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Research Article

Positive Solutions for Two-Point Semipositone Right Focal Eigenvalue Problem

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Krasnoselskii's fixed-point theorem in a cone is used to discuss the existence of positive solutions to semipositone right focal eigenvalue problems $(-1)^{n-p}u^{(n)}(t) = \lambda f(t,u(t), u'(t),\dots,u^{(p-1)}(t))$, $u^{(i)}(0) = 0$, $0 \le i \le p-1$, $u^{(i)}(1) = 0$, $p \le i \le n-1$, where $n \ge 2$, $1 \le p \le n-1$ is fixed, $f : [0,1] \times [0,\infty)^p \to (-\infty,\infty)$ is continuous with $f(t,u_1,u_2,\dots,u_p) \ge -M$ for some positive constant M.

1. Introduction

In recent years, many papers have discussed the existence of positive solutions of right focal boundary value problems, see [1–7]. In 2003, Ma [5] established existence results of positive solutions for the fourth-order semipositone boundary value problems

$$\begin{gathered} u^{(4)}(x) = \lambda f(x,u(x),u'(x)), \\ u(0) = u'(0) = u''(1) = u'''(1) = 0. \end{gathered} \tag{1.1}$$

Motivated by Agarwal and Wong [8] and Ma [5], the purpose of this article is to generalize and complement Ma's work to nth-order right focal eigenvalue problems:

$$(-1)^{n-p}u^{(n)}(t) = \lambda f(t,u(t),u'(t),\dots,u^{(p-1)}(t)) \tag{1.2}$$

with boundary conditions

$$\begin{aligned} u^{(i)}(0) &= 0, \quad 0 \le i \le p-1, \\ u^{(i)}(1) &= 0, \quad p \le i \le n-1, \end{aligned} \tag{1.3}$$

where $n \geq 2$, $1 \leq p \leq n-1$ is fixed, $f : [0,1] \times [0,\infty)^p \to (-\infty,\infty)$ is continuous with $f(t,u_1,u_2,\dots,u_p) \geq -M$ for some positive constant M.

We say that $u(t)$ is positive solution of BVP (1.2), (1.3) if $u(t) \in C^n[0,1]$ is solution of BVP (1.2), (1.3) and $u^{(i)}(t) > 0$, $t \in (0,1)$, $i = 0,1,\dots,p-1$.

For other related works with focal boundary value problem, we refer to recent contributions of Agarwal [1], Agarwal et al. [2], Boey and Wong [3], He and Ge [4], and Wong and Agarwal [6, 7].

The outline of the paper is as follows: in Section 2, we will present some lemmas which will be used in the proof of main results. In Section 3, by using Krasnoselskii's fixed-point theorem in a cone, we offer criteria for the existence of a positive solution and two positive solutions of BVP (1.2), (1.3).

2. Some preliminaries

In order to abbreviate our discussion, we use C_i $(i = 1,2,3,4,5)$ to denote the following conditions:

(C_1) $f(t,u_1,u_2,\dots,u_p) \in C([0,1] \times [0,\infty)^p,(-\infty,\infty))$ is continuous with $f(t,u_1,u_2,\dots,u_p) \geq -M$ for some positive constant M;

(C_2) there exists constant $0 < \varepsilon < 1$ such that

$$\lim_{u_1,u_2,\dots,u_p \to \infty} \min_{t\in[\varepsilon,1]} \frac{f(t,u_1,u_2,\dots,u_p)+M}{u_p} = \infty; \tag{2.1}$$

(C_3) there exists constant $\alpha > 0$ such that

$$\lim_{u_p \to 0^+} \min_{(t,u_1,u_2,\dots,u_{p-1})\in[0,1]\times[0,\alpha]^{p-1}} \frac{f(t,u_1,u_2,\dots,u_p)}{u_p} = \infty; \tag{2.2}$$

(C_4) there exists constant $\alpha > 0$ such that

$$f(t,u_1,u_2,\dots,u_{p-1},0) > 0, \quad (t,u_1,u_2,\dots,u_{p-1}) \in [0,1] \times [0,\alpha]^{p-1}; \tag{2.3}$$

(C_5) $h(s) = s^{n-p}/(n-p)!$, $D_1 = (\int_0^1 h(s)ds)^{-1}$, $D_2 = (\int_\varepsilon^1 h(s)ds)^{-1}$, where $0 < \varepsilon < 1$ is constant.

Let $B = \{u \in C^{p-1}[0,1] : u^{(i)}(0) = 0,\ 0 \leq i \leq p-2\}$ with the norm $\|u\| = \sup_{t\in[0,1]} |u^{(p-1)}(t)|$. It is easy to prove that B is a Banach space.

Lemma 2.1. *Let*

$$C \equiv \{u \in B : u^{(p-1)}(t) \geq t\|u\|,\ t \in [0,1]\}. \tag{2.4}$$

Then C is a cone in B and for all $u \in C$,

$$\frac{t^{p-i}\|u\|}{(p-i)!} \leq u^{(i)}(t) \leq \|u\|, \quad t \in [0,1],\ i = 0,1,\dots,p-1. \tag{2.5}$$

Proof. For all $u, v \in C$ and for all $\alpha \geq 0$, $\beta \geq 0$, we have

$$\begin{aligned}(\alpha u(t)+\beta v(t))^{(p-1)} &= \alpha u^{(p-1)}(t)+\beta v^{(p-1)}(t)\\ &\geq \alpha t\|u\|+\beta t\|v\|\\ &\geq t\|\alpha u+\beta v\|,\end{aligned} \tag{2.6}$$

so $\alpha u+\beta v \in C$. In addition, if $u \in C$, $-u \in C$, and $u \neq \theta$ (where θ denotes the zero element of B), then

$$\begin{aligned}u^{(p-1)}(t) \geq t\|u\| \geq 0, \quad t \in [0,1],\\ -u^{(p-1)}(t) \geq t\|u\| \geq 0, \quad t \in [0,1].\end{aligned} \tag{2.7}$$

Thus $u^{(p-1)}(t) = 0$, $t \in [0,1]$. It follows that $\|u\| = 0$, which contradicts the assumption. Hence C is a cone in B.

For all $u \in C$, $0 \leq i \leq p-1$, due to Taylor's formula, we have $\xi \in (0,t)$ such that

$$u^{(i)}(t) = u^{(i)}(0)+u^{(i+1)}(0)t+\cdots+\frac{u^{(p-2)}(0)t^{p-i-2}}{(p-i-2)!}+\frac{u^{(p-1)}(\xi)t^{p-i-1}}{(p-i-1)!}. \tag{2.8}$$

It follows from $u \in C$ that for $i = 0,1,\dots,p-1$,

$$\begin{aligned}\|u\| \geq u^{(i)}(t) &= \frac{u^{(p-1)}(\xi)t^{p-i-1}}{(p-i-1)!}\\ &\geq \frac{t\|u\|t^{p-i-1}}{(p-i-1)!} = \frac{t^{p-i}\|u\|}{(p-i-1)!} \geq \frac{t^{p-i}\|u\|}{(p-i)!}.\end{aligned} \tag{2.9}$$

□

Lemma 2.2 [6]. *Let $K(t,s)$ be Green's function of the differential equation $(-1)^{n-p}u^{(n)}(t) = 0$ subject to the boundary conditions (1.3). Then*

$$K(t,s) = \frac{(-1)^{n-p}}{(n-1)!}\begin{cases}\displaystyle\sum_{i=0}^{p-1}\binom{n-1}{i}t^i(-s)^{n-i-1}, & 0 \leq s \leq t \leq 1,\\ \displaystyle-\sum_{i=p}^{n-1}\binom{n-1}{i}t^i(-s)^{n-i-1}, & 0 \leq t \leq s \leq 1,\end{cases}$$
$$\frac{\partial^i}{\partial t^i}K(t,s) \geq 0, \quad (t,s) \in [0,1]\times[0,1],\ 0 \leq i \leq p. \tag{2.10}$$

Lemma 2.3. *Assume that (C_5) holds. Let $k(t,s)$ be Green's function of the differential equation*

$$(-1)^{n-p}u^{(n-p+1)}(t) = 0 \tag{2.11}$$

subject to the boundary conditions

$$u(0) = 0, \qquad u^{(i)}(1) = 0, \quad 1 \leq i \leq n-p. \tag{2.12}$$

Then

$$th(s) \le k(t,s) \le h(s), \quad (t,s) \in [0,1] \times [0,1]. \tag{2.13}$$

Proof. It is clear that

$$k(t,s) = \frac{\partial^{p-1}}{\partial t^{p-1}} K(t,s) = \frac{1}{(n-p)!} \begin{cases} s^{n-p}, & 0 \le s \le t \le 1, \\ s^{n-p} - (s-t)^{n-p}, & 0 \le t \le s \le 1. \end{cases} \tag{2.14}$$

Obviously,

$$th(s) \le \frac{1}{(n-p)!} s^{n-p} \le h(s), \quad 0 \le s \le t \le 1. \tag{2.15}$$

For $0 \le t \le s \le 1$,

$$\begin{aligned} h(s) &\ge \frac{1}{(n-p)!}\left[s^{n-p} - (s-t)^{n-p}\right] \\ &= \frac{1}{(n-p)!}\left[s - (s-t)\right] \sum_{i=0}^{n-p-1} s^{n-p-1-i}(s-t)^i \\ &\ge \frac{1}{(n-p)!} t s^{n-p-1} \\ &\ge \frac{1}{(n-p)!} t s^{n-p} = th(s). \end{aligned} \tag{2.16}$$

Thus,

$$th(s) \le k(t,s) \le h(s), \quad (t,s) \in [0,1] \times [0,1]. \tag{2.17}$$

□

Lemma 2.4. *The boundary value problem*

$$\begin{aligned} (-1)^{(n-p)} u^{(n)}(t) &= 1, \quad t \in [0,1], \\ u^{(i)}(0) &= 0, \quad 0 \le i \le p-1, \\ u^{(i)}(1) &= 0, \quad p \le i \le n-1, \end{aligned} \tag{2.18}$$

has unique solution $w(t) \in C^n[0,1]$ *and*

$$0 \le w^{(i)}(t) \le \frac{t^{p-i}}{(n-p)!(p-i)!}, \quad t \in [0,1],\ 0 \le i \le p-1. \tag{2.19}$$

Proof. It is clear that the boundary value problem

$$\begin{aligned} (-1)^{(n-p)} u^{(n)}(t) &= 1, \quad t \in [0,1], \\ u^{(i)}(0) &= 0, \quad 0 \le i \le p-1, \\ u^{(i)}(1) &= 0, \quad p \le i \le n-1, \end{aligned} \tag{2.20}$$

has unique solution

$$w(t) = \int_0^1 K(t,s)ds, \tag{2.21}$$

where $K(t,s)$ is as in Lemma 2.2.

Obviously, for $0 \le s \le t \le 1$,

$$\frac{1}{(n-p)!}s^{n-p} \le \frac{ts^{n-p-1}}{(n-p-1)!}. \tag{2.22}$$

For $0 \le t \le s \le 1$,

$$\begin{aligned} \frac{1}{(n-p)!}\left[s^{n-p} - (s-t)^{n-p}\right] &= \frac{1}{(n-p)!}[s-(s-t)] \sum_{i=0}^{n-p-1} s^{n-p-1-i}(s-t)^i \\ &\le (n-p)\frac{ts^{n-p-1}}{(n-p)!} = \frac{ts^{n-p-1}}{(n-p-1)!}. \end{aligned} \tag{2.23}$$

So

$$0 \le k(t,s) \le \frac{ts^{n-p-1}}{(n-p-1)!}, \tag{2.24}$$

where $k(t,s)$ is as in Lemma 2.3. Since $w^{(p-1)}(t) = \int_0^1 k(t,s)ds$, then

$$0 \le w^{(p-1)}(t) = \int_0^1 k(t,s)ds \le \int_0^1 \frac{ts^{n-p-1}}{(n-p-1)!}ds = \frac{t}{(n-p)!}. \tag{2.25}$$

Further, since $w^{(i)}(0) = 0$, $0 \le i \le p-1$, we get

$$0 \le w^{(i)}(t) \le \frac{t^{p-i}}{(n-p)!(p-i)!}, \quad t \in [0,1],\ 0 \le i \le p-1. \tag{2.26}$$

□

Lemma 2.5 [8]. *Let E be a Banach space, and let $C \subset E$ be a cone in E. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $T : C \cap (\overline{\Omega}_2 \setminus \Omega_1) \to C$ be a completely continuous operator such that either*

(i) $\|Tu\| \le \|u\|$, $u \in C \cap \partial\Omega_1$, $\|Tu\| \ge \|u\|$, $u \in C \cap \partial\Omega_2$ *or*

(ii) $\|Tu\| \ge \|u\|$, $u \in C \cap \partial\Omega_1$, $\|Tu\| \le \|u\|$, $u \in C \cap \partial\Omega_2$.

Then, T has a fixed point in $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main results

In this section, by using Lemma 2.5, we offer criteria for the existence of positive solutions for two-point semipositone right focal eigenvalue problem (1.2), (1.3).

Theorem 3.1. *Assume (C_1), (C_2), and (C_5) hold. Then BVP (1.2), (1.3) has at least one positive solution if $\lambda > 0$ is small enough.*

Proof. We consider BVP

$$\begin{gathered}(-1)^{n-p}u^{(n)}(t)=\lambda f^*\left(t,u(t)-\phi(t),\dots,u^{(p-1)}(t)-\phi^{(p-1)}(t)\right),\\ u^{(i)}(0)=0,\quad 0\le i\le p-1,\\ u^{(i)}(1)=0,\quad p\le i\le n-1,\end{gathered}\tag{3.1}$$

where

$$\begin{aligned}&\phi(t)=\lambda Mw(t)\quad (w(t)\text{ is as in Lemma 2.4}),\\ &f^*\left(t,u_1,u_2,\dots,u_p\right)=f\left(t,\rho_1,\rho_2,\dots,\rho_p\right)+M,\end{aligned}\tag{3.2}$$

and for all $i=1,2,\dots,p$,

$$\rho_i=\begin{cases}u_i, & u_i\ge 0;\\ 0, & u_i<0.\end{cases}\tag{3.3}$$

We will prove that (3.1) has a solution $u_1(t)$. Obviously, (3.1) has a solution in C if and only if

$$\begin{aligned}u(t)&=\int_0^1 K(t,s)\lambda f^*\left(s,u(s)-\phi(s),u'(s)-\phi'(s),\dots,u^{(p-1)}(s)-\phi^{(p-1)}(s)\right)ds\\ &:=\left(T_1u\right)(t)\end{aligned}\tag{3.4}$$

or

$$\begin{aligned}u^{(p-1)}(t)&=\int_0^1 k(t,s)\lambda f^*\left(s,u(s)-\phi(s),u'(s)-\phi'(s),\dots,u^{(p-1)}(s)-\phi^{(p-1)}(s)\right)ds\\ &:=\left(T_1u\right)^{(p-1)}(t)\end{aligned}\tag{3.5}$$

has a solution in C. From Lemma 2.3, we know that

$$\begin{aligned}&\left(T_1u\right)^{(p-1)}(t)\\ &\quad=\int_0^1 k(t,s)\lambda f^*\left(s,u(s)-\phi(s),u'(s)-\phi'(s),\dots,u^{(p-1)}(s)-\phi^{(p-1)}(s)\right)ds\\ &\quad\le\int_0^1 h(s)\lambda f^*\left(s,u(s)-\phi(s),u'(s)-\phi'(s),\dots,u^{(p-1)}(s)-\phi^{(p-1)}(s)\right)ds,\end{aligned}\tag{3.6}$$

so

$$\left\|T_1u\right\|\le\int_0^1 h(s)\lambda f^*\left(s,u(s)-\phi(s),u'(s)-\phi'(s),\dots,u^{(p-1)}(s)-\phi^{(p-1)}(s)\right)ds.\tag{3.7}$$

From Lemma 2.3 again,

$$\begin{aligned}
&(T_1u)^{(p-1)}(t)\\
&\quad= \int_0^1 k(t,s)\lambda f^*\left(s,u(s)-\phi(s),u'(s)-\phi'(s),\dots,u^{(p-1)}(s)-\phi^{(p-1)}(s)\right)ds\\
&\quad\geq \int_0^1 th(s)\lambda f^*\left(s,u(s)-\phi(s),u'(s)-\phi'(s),\dots,u^{(p-1)}(s)-\phi^{(p-1)}(s)\right)ds\\
&\quad= t\int_0^1 h(s)\lambda f^*\left(s,u(s)-\phi(s),u'(s)-\phi'(s),\dots,u^{(p-1)}(s)-\phi^{(p-1)}(s)\right)ds\\
&\quad\geq t\|T_1u\|.
\end{aligned} \tag{3.8}$$

Hence, $T_1(C)\subseteq C$. Further, it is clear that $T_1 : C\to C$ is completely continuous.

Let

$$\lambda\in(0,\Lambda) \tag{3.9}$$

be fixed, where

$$\Lambda=\min\left\{\frac{2D_1}{M_1},\frac{(n-p)!}{M}\right\}, \tag{3.10}$$

$$M_1=\max\left\{f^*(t,u_1,u_2,\dots,u_p) : (t,u_1,u_2,\dots,u_p)\in[0,1]\times[0,2]^p\right\}. \tag{3.11}$$

We separate the rest of the proof into the following two steps.

Step 1. Let

$$\Omega_1=\{u\in B : \|u\|<2\}. \tag{3.12}$$

From the definition of f^*, we know

$$\begin{aligned}
M_1&=\max\left\{f^*(t,u_1,u_2,\dots,u_p) : (t,u_1,u_2,\dots,u_p)\in[0,1]\times[0,2]^p\right\}\\
&=\max\left\{f^*(t,u_1,u_2,\dots,u_p) : (t,u_1,u_2,\dots,u_p)\in[0,1]\times(-\infty,2]^p\right\}.
\end{aligned} \tag{3.13}$$

It follows from Lemma 2.3 and (C_5) that for all $u\in\partial\Omega_1\cap C$,

$$\begin{aligned}
&(T_1u)^{(p-1)}(t)\\
&\quad= \int_0^1 k(t,s)\lambda f^*\left(s,u(s)-\phi(s),u'(s)-\phi'(s),\dots,u^{(p-1)}(s)-\phi^{(p-1)}(s)\right)ds\\
&\quad\leq \int_0^1 h(s)\lambda M_1\,ds=\lambda M_1D_1^{-1}<2=\|u\|.
\end{aligned} \tag{3.14}$$

Hence,

$$\|T_1u\|\leq\|u\|,\quad u\in\partial\Omega_1\cap C. \tag{3.15}$$

Step 2. From (C_2), we know that there exists $\eta > 2$ (η can be chosen arbitrarily large) such that

$$\sigma := 1 - \frac{\lambda M}{(n-p)!\eta} > 1 - \frac{\lambda M}{2(n-p)!} > \frac{1}{2}, \tag{3.16}$$

and for all $(u_1, u_2, \dots, u_p) \in [(\varepsilon^p \sigma\eta)/p!, \infty)^{p-1} \times [\varepsilon\sigma\eta, \infty)$,

$$\min_{t\in[\varepsilon,1]} \frac{f(t, u_1, u_2, \dots, u_p) + M}{u_p} \geq \frac{2D_2}{\lambda\varepsilon} \geq \frac{D_2}{\lambda\varepsilon\sigma}. \tag{3.17}$$

Then, for all $(t, u_1, u_2, \dots, u_p) \in [\varepsilon, 1] \times [(\varepsilon^p\sigma\eta)/p!, \eta]^{p-1} \times [\varepsilon\sigma\eta, \eta]$,

$$f(t, u_1, u_2, \dots, u_p) + M \geq \frac{D_2 u_p}{\lambda\varepsilon\sigma} \geq \frac{D_2\eta}{\lambda}. \tag{3.18}$$

It follows from Lemmas 2.1 and 2.4 that for $u \in C$ and $\|u\| = \eta$,

$$\begin{aligned} u^{(i)}(t) - \phi^{(i)}(t) &= u^{(i)}(t) - \lambda M w^{(i)}(t) \\ &\geq u^{(i)}(t) - \frac{\lambda M t^{p-i}}{(n-p)!(p-i)!} \\ &\geq u^{(i)}(t) - \frac{\lambda M u^{(i)}(t)}{(n-p)!\eta} \\ &= \left[1 - \frac{\lambda M}{(n-p)!\eta}\right] u^{(i)}(t) \\ &\geq \left[1 - \frac{\lambda M}{(n-p)!\eta}\right] \frac{t^{p-i}\eta}{(p-i)!} \\ &= \sigma \frac{t^{p-i}\eta}{(p-i)!}, \quad t \in [0,1] \quad (\text{by } (3.16)) \\ &\geq \begin{cases} \dfrac{\varepsilon^p\sigma\eta}{p!}, & 0 \leq i \leq p-2,\ t \in [\varepsilon, 1], \\ \varepsilon\sigma\eta, & i = p-1,\ t \in [\varepsilon, 1]. \end{cases} \end{aligned} \tag{3.19}$$

Using Lemma 2.3 and (3.18), we know that

$$\begin{aligned} &(T_1 u)^{(p-1)}(1) \\ &\quad = \int_0^1 k(1,s)\lambda f^*(s, u(s) - \phi(s), u'(s) - \phi'(s), \dots, u^{(p-1)}(s) - \phi^{(p-1)}(s))\, ds \\ &\quad \geq \int_\varepsilon^1 h(s)\lambda \frac{D_2\eta}{\lambda} ds = \int_\varepsilon^1 h(s) D_2 \eta\, ds = \eta = \|u\|. \end{aligned} \tag{3.20}$$

Hence, let

$$\Omega_2 = \{u \in B : \|u\| < \eta\}, \tag{3.21}$$

then

$$\|T_1 u\| \geq \|u\|, \quad u \in \partial\Omega_2 \cap C. \tag{3.22}$$

Thus, it follows from the first part of Lemma 2.5 that $T_1(u) = u$ has one fixed point $\overline{u}(t)$ in C, such that $2 \leq \|\overline{u}\| \leq \eta$.

Let

$$u_1(t) = \overline{u}(t) - \phi(t). \tag{3.23}$$

From Lemmas 2.1, 2.4, and (3.16), we know that for $i = 0, 1, \ldots, p-1$,

$$\begin{aligned} u_1^{(i)}(t) &= \overline{u}^{(i)}(t) - \phi^{(i)}(t) \\ &= \overline{u}^{(i)}(t) - \lambda M w^{(i)}(t) \\ &\geq \overline{u}^{(i)}(t) - \frac{\lambda M t^{p-i}}{(n-p)!(p-i)!} \\ &\geq \overline{u}^{(i)}(t) - \frac{\lambda M \overline{u}_1^{(i)}(t)}{2(n-p)!} \\ &= \left[1 - \frac{\lambda M}{2(n-p)!}\right]\overline{u}^{(i)}(t) \\ &\geq \left[1 - \frac{\lambda M}{2(n-p)!}\right]\frac{2t^{p-i}}{(p-i)!} \\ &> \frac{t^{p-i}}{(p-i)!} > 0, \quad t \in (0,1]. \end{aligned} \tag{3.24}$$

This implies that

$$u_1^{(i)}(t) > 0, \quad t \in (0,1],\ i = 0, 1, \ldots, p-1. \tag{3.25}$$

Further, we get

$$\begin{aligned} (-1)^{n-p} u_1^{(n)}(t) &= (-1)^{n-p}\overline{u}^{(n)}(t) - \lambda M \\ &= \lambda f^*\left(t, \overline{u}(t) - \phi(t), \overline{u}'(t) - \phi'(t), \ldots, \overline{u}^{(p-1)}(t) - \phi^{(p-1)}(t)\right) - \lambda M \\ &= \lambda f\left(t, \overline{u}(t) - \phi(t), \overline{u}'(t) - \phi'(t), \ldots, \overline{u}^{(p-1)}(t) - \phi^{(p-1)}(t)\right) \\ &= \lambda f\left(t, u_1(t), u_1'(t), \ldots, u_1^{(p-1)}(t)\right). \end{aligned} \tag{3.26}$$

So, $u_1(t) = \overline{u}(t) - \phi(t)$ is a positive solution of BVP (1.2), (1.3).

Thus, for $\lambda \in (0, \Lambda)$, BVP (1.2), (1.3) has at least one positive solution. □

Theorem 3.2. *Assume (C_1), (C_2), (C_3), and (C_5) hold. Then BVP (1.2), (1.3) has at least two positive solutions if $\lambda > 0$ is small enough.*

Proof. It follows from Theorem 3.1 that, for $\lambda \in (0,\Lambda)$, where Λ is as in (3.10), BVP (1.2), (1.3) has positive solution $u_1(t)$ such that

$$||u_1|| > 1. \tag{3.27}$$

Next, we will find the second positive solution. From (C_3), we know that there exists $a \in (0,\infty)$ such that

$$f(t,u_1,u_2,\dots,u_p) \geq 0, \quad (t,u_1,u_2,\dots,u_p) \in [0,1]\times[0,a]^p. \tag{3.28}$$

We consider the following BVP:

$$\begin{gathered}(-1)^{(n-p)}u^{(n)}(t) = \lambda f^{**}(t,u(t),u'(t),\dots,u^{(p-1)}), \quad t\in[0,1],\\ u^{(i)}(0)=0, \quad 0\leq i\leq p-1,\\ u^{(i)}(1)=0, \quad p\leq i\leq n-1,\end{gathered} \tag{3.29}$$

where

$$\begin{gathered}f^{**}(t,u_1,u_2,\dots,u_p) = f(t,\rho_1,\rho_2,\dots,\rho_p),\\ \rho_i = \begin{cases} u_i, & u_i\in[0,a],\\ a, & u_i\in(a,\infty),\end{cases} \quad i=1,2,\dots,p.\end{gathered} \tag{3.30}$$

It is easy to prove that (3.29) has a solution in C if and only if operator

$$u(t) = \int_0^1 K(t,s)\lambda f^{**}(s,u(s),u'(s),\dots,u^{(p-1)}(s))\,ds := (T_2u)(t) \tag{3.31}$$

or

$$u^{(p-1)}(t) = \int_0^1 k(t,s)\lambda f^{**}(s,u(s),u'(s),\dots,u^{(p-1)}(s))\,ds = (T_2u)^{(p-1)}(t) \tag{3.32}$$

has a fixed point in C. Moreover, it is easy to check that $T_2 : C\to C$ is completely continuous.

Let

$$\begin{gathered}H = \min\{1,a\},\\ \Lambda_1 = \min\left\{\Lambda, \frac{D_1H}{M_2}\right\},\end{gathered} \tag{3.33}$$

where Λ is as in (3.10) and

$$M_2 := \max\{f^{**}(t,u_1,u_2,\dots,u_p) : (t,u_1,u_2,\dots,u_p)\in[0,1]\times[0,a]^p\}. \tag{3.34}$$

Let

$$\lambda \in (0, \Lambda_1) \tag{3.35}$$

be fixed.

Let

$$\Omega_3 = \{u \in B : \|u\| < H\}. \tag{3.36}$$

Then for $u \in C \cap \partial\Omega_3$, we have from Lemma 2.3 and (C_5) that

$$\begin{aligned}
(T_2 u)^{(p-1)}(t) &= \lambda \int_0^1 k(t,s) f^{**}\left(t, u(s), u'(s), \dots, u^{(p-1)}(s)\right) ds \\
&\le \lambda \int_0^1 h(s) f^{**}\left(t, u(s), u'(s), \dots, u^{(p-1)}(s)\right) ds \\
&\le \lambda D_1^{-1} M_2 < H.
\end{aligned} \tag{3.37}$$

Therefore,

$$\left\|T_2 u\right\| \le \|u\|, \quad u \in C \cap \partial\Omega_3. \tag{3.38}$$

From (C_3), there exist η, r_0, where $\lambda\eta \int_0^1 sh(s)ds > 1$ with $r_0 < H$ such that

$$f^{**}(t, u_1, u_2, \dots, u_p) \ge \eta u_p, \quad (t, u_1, u_2, \dots, u_p) \in [0,1] \times [0, r_0]^p. \tag{3.39}$$

For $u \in C$ and $\|u\| = r_0$, we have from Lemma 2.3 and (3.39) that

$$\begin{aligned}
(T_2 u)^{(p-1)}(1) &= \lambda \int_0^1 k(1,s) f^{**}\left(s, u(s), u'(s), \dots, u^{(p-1)}(s)\right) ds \\
&= \lambda \int_0^1 h(s) f^{**}\left(s, u(s), u'(s), \dots, u^{(p-1)}(s)\right) ds \\
&\ge \lambda \int_0^1 h(s) \eta u^{(p-1)}(s) ds \\
&\ge \lambda \int_0^1 h(s) \eta s \|u\| ds \quad \text{(by the definition of } C\text{)} \\
&= \lambda\eta \int_0^1 sh(s) ds \|u\| \\
&> \|u\|.
\end{aligned} \tag{3.40}$$

Thus, let

$$\Omega_4 = \{u \in B : \|u\| < r_0\}, \tag{3.41}$$

then

$$\left\|T_2 u\right\| \ge \|u\|, \quad u \in C \cap \partial\Omega_4. \tag{3.42}$$

Therefore, it follows from the first part of Lemma 2.5 that BVP (3.29) has a solution u_2 such that

$$r_0 \leq \|u_2\| \leq H. \tag{3.43}$$

From the definition of f^{**} and Lemma 2.1, we know that u_2 is positive solution of BVP (1.2), (1.3).

Thus, from (3.27), (3.33), and (3.43), we find that for $\lambda \in (0, \Lambda_1)$, BVP (1.2), (1.3) has two distinct positive solutions u_1 and u_2. □

Corollary 3.3. *Assume (C_1), (C_2), (C_4), and (C_5) hold. Then BVP (1.2), (1.3) has at least two positive solutions if $\lambda > 0$ is small enough.*

Proof. It is easy to prove from (C_4) that (C_3) holds. By using Theorem 3.2, we know that the result holds. □

Remark 3.4. By letting $n = 4$, $p = 2$ in Theorem 3.1 and Corollary 3.3, we get Ma [5, Theorems 1 and 2].

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