infty^{-2/q_0} + \lambda C_4||U_\lambda||_\infty^{-(2^*-q_0(p-1))/q_0} + \lambda C_2||h||_{q_0}||U_\lambda||_\infty^{-1}, \tag{6.30}$$

where C_2, C_3, and C_4 are constants independent of λ. Now, let $\lambda \to 0^+$ and by $\lim_{\lambda\to 0^+}\|U_\lambda\|_\infty = +\infty$, then we obtain a contradiction. Hence, $\{U_\lambda : \lambda \in (0,\lambda^*)\}$ is unbounded in $H_0^1(\Omega)$ and $\lim_{\lambda\to 0^+}\|U_\lambda\| = +\infty$. This completes the proof of Proposition 6.6. □

In order to get bifurcation results, we need the following bifurcation theorem which can be found in Crandall and Rabinowitz [6].

Theorem B. *Let X, Y be Banach space. Let $(\overline{\lambda},\overline{x}) \in \mathbb{R}\times X$ and let F be a continuously differentiable mapping of an open neighborhood of $(\overline{\lambda},\overline{x})$ into Y. Let the null space $N(F_x(\overline{\lambda},\overline{x})) = \operatorname{span}\{x_0\}$ be one-dimensional and* $\operatorname{codim} R(F_x(\overline{\lambda},\overline{x})) = 1$. *Let $F_\lambda(\overline{\lambda},\overline{x}) \notin R(F_x(\overline{\lambda},\overline{x}))$. If Z is the complement of* $\operatorname{span}\{x_0\}$ *in X, then the solutions of $F(\lambda,x) = F(\overline{\lambda},\overline{x})$ near $(\overline{\lambda},\overline{x})$ form a curve $(\lambda(s),x(s)) = (\overline{\lambda}+\tau(s),\overline{x}+sx_0+z(s))$, where $s \to (\tau(s),z(s)) \in \mathbb{R}\times Z$ is a continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$.*

Proof of Theorems 1.1 and 1.2. First, we consider the case $\Omega = \mathbb{S}\setminus\overline{D}$. Theorem 1.1 now follows from Lemmas 4.4, 4.5, 6.4, and Theorem 5.5. The conclusions (i) and (ii) of Theorem 1.2 follow immediately from Lemma 4.5, and Propositions 6.5, 6.6. Now we are going to prove that $(\lambda^*, u_{\lambda^*})$ is a bifurcation point in $C^{2,\alpha}(\Omega)\cap H^2(\Omega)$ by using an idea in [13]. We also assume that $h(x)$ is in $C^\alpha(\Omega)\cap L^2(\Omega)$ and define

$$F : \mathbb{R}^1 \times C^{2,\alpha}(\Omega)\cap H^2(\Omega) \longrightarrow C^\alpha(\Omega)\cap L^2(\Omega) \tag{6.31}$$

by

$$F(\lambda,u) = \Delta u - u + \lambda f(u^+) + \lambda h, \tag{6.32}$$

where $C^{2,\alpha}(\Omega)\cap H^2(\Omega)$ and $C^{\alpha}(\Omega)\cap L^2(\Omega)$ are endowed with the natural norm; then they become Banach spaces. It can be easily verified that $F(\lambda,u)$ is differentiable. From Lemma 6.2 and Remark 6.3, we know that

$$F_u(\lambda,u_\lambda)w=\Delta w-w+\lambda f'(u_\lambda)w \tag{6.33}$$

is an isomorphism of $\mathbb{R}^1\times C^{2,\alpha}(\Omega)\cap H^2(\Omega)$ onto $C^{\alpha}(\Omega)\cap L^2(\Omega)$. It follows from implicit function theorem that the solutions of $F(\lambda,u)=0$ near (λ,u_λ) are given by a continuous curve.

Now we are going to prove that $(\lambda^*,u_{\lambda^*})$ is a bifurcation point of F. We show first that at the critical point $(\lambda^*,u_{\lambda^*})$, Theorem B applies. Indeed, from Lemma 6.4, problem (6.18) has a solution $\phi_1>0$ in Ω. By the standard elliptic regular theory, we have that $\phi_1\in C^{2,\alpha}(\Omega)\cap H^2(\Omega)$ if $h\in C^{\alpha}(\Omega)\cap L^2(\Omega)$. Thus $F_u(\lambda^*,u_{\lambda^*})\phi=0$, $\phi\in C^{2,\alpha}(\Omega)\cap H^2(\Omega)$ has a solution $\phi_1>0$. This implies that $N(F_u(\lambda^*,u_{\lambda^*}))=\operatorname{span}\{\phi_1\}=1$ is one dimensional and codim $R(F_u(\lambda^*,u_{\lambda^*}))=1$ by the Fredholm alternative. It remains to check that $F_\lambda(\lambda^*,u_{\lambda^*})\notin R(F_u(\lambda^*,u_{\lambda^*}))$.

Assuming the contrary would imply existence of $v\not\equiv 0$ such that

$$\Delta v-v+\lambda^* f'(u_{\lambda^*})v=f(u_{\lambda^*})+h,\quad v\in H_0^1(\Omega). \tag{6.34}$$

From $F_u(\lambda^*,u_{\lambda^*})\phi_1=0$, we conclude that $\int_\Omega(f(u_{\lambda^*}(x))+h(x))\phi_1(x)dx=0$. This is impossible because $f(t)>0$, for $t>0$, $u_{\lambda^*}(x)>0$, $h(x)\geq 0$, $h(x)\not\equiv 0$ and $\phi_1(x)>0$ for $x\in\Omega$.

Applying Theorem B, we conclude that $(\lambda^*,u_{\lambda^*})$ is a bifurcation point near which the solution of $(1.1)_\lambda$ forms a curve $(\lambda^*+\tau(s),u_{\lambda^*}+s\phi_1+z(s))$ with s near $s=0$ and $\tau(0)=\tau'(0)=0$, $z(0)=z'(0)=0$. We claim that $\tau''(0)<0$ which implies that the bifurcation curve turns strictly to the left in (λ,u) plane. In order to obtain that $\tau''(0)<0$, we need the following lemma. □

Lemma 6.7. *For $R>0$, let $\Omega_R=\{x=(y,z)\in\Omega:|z|<R\}=(\omega\times B_R)\setminus\overline{D}$, where $B_R=\{z\in\mathbb{R}^n:|z|<R\}$. Suppose conditions (f1)–(f6) hold, then*

$$\int_\Omega f''(u_{\lambda^*})\phi_1^3dx<+\infty. \tag{6.35}$$

Proof. Since $u_{\lambda^*}(x)\to 0$ as $|x|\to\infty$, and by conditions (f1) and (f3), we have that there is $R_1>0$ such that

$$0=\Delta\phi_1-\phi_1+\lambda^* f'(u_{\lambda^*})\phi_1\leq\Delta\phi_1-\frac{1}{4}\phi_1\quad\text{for } y\in\omega,\ |z|\geq R_1. \tag{6.36}$$

It is well known that the Dirichlet equation $\Delta w-(1/4)w=-w^p$ in $\mathbb{S}$ has a positive ground-state solution, denoted by $\overline{w}$ (see [14] and the references there). We can modify the proof in Hsu [11] and obtain that for any $\varepsilon>0$ with $0<\varepsilon<1/4+\lambda_1$, there exist constants $C_\varepsilon>0$ and $R_2>0$ such that

$$\overline{w}(y,z)\leq C_\varepsilon\varphi(y)\exp\left(-\sqrt{\frac{1}{4}+\lambda_1-\varepsilon}|z|\right)\quad\text{for } y\in\omega,\ |z|\geq R_2, \tag{6.37}$$

where φ is the first positive eigenfunction of the Dirichlet problem $-\Delta\varphi = \lambda_1\varphi$ in ω. Now, let $\varepsilon = (1/2)\lambda_1$. Since $\Delta\overline{w} - (1/4)\overline{w} = -\overline{w}^p \le 0$ in $\mathbb{S}$, hence by the maximum principle we obtain that there exist constants $C_1 > 0$ and $R_3 > 0$ such that

$$\phi_1(x) \le C_1\varphi(y)\exp\left(-\frac{1}{2}\sqrt{1+2\lambda_1}|z|\right) \quad \text{for } y \in \omega,\ |z| \ge R_3. \tag{6.38}$$

By condition (f6), (3.9), (6.38), and $u_{\lambda^*}(x) \to 0$ as $|x| \to \infty$, we have that there exist constants $C_2 > 0$ and $R_0 \ge R_1 + R_2 + R_3$ such that $D \subset\subset \omega \times B_{R_0}$ and

$$\begin{aligned} f''(u_{\lambda^*}) &\le C_2 u_{\lambda^*}^{q_1-1} \\ u_{\lambda^*}^{-1}(x)\phi_1^2(x) &\le C_2 \end{aligned} \quad \text{for } x = (y,z) \in \Omega \setminus \Omega_{R_0}, \tag{6.39}$$

where $0 < q_1 < 4/(N-2)$ if $N \ge 3$, $q_1 > 0$ if $N = 2$.

By the strong maximum principle and modifying the proof in Lemma 3.6(i), we have that $u_{\lambda^*}^{-1}\phi_1 \in C^1(\overline{\Omega})$ and $u_{\lambda^*}^{-1}\phi_1 > 0$ on $\overline{\Omega}$. Therefore, there exists $C_3 > 0$ such that

$$u_{\lambda^*}^{-1}(x)\phi_1(x) \le C_3 \quad \text{for } x \in \Omega_{R_0}. \tag{6.40}$$

Since $u_{\lambda^*} \equiv 0$ on $\mathbb{U} = \partial D \bigcup (\partial\omega \times \overline{B_{R_0}})$ and u_{λ^*} is uniformly continuous on $\overline{\Omega_{R_0}}$, and by conditions (f5) and (f6), there exist $\delta > 0$ and C_4 such that

$$\begin{aligned} f''(u_{\lambda^*}) &\le C_4 u_{\lambda^*}^{q_1-1} \quad \text{for } x \in \mathbb{U}_\delta, \\ f''(u_{\lambda^*}) &\le C_4 \quad \text{for } x \in \Omega_{R_0} \setminus \mathbb{U}_\delta, \end{aligned} \tag{6.41}$$

where $\mathbb{U}_\delta = \mathbb{U}^\delta \bigcap \overline{\Omega_{R_0}}$, $\mathbb{U}^\delta$ is a δ-tubular neighborhood of $\mathbb{U}$, $0 < q_1 < 4/(N-2)$ if $N \ge 3$, $q_1 > 0$ if $N = 2$.

From (6.38)–(6.41) and the Hölder inequality, we derive that

$$\begin{aligned} \int_\Omega f''(u_{\lambda^*})\phi_1^3 dx &= \int_{\mathbb{U}_\delta} f''(u_{\lambda^*})\phi_1^3 dx + \int_{\Omega_{R_0}\setminus\mathbb{U}_\delta} f''(u_{\lambda^*})\phi_1^3 dx + \int_{\Omega\setminus\Omega_{R_0}} f''(u_{\lambda^*})\phi_1^3 dx \\ &\le \int_{\mathbb{U}_\delta} C_4 u_{\lambda^*}^{q_1-1}\phi_1^3 dx + \int_{\Omega_{R_0}\setminus\mathbb{U}_\delta} C_4\phi_1^3 dx + \int_{\Omega\setminus\Omega_{R_0}} C_2 u_{\lambda^*}^{q_1-1}\phi_1^3 dx \\ &\le C_3C_4\int_{\mathbb{U}_\delta} u_{\lambda^*}^{q_1}\phi_1^2 dx + C_5 + C_2^2\int_{\Omega\setminus\Omega_{R_0}} u_{\lambda^*}^{q_1}\phi_1 dx \\ &\le C_6 + C_2^2||u_{\lambda^*}||_{q_1+2}^{q_1}||\phi_1||_{(q_1+2)/2} \le C. \end{aligned} \tag{6.42}$$

□

Since $\lambda = \lambda^* + \tau(s)$, $u = u_{\lambda^*} + s\phi_1 + z(s)$ in

$$-\Delta u + u - \lambda f(u) - \lambda h = 0, \quad u > 0,\ u \in C^{2,\alpha}(\Omega) \cap H^2(\Omega). \tag{6.43}$$

Differentiating (6.43) in s twice, we have

$$-\Delta u_{ss} + u_{ss} - \lambda f'(u)u_{ss} - 2\lambda_s f'(u)u_s - \lambda f''(u)(u_s)^2 - \lambda_{ss}(f(u)+h) = 0. \tag{6.44}$$

Setting here $s = 0$ and using the facts that $\tau'(0) = 0$, $u_s = \phi_1(x)$ and $u = u_{\lambda^*}$ as $s = 0$, we obtain

$$-\Delta u_{ss} + u_{ss} - \lambda^* f'(u_{\lambda^*})u_{ss} - \lambda^* f''(u_{\lambda^*})\phi_1^2 - \tau''(0)(f(u_{\lambda^*}) + h) = 0. \tag{6.45}$$

Multiplying $F_u(\lambda^*, u_{\lambda^*})\phi_1 = 0$ by u_{ss} and (6.45) by ϕ_1, integrating and subtracting the result, and by (6.35), we obtain

$$\int_\Omega \lambda^* f''(u_{\lambda^*})\phi_1^3 dx + \tau''(0)\int_\Omega (f(u_{\lambda^*}) + h)\phi_1 dx = 0, \tag{6.46}$$

which immediately gives $\tau''(0) < 0$. Thus

$$\begin{aligned} u_\lambda &\longrightarrow u_{\lambda^*} \quad \text{in } C^{2,\alpha}(\Omega) \cap H^2(\Omega) \text{ as } \lambda \longrightarrow \lambda^*, \\ U_\lambda &\longrightarrow u_{\lambda^*} \quad \text{in } C^{2,\alpha}(\Omega) \cap H^2(\Omega) \text{ as } \lambda \longrightarrow \lambda^*, \end{aligned} \tag{6.47}$$

and we complete the proof of Theorem 1.2 for $\Omega = \mathbb{S} \setminus \overline{D}$.

With the same argument, we also have that Theorems 1.1 and 1.2 hold for $\Omega = \mathbb{R}^N \setminus \overline{D}$.

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