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Research Article

Reverse Smoothing Effects, Fine Asymptotics, and Harnack Inequalities for Fast Diffusion Equations

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We investigate local and global properties of positive solutions to the fast diffusion equation $u_t = \Delta u^m$ in the good exponent range $(d-2)_+/d < m < 1$, corresponding to general nonnegative initial data. For the Cauchy problem posed in the whole Euclidean space $\mathbb{R}^d$, we prove sharp local positivity estimates (weak Harnack inequalities) and elliptic Harnack inequalities; also a slight improvement of the intrinsic Harnack inequality is given. We use them to derive sharp global positivity estimates and a global Harnack principle. Consequences of these latter estimates in terms of fine asymptotics are shown. For the mixed initial and boundary value problem posed in a bounded domain of $\mathbb{R}^d$ with homogeneous Dirichlet condition, we prove weak, intrinsic, and elliptic Harnack inequalities for intermediate times. We also prove elliptic Harnack inequalities near the extinction time, as a consequence of the study of the fine asymptotic behavior near the finite extinction time.

1. Introduction

In this paper, we are interested in the questions of boundedness, positivity, and regularity of the solutions of fast diffusion equations. Though the arguments have a more general scope, two settings will be considered in order to obtain sharp results: in one of them, the Cauchy problem is considered in the whole space

$$\begin{gathered} u_t = \Delta(u^m) \quad \text{in } Q = (0,+\infty)\times\mathbb{R}^d, \\ u(0,x) = u_0(x) \quad \text{in } \mathbb{R}^d, \end{gathered} \tag{1.1}$$

and in the range $(d-2)_+/d = m_c < m < 1$. In the second option, the mixed Cauchy-Dirichlet problem is considered in bounded domains with smooth boundary

$$\begin{gathered} u_t = \Delta(u^m) \quad \text{in } Q = (0,+\infty)\times\Omega, \\ u(0,x) = u_0(x) \quad \text{in } \Omega, \\ u(t,x) = 0 \quad \text{for } t>0,\ x\in\partial\Omega, \end{gathered} \tag{1.2}$$

and in the range $(d-2)_+/(d+2) = m_s < m < 1$. In both cases, nonnegative solutions are considered. The restrictions on the exponent range are not a matter of convenience.

It is well known that the solutions of the heat equation $u_t = \Delta u$ posed in the whole space with nonnegative data at $t=0$ become positive and smooth for all positive times and all points of space. The same positivity property is true in many other settings, for example, for nonnegative solutions posed in a bounded space domain with zero boundary conditions. Such properties of positivity and smoothness are shared by the fast diffusion equation

$$u_t = \Delta u^m, \quad 0 < m < 1, \tag{1.3}$$

but this happens under certain conditions on the exponent and data and with quite different quantitative estimates.

The question of boundedness is closely related to existence theory and has been much investigated in the whole exponent range $m \in \mathbb{R}$. A comprehensive account can be found in works of one of the authors (see [1–3]). The smoothing effect explained there is usually expressed in the form

$$\|u(t)\|_\infty \le \frac{C\|u_0\|_1^\sigma}{t^\alpha}, \tag{1.4}$$

where $t>0$ and all the L^p are taken over the whole domain Ω or $\mathbb{R}^d$. The positive constants C, σ, and α depend only on m, d. The analysis shows that the FDE maps initial data, possibly unbounded, to bounded solutions if m is larger than a first-critical exponent $m_c = (d-2)_+/d$. The situation becomes quite involved, and interesting, for subcritical m.

A natural problem that we will address here arises next: *starting from nonnegative initial data, do we obtain strictly positive solutions, at least locally?* This positivity property is strictly related to Harnack inequalities, as we will see. If we express the positivity result in terms of L^p norms, we are led to the case of negative exponents and of course the quantities

$$\|f\|_{-p} = \left[\int_\Omega f(x)^{-p}dx\right]^{-1/p} \tag{1.5}$$

are no more norms in the usual sense. But there is a nice well-known result, which helps us to better understand the nature of such lower bounds:

$$\inf_{x\in\Omega} |f(x)| = \|f\|_{-\infty} = \lim_{p\to\infty} \|f\|_{-p}. \tag{1.6}$$

The aim of this paper is to present from a unified point of view some results and techniques recently discovered by the authors and described in whole detail in [4, 5], and also to discuss some new ideas to attack some open problems related to Harnack inequalities. Let us present the lower bounds that we obtain. We take the case of the Cauchy problem posed in the whole space: in Theorem 2.1, we prove that

$$\inf_{x\in B_R(x_0)} u(t,x) \geq \overline{M}_R(x_0) H\left(\frac{t}{t_c}\right) > 0. \tag{1.7}$$

Here, $\overline{M}_R(x_0)$ is the average initial mass in the ball $B_R(x_0)$, H is an explicit function of time relative to the characteristic time t_c, which is loosely speaking, a time that we have to wait in order to let the regularization to take place, and is calculated in terms of the initial data. For $t \geq t_c$, the above lower bound can be rewritten as

$$||u(t)||_{L^{-\infty}(B_R(x_0))} \geq K_{m,d}||u_0||^{2\vartheta}_{L^1(B_R(x_0))} t^{-d\vartheta}, \tag{1.8}$$

that is, exactly the reverse of the standard smoothing effect above, thought as L^1–L^∞ regularization, and expressed as a local L^1–$L^{-\infty}$ smoothing effect (over balls); for this reason, we call it *reverse smoothing effect*.

Putting together the direct and reverse smoothing effects, we obtain the intrinsic and elliptic Harnack inequalities and thus as a consequence, we have a quite simple proof of the Hölder continuity of the solution, which has been first proved by DiBenedetto et al., see, for example, [6, 7], by entirely different techniques.

When dealing with elliptic problems, our positivity result, or reverse smoothing effect, is also known as weak Harnack inequality or half Harnack. Indeed, nothing is more natural than this terminology since this easily implies intrinsic Harnack inequality as a corollary, compare Theorems 6.2 and 6.4. Moreover, the combination of direct and reverse smoothing effects implies a Harnack inequality of elliptic type, compare Theorems 6.1 and 6.5, namely, we compare the supremum and infimum of the solution at the same time.

Another issue that we address is the extension to the whole space (or domain) of local positivity properties. This leads to the global Harnack principle, GHP, that is, to accurate upper and lower bounds in terms of some special (sub/super) solutions. In the case of the whole space, the super- and subsolutions are Barenblatt functions. In the case of bounded domains, the global Harnack principle was first proved in [7], and the super- and subsolutions were related to the solution obtained by separation of variables.

We also investigate the connection between the global Harnack principle and the fine asymptotic behavior, first introduced by one of the authors in [1], in terms of uniform convergence in relative error, shortly CRE. We show that the GHP implies CRE both in the case of $\mathbb{R}^d$ and in the case of bounded domains.

Finally, we show in the case of bounded domains that the convergence in relative error implies elliptic Harnack inequalities for times near the extinction time, thus completing the panorama of the validity of Harnack inequalities in the case of bounded domains.

Open problems. These ideas lead to further possible interesting generalizations which are actually under investigation. For example, we can consider the case in which the problem is posed on a Riemannian manifold, and the operator is the Laplace-Beltrami operator, or when it is replaced by a more general elliptic operator, possibly with measurable coefficients. The methods we present here may open new directions to solve the problem of Harnack inequalities for more general nonlinear diffusion equations for a larger range of exponents m.

Notation. In the sequel, the letters a_i, b_i, C_i, K, k_i, λ_i, μ are used to denote universal positive constants that depend only on m and d. The constant ϑ is fixed to the value $\vartheta = 1/(2-d(1-m)) > 0$.

2. Positivity results for the fast diffusion equation

We start our analysis by considering the problem of estimating the positivity of solutions of the FDE, both in the case of the Cauchy problem posed in the whole $\mathbb{R}^d$ space and in the case of the mixed Cauchy-Dirichlet problem posed in a domain of $\mathbb{R}^d$. In both cases, we analyze local and global positivity estimates. In view of the remarks of the introduction, the local positivity estimates can be considered as a reverse smoothing effect and are independent of the choice of some explicit (sub-/super-) solutions. Vice versa, when we deal with global positivity, we make use of some special (sub-/super-) solutions. For a complete discussion of these results, we refer to our paper [5].

2.1. Local and global positivity estimates in $\mathbb{R}^d$. Let us prove quantitative positivity estimates for the Cauchy problem posed in the whole Euclidean space $\mathbb{R}^d$:

$$\begin{gathered} u_t = \Delta(u^m) \quad \text{in } Q = (0,+\infty)\times\mathbb{R}^d, \\ u(0,x) = u_0(x) \quad \text{in } \mathbb{R}^d, \end{gathered} \tag{2.1}$$

in the range $(d-2)_+/d = m_c < m < 1$. We then derive elliptic Harnack inequalities. In the results, we fix a point $x_0 \in \mathbb{R}^d$ and consider different balls $B_R = B_R(x_0)$ with $R > 0$. We introduce the following measures of the local mass:

$$M_R(x_0) = \int_{B_R} u_0(x)\,dx, \qquad \overline{M}_R(x_0) = \frac{M_R}{R^d}. \tag{2.2}$$

More precisely, we should write $M_R(u_0,x_0)$, $\overline{M}_R(u_0,x_0)$, but we will even drop the variable x_0 when no confusion is feared. This is the intrinsic positivity result that shows in a quantitative way that solutions are positive for all $(x,t)\in Q$. This type of results is also called *weak Harnack inequality*, and also *half Harnack inequality* or *lower Harnack inequality*, meaning that it is half of the full pointwise comparison that Harnack inequalities imply.

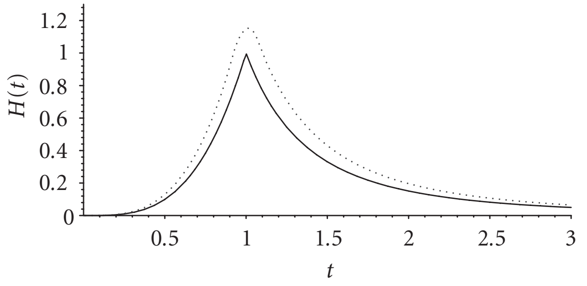

Figure 2.1. Approximative graphic of the functions $u(t,x)$ (dotted line) and $H(t)$ (solid line).

Theorem 2.1 (local positivity on $\mathbb{R}^d$). *There exists a positive function $H(t)$ such that for any $t>0$ and $R>0$ the following bound holds true for all continuous nonnegative solutions u to (2.1) with $m_c<m<1$:*

$$\inf_{x\in B_R(x_0)} u(t,x) \geq \overline{M}_R(x_0)H\left(\frac{t}{t_c}\right). \tag{2.3}$$

Function $H(\eta)$ is positive and takes the precise form

$$H(\eta)=\begin{cases} K\eta^{-d\vartheta} & \text{for } \eta\geq 1,\\ K\eta^{1/(1-m)} & \text{for } \eta\leq 1.\end{cases} \tag{2.4}$$

The characteristic time is given by

$$t_c = CM_R^{1-m}R^{1/\vartheta}. \tag{2.5}$$

Constants $C,K>0$ depend only on m and d.

Figure 2.1 gives an idea of the positivity result, in particular the change of the behavior of the general lower profile as a function of time. It shows the importance of the critical time t_c. For the sake of simplicity, we consider $t_c=1$.

Proof. We skip the proof of this result, given in [5], since it is similar to the proof of the problem posed in a bounded domain, that we will present in Theorem 2.5; that case which presents the extra difficulty caused by the phenomenon of extinction in finite time. Instead, we concentrate on a number of observations. □

(1) *Characteristic time.* Notice that t_c is an increasing function of M_R and R. This is in contrast with the porous medium case $m>1$ where it can be shown that t_c decreases with M_R (see, e.g., [8] or [3, Chapter 4]).

(2) *Minimax problem.* Suppose that we want to obtain the best of the lower bounds when t varies. This happens for $t/t_c\approx 1$ and the value is

$$u(t_c,0)\geq C_3M_RR^{-d}, \tag{2.6}$$

which is just proportional to the average. At this time also the maximum is controlled by the average (see the upper estimate).

(3) The behavior of H is optimal in the limits $t \gg 1$ and $t \approx 0$ as the Barenblatt solutions show. If we perform the explicit computation for the Barenblatt solution in the worst case where the mass is placed on the border of the ball B_{R_0}, it gives (see (2.8))

$$\mathcal{B}(0,t) = \frac{M_R^{2\vartheta} t^{1/(1-m)}}{\left(b_1 t^{2\vartheta} + b_2 t_c^{2\vartheta}\right)^{1/(1-m)}}. \tag{2.7}$$

The consideration of the Barenblatt solutions as an example leads us to examine what is the form of the positivity estimate when we move far away from a ball in space. Indeed, we can get a global estimate by carefully inserting a Barenblatt solution with small mass below our solution. Let us recall that the Barenblatt solution of mass M is given by the formula

$$\mathcal{B}(t,x;M) = \frac{t^{1/(1-m)}}{\left[b_1 t^{2\vartheta}/M^{2\vartheta(1-m)} + b_2 |x|^2\right]^{1/(1-m)}}, \tag{2.8}$$

and also that

$$t_c = C M_R^{(1-m)} R^{1/\vartheta}. \tag{2.9}$$

The following theorem can be viewed as a weak global Harnack principle, since it leads to the global Harnack principle, which will be derived in the next section. Notice that the parameters of the Barenblatt subsolution have a different form in the two cases $t \geq t_c$ and $0 < t < t_c$.

Theorem 2.2 (global positivity in $\mathbb{R}^d$). (I) *There exist* $\tau_1 \in (0, t_c)$ *and* $M_c > 0$ *such that for all* $x \in \mathbb{R}^d$ *and* $t \geq t_c$,

$$u(t,x) \geq \mathcal{B}(t - \tau_1, x; M_c), \tag{2.10}$$

where $\tau_1 = \lambda t_c$ *and* $M_c = k M_R$ *for some universal constants* $\lambda, k > 0$ *which depend only on* m *and* d. (II) *For any* $0 < \varepsilon < t_c$, *the global lower bound is valid for* $t \geq \varepsilon$,

$$u(t,x) \geq \mathcal{B}(t - \tau(\varepsilon), x; M(\varepsilon)), \tag{2.11}$$

with $\tau(\varepsilon) = \lambda\varepsilon$ *and*

$$M(\varepsilon) = \left(\frac{\varepsilon}{t_c}\right)^{1/(1-m)} M_c = k_1 \left(\frac{\varepsilon}{R^{1/\vartheta}}\right)^{1/(1-m)}. \tag{2.12}$$

Proof. The proof presented here has been taken from [5]. The main result is the first, the point of stating (II) is to have an estimate for small times (with a smaller time shift) at the price of having a subsolution with smaller mass. Let us point out that the last constant $k_1 = kC^{-1/(1-m)}$. We divide the proof in a number of steps; the proof of (I) consists of steps (i)–(iii). (i) Let us first argue for $x \in B_R(0)$ at time $t = t_c$. As a consequence of our local estimate (2.1) at $t = t_c$, one gets

$$u(t_c, x) \geq K \frac{M_R}{R^d} \tag{2.13}$$

for all $|x| \le R$. Hence, (2.10) is implied in this region by the inequality

$$K\frac{M_R}{R^d} \ge \mathcal{B}(t_c-\tau_1,x;M_c) = \frac{(t_c-\tau_1)^{1/(1-m)}}{\left[b_1(t_c-\tau_1)^{2\vartheta}/M_c^{2\vartheta(1-m)}+b_2|x|^2\right]^{1/(1-m)}}. \tag{2.14}$$

Now we choose $\tau_1 = \lambda t_c$ with a certain $\lambda \in (0,1)$. We put $\mu = 1-\lambda \in (0,1)$ so that $t_c - \tau_1 = \mu t_c$. With this choice, (2.14) is equivalent to

$$\frac{b_1(\mu t_c)^{2\vartheta}}{M_c^{2\vartheta(1-m)}} + b_2|x|^2 \ge \frac{R^{d(1-m)}\mu t_c}{M_R^{1-m}K^{1-m}} \tag{2.15}$$

putting $x=0$ and using the value of t_c, it is implied by the condition

$$M_c = kM_R, \quad k \le b_1^{1/(2\vartheta(1-m))}K^{1/2\vartheta}(\mu C)^{d/2}. \tag{2.16}$$

(ii) We now extend the comparison to the region $|x| \ge R$, again at time $t = t_c$. We take as a domain of comparison the exterior space-time domain

$$S = (\tau_1,t_c) \times \{x \in \mathbb{R}^d : |x| > R\}. \tag{2.17}$$

Both functions in estimate (2.10) are solutions of the same equation, hence we need only to compare them on the parabolic boundary. Comparison at the initial time $t = \tau_1$ is clear since $B(t_c-\tau_1,x;M_c)$ vanishes. The comparison on the lateral boundary where $|x| = R$ and $\tau_1 \le t \le t_c$ amounts to

$$K\frac{M_R}{R^d}\left(\frac{t}{t_c}\right)^{1/(1-m)} \ge \frac{(t-\tau_1)^{1/(1-m)}}{\left[b_1(t-\tau_1)^{2\vartheta}/M_c^{2\vartheta(1-m)}+b_2R^2\right]^{1/(1-m)}}. \tag{2.18}$$

Raising to the power $(1-m)$ and using the value of t_c, we get

$$\frac{K^{1-m}t}{R^2C} \ge \frac{t-\tau_1}{b_1(t-\tau_1)^{2\vartheta}/M_c^{2\vartheta(1-m)}+b_2R^2}, \tag{2.19}$$

or

$$K^{1-m}\frac{b_1(t-\tau_1)^{2\vartheta}}{M_c^{2\vartheta(1-m)}} + K^{1-m}b_2R^2 \ge \left(1-\frac{\tau_1}{t}\right)R^2C. \tag{2.20}$$

If we have fixed τ_1 as before and if we define $M_c = kM_R$ with $k = k(m,d)$ small enough, this inequality is true for $\tau_1 \le t \le t_c$. (iii) Using now the maximum principle in S, the proof of (2.10) is thus complete for $t = t_c$ in the exterior region. Since the comparison holds in the interior region by step (i), we get a global estimate at $t = t_c$. (iv) We now prove part (II) of the theorem. We only need to prove it at $t = \varepsilon$. We recall that λ and M_c are as defined in part (I). We know that

$$t_c - \tau_1 = \mu t_c, \quad \text{with } \mu \in (0,1). \tag{2.21}$$

Using the Bénilan-Crandall estimate, we have for $0 < t < t_c$

$$u(t,x) \geq u(t_c,x)\frac{t^{1/(1-m)}}{t_c^{1/(1-m)}}, \tag{2.22}$$

together with the above estimate (2.10), we can see that

$$\begin{aligned} u(t,x) &\geq u(t_c,x)\frac{t^{1/(1-m)}}{t_c^{1/(1-m)}} \geq \frac{t^{1/(1-m)}}{t_c^{1/(1-m)}}\mathcal{B}(t_c-\tau_1,x;M_c) \\ &= \frac{t^{1/(1-m)}}{t_c^{1/(1-m)}}\frac{(\mu t_c)^{1/(1-m)}}{\left[b_1(\mu t_c)^{2\vartheta}/M_c^{2\vartheta(1-m)}+b_2|x|^2\right]^{1/(1-m)}} \\ &= \frac{(\mu t)^{1/(1-m)}}{\left[b_1(\mu t)^{2\vartheta}/M_c^{2\vartheta(1-m)}t^{2\vartheta}t_c^{-2\vartheta}+b_2|x|^2\right]^{1/(1-m)}} \\ &= \mathcal{B}\left(\mu t,x;\frac{M_c t^{1/(1-m)}}{t_c^{1/(1-m)}}\right) = \mathcal{B}(t-\tau,x;M_c(t)) \end{aligned} \tag{2.23}$$

once one lets $t-\tau=\mu t$ and M_c as above. The proof of (2.11) is thus complete. □

A consequence of this result is the following lower asymptotic behavior that is peculiar of the FDE evolution.

Corollary 2.3. *Under the same hypothesis of Theorem 2.2,*

$$\liminf_{|x|\to\infty} u(t,x)|x|^{2/(1-m)} \geq c(m,d)t^{1/(1-m)}. \tag{2.24}$$

The constant $c(m,d)=(2m/\vartheta(1-m))^{1/(1-m)}$ of the Barenblatt solution is sharp.

This result has been proved by Herrero and Pierre (see [9, Theorem 2.4]) by similar methods. Here, it easily follows from the estimates of Theorem 2.2 which provides an exact lower bound for all times, not only for large times.

Remarks 2.4. (1) In order to complement the previous lower estimates, let us review what is known about estimates from above. These depend on the behavior of the initial data as $|x|\to\infty$. Recall only that constant data produce the constant solution that does not decay. Under the decay assumption on the initial datum $u_0 \in L^1_{\text{loc}}(\mathbb{R}^d)$

$$\int_{|y-x|\leq|x|/2} |u_0(y)|\,dy = O(|x|^{d-2/(1-m)}) \quad \text{as } |x| \longrightarrow \infty, \tag{2.25}$$

it has been proved by entirely different methods in [1] that

$$\lim_{|x|\to\infty} u(t,x)|x|^{2/(1-m)} \leq c(m,d)(t+S)^{1/(1-m)}, \tag{2.26}$$

where $S>0$ depends on the constant in the bound (2.25) as $|x|\to\infty$. The time shift S is needed in the asymptotic behavior of u as $|x|\to\infty$. Actually, when the initial datum has an exact decay at infinity, $u_0 \sim a|x|^{-2/(1-m)}$, we have more

$$\lim_{|x|\to\infty} u(t,x)|x|^{2/(1-m)} = C(t+S)^{1/(1-m)}, \tag{2.27}$$

with $C = 2m/\vartheta(1-m)$ and $S = a^{1-m}/C$, and this cannot be improved as the delayed Barenblatt solutions show. Moreover, there exists a t_0 such that u^{1-m} is convex as a function of x for $t > t_0$, compare [10].

(2) In comparison with the upper bounds, we have shown that global lower estimates need a time shift τ (in the other direction, explicitly calculated), but in the limit we can put $\tau = 0$, as one can see above. Moreover, the behavior at infinity is independent of the mass (a fact that is false for the heat equation), hence all Barenblatt solutions with different free constant b_1 behave in the same way in the limit as $|x| \to \infty$, compare [1].

(3) We can also get better results if we consider radially symmetric initial data (always in our range of parameters $m_c < m < 1$), compare [11].

2.2. Local and global positivity estimates on domains. In this section, we will prove local positivity estimates (weak Harnack) and elliptic Harnack inequalities for the fast diffusion equation in the range $(d-2)_+/d = m_c < m < 1$ in a Euclidean domain $\Omega \subset \mathbb{R}^d$,

$$\begin{aligned} u_t &= \Delta(u^m) \quad \text{in } Q = (0,+\infty)\times\Omega,\\ u(0,x) &= u_0(x) \quad \text{in } \Omega,\\ u(t,x) &= 0 \quad \text{for } t>0,\ x\in\partial\Omega, \end{aligned} \tag{2.28}$$

where $\Omega \subset \mathbb{R}^d$ is an open-connected domain with sufficiently smooth boundary. Since we are interested in lower estimates, by comparison, we may assume that Ω is bounded without loss of generality. In the case of bounded domains, an extra difficulty appears: the extinction in finite time, for example, there exists a time $T > 0$ such that $u(t,x) \equiv 0$ for any $t \geq T$ and $x \in \Omega$. In the proof of Theorem 2.5, we prove a lower bound for such extinction time in terms of the volume of the domain. This will in particular show that in the case of the whole $\mathbb{R}^d$, solutions do not extinguish in finite time. This is the intrinsic positivity result that shows in a quantitative way that solutions are positive for all $(x,t) \in Q$. In the result, we fix a point $x_0 \in \Omega$ and consider different balls $B_R = B_R(x_0)$ with $R > 0$, included in Ω. It is a version of Theorem 2.1 in the case of the mixed Cauchy-Dirichlet problem on domains.

Theorem 2.5 (local positivity on domains). *Let u be a continuous nonnegative solution to (2.28), with $m_c < m < 1$. There exist times $0 < t_c^* < T_c \leq T$, where T is the finite extinction time, and a positive function $H(t)$ such that for any $t \in (0, T_c)$ and $R > 0$ such that*

$$R \leq \Lambda\,\mathrm{dist}\,(x_0, \partial\Omega), \tag{2.29}$$

the following bound holds true:

$$\inf_{x\in B_R} u(t,x) \geq \overline{M}_R H\left(\frac{t}{t_c^*}\right), \tag{2.30}$$

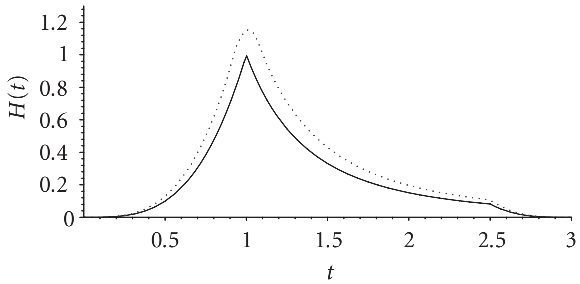

Figure 2.2. Approximative graphic of the functions $u(t,x)$ (dotted line) and $H(t)$ (solid line).

where $\overline{M}_R = M_R/R^d$, $M_R = \int_{B_R} u_0(x)dx$. *Function* $H(t)$ *is positive and takes the precise form*

$$H(\eta) = \begin{cases} K\eta^{-d\vartheta} & \text{for } 1 \le \eta \le \dfrac{T_c}{t_c^*}, \\ K\eta^{1/(1-m)} & \text{for } \eta \le 1. \end{cases} \tag{2.31}$$

The times $0 < t_c^* \le T_c \le T$ *are given by*

$$\begin{gathered} t_c^* = \tau_c (2R)^{1/d\vartheta} M_R^{1-m}, \\ T_c = \tau_c' [\operatorname{dist}(x_0, \partial\Omega) - 2R] M_R^{1-m}. \end{gathered} \tag{2.32}$$

Constants C, K, τ_c, τ_c', $\Lambda > 0$ *depend only on* d *and* m.

Figure 2.2 gives an idea of the positivity result, in particular the change of the behavior of the general lower profile, in function of time, showing the importance of both the lower critical time t_c and the upper critical time T_c. For the sake of simplicity, we consider $t_c = 1$ and $T_c = 2.5$, while the extinction time is taken as $T = 3$.

Proof. The proof presented here has been taken from [5]. It is a combination of several steps. Without loss of generality, we assume that $x_0 = 0$. Different positive constants that depend on m and d are denoted by C_i. The precise values we get for C, K, τ_c, τ_c', and Λ are given at the end of the proof.

Reduction. By comparison, we may assume that $\operatorname{supp}(u_0) \subset B_R(0)$. Indeed, a general $u_0 \ge 0$ is greater than $u_0\eta$, η being a suitable cutoff function compactly supported in B_R and less than one. If v is the solution of the fast diffusion equation with initial data $u_0\eta$ (existence and uniqueness are well known in this case), we obtain

$$\int_{B_R} u(0,x)dx \ge \int_{B_R} u_0(x)\eta(x)dx = M_R \tag{2.33}$$

and if the statement holds true for v, then

$$\inf_{x\in B_R} u(t,x) \ge \inf_{x\in B_R} v(t,x) \ge H\left(\frac{t}{t_c}\right)\overline{M}_R. \tag{2.34}$$

Lower bounds on the extinction time. In order to get a lower bound for the extinction time in terms of local mass information, we use a property which can be labeled as weak conservation of mass, and has been proved in [9, Lemma 3.1]. It reads: for any $R, r > 0$ and $s, t \geq 0$, one has

$$\int_{B_{2R}} u(s,x)\,dx \leq C_3\left[\int_{B_{2R+r}} u(t,x)\,dx + \frac{|s-t|^{1/(1-m)}}{r^{(2-d(1-m))/(1-m)}}\right]. \tag{2.35}$$

Now letting $t = T$, so that $u(T,x) = 0$, and $s = 0$ so that $\int_{B_{2R}} u(0,x)\,dx = M_R$, we get

$$T \geq \frac{M_R^{1-m} r^{1/\vartheta}}{C_3^{1-m}} \geq \frac{M_R^{1-m}\left[\operatorname{dist}(0,\partial\Omega) - 2R\right]^{1/\vartheta}}{C_3^{1-m}}, \tag{2.36}$$

since $r \in (0, \operatorname{dist}(0,\partial\Omega) - 2R)$.

A priori estimates. The second step again is similar to the analogous step in the proof of Theorem 2.1, so we will omit the details. We rewrite the well-known smoothing effect (see, e.g., [3]), after an integration over $B_{2^b R}$, in the form

$$\int_{B_{2^b R}} u(t,x)\,dx \leq C_2 M_R^{2\vartheta} R^d t^{-d\vartheta}, \tag{2.37}$$

since u_0 is nonnegative and supported in B_R. Here $C_2 = C_1 2^{bd}\omega_d$.

Integral estimate. Again in this step we are going to use the estimate (2.35). We let $s = 0$ and we rewrite it in a form more useful to our purposes (remember that $M_{2R} = M_R$ since u_0 is supported in B_R):

$$\int_{B_{2R+r}} u(t,x)\,dx \geq \frac{M_R}{C_3} - \frac{t^{1/(1-m)}}{r^{1/\theta(1-m)}}, \tag{2.38}$$

we now remark that r and R are such that $B_{2R+r} \subset \Omega$.

Aleksandrov principle. The fourth step consists in using the well-known reflection principle in a slightly different form (see Proposition A.1 and formula (A.5) in the appendix for more details). This principle reads

$$\int_{B_{2R+r}\setminus B_{2^b R}} u(t,x)\,dx \leq A_d r^d u(t,0), \tag{2.39}$$

where A_d and $b = 2 - 1/d$ are chosen as in (A.5) in the appendix, and one has to remember the condition $r \geq (2^{(d-1)/d} - 1)2R$.

We now put together all the previous calculations:

$$\begin{aligned}\int_{B_{2R+r}} u(t,x)\,dx &= \int_{B_{2^b R}} u(t,x)\,dx + \int_{B_{2R+r}\setminus B_{2^b R}} u(t,x)\,dx \\ &\leq C_2 M_R^{2\vartheta} R^d t^{-d\vartheta} + A_d r^d u(t,0).\end{aligned} \tag{2.40}$$

This follows by (2.37) and (2.39). Now we are going to use (2.38) to obtain

$$\frac{M_R}{C_3} - \frac{t^{1/(1-m)}}{r^{1/\theta(1-m)}} \leq \int_{B_{2R+r}} u(t,x)\,dx \leq \frac{C_2 M_R^{2\vartheta} R^d}{t^{d\vartheta}} + A_d r^d u(t,0). \tag{2.41}$$

And finally we obtain

$$u(t,0) \geq \frac{1}{A_d}\left[\left(\frac{M_R}{C_3} - \frac{C_2 M_R^{2\vartheta} R^d}{t^{d\vartheta}}\right)\frac{1}{r^d} - \frac{t^{1/(1-m)}}{r^{2/(1-m)}}\right] = \frac{1}{A_d}\left[\frac{A(t)}{r^d} - \frac{B(t)}{r^{2/(1-m)}}\right]. \tag{2.42}$$

Now we would like to obtain the claimed estimate for $t > t_c^*$. To this end, we seek whether $A(t)$ is positive:

$$A(t) = \frac{M_R}{C_3} - C_2 \frac{M_R^{2\vartheta} R^d}{t^{d\vartheta}} > 0 \Longleftrightarrow t > \left(C_3 C_2\right)^{1/(d\vartheta)} M_R^{1-m} R^{1/\vartheta} = t_c^*. \tag{2.43}$$

Now we have to check if $t_c^* \leq T$. By (2.36), one knows that a sufficient condition is that $t_c^* \leq T_c = C_3^{m-1} M_R^{1-m} [\mathrm{dist}(0,\partial\Omega) - 2R]^{1/\vartheta} \leq T$, that is,

$$R \leq \frac{\mathrm{dist}(0,\partial\Omega)}{2 + C_3^{1-m+1/d\vartheta} C_2^{1/d\vartheta}}. \tag{2.44}$$

Now, assuming that $t \in (t_c^*, T_c)$ is temporarily fixed, we optimize the function

$$f(r) = \frac{1}{A_d}\left[\frac{A(t)}{r^d} - \frac{B(t)}{r^{2/(1-m)}}\right] \tag{2.45}$$

with respect to $r = r(t) \in (0, \mathrm{dist}(0,\partial\Omega) - 2R)$ and we obtain that it attains its maximum in $r = r_{\max}(t)$:

$$r_{\max}(t) = \left[\frac{2}{d(1-m)}\right]^{\vartheta(1-m)} t^{\vartheta}\left[\frac{M_R}{C_3} - \frac{C_2 M_R^{2\vartheta} R^d}{t^{d\vartheta}}\right]^{-\vartheta(1-m)}. \tag{2.46}$$

At this point, it is necessary to check the conditions

$$\left(2^{(d-1)/d} - 1\right) 2R < r_{\max}(t) < \mathrm{dist}(0,\partial\Omega) - 2R. \tag{2.47}$$

To this end, it is useful to get a simpler parametrization of the time interval (t_c^*, T_c), indeed

$$t_\alpha = \alpha t_c^* = \alpha \left(C_3 C_2\right)^{1/(d\vartheta)} M_R^{1-m} R^{1/\vartheta} \tag{2.48}$$

maps the time interval (t_c^*, T_c) into $(1, \alpha_c)$, where

$$\begin{aligned} \alpha_c &= \frac{T_c}{t_c^*} = C_3^{1-m+1/d\vartheta} C_2^{1/d\vartheta}\left(\frac{\mathrm{dist}(0,\partial\Omega)}{R} - 2\right), \\ r_{\max}(t_\alpha) &= \left(\frac{2}{d(1-m)}\right)^{\vartheta(1-m)} C_3^{1-m+1/d\vartheta} C_2^{1/d\vartheta} \frac{\alpha^{\vartheta}}{\left(1-\alpha^{-d\vartheta}\right)^{\vartheta(1-m)}} R. \end{aligned} \tag{2.49}$$

Now optimizing this function with reflect to $\alpha \in (1, \alpha_c)$ will lead to the value

$$\alpha_{\min} = 1 + \vartheta d(1-m) \tag{2.50}$$

and in order to guarantee the fact that $\alpha_{\min} \le \alpha_c$, we impose the condition

$$R \le \frac{\operatorname{dist}(0,\partial\Omega)}{2 + \left(\left(1+\vartheta d(1-m)\right)C_3^{1-m+1/d\vartheta}C_2^{1/d\vartheta}\right)^{\vartheta}}. \tag{2.51}$$

Moreover, it is tedious but straightforward to verify that

$$\left(2^{(d-1)/d} - 1\right)2R < r_{\max}\left(t_{\alpha_c}\right) \le \operatorname{dist}\left(x_0,\partial\Omega\right) - 2R, \tag{2.52}$$

the first inequality becomes nothing else but a lower bound on the constants C_2 and C_3, but since they are constants used in upper estimates, they can be chosen arbitrarily large. The second inequality is guaranteed by the hypothesis $R \le \Lambda \operatorname{dist}(0,\partial\Omega)$. Now going back to the standard time parametrization, we proved that

$$f\left(r_{\max}(t)\right) = A_d \frac{\left[d(1-m)\right]^{2\vartheta-1}}{2^{2\vartheta}\vartheta}\left[\frac{1}{C_3} - C_2\frac{M_R^{2\vartheta-1}R^d}{t^{d\vartheta}}\right]^{2\vartheta}\frac{M_R^{2\vartheta}}{t^{d\vartheta}} > 0 \tag{2.53}$$

for all $t \in (t_{\alpha_{\min}}, T_c) \subset (t_c^*, T)$. We thus found the estimate

$$u(t,0) \ge A_d \frac{\left[d(1-m)\right]^{2\vartheta-1}}{2^{2\vartheta}\vartheta}\left[\frac{1}{C_3} - C_2\frac{M_R^{2\vartheta-1}R^d}{t^{d\vartheta}}\right]^{2\vartheta}\frac{M_R^{2\vartheta}}{t^{d\vartheta}} = K_1 A(t)\frac{M_R^{2\vartheta}}{t^{d\vartheta}}, \tag{2.54}$$

a straightforward calculation shows that the function

$$A(t) = \left[\frac{1}{C_3} - C_2\frac{M_R^{2\vartheta-1}R^d}{t^{d\vartheta}}\right]^{2\vartheta} \tag{2.55}$$

is nondecreasing in time, thus if $t \ge t_{\alpha_{\min}}$,

$$A(t) \ge A\left(t_{\alpha_{\min}}\right) = \left(\frac{1-\left(1+\vartheta d(1-m)\right)^{-d\vartheta}}{2C_3}\right)^{2\vartheta} \tag{2.56}$$

and finally we obtain

$$u(t,0) \ge K_1 A(t)\frac{M_R^{2\vartheta}}{t^{d\vartheta}} \ge K_1 A\left(t_{\alpha_{\min}}\right)\frac{M_R^{2\vartheta}}{t^{d\vartheta}}. \tag{2.57}$$

So we proved that

$$u(t,0) \ge K\frac{M_R^{2\vartheta}}{t^{d\vartheta}} \tag{2.58}$$

for $t \in (t_{\alpha_{\min}}, T_c)$, with

$$K = \frac{A_d}{\left(2C_3\right)^{2\vartheta}}\frac{\left[d(1-m)\right]^{2\vartheta-1}}{2^{2\vartheta}\vartheta}\left[1-\left(1+\vartheta d(1-m)\right)^{-d\vartheta}\right]^{2\vartheta}. \tag{2.59}$$

From the center to the infimum. Now we want to obtain a positivity estimate for the infimum of the solution u in the ball $B_R = B_R(0)$. Suppose that the infimum is attained in some point $x_m \in \overline{B_R}$, so that $\inf_{x\in B_R} u(t,x) = u(t,x_m)$, then one can apply (2.58) to this point and obtain

$$u(t,x_m) \geq K\frac{M_{2R}^{2\vartheta}(x_m)}{t^{d\vartheta}} \tag{2.60}$$

for $t_{\alpha_{\min}}(x_m) < t < T_c(x_m) < T$. Since the point $x_m \in \overline{B_R(0)}$, then it is clear that $B_R(0) \subset B_{2R}(x_m) \subset B_{4R}(0)$ and this leads to the equality

$$M_{2R}(x_m) = M_R(0) = M_{4R}(0) \tag{2.61}$$

since $M_\varrho(y) = \int_{B_\varrho(y)} u_0(x)dx$, $\operatorname{supp}(u_0) \subset B_R(0)$ and $u_0 \geq 0$. These equalities will imply then that the times

$$\begin{aligned} t_{\alpha_{\min}}(x_m) &= (1+\vartheta d(1-m))(C_3C_2)^{1/d\vartheta}(2R)^{1/\vartheta}M_{2R}(x_m) \\ &= (1+\vartheta d(1-m))(C_3C_2)^{1/d\vartheta}(2R)^{1/\vartheta}M_R(0) = t^*_{\min}(0) \geq t_{\alpha_{\min}}(0), \end{aligned} \tag{2.62}$$

$$\begin{aligned} T_c(x_m) &= C_3^{m-1}[\operatorname{dist}(0,\partial\Omega) - 4R]^{1/\vartheta}M_{2R}^{1-m}(x_m) \\ &= C_3^{m-1}[\operatorname{dist}(0,\partial\Omega) - 4R]^{1/\vartheta}M_R^{1-m}(0) \leq T_c(0). \end{aligned} \tag{2.63}$$

Thus, we have found that

$$\inf_{x\in B_R(0)} u(t,x) = u(t,x_m) \geq K\frac{M_R^{2\vartheta}(x_m)}{t^{d\vartheta}} = K\frac{M_R^{2\vartheta}(0)}{t^{d\vartheta}} = K\frac{t^{*d\vartheta}_{\min}(0)}{t^{d\vartheta}}\frac{M_R^{2\vartheta}(0)}{t^{*d\vartheta}_{\min}(0)} \tag{2.64}$$

for $t_c^* = t^*_{\min}(0) < t < T_c(0) < T$, which is exactly (2.30).

The last step consists in obtaining a lower estimate when $0 \leq t \leq t_c^*$.

To this end, we consider the fundamental estimate of Bénilan and Crandall [12]

$$u_t(t,x) \leq \frac{u(t,x)}{(1-m)t}. \tag{2.65}$$

This easily implies that the function

$$u(t,x)t^{-1/(1-m)} \tag{2.66}$$

is nonincreasing in time, thus, for any $t \in (0,t_c)$, we have

$$u(t,x) \geq u(t_c^*,x)\frac{t^{1/(1-m)}}{t_c^{*1/(1-m)}}. \tag{2.67}$$

We can now prove (2.30) for intermediate times, that is, $0 < t < t_c^*$, simply by applying the estimate (2.64) with $t = t_c^*$ to the right-hand side of the above estimate (2.67). Notice that (2.64) holds for any $t \geq t_c^*$. The proof of formula (2.30) is complete in all cases.

The values of the constants K and C are given by

$$K = \frac{A_d}{(2C_3)^{2\vartheta}} \frac{[d(1-m)]^{2\vartheta-1}}{2^{2\vartheta}\vartheta} \frac{[1-(1+\vartheta d(1-m))^{-d\vartheta}]^{2\vartheta}}{2^d C_3 C_2 (1+\vartheta d(1-m))},$$
$$C = C_3^{1-m+1/d\vartheta} C_2^{1/d\vartheta},$$
$$\tau_c = (1+\vartheta d(1-m))(C_3C_2)^{1/d\vartheta}, \tag{2.68}$$
$$\tau_c' = \frac{1}{C_3^{1-m}},$$
$$\Lambda = \min\left(\frac{1}{(2+C)}, \frac{1}{2+((1+\vartheta d(1-m))C)^{\vartheta}}\right).$$

The proof is complete. □

Global positivity on domains. The global positivity in this setup has been proved first by DiBenedetto et al. [7] in the form of the global Harnack principle that we will discuss in the following section.

3. Global Harnack principle on the whole space and relative error estimates

Under a further control on the initial data, we can transform the local Harnack principle into a global version. The *global Harnack principle*, which is the natural extension of Harnack inequalities to a global point of view, is indeed nothing else than a *global sharp upper and lower bound* in terms of a Barenblatt solution shifted in time and possibly with different mass. The range of the parameter m is always $m_c < m < 1$. We recall that b_i, λ_1, k_1, and C_i are constants that depend only on m and d, while the rest of the parameters depend also on the data as expressed.

THEOREM 3.1 (global Harnack principle). *Let $u_0 \in L^1(\mathbb{R}^d)$, $u_0 \geq 0$, and*

$$u_0(x)|x|^{2/(1-m)} \leq A \tag{3.1}$$

for $|x| \geq R_0$. Then, for any time $\varepsilon > 0$, there exist constants τ_1, τ_2, M_1, and M_2, such that for any $(t,x) \in (\varepsilon,\infty)\times\mathbb{R}^d$, the following upper and lower bounds hold:

$$\mathcal{B}(t-\tau_1,x;M_1) \leq u(t,x) \leq \mathcal{B}(t+\tau_2,x;M_2), \tag{3.2}$$

where $\tau_1 = \lambda_1\varepsilon$, $\tau_2 = \tau(\varepsilon,A,t_s)$, $M_1 = M(\varepsilon)$ as given in Theorem 2.2, and $M_2 = k_2(\varepsilon,A,\tau_2)M_\infty$, while

$$t_c = CM_R^{1-m}R^{1/\vartheta}, \qquad t_s = C_5 M_\infty^{1-m} R_0^{1/\vartheta}. \tag{3.3}$$

Proof. The detailed proof can be found in [5]. It is based on a quite delicate analysis of the properties of the solution and the size of the Barenblatt solutions in different parts of the space-time domain. Convenient parabolic comparisons are then used to arrive at the result. □

Asymptotic behavior and relative error estimates in $\mathbb{R}^d$. The second author has proved in [1] the so-called relative error estimates (REE) for the FDE in the same range of parameters, namely,

$$\lim_{t\to\infty}\left\|\frac{u(t,\cdot)-\mathcal{B}(t,\cdot;M)}{\mathcal{B}(t,\cdot;M)}\right\|_\infty = 0, \tag{3.4}$$

where $\mathcal{B}$ is the Barenblatt solution with the same mass (the result is independent of a possible shift in time or space). This is related to our Theorem 3.1 as follows: for every $\varepsilon > 0$, we can find a Barenblatt solution with mass $M_1(\varepsilon) < M_\infty$ and another one with mass $M_2(\varepsilon) > M_\infty$ that serve as lower bound, respectively, upper bound for the solution for all times $t \geq \varepsilon$. It is clear from the maximum principle that $M_1(\varepsilon)$ increases with time while $M_2(\varepsilon)$ decreases. The asymptotic result says that

$$\lim_{\varepsilon\to\infty} M_1(\varepsilon) = \lim_{\varepsilon\to\infty} M_2(\varepsilon) = M_\infty. \tag{3.5}$$

Theorem 3.1 adds to this asymptotic statement a more precise quantitative information that is valid not only for large times, but also for arbitrary small times. The solution thus inherits positivity and boundedness properties directly from the Barenblatt solutions that serve as upper and lower bounds from the very beginning. Usually, it is said that the Barenblatt solution of the nonlinear equations is a "poor cousin" of the fundamental solution of the heat equation since there is no representation formula as in the linear case. The above results show that in the good fast diffusion range $m_c < m < 1$, it is a stronger model in some respects. Thus, a consequence of this powerful global Harnack principle, obviously valid for the Barenblatt solutions, is that the behavior at infinity (i.e., for $|x| \to \infty$ and/or $t \to \infty$) of the Barenblatt solution is always the same, independent of the mass. This uniformity property is not shared by the heat equation nor by the porous medium equation and it shows how much more effectively the fast diffusion process regularizes the initial data.

Different behavior in the cases $m \notin (m_c, 1)$. In the above considerations, it is essential that the range of parameters is $m_c < m < 1$, since when $m \notin (m_c, 1)$, different phenomena hold. We refer to [3] for a detailed and exhaustive exposition and as a source for more complete bibliography. Let us discuss here the question of possible uniform lower bounds. The following result is proved in [5].

Proposition 3.2. *Locally uniform positivity estimates and a posteriori any kind of Harnack inequalities are false for general initial data.*

This quite simple example shows that the range of parameters we consider in this paper is optimal from below, if we want the initial datum u_0 to be as general as possible.

Let us now comment that the results discussed above have been motivated by similar properties of the heat equation flow. It has to be noted that there are slight differences in favor of the fast diffusion case. Indeed, if one considers as initial datum $u_0 = \delta_y$, then it is easy to see that the shifted fundamental solution of the linear heat equation

$$E_y(t,x) = (4\pi t)^{-d/2} e^{-|x-y|^2/t} \tag{3.6}$$

does not satisfy the condition

$$c_1 E_0(t,x) \le E_y(t,x) \le c_2 E_0(t,x) \tag{3.7}$$

for some universal constants $c_i > 0$, which is, however, satisfied by the Barenblatt solutions if $m_c < m < 1$.

4. Global Harnack principle on bounded domains

In this section, we will enlarge a bit the range of the parameter m, namely, we will consider

$$\frac{(d-2)_+}{d+2} = m_s < m < 1, \tag{4.1}$$

with $m_s \le m_c$ (note that m_s is the inverse of the usual Sobolev exponent). Passing now from the local to the global point of view, we should mention that the global Harnack principle in the case of bounded domains has been proved by DiBenedetto et al. [7]. They investigate some regularity properties of the FDE problem posed on bounded domains. We quote hereafter their [7, Theorem 1.1] for convenience of the reader and since it will be used in the sequel, for its relation with the fine asymptotic behavior, near the extinction time.

Theorem 4.1 (global Harnack principle on bounded domains) [7]. *For any* $\varepsilon \in (0,T)$, *there exist constants* λ, Λ *depending only upon* d, m, $\|u_0\|_{1+m}$, $\|\nabla u_0^m\|_2$, $\mathrm{diam}(\Omega)$, $\partial\Omega$, *and* ε, *such that for all* $(t,x) \in (0,T)\times\Omega$, $t > \varepsilon$

$$\lambda\, \mathrm{dist}(x,\partial\Omega)^{1/m}(T-t)^{1/(1-m)} \le u(t,x) \le \Lambda\, \mathrm{dist}(x,\partial\Omega)^{1/m}(T-t)^{1/(1-m)}. \tag{4.2}$$

This global Harnack principle also gives further regularity of the solutions (namely space analyticity and time Hölder continuity), and holds on bounded domains depending on some further global regularity of the initial datum. The difference between the $\mathbb{R}^d$ case and the bounded domain case is that in the case of whole space $\mathbb{R}^d$ the general solution $u(x,t)$ is estimated from above and from below in terms of the Barenblatt solution, while in the case of a bounded domain, it is bounded between $d(x)^{1/m}(T-t)^{1/(1-m)}$, which is essentially the solution obtained by separation of variables.

We conclude this topic section by saying that the global version of the elliptic Harnack inequality is the global Harnack principle, that is, nothing more than an accurate lower and upper bound with the same "comparison function," both in the case of the whole space and in the case of bounded domains. As far as we know, it is an interesting open problem to find such global principle in unbounded domains.

5. Convergence in relative error on a domain

Our next interest is the asymptotic behavior of nonnegative solutions of the fast diffusion equation (FDE) near the extinction time. More precisely, we consider the initial and boundary value problem

$$\begin{aligned} u_t &= \Delta(u^m) \quad \text{in } (0,+\infty)\times\Omega, \\ u(0,x) &= u_0(x) \quad \text{in } \Omega, \\ u(t,x) &= 0 \quad \text{for } t>0,\ x\in\partial\Omega \end{aligned} \tag{5.1}$$

posed in a bounded connected domain $\Omega\subset\mathbb{R}^d$ with sufficiently smooth boundary; as we have said, $m_s<m<1$. We assume that the initial data u_0 is bounded and nonnegative. We recall that the above problem possesses a unique weak solution $u\geq 0$, that is, defined and positive for some time interval $0<t<T$ and vanishes at a time $T=T(m,d,u_0)>0$, which is called the (finite) extinction time, compare [13, 7]. Note that the conditions on the initial data can be relaxed into $u_0\in L^p(\Omega)$ for some $p>p_c$ where $p_c=\max\{1,d(1-m)/2\}$ in view of the L^p–L^∞ smoothing effect

$$||u(t)||_\infty \leq C_{m,d}||u_0||_p^\gamma t^{-\alpha} \quad \text{for any } t>0, \tag{5.2}$$

valid for $p>p_c$ with $\gamma,\alpha>0$ depending only on m, d, p (see [3] for further details on this issue).

The asymptotic profiles and the associated elliptic problem. We want to investigate the precise behavior of the solution near the extinction time. For this purpose, it is convenient to transform the above problem by the known method of rescaling and time transformation. If we put

$$u(t,x)=(T-t)^{1/(1-m)}w(\tau,x), \quad \tau=\log\left(\frac{T}{(T-t)}\right), \tag{5.3}$$

then, problem (5.1) is mapped into

$$\begin{aligned} w_\tau &= \Delta(w^m)+\frac{w}{1-m} \quad \text{in } (0,+\infty)\times\Omega, \\ w(0,x) &= T^{-1/(1-m)}u_0(x) \quad \text{in } \Omega, \\ w(\tau,x) &= 0 \quad \text{for } \tau>0,\ x\in\partial\Omega. \end{aligned} \tag{5.4}$$

The transformation can also be expressed as

$$w(\tau,x)=\frac{u(T-Te^{-\tau},x)}{(Te^{-\tau})^{1/(1-m)}}, \tag{5.5}$$

and the time interval $0<t<T$ becomes $0<\tau<\infty$. In a celebrated paper, Berryman and Holland [13] reduced the study of the behavior near T of the solutions of problem (5.1) to the study of the possible stabilization of the solutions of the transformed evolution problem (5.4). In fact, it can be proved (see below) that the solutions of the latter problem stabilize towards the solutions of the associated stationary problem, which is the elliptic problem

$$-\Delta(S^m) = \frac{1}{1-m}S \quad \text{in } \Omega, \qquad S(x) = 0 \quad \text{for } x \in \partial\Omega, \tag{5.6}$$

where $m_s < m < 1$ and Ω are as before.

We will prove that the solutions of problem (5.4) have as ω-limits nontrivial solutions of problem (5.6), and also that the convergence takes place in the weighted uniform sense that we will explain next. Every solution S to the elliptic problem produces a separable solution $\mathcal{U}$ of the original FDE of the form

$$\mathcal{U}(t,x) = S(x)(T-t)^{1/(1-m)} \tag{5.7}$$

which corresponds to the initial datum $\mathcal{U}(0,x) = T^{1/(1-m)}S(x)$. In this context, we can fix T at will, and we will write $\mathcal{U}_T$ for definiteness.

The elliptic problem. The question of existence, regularity, and uniqueness for the Dirichlet elliptic problem is well understood in its basic features, in the range of parameters under consideration.

Existence of positive classical solutions. If $0 < m < 1$ for $d \le 2$ or if $(d-2)/(d+2) = m_s < m < 1$ for $d \ge 3$, then there exist positive classical solutions to equation (5.6) (see, e.g., [13] and references quoted therein, and also [7]).

Uniqueness. In the supercritical case $m > m_s$ that we consider, *the geometry of Ω plays a role in the uniqueness problem*. Indeed, if $d = 1$ or if $d \ge 2$ and Ω is a ball, then the solution is unique (see [14]). Moreover, if $d \ge 2$ and Ω is an annulus, then the solution is unique in the class of positive radial solutions (see [15]). However, there are cases in which the solution is not unique, see; for example, [16, 15].

Regularity and boundary behavior. Since the solutions of problem (5.6) are stationary solutions of problem (5.1), estimates (4.2) give us the following estimates for the behavior of S:

$$\lambda d(x)^{1/m} \le S(x) \le \Lambda d(x)^{1/m}. \tag{5.8}$$

Dynamical system approach: ω-limits. For convenience of the reader, we introduce now some basic ideas from the dynamical system approach. Basically, this approach consists of viewing the solution as an orbit in a functional space and considering the points to which it accumulates as time goes to infinity. We suggest to the interested reader the books [17, 18] for this approach. Note that the approach is applied to the rescaled solutions that have nontrivial asymptotics.

Definition 5.1. The positive *semiorbit* of a solution $w(\tau,x)$ starting at time t_0 if the family

$$\gamma(w;\tau_0) = \{w(\tau) : \tau \ge \tau_0\}, \tag{5.9}$$

where $w(\tau) = w(\tau,\cdot)$ is viewed as an element of a suitable space X of functions in Ω.

Hopefully, X will be a Banach space or a closed convex subset thereof. With the previous estimates, the semiorbit is a relatively compact subset of L^p, with $1 \le p \le \infty$, which can be taken as X, since the semiorbit is uniformly bounded in L^∞. In any case, for every sequence τ_j, there is a subsequence along which

$$w(\tau_{j_k}) \longrightarrow f \quad \text{in } L^p \text{strong, with } p \in [1,\infty]. \tag{5.10}$$

Definition 5.2. The set of all possible limits of a semiorbit along sequences $\tau_j \to \infty$ is called *ω-limit* of the orbit

$$\mathcal{L}(w) = \{f \in L^p : \exists \tau_j \longrightarrow \infty,\ w(\tau_j) \longrightarrow f \text{ in } L^p \text{ strong}\}. \tag{5.11}$$

An alternative way of writing this definition is

$$\mathcal{L}(w) = \bigcap_{\tau>0} \overline{\bigcup_{\tau \ge \tau_0} \gamma(w;\tau)}, \tag{5.12}$$

where the overline denotes the closure.

It is well known that the ω-limit is a closed and connected set in X. We now revisit the well-known result by Berryman and Holland [13] who proved convergence in $W^{1,2}$ by similar methods, based on Lyapunov functional techniques. Our first result is a little improvement in the sense that in [13] the authors do not prove uniform convergence for dimensions higher than 1.

Theorem 5.3 (uniform convergence to the ω-limit) [4]. *The ω-limit set $\mathcal{L}(u)$ of a rescaled solution of the parabolic problem (5.1) is contained in the set $\mathcal{S}$ of a solution of the elliptic problem (5.6) and the convergence takes place uniformly in Ω as $t \to T^-$.*

Remark 5.4. In case the solution of the elliptic problem is unique, the ω-limit consists of a single point $\mathcal{L}(w) = S$ given by such unique solution. In this case, the convergence is unconditioned (i.e., as $t \to T$), since along any sequence $t_j \to T$ (i.e., $\tau_j \to \infty$), we have convergence to the same point $\mathcal{L}(w) = S$, in all $L^p(\Omega)$ spaces with $p \in [1,\infty]$.

Relative error convergence. We are now ready to address the main issue of this section, that is, the relative error convergence estimates (REC) which are nothing else but uniform estimates as $t \to T^-$ of the quotient of the solution to the FDE u divided by a separable solution $\mathcal{U}_T$ to the same problem, T being the finite extinction time. The formulation of the result depends on whether the elliptic problem has multiple solutions so that the ω-limit set $\mathcal{L}(w)$ of Theorem 5.3 may consist of many points, or the solution to this problem is unique. The general result is as follows.

Theorem 5.5. *Let u be the solution to the problem (5.1). Then,*

$$\liminf_{t\to T^-} \inf_{S\in\mathcal{L}} \left\| \frac{u(t,\cdot)}{S(\cdot)(T-t)^{1/(1-m)}} - 1 \right\|_{L^\infty(\Omega)} = 0, \tag{5.13}$$

where S is a point of the ω-limit set $\mathcal{L}$, included in the set $\mathcal{S}$ of positive classical solution to the Dirichlet elliptic problem (5.6).

This type of convergence is what we call *uniform relative-error convergence* (REC for short), and it is our main contribution to the subject of fine asymptotics. To understand better the meaning of this terminology, it will be convenient to introduce the weighted distance to the set $\mathcal{S}$

$$d_\infty(f,\mathcal{S}) = \inf_{S\in\mathcal{S}} \left\| \frac{f(\cdot)}{S(\cdot)} - 1 \right\|_{L^\infty(\Omega)}. \tag{5.14}$$

This peculiar distance (which gives a topology strictly finer than the standard L^∞ norm) is zero if and only if f is a point of $\mathcal{S}$. Theorem 5.5 says that the relative distance between the trajectory $f(t) = u(t)(T-t)^{-1/(1-m)}$ and the ω-limit set $\mathcal{S}$,

$$d_\infty\left(\frac{u(t)}{(T-t)^{1/(1-m)}},\mathcal{S}\right) \longrightarrow 0, \quad \text{as } t \longrightarrow T, \tag{5.15}$$

is going to zero uniformly in space variables as $t \to T$. Loosely speaking, taking into account the behavior of both u and S near the boundary, what we say is that, if $d(x)$ denotes distance to the boundary, then

$$\frac{u(t,x) - S(x)(T-t)^{1/(1-m)}}{d(x)(T-t)^{1/(1-m)}} \longrightarrow 0 \tag{5.16}$$

converges to zero uniformly in $x \in \Omega$ as $t \to T$. We also state a particular case of the above theorem, the case where the Dirichlet elliptic problem (5.6) has a unique positive classical solution $S(x)$. In that case, the result takes the simpler form.

Theorem 5.6. *Let u be the solution to problem (5.1). Then,*

$$\lim_{t\to T^-} \left\| \frac{u(t,\cdot)}{\mathcal{U}(t,\cdot)} - 1 \right\|_{L^\infty(\Omega)} = 0, \tag{5.17}$$

where $\mathcal{U}$ is the separable solution (5.7) of the form

$$\mathcal{U}(t,x) = S(x)(T-t)^{1/(1-m)}. \tag{5.18}$$

Let us now choose a parametrization of the set $\mathcal{S}$ of solutions to the elliptic problem (5.6), $\mathcal{S} = \{S_\alpha\}_{\alpha\in A}$. Then, $\mathcal{L}(u) \subset \mathcal{S}$ and both sets are possibly not equal. $\mathcal{L}$ inherits the parametrization of $\mathcal{S}$, thus, for any u solution to the problem (5.1), there exists an $A' = A'(u) \subset A$ such that

$$\mathcal{L}(u) = \{S_\alpha\}_{\alpha\in A'}. \tag{5.19}$$

It is worth noting that when the ω-limit consists of one point, that is, when the elliptic problem (5.6) possesses a unique solution, things are simpler and the parametrization below is trivial, that is, the set $A = A' = \{\alpha_1\}$, thus we will omit the subindexes α when no confusion is feared.

We now state a different version of the REC theorem in the general case in which the elliptic problem has multiple solutions, so that the ω-limit set $\mathcal{L}(w)$ of Theorem 5.3 may consist of many points.

Corollary 5.7. *With the same hypothesis as in Theorem 5.5, for any* $\varepsilon > 0$ *there exist* $t_\varepsilon > 0$ *and a function* $\alpha(t) \in A'$, *defined for any* $t_\varepsilon < t < T$ *such that*

$$\left\| \frac{u(t)}{S_{\alpha(t)}} - 1 \right\|_\infty \leq \varepsilon, \quad \text{for any } t_\varepsilon < t < T. \tag{5.20}$$

This corollary is important since it allows to prove elliptic Harnack inequality near the extinction time, also in the case when the ω-limit set consists of many points, as we will see in a subsequent section. This will show also that the regularity properties of the solution are somehow independent of the exact profile of the solution close to the extinction time.

Comments. The above results improve on the celebrated result of Berryman and Holland [13], where it was shown that the asymptotic profile for the solution to (5.1) is given by the separable solution $\mathcal{U}$, but convergence was proved only for some special classes of initial data and in some Sobolev spaces; that convergence in general does not imply uniform convergence up to the boundary. Our result of convergence in relative error is stronger than uniform convergence because of the fine behavior near the boundary; moreover, it easily implies elliptic Harnack inequalities near finite extinction time T.

It is also worth noticing that the convergence result can be viewed as a concrete S-theorem, in the spirit of [17], applied to the problem under consideration. The proof borrows the main lines of the proofs of the same result for the PME as done, for example, in [2].

Let us also note that the convergence in relative error cannot be true for obvious reasons when the profiles have moving free boundaries, like in the porous medium case or its rescaled version (nonlinear Fokker-Planck equation), and the problem is posed in free space. This is due to the fact that the interfaces do not match exactly, so that the quotient $u/\mathcal{U}$ may be infinite. As a final note on this issue, let us point out that our REC result shows convergence in time to the ω-limit, but the question of establishing precise rates is not investigated. This is a natural further question that deserves attention.

6. Harnack inequalities

We finally come to one of the most important aims of this paper. We want to show how an intelligent combination of direct and reverse smoothing effects implies easily an optimal Harnack inequality, whose form changes in time.

As a precedent, DiBenedetto and Kwong proved an intrinsic Harnack inequality (see [19, Theorem 2.1]): *there exist constants* $0 < \delta < 1$ *and* $C > 1$ *depending on* d *and* m *such that for every point* $P_0 = (t_0, x_0) \in Q_T$, $Q_T = (0,T) \times \Omega$,

$$\inf_{x \in B_R} u(t_0 + \theta, x) \geq C u(t_0, x_0) \tag{6.1}$$

provided $u(t_0, x_0)$ *is strictly positive and*

$$(t_0 - \tau, t_0 + \tau) \times B_R(x_0) \subset Q_T, \quad \tau = u(t_0, x_0)^{1-m} R^2. \tag{6.2}$$

The constant $\theta = \delta\tau$ depends on the positive value of u at P_0. It is a local property and thus it holds both for the case of the whole space and for the domain case. Our local positivity results, Theorems 2.1 and 2.5, support quantitatively the above intrinsic Harnack inequality, proving its validity in another aspect.

We will present intrinsic and elliptic forms of the Harnack inequality. We can say that the intrinsic Harnack inequality, which compares values of the solution in different times, is the only Harnack possible for small times, while for intermediate times, the elliptic Harnack inequality seems to be the best one: it is stronger, since it easily implies the Intrinsic.

6.1. Harnack inequalities for the FDE on $\mathbb{R}^d$. We now show that the positivity result implies a full local Harnack inequality on the whole Euclidean space. We will see that once again, the critical time

$$t_c = C_{m,d} M_R^{(1-m)} R^{1/\vartheta} \tag{6.3}$$

plays a role, indeed, the form of the Harnack inequality changes when dealing with times smaller or larger than t_c. First we deal with the case of large times, namely, $t > t_c$: we will consider $u_0 \in L^1(\mathbb{R}^d)$, $u_0 \geq 0$ and we let

$$M_\infty = \int_{\mathbb{R}^d} u_0(x)dx, \qquad M_R = \int_{B_R} u_0(x)dx \tag{6.4}$$

for some $R > 0$, $x_0 \in \mathbb{R}^d$.

Theorem 6.1 (elliptic Harnack inequality). *Let $u(t,x)$ satisfy the same hypothesis as Theorem 2.1. If moreover $u_0 \in L^1(\mathbb{R}^d)$, there exists a positive constant $\mathcal{H}$, depending only on m and d on the ratio M_R/M_∞, such that for any $t \geq t_c(M_R,R)$,*

$$\sup_{x\in B_R} u(t,x) \leq \mathcal{H} \inf_{x\in B_R} u(t,x). \tag{6.5}$$

If moreover u_0 is supported in B_R, then the constant $\mathcal{H}$ is universal and depends only on m and d.

Proof. First we remark that the exact expression for t_c is given in Theorem 2.1. The well known smoothing effect can be rewritten in an equivalent form

$$\sup_{x\in B_R} u(t,x) \leq C_1 M_\infty^{2\vartheta} t^{-d\vartheta} = C_1 \left[\frac{M_\infty}{M_R}\right]^{2\vartheta} M_R^{2\vartheta} t^{-d\vartheta}. \tag{6.6}$$

Using now the reverse smoothing effect when $t > t_c$, we get

$$\inf_{x\in B_R} u(t,x) \geq K M_R^{2\vartheta} t^{-d\vartheta} \geq K C_1^{-1} \left[\frac{M_R}{M_\infty}\right]^{2\vartheta} \sup_{x\in B_R} u(t,x), \tag{6.7}$$

that is, (6.5) with $\mathcal{H} = K^{-1} C_1 [M_\infty/M_R]^{2\vartheta}$. This concludes the proof. $\square$

The above elliptic Harnack inequality holds for times larger than the critical time t_c and it strongly depends on the sharp lower bounds of Theorem 2.1, for $t > t_c$. In the case of small times, the lower bound changes its shape and we recover the intrinsic Harnack inequality of [19] by different methods and with a little improvement: the intrinsic Harnack inequality (6.18) holds for any positive time, namely, we have the following

Theorem 6.2 (intrinsic Harnack inequality). *Let $u(t,x)$ satisfy the same hypothesis as Theorem 2.1, and let $R > 0$. There exist constants $0 < \delta < 1$ and $C > 1$ depending on m and d such that for every $0 < t_0 \le t_c$,*

$$\inf_{x \in B_R} u(t_0 + \theta, x) \ge C u(t_0, x_0), \tag{6.8}$$

where

$$\theta = \delta\tau, \quad \tau = u(t_0, x_0)^{1-m} R^2 > 0. \tag{6.9}$$

Proof. Let us consider the lower bound of Theorem 2.1 in the case $0 < t < t_c$:

$$u(t,x) \ge \frac{t^{1/(1-m)}}{t_c^{*\,1/(1-m)}} u(t_c^*, x) \ge \frac{t^{1/(1-m)}}{t_c^{*\,1/(1-m)}} \frac{M_R^{2\vartheta}}{t_c^{*,d\vartheta}} = (C_3 C_2)^{2/d(1-m)} \frac{t^{1/(1-m)}}{R^{2/(1-m)}} \tag{6.10}$$

since

$$t_c^* = (C_3 C_2)^{1/(d\vartheta)} M_R^{1-m} R^{1/\vartheta}, \quad \vartheta = \frac{1}{(2 - d(1-m))}. \tag{6.11}$$

We remark that we can take $C_3 C_2 > 1$, as remarked in the proof of Theorem 2.2, and such intrinsic lower bound is independent of M_R, and thus of u_0. Indeed, the choices

$$t = t_0 + \theta, \quad \theta = \delta\tau, \quad \tau = u(t_0, x_0)^{1-m} R^2 > 0, \tag{6.12}$$

where $\delta \in (0,1)$ can be chosen in such a way that $t_0 + \delta\tau \le t_c^*$ and the positivity of τ is guaranteed by the positivity estimates. We then get

$$\begin{aligned} u(t_0 + \theta, x) &\ge (C_3 C_2)^{2/d(1-m)} \frac{(t_0 + \theta)^{1/(1-m)}}{R^{2/(1-m)}} \\ &> (C_3 C_2)^{2/d(1-m)} \frac{\theta^{1/(1-m)}}{R^{2/(1-m)}} \\ &= (C_3 C_2)^{2/d(1-m)} \delta^{1/(1-m)} u(t_0, x_0). \end{aligned} \tag{6.13}$$

This concludes the proof. □

Remark 6.3. We now show how the elliptic Harnack inequality implies the intrinsic Harnack inequality when $t > t_c$. From the above proofs, one can prove a stronger version of the elliptic Harnack inequality. Indeed,

$$\begin{aligned}\sup_{x\in B_R} u(t-\omega,x) \le \sup_{x\in\Omega} u(t-\omega,x) &\le C_1 \frac{M_\infty^{2\vartheta}}{(t-\omega)^{d\vartheta}} \\ &= \frac{C_1}{K}\left[\frac{M_\infty}{M_R}\right]^{2\vartheta}\left[\frac{t}{t-\omega}\right]^{d\vartheta} M_R^{2\vartheta} t^{-d\vartheta} \\ &\le \frac{C_1}{K}\left[\frac{M_\infty}{M_R}\right]^{2\vartheta}\left[\frac{t}{t-\omega}\right]^{d\vartheta} \inf_{x\in B_R} u(t,x) \\ &= \frac{C_1}{K}\left[\frac{1}{1-\sigma}\right]^{d\vartheta}\left[\frac{M_\infty}{M_R}\right]^{2\vartheta} \inf_{x\in B_R} u(t,x)\end{aligned} \tag{6.14}$$

since we took $\omega = \sigma t$, with $\sigma \in [0,1)$. Thus, it can be rewritten as

$$\inf_{x\in B_R} u(t+\omega,x) \ge \mathcal{H} \sup_{x\in B_R} u(t,x) \tag{6.15}$$

and in particular, we can choose $\sigma \in [0,1)$ such that $\omega = \sigma t = \delta u(t_0,x_0)^{1-m}R^2 = \theta$, as in the above intrinsic Harnack inequality (6.18).

Finally, we remark that the Harnack inequality (6.15) holds for any $\sigma \in [0,1)$, thus it compares the infimum and supremum of u at different times, and by the monotonicity of the L^∞ norm, we can compare the supremum and infimum of u at any two different times $t_1, t_2 \in (0,\infty)$. This inequality is sometimes called the backward Harnack inequality, in the case of the heat equation.

Panorama. At this point, it is convenient to make a summary for the Cauchy problem in $\mathbb{R}^d$. The role of the critical time t_c is to split the time axis into two parts, showing the range of validity of different Harnack inequalities and behaviors of the solutions.

(i) *Small times,* $0 < t < t_c$: intrinsic Harnack inequalities [19] and Theorem 6.2. The validity for all positive times is guaranteed by our local positivity result. The constants do not depend on u_0. There is Hölder continuity, implied by Harnack inequalities.

(ii) *Large times, $t > t_c$*: elliptic Harnack inequalities. The constant may depend on the initial datum. Elliptic Harnack inequalities imply intrinsic Harnack inequalities and Hölder continuity.

(iii) *All positive times,* for any $\varepsilon > 0$ and for all $t > \varepsilon$: global Harnack principle that implies the convergence in relative error.

(iv) *Asymptotics, $t \to \infty$*: uniform convergence in relative error.

6.2. Harnack inequalities for FDE on a domain. In this section, we analyze the case of the mixed Cauchy-Dirichlet problem on a bounded domain. We find an extra difficulty, due to the presence of the finite extinction time. We start by recalling the result of [7], where the following rather peculiar property of the solutions of problem (2.28) is found as a consequence of the global Harnack principle on domains,

$$u(t_0,x_0) \ge \gamma_0 \sup_{|x-x_0|<R} u(t_0,x), \tag{6.16}$$

valid for an $R > 0$, so small that the box

$$(t_0 - \tau, t_0 + \tau) \times B_R(x_0) \subset Q_T, \quad \tau = u(t_0, x_0)^{1-m} R^2, \tag{6.17}$$

but again the box depends on the positivity value of u in the point (t_0, x_0). It resembles our elliptic Harnack inequality, but again it has to be supported by a positivity result to hold in full generality.

Analogously to what we did before, we can prove the elliptic Harnack inequality for intermediate times in the case of bounded domains. Our main result takes the form of a precise lower estimate for the values in question, and will thus ensure that such intrinsic Harnack inequality will hold for all positive times not too close to the extinction time. We also prove an elliptic Harnack inequality for intermediate times, that is, for $t \in I = [t_c, T_c]$ with $0 < t_c < T_c < T$, where t_c and T_c are computed in terms of the initial datum, which follows from our sharp result on positivity. Note that this allows to calculate explicitly all the constants. As before, we can say that our positivity results somehow "support" the results of [7], in the sense that we ensure positivity in a quantitative way, and a posteriori their result holds true for times not too close to the extinction time.

In this section, we prove intrinsic and elliptic Harnack inequalities, in the whole interval $(0, T)$, in analogy to what has been done in the whole space. We point out that for times close to the extinction time, an elliptic Harnack inequality is still valid, thanks to the accurate asymptotic information given by the relative error estimates, compare Theorems 5.5 and 5.6.

Theorem 6.4 (intrinsic Harnack inequality). *Let $u(t,x)$ and $R > 0$ satisfy the same hypothesis as Theorem 2.1. There exist constants $0 < \delta < 1$ and $C > 1$ depending on m and d such that for every $0 < t_0 \le t_c$,*

$$\inf_{x \in B_R} u(t_0 + \theta, x) \ge C u(t_0, x_0), \tag{6.18}$$

where

$$\theta = \delta\tau, \quad \tau = u(t_0, x_0)^{1-m} R^2 > 0. \tag{6.19}$$

Proof. The proof is formally the same as in Theorem 6.2. □

Theorem 6.5 (elliptic Harnack inequality for intermediate times). *Let $u(t,x)$ and $R > 0$ satisfy the same hypothesis as Theorem 2.5. If moreover $u_0 \in L^1(\Omega)$, then there exists a positive constant $\mathcal{H}$, depending only on m, d, and on the ratio*

$$\frac{M_\Omega}{M_{R_0}} = \frac{\int_\Omega u_0(x)dx}{\int_{B_{R_0}} u_0(x)dx}, \tag{6.20}$$

such that for any $t_c^* < t < T_c < T$,

$$\sup_{x \in B_{R_0}} u(t,x) \le \mathcal{H} \inf_{x \in B_{R_0}} u(t,x). \tag{6.21}$$

If moreover u_0 is supported in B_{R_0}, then the constant $\mathcal{H}$ is universal and depends only on m and d.

Proof. The proof is formally the same as in Theorem 6.1, since the upper bounds are the same, (once one replaces M_∞ with M_Ω) and uses (2.30) when $t_c^* < t < T_c$. □

At this point, it is natural to ask if there holds a Harnack inequality for times close to the extinction time, and of which kind. The answer is that there holds an elliptic Harnack inequality, but the proof is different from the above ones, since it relies on the fine asymptotic behavior close to the extinction time. We already discussed the question of the asymptotic profile near the extinction time, and we showed the convergence in relative error. This very strong convergence to the solution obtained by separation of variables, also transports some other regularity properties, from the elliptic problem to the parabolic one, namely, the validity of the following Harnack inequality for the elliptic problem (5.6) implies the validity of an elliptic Harnack inequality for the solution to the parabolic problem (5.1).

Theorem 6.6 (Harnack inequalities for the elliptic problem (5.6)). *Let $0 \le S \in W_0^{1,2}(\Omega)$ be a solution to the elliptic Dirichlet problem (5.6). Then, if the ball $B_{4R}(x_0) \subset \Omega$,*

$$\sup_{B_R(x_0)} S \le H \inf_{B_R(x_0)} S, \tag{6.22}$$

where H is a positive constant depending on m, d, and R.

Proof. See, for example, [20]. □

Also the range of the parameter m can be enlarged a bit, namely, we will consider

$$\frac{d-2}{d+2} = m_s < m < 1. \tag{6.23}$$

Harnack inequality via relative error estimates. The last part of the paper is devoted to derive an elliptic Harnack inequality for the evolution trajectories near the extinction time, showing that regularity properties of the solution to the parabolic problem are somehow inherited from the elliptic problem via the strong convergence in relative error and are independent from the exact profile near the extinction time. The relative error estimate in the form of Corollary 5.7 implies the elliptic Harnack inequality as an easy corollary.

Theorem 6.7 (elliptic Harnack inequality near extinction time). *Let u be a solution to the problem (5.1). Then, for any $\varepsilon \in (0,1)$, there exists a time $t_\varepsilon \in (0,T)$ such that the following elliptic Harnack inequality holds for any ball $B_{4R} = B_{4R}(x_0) \subset \Omega$ and for any $t \in [t_\varepsilon, T)$:*

$$\sup_{x \in B_R} u(t,x) \le \frac{1+\varepsilon}{1-\varepsilon} H \inf_{x \in B_R} u(t,x), \tag{6.24}$$

where H is the positive constant in the Harnack inequality for problem (5.6).

Proof. By the relative error estimate of Theorem 5.5, we easily obtain that for any $\varepsilon \in (0,1)$, there exists a time $t_\varepsilon \in (0,T)$ such that for any $t \ge t_\varepsilon$,

$$\begin{aligned}(1-\varepsilon)S(x)(T-t)^{1/(1-m)} &= \mathcal{U}(t,x)(1-\varepsilon) \le u(t,x)\\ &\le (1+\varepsilon)\mathcal{U}(t,x) = (1+\varepsilon)S(x)(T-t)^{1/(1-m)},\end{aligned} \tag{6.25}$$

where $S = S_{\alpha(t)} \in \mathcal{S}$, that is, a suitable solution to the elliptic problem (5.6). Thus we have

$$\begin{aligned}\sup_{x\in B_R} u(t,x) &\le (1+\varepsilon)(T-t)^{1/(1-m)} \sup_{x\in B_R} S(x)\\ &\le (1+\varepsilon)(T-t)^{1/(1-m)} H \inf_{x\in B_R} S(x)\\ &\le \frac{1+\varepsilon}{1-\varepsilon} H \inf_{x\in B_R} u(t,x),\end{aligned} \tag{6.26}$$

where in the second step we used the Harnack inequality (6.22), valid for any solution $S(x)$ to the elliptic problem (5.6), with constant H which depends only on m, d, and R. The proof is now complete. □

Panorama. At this point, it is convenient to make a panorama for the mixed Cauchy-Dirichlet problem on a domain $\Omega \subset \mathbb{R}^d$. The rule of the critical time t_c is again to split the time axis, showing the range of validity of different Harnack inequalities and behaviors of the solutions. Another critical time T_c appears, with $t_c \le T_c < T$, between t_c and the extinction time T, thus the time interval $(0,T)$ is split into three parts.

(i) *Small times,* $0 < t < t_c$: intrinsic Harnack inequalities [19] and Theorem 6.4. The validity for all positive times is guaranteed by our local positivity result. The constants do not depend on u_0. There is, Hölder continuity, implied by Harnack inequalities.

(ii) *Intermediate times,* $t_c < t < T_c$: elliptic Harnack inequalities. The constant may depend on the initial datum. Elliptic Harnack inequalities imply intrinsic Harnack inequalities and Hölder continuity.

(iii) *Near extinction time,* $T_c < t < T$: elliptic Harnack inequalities, as consequence of the Convergence in relative error. The constants do not depend on the initial datum. Elliptic Harnack inequalities imply intrinsic Harnack inequalities and Hölder continuity.

(iv) *All positive times,* for any $\varepsilon > 0$ and for all $t > \varepsilon$: global Harnack principle, [7, Theorem 4.1]

(v) *Asymptotics,* $t \to \infty$: uniform convergence in relative error, Theorems 5.5 and 5.6.

Appendix

Here we prove the reflection principle of Aleksandrov in a slightly different form, more useful to our purposes. Other forms of the same principle in different settings can be found, for example, in [17, Proposition 2.24, page 51] or in [21, Lemma 2.2]. We also notice that it is sufficient to consider the Dirichlet problem on a suitable ball in order to achieve the stated positivity results, namely, consider

$$\begin{gathered} u_t = \Delta(u^m) \quad \text{in } (0,T)\times B_{4R}(0),\\ u(0,x) = u_0(x) \quad \text{in } B_{4R}(0),\\ u(t,x) = 0 \quad \text{for } 0<t<T,\ x\in\partial B_{4R}(0)\end{gathered} \tag{A.1}$$

with $\operatorname{supp}(u_0) \subset B_R(0) \subset B_{4R}(0) \subset \Omega$, where $T > 0$ is the finite extinction time. Let u_B denote the solution to the above problem (A.1), while let u_Ω denote the solution to the problem (2.28). It is clear then that u_B is a subsolution to the problem (2.28) so that $u_B \le u_\Omega$ and thus local positivity result for u_B will imply local positivity result for u_Ω. Note however that since the solutions have extinction in finite time and u_B disappears before u_Ω, we are renouncing to obtain estimates near the extinction time of u_Ω.

Proposition A.1 (local Aleksandrov's reflection principle). *Let $B_{\lambda R}(x_0) \subset \mathbb{R}^d$ be an open ball with center in $x_0 \in \mathbb{R}^d$ of radius λR with $R > 0$ and $\lambda > 2$. Let u be a solution to the problem*

$$\begin{aligned} u_t = \Delta(u^m) \quad & \text{in } (0,+\infty) \times B_{\lambda R}(x_0), \\ u(0,x) = u_0(x) \quad & \text{in } B_{\lambda R}(x_0), \\ u(t,x) = 0 \quad & \text{for } t > 0,\ x \in \partial B_{\lambda R}(x_0) \end{aligned} \tag{A.2}$$

with $\operatorname{supp}(u_0) \subset B_R(x_0)$. *Then, for any $t > 0$, one has*

$$u(t,x_0) \ge u(t,x_2) \tag{A.3}$$

for any $t > 0$ and for any $x_2 \in D_{\lambda,R}(x_0) = B_{\lambda R}(x_0) \setminus B_{2R}(x_0)$. Hence,

$$u(t,x_0) \ge \left|D_{\lambda,R}(x_0)\right|^{-1} \int_{D_{\lambda,R}(x_0)} u(t,x)dx = ⨍_{D_{\lambda,R}(x_0)} u(t,x)dx. \tag{A.4}$$

Proof. A detailed proof can be found in the appendix of [5]. □

Remark A.2. Formula (A.4) can be viewed as a *local mean value inequality*, it has been derived here from the Aleksandrov principle, but it is interesting by itself and moreover it is independent of the range of m: one can apply the same argument to any $m > 0$. Loosely speaking, formula (A.4) states that for the solutions of diffusion equations, their average on an annulus at a time $t > 0$ is smaller than their value taken at the same time and in the center of the ball where mass was concentrated at the beginning. This property is crucial in the proof of the positivity estimates and, a posteriori, of the Harnack inequality. We used this mean value inequality (A.4) in a slightly different form

$$\int_{B_{R+r}(x_0)\setminus B_R(x_0)} u(t,x)dx \le A_d r^d u(t,x_0) \tag{A.5}$$

with $r \ge \mu R$, $\mu > 1$, and a suitable positive constant $A_{d,\mu}$. This inequality can easily be obtained from (A.4), noticing that for $r \ge \mu R$, one has

$$(R+r)^d \le c_1(R^d + r^d) \tag{A.6}$$

for a constant c_1, that depends on d and $\mu > 1$. Then, we get $(R+r)^d - R^d \le (c_1 - 1)R^d + r^d \le c_2 r^d$, so that

$$\left|B_{2R+r}(x_0) \setminus B_{2^b R}(x_0)\right| = \omega_d\left[(R+r)^d - R^d\right] \le A_d r^d \tag{A.7}$$

with $A_d = \omega_d c_2$, where ω_d is the volume of the unit ball in $\mathbb{R}^d$.

Final remarks. Open problems. If we consider solutions to other more general diffusion equations to prove inequality (A.4), we will automatically prove positivity and Harnack inequalities, provided that there is a direct smoothing effect. We point out some directions which are actually under investigation by the authors and collaborators. *The Subcritical case:* we consider the same problems of this paper, that is, local and global positivity and Harnack inequalities, but in the bad range, namely, $0 \le m < m_c$. This range includes also the cases of logarithmic diffusion, for example, $m \to 0$. *The coefficient case:* we want to prove positivity and Harnack inequalities for solution to the FDE both on $\mathbb{R}^d$, problem (2.1) and on domains, problem (2.28), in the more general case, when we replace the Laplacian with an elliptic operator with measurable coefficient. *Riemannian Manifolds:* we consider the above mentioned problems on a Riemannian manifold, in which case the Laplacian is meant as the Laplace-Beltrami operator. The strategy to prove the problem would be the same: from (A.4), we get the local positivity result, or reverse smoothing effect, that we combine with the well-known (direct) smoothing effects.

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