

*Research Article*

**Blow up of the Solutions of Nonlinear Wave Equation**

Svetlin Georgiev Georgiev

Received 14 March 2007; Accepted 26 May 2007

Recommended by Peter Bates

We construct for every fixed  $n \geq 2$  the metric  $g_s = h_1(r)dt^2 - h_2(r)dr^2 - k_1(\omega)d\omega_1^2 - \dots - k_{n-1}(\omega)d\omega_{n-1}^2$ , where  $h_1(r)$ ,  $h_2(r)$ ,  $k_i(\omega)$ ,  $1 \leq i \leq n - 1$ , are continuous functions,  $r = |x|$ , for which we consider the Cauchy problem  $(u_{tt} - \Delta u)_{g_s} = f(u) + g(|x|)$ , where  $x \in \mathbb{R}^n$ ,  $n \geq 2$ ;  $u(1, x) = u_o(x) \in L^2(\mathbb{R}^n)$ ,  $u_t(1, x) = u_1(x) \in \dot{H}^{-1}(\mathbb{R}^n)$ , where  $f \in \mathcal{C}^1(\mathbb{R}^1)$ ,  $f(0) = 0$ ,  $a|u| \leq f'(u) \leq b|u|$ ,  $g \in \mathcal{C}(\mathbb{R}^+)$ ,  $g(r) \geq 0$ ,  $r = |x|$ ,  $a$  and  $b$  are positive constants. When  $g(r) \equiv 0$ , we prove that the above Cauchy problem has a nontrivial solution  $u(t, r)$  in the form  $u(t, r) = v(t)\omega(r)$  for which  $\lim_{t \rightarrow 0} \|u\|_{L^2([0, \infty))} = \infty$ . When  $g(r) \neq 0$ , we prove that the above Cauchy problem has a nontrivial solution  $u(t, r)$  in the form  $u(t, r) = v(t)\omega(r)$  for which  $\lim_{t \rightarrow 0} \|u\|_{L^2([0, \infty))} = \infty$ .

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**1. Introduction**

In this paper, we study the properties of the solutions of the Cauchy problem

$$(u_{tt} - \Delta u)_{g_s} = f(u) + g(|x|), \quad x \in \mathbb{R}^n, \quad n \geq 2, \tag{1}$$

$$u(1, x) = u_o(x) \in L^2(\mathbb{R}^n), \quad u_t(1, x) = u_1(x) \in \dot{H}^{-1}(\mathbb{R}^n), \tag{2}$$

where  $g_s$  is the metric

$$g_s = h_1(r)dt^2 - h_2(r)dr^2 - k_1(\omega)d\omega_1^2 - \dots - k_{n-1}(\omega)d\omega_{n-1}^2, \tag{1.1}$$

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the functions  $h_1(r), h_2(r)$  satisfy the conditions

$$\begin{aligned}
 & h_1(r), h_2(r) \in \mathcal{C}^1([0, \infty)), \quad h_1(r) > 0, \quad h_2(r) \geq 0 \quad \forall r \in [0, \infty), \\
 & \int_0^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} d\tau < \infty, \quad \int_0^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \sqrt{h_1(\tau)h_2(\tau)} d\tau ds < \infty, \\
 & \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} C_1 + C_2 \sqrt{h_1(\tau)h_2(\tau)} \right) d\tau \right)^{1/2} ds \right)^2 dr < \infty, \\
 & \quad C_1, C_2 \text{ are arbitrary nonnegative constants,} \\
 & \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} C_1 + C_2 \sqrt{h_1(\tau)h_2(\tau)} \right) d\tau \right) ds \right)^2 dr < \infty, \\
 & \quad C_1, C_2 \text{ are arbitrary nonnegative constants,} \\
 & \max_{r \in [0, \infty)} \sqrt{h_1(r)h_2(r)} < \infty, \\
 & \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} d\tau ds \right)^2 dr < \infty, \\
 & \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty h_1(\tau)h_2(\tau) d\tau \right)^{1/2} ds \right)^2 dr < \infty, \\
 & \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \frac{h_2(\tau)}{h_1(\tau)} d\tau \right)^{1/2} ds \right)^2 dr < \infty, \\
 & \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} ds \right)^2 dr < \infty,
 \end{aligned} \tag{i1}$$

$k_i(\omega) \in \mathcal{C}^1([0, 2\pi] \times \cdots \times [0, 2\pi])$ ,  $i = 1, \dots, n-1$ ,  $f \in \mathcal{C}^1(\mathbb{R}^1)$ ,  $f(0) = 0$ ,  $a|u| \leq f'(u) \leq b|u|$ ,  $a$  and  $b$  are positive constants,  $g \in \mathcal{C}(\mathbb{R}^1)$ ,  $g(|x|) \geq 0$  for  $|x| \in [0, \infty)$ . (In Section 2 we will give example for such metric  $g_s$ .)

We search a solution  $u = u(t, r)$  to the Cauchy problem (1), (2). Therefore, if the Cauchy problem (1), (2) has such solution, it will satisfy the Cauchy problem

$$\frac{1}{h_1(r)} u_{tt} - \frac{1}{\sqrt{h_1(r)h_2(r)}} \partial_r \left( \frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}} u_r \right) = f(u) + g(r), \tag{1.2}$$

$$u(1, r) = u_0 \in L^2([0, \infty)), \quad u_t(1, r) = u_1 \in \dot{H}^{-1}([0, \infty)). \tag{1.3}$$

In this paper, we will prove that the Cauchy problem (1), (2) has nontrivial solution  $u = u(t, r)$  for which

$$\lim_{t \rightarrow 0} \|u\|_{L^2([0, \infty))} = \infty. \tag{1.4}$$

Our main results are the following.

**THEOREM 1.1.** *Suppose  $n \geq 2$  is fixed,  $h_1(r), h_2(r)$  satisfy the conditions (i1),  $g \equiv 0$ ,  $f \in \mathcal{C}^1(\mathbb{R}^1)$ ,  $f(0) = 0$ ,  $a|u| \leq f'(u) \leq b|u|$ ,  $a$  and  $b$  are positive constants. Then the homogeneous problem of Cauchy (1), (2) has nontrivial solution  $u = u(t, r) \in \mathcal{C}((0, 1]L^2([0, \infty)))$  for which*

$$\lim_{t \rightarrow 0} \|u\|_{L^2([0, \infty))} = \infty. \quad (1.5)$$

**THEOREM 1.2.** *Suppose  $n \geq 2$  is fixed,  $h_1(r), h_2(r)$  satisfy the conditions (i1). Suppose also that  $a$  and  $b$  are fixed positive constants,  $a \leq b$ ,  $f \in \mathcal{C}^1(\mathbb{R}^1)$ ,  $f(0) = 0$ ,  $a|u| \leq f'(u) \leq b|u|$ ,  $b/2 \geq f(1) \geq a/2$ ,  $g \neq 0$ ,  $g \in \mathcal{C}([0, \infty))$ ,  $g(r) \geq 0$  for every  $r \geq 0$ ,  $g(r) \leq b/2 - f(1)$  for every  $r \in [0, \infty)$ . Then the nonhomogeneous problem of Cauchy (1), (2) has nontrivial solution  $u = u(t, r) \in \mathcal{C}((0, 1]L^2([0, \infty)))$  for which*

$$\lim_{t \rightarrow 0} \|u\|_{L^2([0, \infty))} = \infty. \quad (1.6)$$

When  $g_s$  is the Minkowski metric and  $u_0, u_1 \in \mathcal{C}_0^\infty(\mathbb{R}^3)$  in [1] (see also [2, Section 6.3]), it is proved that there exists  $T > 0$  and a unique local solution  $u \in \mathcal{C}^2([0, T] \times \mathbb{R}^3)$  for the Cauchy problem

$$\begin{aligned} (u_{tt} - \Delta u)_{g_s} &= f(u), \quad f \in \mathcal{C}^2(\mathbb{R}), \quad t \in [0, T], \quad x \in \mathbb{R}^3, \\ u|_{t=0} &= u_0, \quad u_t|_{t=0} = u_1, \end{aligned} \quad (1.7)$$

for which

$$\sup_{t < T, x \in \mathbb{R}^3} |u(t, x)| = \infty. \quad (1.8)$$

When  $g_s$  is the Minkowski metric,  $1 \leq p < 5$  and initial data are in  $\mathcal{C}_0^\infty(\mathbb{R}^3)$  in [1] (see also [2, Section 6.3]), it is proved that the initial value problem

$$\begin{aligned} (u_{tt} - \Delta u)_{g_s} &= u|u|^{p-1}, \quad t \in [0, T], \quad x \in \mathbb{R}^3, \\ u|_{t=0} &= u_0, \quad u_t|_{t=0} = u_1 \end{aligned} \quad (1.9)$$

admits a global smooth solution.

When  $g_s$  is the Minkowski metric and initial data are in  $\mathcal{C}_0^\infty(\mathbb{R}^3)$  in [3] (see also [2, Section 6.3]) it is proved that there exists a number  $\epsilon_0 > 0$  such that for any data  $(u_0, u_1) \in \mathcal{C}_0^\infty(\mathbb{R}^3)$  with  $E(u(0)) < \epsilon_0$ , the initial value problem

$$\begin{aligned} (u_{tt} - \Delta u)_{g_s} &= u^5, \quad t \in [0, T], \quad x \in \mathbb{R}^3, \\ u|_{t=0} &= u_0, \quad u_t|_{t=0} = u_1 \end{aligned} \quad (1.10)$$

admits a global smooth solution.

When  $g_s$  is the Reissner-Nordström metric in [4], it is proved that the Cauchy problem

$$\begin{aligned} (u_{tt} - \Delta u)_{g_s} + m^2 u &= f(u), \quad t \in [0, 1], \quad x \in \mathbb{R}^3, \\ u(1, x) &= u_0 \in \dot{B}_{p,p}^\gamma(\mathbb{R}^3), \quad u_t(1, x) = u_1 \in \dot{B}_{p,p}^{\gamma-1}(\mathbb{R}^3), \end{aligned} \quad (1.11)$$

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where  $m \neq 0$  is constant and  $f \in \mathcal{C}^2(\mathbb{R}^1)$ ,  $a|u| \leq |f^{(l)}(u)| \leq b|u|$ ,  $l = 0, 1$ ,  $a$  and  $b$  are positive constants, has unique nontrivial solution  $u(t, r) \in \mathcal{C}((0, 1] \dot{B}_{p,p}^\gamma(\mathbb{R}^+))$ ,  $r = |x|$ ,  $p > 1$ , for which

$$\lim_{t \rightarrow 0} \|u\|_{\dot{B}_{p,p}^\gamma(\mathbb{R}^+)} = \infty. \quad (1.12)$$

When  $g_s$  is the Minkowski metric in [5], it is proved that the Cauchy problem

$$\begin{aligned} (u_{tt} - \Delta u)_{g_s} &= f(u), \quad t \in [0, 1], \quad x \in \mathbb{R}^3, \\ u(1, x) &= u_0, \quad u_t(1, x) = u_1 \end{aligned} \quad (1.13)$$

has global solution. Here  $f \in \mathcal{C}^2(\mathbb{R})$ ,  $f(0) = f'(0) = f''(0) = 0$ ,

$$|f''(u) - f''(v)| \leq B|u - v|^{q_1} \quad (1.14)$$

for  $|u| \leq 1$ ,  $|v| \leq 1$ ,  $B > 0$ ,  $\sqrt{2} - 1 < q_1 \leq 1$ ,  $u_0 \in \mathcal{C}_0^5(\mathbb{R}^3)$ ,  $u_1 \in \mathcal{C}_0^4(\mathbb{R}^3)$ ,  $u_0(x) = u_1(x) = 0$  for  $|x - x_0| > \rho$ ,  $x_0$  and  $\rho$  are suitable chosen.

When  $g_s$  is the Reissner-Nordström metric,  $n = 3$ ,  $p > 1$ ,  $q \geq 1$ ,  $\gamma \in (0, 1)$  are fixed constants,  $f \in \mathcal{C}^1(\mathbb{R}^1)$ ,  $f(0) = 0$ ,  $a|u| \leq f'(u) \leq b|u|$ ,  $g \in \mathcal{C}(\mathbb{R}^+)$ ,  $g(|x|) \geq 0$ ,  $g(|x|) = 0$  for  $|x| \geq r_1$ ,  $a$  and  $b$  are positive constants,  $r_1 > 0$  is suitable chosen, in [6], it is proved that the initial value problem (1), (2) has nontrivial solution  $u \in \mathcal{C}((0, 1] \dot{B}_{p,q}^\gamma(\mathbb{R}^+))$  in the form

$$u(t, r) = \begin{cases} v(t)\omega(r), & \text{for } r \leq r_1, \quad t \in [0, 1], \\ 0, & \text{for } r \geq r_1, \quad t \in [0, 1], \end{cases} \quad (1.15)$$

where  $r = |x|$ , for which  $\lim_{t \rightarrow 0} \|u\|_{\dot{B}_{p,q}^\gamma(\mathbb{R}^+)} = \infty$ .

The paper is organized as follows. In Section 2, we will prove some preliminary results. In Section 3, we will prove Theorem 1.1. In Section 4, we will prove Theorem 1.2. In the appendix we will prove some results which are used for the proof of Theorems 1.1 and 1.2.

### 2. Preliminary results

**PROPOSITION 2.1.** *Let  $h_1(r)$ ,  $h_2(r)$  satisfy the conditions (i1),  $f \in \mathcal{C}(-\infty, \infty)$ ,  $g \equiv 0$ . If for every fixed  $t \in [0, 1]$  the function  $u(t, r) = v(t)\omega(r)$ , where  $v(t) \in \mathcal{C}^4([0, 1])$ ,  $v(t) \neq 0$  for every  $t \in [0, 1]$ ,  $\omega(r) \in \mathcal{C}^2([0, \infty))$ ,  $\omega(\infty) = \omega'(\infty) = 0$ , satisfies (1), then the function  $u(t, r) = v(t)\omega(r)$  satisfies the integral equation*

$$u(t, r) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(\tau)}{v(\tau)} u(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} f(u) \right) d\tau ds \quad (1^*)$$

for every fixed  $t \in [0, 1]$ .

*Proof.* Suppose that  $t \in [0, 1]$  is fixed and the function  $u(t, r) = v(t)\omega(r)$ ,  $v(t) \in \mathcal{C}^4([0, 1])$ ,  $v(t) \neq 0$  for every  $t \in [0, 1]$ ,  $\omega(r) \in \mathcal{C}^2([0, \infty))$ ,  $\omega(\infty) = \omega'(\infty) = 0$ , satisfies (1). Then for

every fixed  $t \in [0, 1]$  and for  $r \in [0, \infty)$  we have

$$\begin{aligned}
 u_{tt}(t, r) &= \frac{v''(t)}{v(t)} u(t, r), \\
 \frac{1}{h_1(r)} \frac{v''(t)}{v(t)} u(t, r) - \frac{1}{\sqrt{h_1(r)h_2(r)}} \partial_r \left( \frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}} u_r(t, r) \right) &= f(u), \\
 \frac{1}{\sqrt{h_1(r)h_2(r)}} \partial_r \left( \frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}} u_r(t, r) \right) &= \frac{1}{h_1(r)} \frac{v''(t)}{v(t)} u(t, r) - f(u), \\
 \partial_r \left( \frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}} u_r(t, r) \right) &= \sqrt{\frac{h_2(r)}{h_1(r)}} \frac{v''(t)}{v(t)} u(t, r) - \sqrt{h_1(r)h_2(r)} f(u).
 \end{aligned} \tag{2.1}$$

Now we integrate the last equality from  $r$  to  $\infty$  here we suppose that  $u_r(t, r) = v(t)\omega'(r)$ ,  $u_r(t, \infty) = v(t)\omega'(\infty) = 0$ , then we get

$$\begin{aligned}
 - \frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}} u_r(t, r) &= \int_r^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} f(u) \right) d\tau, \\
 - \sqrt{\frac{h_1(r)}{h_2(r)}} u_r(t, r) &= \int_r^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} f(u) \right) d\tau, \\
 - u_r(t, r) &= \sqrt{\frac{h_2(r)}{h_1(r)}} \int_r^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} f(u) \right) d\tau.
 \end{aligned} \tag{2.2}$$

Now we integrate the last equality from  $r$  to  $\infty$ ; we use that  $u(t, \infty) = v(t)\omega(\infty) = 0$ , then we get

$$u(t, r) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} f(u) \right) d\tau ds, \tag{2.3}$$

that is, for every fixed  $t \in [0, 1]$  if the function  $u(t, r) = v(t)\omega(r)$  satisfies (1), then the function  $u(t, r) = v(t)\omega(r)$  satisfies the integral equation (1\*). Here  $v(t) \in \mathcal{C}^4([0, 1])$ ,  $v(t) \neq 0$  for every  $t \in [0, 1]$ ,  $\omega(r) \in \mathcal{C}^2([0, \infty))$ ,  $\omega(\infty) = \omega'(\infty) = 0$ .  $\square$

**PROPOSITION 2.2.** *Let  $h_1(r)$ ,  $h_2(r)$  satisfy the conditions (i1),  $f \in \mathcal{C}(-\infty, \infty)$ ,  $g \equiv 0$ . If for every fixed  $t \in [0, 1]$  the function  $u(t, r) = v(t)\omega(r)$ , where  $v(t) \in \mathcal{C}^4([0, 1])$ ,  $v(t) \neq 0$  for every  $t \in [0, 1]$ ,  $\omega(r) \in \mathcal{C}^2([0, \infty))$ ,  $\omega(\infty) = \omega'(\infty) = 0$ , satisfies the integral equation (1\*) then the function  $u(t, r) = v(t)\omega(r)$  satisfies (1) for every fixed  $t \in [0, 1]$ .*

*Proof.* Let  $t \in [0, 1]$  be fixed and let the function  $u(t, r) = v(t)\omega(r)$ , where  $v(t) \in \mathcal{C}^4([0, 1])$ ,  $v(t) \neq 0$  for every  $t \in [0, 1]$ ,  $\omega(r) \in \mathcal{C}^2([0, \infty))$ ,  $\omega(\infty) = \omega'(\infty) = 0$ , satisfy the integral equation (1\*). From here and from  $f \in \mathcal{C}(-\infty, \infty)$ , for every fixed  $t \in [0, 1]$  we have

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$u(t, r) \in \mathcal{C}^2([0, \infty))$  and

$$\begin{aligned}
 u_r(t, r) &= -\sqrt{\frac{h_2(r)}{h_1(r)}} \int_r^\infty \left( \sqrt{\frac{h_2(r)}{h_1(r)}} \frac{v''(t)}{v(t)} u(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} f(u) \right) d\tau, \\
 \sqrt{\frac{h_1(r)}{h_2(r)}} u_r(t, r) &= -\int_r^\infty \left( \sqrt{\frac{h_2(r)}{h_1(r)}} \frac{v''(t)}{v(t)} u(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} f(u) \right) d\tau, \\
 \frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}} u_r(t, r) &= -\int_r^\infty \left( \sqrt{\frac{h_2(r)}{h_1(r)}} \frac{v''(t)}{v(t)} u(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} f(u) \right) d\tau, \\
 \partial_r \left( \frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}} u_r(t, r) \right) &= \sqrt{\frac{h_2(r)}{h_1(r)}} \frac{v''(t)}{v(t)} u(t, r) - \sqrt{h_1(r)h_2(r)} f(u), \\
 \sqrt{\frac{h_2(r)}{h_1(r)}} \frac{v''(t)}{v(t)} u(t, r) - \partial_r \left( \frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}} u_r(t, r) \right) &= \sqrt{h_1(r)h_2(r)} f(u), \\
 \frac{1}{h_1(r)} \frac{v''(t)}{v(t)} u(t, r) - \frac{1}{\sqrt{h_1(r)h_2(r)}} \partial_r \left( \frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}} u_r(t, r) \right) &= f(u).
 \end{aligned} \tag{2.4}$$

Since for every fixed  $t \in [0, 1]$  we have

$$\frac{v''(t)}{v(t)} u(t, r) = u_{tt}(t, r), \tag{2.5}$$

we get

$$\frac{1}{h_1(r)} u_{tt}(t, r) - \frac{1}{\sqrt{h_1(r)h_2(r)}} \partial_r \left( \frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}} u_r(t, r) \right) = f(u), \tag{2.6}$$

that is, for every fixed  $t \in [0, 1]$  if the function  $u(t, r) = v(t)\omega(r)$ , where  $v(t) \in \mathcal{C}^4([0, 1])$ ,  $v(t) \neq 0$  for every  $t \in [0, 1]$ ,  $\omega(r) \in \mathcal{C}^2([0, \infty))$ ,  $\omega(\infty) = \omega'(\infty) = 0$ , satisfies (1\*), then it satisfies (1) for every fixed  $t \in [0, 1]$ .  $\square$

**PROPOSITION 2.3.** *Let  $h_1(r), h_2(r)$  satisfy the conditions (i1),  $f \in \mathcal{C}(-\infty, \infty)$ ,  $g \in \mathcal{C}([0, \infty))$ ,  $g(r) \geq 0$  for every  $r \geq 0$ . If for every fixed  $t \in [0, 1]$  the function  $u(t, r) = v(t)\omega(r)$ , where  $v(t) \in \mathcal{C}^4([0, 1])$ ,  $v(t) \neq 0$  for every  $t \in [0, 1]$ ,  $\omega(r) \in \mathcal{C}^2([0, \infty))$ ,  $\omega(\infty) = \omega'(\infty) = 0$ , satisfies (1), then the function  $u(t, r) = v(t)\omega(r)$  satisfies the integral equation*

$$u(t, r) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} (f(u) + g(r)) \right) d\tau ds \tag{1**}$$

for every fixed  $t \in [0, 1]$ .

*Proof.* Let  $t \in [0, 1]$  be fixed and let the function  $u(t, r) = v(t)\omega(r)$ ,  $v(t) \in \mathcal{C}^4([0, 1])$ ,  $v(t) \neq 0$  for every  $t \in [0, 1]$ ,  $\omega(r) \in \mathcal{C}^2([0, \infty))$ ,  $\omega(\infty) = \omega'(\infty) = 0$ , satisfy (1). Then for

every fixed  $t \in [0, 1]$  and for  $r \in [0, \infty)$  we have

$$\begin{aligned}
 u_{tt}(t, r) &= \frac{v''(t)}{v(t)} u(t, r), \\
 \frac{1}{h_1(r)} \frac{v''(t)}{v(t)} u(t, r) - \frac{1}{\sqrt{h_1(r)h_2(r)}} \partial_r \left( \frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}} u_r(t, r) \right) &= (f(u) + g(r)), \\
 \frac{1}{\sqrt{h_1(r)h_2(r)}} \partial_r \left( \frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}} u_r(t, r) \right) &= \frac{1}{h_1(r)} \frac{v''(t)}{v(t)} u(t, r) - (f(u) + g(r)), \\
 \partial_r \left( \frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}} u_r(t, r) \right) &= \sqrt{\frac{h_2(r)}{h_1(r)}} \frac{v''(t)}{v(t)} u(t, r) - \sqrt{h_1(r)h_2(r)} (f(u) + g(r)).
 \end{aligned} \tag{2.7}$$

Now we integrate the last equality from  $r$  to  $\infty$ ; here we suppose that  $u_r(t, r) = v(t)\omega'(r)$ ,  $u_r(t, \infty) = v(t)\omega'(\infty) = 0$ , then we get

$$\begin{aligned}
 -\frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}} u_r(t, r) &= \int_r^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} (f(u) + g(r)) \right) d\tau, \\
 -\sqrt{\frac{h_1(r)}{h_2(r)}} u_r(t, r) &= \int_r^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} (f(u) + g(r)) \right) d\tau, \\
 -u_r(t, r) &= \sqrt{\frac{h_2(r)}{h_1(r)}} \int_r^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} (f(u) + g(r)) \right) d\tau.
 \end{aligned} \tag{2.8}$$

Now we integrate the last equality from  $r$  to  $\infty$ ; we suppose that  $u(t, \infty) = v(t)\omega(\infty) = 0$ , then we get

$$u(t, r) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} (f(u) + g(r)) \right) d\tau ds, \tag{2.9}$$

that is, for every fixed  $t \in [0, 1]$  if the function  $u(t, r) = v(t)\omega(r)$  satisfies (1), then the function  $u(t, r) = v(t)\omega(r)$  satisfies the integral equation (1\*\*). Here  $v(t) \in \mathcal{C}^4([0, 1])$ ,  $v(t) \neq 0$  for every  $t \in [0, 1]$ ,  $\omega(r) \in \mathcal{C}^2([0, \infty))$ ,  $\omega(\infty) = \omega'(\infty) = 0$ .  $\square$

**PROPOSITION 2.4.** *Let  $h_1(r)$ ,  $h_2(r)$  satisfy the conditions (i1),  $f \in \mathcal{C}(-\infty, \infty)$ ,  $g \in \mathcal{C}([0, \infty))$ ,  $g(r) \geq 0$  for every  $r \geq 0$ . If for every fixed  $t \in [0, 1]$  the function  $u(t, r) = v(t)\omega(r)$ , where*

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$v(t) \in \mathcal{C}^4([0, 1])$ ,  $v(t) \neq 0$  for every  $t \in [0, 1]$ ,  $\omega(r) \in \mathcal{C}^2([0, \infty))$ ,  $\omega(\infty) = \omega'(\infty) = 0$ , satisfies the integral equation (1\*\*), then the function  $u(t, r) = v(t)\omega(r)$  satisfies (1) for every fixed  $t \in [0, 1]$ .

*Proof.* Let  $t \in [0, 1]$  be fixed and let the function  $u(t, r) = v(t)\omega(r)$ , where  $v(t) \in \mathcal{C}^4([0, 1])$ ,  $v(t) \neq 0$  for every  $t \in [0, 1]$ ,  $\omega(r) \in \mathcal{C}^2([0, \infty))$ ,  $\omega(\infty) = \omega'(\infty) = 0$ , satisfy the integral equation (1\*\*). From here and from  $f \in \mathcal{C}(-\infty, \infty)$ ,  $g \in \mathcal{C}([0, \infty))$ , for every fixed  $t \in [0, 1]$  we have  $u(t, r) \in \mathcal{C}^2([0, \infty))$  and

$$\begin{aligned}
 u_r(t, r) &= -\sqrt{\frac{h_2(r)}{h_1(r)}} \int_r^\infty \left( \sqrt{\frac{h_2(r)}{h_1(r)}} \frac{v''(t)}{v(t)} u(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} (f(u) + g(r)) \right) d\tau, \\
 \sqrt{\frac{h_1(r)}{h_2(r)}} u_r(t, r) &= -\int_r^\infty \left( \sqrt{\frac{h_2(r)}{h_1(r)}} \frac{v''(t)}{v(t)} u(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} (f(u) + g(r)) \right) d\tau, \\
 \frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}} u_r(t, r) &= -\int_r^\infty \left( \sqrt{\frac{h_2(r)}{h_1(r)}} \frac{v''(t)}{v(t)} u(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} (f(u) + g(r)) \right) d\tau, \\
 \partial_r \left( \frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}} u_r(t, r) \right) &= \sqrt{\frac{h_2(r)}{h_1(r)}} \frac{v''(t)}{v(t)} u(t, r) - \sqrt{h_1(r)h_2(r)} (f(u) + g(r)), \\
 \sqrt{\frac{h_2(r)}{h_1(r)}} \frac{v''(t)}{v(t)} u(t, r) - \partial_r \left( \frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}} u_r(t, r) \right) &= \sqrt{h_1(r)h_2(r)} (f(u) + g(r)), \\
 \frac{1}{h_1(r)} \frac{v''(t)}{v(t)} u(t, r) - \frac{1}{\sqrt{h_1(r)h_2(r)}} \partial_r \left( \frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}} u_r(t, r) \right) &= f(u) + g(r).
 \end{aligned} \tag{2.10}$$

Since for every fixed  $t \in [0, 1]$  we have

$$\frac{v''(t)}{v(t)} u(t, r) = u_{tt}(t, r), \tag{2.11}$$

we get

$$\frac{1}{h_1(r)} u_{tt}(t, r) - \frac{1}{\sqrt{h_1(r)h_2(r)}} \partial_r \left( \frac{h_1(r)}{\sqrt{h_1(r)h_2(r)}} u_r(t, r) \right) = f(u) + g(r), \tag{2.12}$$

that is, for every fixed  $t \in [0, 1]$  if the function  $u(t, r) = v(t)\omega(r)$ , where  $v(t) \in \mathcal{C}^4([0, 1])$ ,  $v(t) \neq 0$  for every  $t \in [0, 1]$ ,  $\omega(r) \in \mathcal{C}^2([0, \infty))$ ,  $\omega(\infty) = \omega'(\infty) = 0$ , satisfies (1\*\*), then it satisfies (1) for every fixed  $t \in [0, 1]$ .  $\square$

For fixed  $n \geq 2$ ,  $h_1(r)$ ,  $h_2(r)$  which satisfy the conditions (i1) and fixed positive constants  $a$  and  $b$ , we suppose that the positive constants  $c, d, A, B, A_1, A_2$  satisfy the conditions

$$c \leq d, \quad A \geq B, \quad A_1 \leq A_2, \quad A_1 - \frac{b}{2B} > 0,$$

$$\sqrt{\frac{h_2(r)}{h_1(r)}} A_1 - \frac{b}{B} \sqrt{h_1(r)h_2(r)} \geq 0 \quad \text{for every } r \in [0, \infty),$$

$$\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 - \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right) d\tau ds \geq 1 \quad \text{for } r \in [c, d], \quad (H1)$$

$$\int_1^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( A_1 \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} - \frac{b}{B} \sqrt{h_1(\tau)h_2(\tau)} \right) d\tau ds \geq \frac{A_1}{10^{10}}$$

$$\max_{r \in [0, \infty)} \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} \right) d\tau ds \leq 1, \quad (H2)$$

$$\max_{r \in [0, \infty)} \sqrt{\frac{h_2(r)}{h_1(r)}} \int_r^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} \right) d\tau \leq 1,$$

$$\int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} \right) d\tau \right)^{1/2} ds \right)^2 dr < \frac{1}{7}, \quad (H3)$$

$$\int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \frac{b}{B} \sqrt{h_1(\tau)h_2(\tau)} \right) d\tau ds \right)^2 dr < \infty,$$

$$\int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) d\tau \right)^{1/2} ds \right)^2 dr < 1. \quad (H4)$$

*Example 2.5.* Let  $0 < \epsilon \ll 1/3$  be enough small,  $n \geq 2$  is fixed. We choose  $c > 0$ ,  $d > 0$ ,  $c \leq d < \infty$  such that for every  $r \in [c, d]$  we have

$$\frac{\pi}{4} \leq \arctg(d+1-r)^3, \quad \arctgd^3 < \frac{\pi}{3}. \quad (2.13)$$

Let also  $b = 8\epsilon^3$ ,  $a = 4\epsilon^3$ ,  $A = 60$ ,  $B = 40$ ,  $A_1 = \epsilon^3$ ,  $A_2 = 2\epsilon^3$ . Let

$$h_1(r) = \left( \frac{B}{b} \left( -1 + \sqrt{1 + 2 \frac{A_1 b}{B}} \right) \right)^2, \quad h_2(r) = \frac{144(d+1-r)^4}{[(d+1-r)^6 + 1]^2}. \quad (2.14)$$

We note that the functions  $h_1(r)$  and  $h_2(r)$  satisfy all conditions of (i1) and

$$\begin{aligned} & \frac{A_1}{\sqrt{h_1(r)}} - \frac{b}{2B}\sqrt{h_1(r)} = 1, \\ & \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 - \frac{b}{2B}\sqrt{h_1(\tau)h_2(\tau)} \right) d\tau ds \\ & \geq \int_r^{d+1} \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^{d+1} \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 - \frac{b}{2B}\sqrt{h_1(\tau)h_2(\tau)} \right) d\tau ds \geq 1 \quad \text{for } r \in [c, d]. \end{aligned} \tag{2.15}$$

We note that  $\sqrt{h_1(r)} \sim 1,6$ .

For fixed  $n \geq 2$ ,  $h_1(r)$ ,  $h_2(r)$ , which satisfy the conditions (i1), the constants  $a, b, c, d, A, B, A_1, A_2$  are fixed which satisfy the conditions (H1), ..., (H4), then we suppose that the function  $v(t)$  is fixed function and satisfies the conditions

$$v(t) \in \mathcal{C}^4([0, 1]), \quad \frac{v''(t)}{v(t)} > 0, \quad v(t) > 0, \quad \forall t \in [0, 1], \tag{H5}$$

$$A_1 \leq \frac{v''(t)}{v(t)} \leq A_2, \quad v'''(1) = 0, \quad v'(1) = 0, \tag{H6}$$

$$\lim_{t \rightarrow 0} \left( \frac{v''(t)}{v(t)} - \frac{a}{2} \right) = +0. \tag{H7}$$

*Example 2.6.* Let  $a, b, c, d, A_1, A_2, B, A$  be the constants from the above example. Then  $a/2 = A_2$  and

$$v(t) = C \left( e^{\sqrt{A_2}(t-1)} + e^{-\sqrt{A_2}(t-1)} \right), \tag{2.16}$$

where  $C$  is arbitrary positive constant, satisfying the hypotheses (H5), (H6), (H7).

Here and below we suppose that  $v(t)$  is fixed function which satisfies the conditions (H5), ..., (H7).

When  $g(r) \equiv 0$  we put

$$\begin{aligned} u_\circ := v(1)\omega(r) &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} v''(1)\omega(\tau) - \sqrt{h_1(\tau)h_2(\tau)} f(v(1)\omega(\tau)) \right) d\tau ds, \\ u_1 &\equiv 0. \end{aligned} \tag{1'}$$

In Section 3, we will prove that (1') has unique nontrivial solution  $\omega(r) \in L^2([0, \infty))$ .

When  $g(r) \neq 0$  we put

$$\begin{aligned} u_\circ &:= v(1)\omega(r) \\ &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} v''(1)\omega(\tau) - \sqrt{h_1(\tau)h_2(\tau)}(f(v(1)\omega(\tau)) + g(\tau)) \right) d\tau ds, \\ u_1 &\equiv 0. \end{aligned} \tag{1''}$$

In Section 4, we will prove that (1'') has unique nontrivial solution  $\omega(r) \in L^2([0, 1])$ .

### 3. Proof of Theorem 1.1

#### 3.1. Local existence of nontrivial solutions of homogeneous Cauchy problem (1), (2).

In this section, we will prove that the homogeneous Cauchy problem (1), (2) has nontrivial solution in the form  $u(t, r) = v(t)\omega(r)$ .

For fixed function  $v(t)$ , which satisfies the conditions (H5), (H6), and (H7) we consider the integral equation

$$u(t, r) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} f(u) \right) d\tau ds. \tag{1^*}$$

**THEOREM 3.1.** *Let  $n \geq 2$  be fixed, let  $h_1(r)$ ,  $h_2(r)$  fixed, which satisfy the conditions (i1), let the positive constants  $a$ ,  $b$  be fixed,  $a \leq b$ , let the positive constants  $c$ ,  $d$ ,  $A$ ,  $B$ ,  $A_1$ ,  $A_2$  be fixed which satisfy the conditions (H1), ..., (H4) and  $f \in \mathcal{C}^1((-\infty, \infty))$ ,  $f(0) = 0$ ,  $a|u| \leq f'(u) \leq b|u|$ . Let also  $v(t)$  be fixed function which satisfies the conditions (H5), ..., (H7). Then (1\*) has unique nontrivial solution  $u(t, r) = v(t)\omega(r)$  for which  $u(t, r) \in \mathcal{C}([0, 1] \times [0, \infty))$ ,  $u(t, r) \leq 1/B$  for every  $t \in [0, 1]$  and for every  $r \in [0, \infty)$ ,  $u(t, r) \geq 1/A$  for every  $t \in [0, 1]$  and for every  $r \in [c, d]$ ,  $u(t, r) \geq 0$  for every  $t \in [0, 1]$  and for every  $r \in [0, \infty)$ ,  $u(t, \infty) = u_r(t, \infty) = 0$  for every  $t \in [0, 1]$ ,  $u(t, r) \in C([0, 1]L^2([0, \infty)))$ .*

*Proof.* Let  $M$  be the set

$$\begin{aligned} M = \left\{ u(t, r) : u(t, r) \in \mathcal{C}([0, 1] \times [0, \infty)), u(t, \infty) = u_r(t, \infty) = 0 \forall t \in [0, 1], \right. \\ \left. u(t, r) \geq \frac{1}{A} \text{ for } t \in [0, 1], r \in [c, d], u(t, r) \leq \frac{1}{B} \forall t \in [0, 1], \forall r \in [0, \infty), \right. \\ \left. u(t, r) \geq 0 \forall t \in [0, 1], \forall r \in [0, \infty), u(t, r) \in L^2([0, \infty)) \text{ for every } t \in (0, 1] \right\}. \end{aligned} \tag{3.1}$$

Let  $t \in [0, 1]$  be fixed. We define the operator  $L$  as follows:

$$L(u)(t, r) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} f(u) \right) d\tau ds \tag{3.2}$$

for  $u \in M$ . First we will see that  $L : M \rightarrow M$ . Let  $u \in M$ . Then the following holds.

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(1) Since  $v(t) \in \mathcal{C}^4([0,1])$ ,  $u(t,r) \in \mathcal{C}([0,1] \times [0,\infty))$ , and  $f \in \mathcal{C}(-\infty,\infty)$  and from (i1) we have that  $L(u) \in \mathcal{C}([0,1] \times [0,\infty))$ . Also we have

$$\begin{aligned} L(u)|_{r=\infty} &= 0, \\ \frac{\partial}{\partial r} L(u) &= -\sqrt{\frac{h_2(r)}{h_1(r)}} \int_r^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} f(u) \right) d\tau, \\ \frac{\partial}{\partial r} L(u)|_{r=\infty} &= 0. \end{aligned} \quad (3.3)$$

(2) Now we will prove that for every fixed  $t \in [0,1]$  and for every  $r \in [0,\infty)$  we have that  $L(u) \geq 0$ . Really,

$$\begin{aligned} L(u) &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u - \sqrt{h_1(\tau)h_2(\tau)} f(u) \right) d\tau ds \\ &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \min_{t \in [0,1]} \frac{v''(t)}{v(t)} u - \sqrt{h_1(\tau)h_2(\tau)} f(u) \right) d\tau ds \\ &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 u - \sqrt{h_1(\tau)h_2(\tau)} f(u) \right) d\tau ds. \end{aligned} \quad (3.4)$$

Now we suppose that for every fixed  $t \in [0,1]$  and for every  $r \in [0,\infty)$  we have  $u(t,r) \geq 0$ ; from here  $f'(u) \leq bu$ , since  $f(0) = 0$  and  $u(t,r) \leq 1/B$  for every  $t \in [0,1]$  and for every  $r \in [0,\infty)$  we get

$$\begin{aligned} f(u) &\leq \frac{b}{2} u^2 \leq \frac{b}{2B} u(t,r) \\ &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 u - \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} u \right) d\tau ds \\ &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 - \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right) u d\tau ds \\ &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \min_{\tau \in [0,\infty)} \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 - \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right) u d\tau ds. \end{aligned} \quad (3.5)$$

From (H1) we have that

$$\min_{\tau \in [0,\infty)} \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 - \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right) \geq 0 \quad (3.6)$$

and since  $u(t, r) \geq 0$  for every fixed  $t \in [0, 1]$  and for every  $r \in [0, \infty)$  we get

$$\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \min_{\tau \in [0, \infty)} \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 - \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right) u \, d\tau \, ds \geq 0, \quad (3.7)$$

that is, for every fixed  $t \in [0, 1]$  and for every  $r \in [0, \infty)$  we have

$$L(u) \geq 0. \quad (3.8)$$

(3) Now we will see that for every fixed  $t \in [0, 1]$  and for every  $r \in [c, d]$  we have  $L(u) \geq 1/A$ . Really, for every  $r \in [c, d]$  we have

$$\begin{aligned} L'(u) &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} - \sqrt{h_1(\tau)h_2(\tau)} f'(u) \right) d\tau \, ds \\ &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 - \frac{b}{B} \sqrt{h_1(\tau)h_2(\tau)} \right) d\tau \, ds \geq 0. \end{aligned} \quad (3.9)$$

(See (H1).) From here, since for every  $r \in [c, d]$  we have  $u(t, r) \geq 1/A$ , we get

$$\begin{aligned} L(u) &\geq L\left(\frac{1}{A}\right) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \frac{1}{A} - \sqrt{h_1(\tau)h_2(\tau)} f\left(\frac{1}{A}\right) \right) d\tau \, ds \\ &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{A_1}{A} - \frac{b}{2A^2} \sqrt{h_1(\tau)h_2(\tau)} \right) d\tau \, ds \\ &= \frac{1}{A} \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 - \frac{b}{2A} \sqrt{h_1(\tau)h_2(\tau)} \right) d\tau \, ds \\ &\geq \frac{1}{A} \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 - \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right) d\tau \, ds \geq \frac{1}{A}. \end{aligned} \quad (3.10)$$

(Here we use (H1).) Consequently, for every fixed  $t \in [0, 1]$  and for every  $r \in [c, d]$  we have that

$$L(u) \geq \frac{1}{A}. \quad (3.11)$$

(4) Now we will prove that for every fixed  $t \in [0, 1]$  and for every  $r \in [0, \infty)$  we have that

$$L(u) \leq \frac{1}{B}. \quad (3.12)$$

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Really, for every fixed  $t \in [0, 1]$  and for every  $r \in [0, \infty)$  we have

$$\begin{aligned}
 L(u) &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u - \sqrt{h_1(\tau)h_2(\tau)} f(u) \right) d\tau ds \\
 &\leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u + \sqrt{h_1(\tau)h_2(\tau)} |f(u)| \right) d\tau ds \\
 &\leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \max_{t \in [0,1]} \frac{v''(t)}{v(t)} |u| + \sqrt{h_1(\tau)h_2(\tau)} |f(u)| \right) d\tau ds.
 \end{aligned} \tag{3.13}$$

Now we suppose that

$$\begin{aligned}
 |f(u)| &\leq \frac{b}{2} u^2 \leq \frac{b}{2B} u, \quad \max_{t \in [0,1]} \frac{v''(t)}{v(t)} \leq A_2, \\
 \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 u + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} u \right) d\tau ds \\
 &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} \right) u d\tau ds;
 \end{aligned} \tag{3.14}$$

here we use

$$\begin{aligned}
 u &\leq \frac{1}{B} \leq \frac{1}{B} \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} \right) d\tau ds \\
 &\leq \frac{1}{B} \max_{r \in [0, \infty)} \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} \right) d\tau ds \leq \frac{1}{B}.
 \end{aligned} \tag{3.15}$$

In the last inequality, we use (H2). Consequently, for every fixed  $t \in [0, 1]$  and for every  $r \in [0, \infty)$  we have

$$L(u) \leq \frac{1}{B}. \tag{3.16}$$

(5) Now we will prove that for every fixed  $t \in [0, 1]$  we have that  $L(u) \in L^2([0, \infty))$ . Really, for every fixed  $t \in [0, 1]$  after we use the inequality (3.13) we get

$$\begin{aligned}
 \int_0^\infty |L(u)|^2 dr \\
 \leq \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} \right) |u| d\tau ds \right)^2 dr.
 \end{aligned} \tag{3.17}$$

Now we use the Hölder inequality

$$\begin{aligned}
& \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} \right) d\tau \right)^2 \right)^{1/2} \\
& \quad \times \left( \int_s^\infty |u|^2 d\tau \right)^{1/2} ds \Big)^2 dr \\
& \leq \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} \right) d\tau \right)^2 \right)^{1/2} \\
& \quad \times \left( \int_0^\infty |u|^2 d\tau \right)^{1/2} ds \Big)^2 dr \\
& = \|u\|_{L^2([0,\infty))}^2 \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} \right) d\tau \right)^2 \right)^{1/2} ds \Big)^2 dr \\
& \leq \|u\|_{L^2([0,\infty))}^2 \frac{1}{7}.
\end{aligned} \tag{3.18}$$

(In the last inequality, we use the condition (H3).) Consequently, for every fixed  $t \in [0, 1]$  we have  $L(u) \in L^2([0, \infty))$ .

From (1), (2), (3), (4), and (5) we get that for every fixed  $t \in [0, 1]$  we have

$$L : M \longrightarrow M. \tag{3.19}$$

Now we will prove that the operator  $L : M \rightarrow M$  is contractive operator. Let  $u_1$  and  $u_2$  be two elements of the set  $M$ . Then, for every fixed  $t \in [0, 1]$  we have

$$\begin{aligned}
& |L(u_1) - L(u_2)| \\
& = \left| \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} (u_1 - u_2) - \sqrt{h_1(\tau)h_2(\tau)} (f(u_1) - f(u_2)) \right) d\tau ds \right| \\
& \leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} |u_1 - u_2| + \sqrt{h_1(\tau)h_2(\tau)} |f(u_1) - f(u_2)| \right) d\tau ds,
\end{aligned} \tag{3.20}$$

then from the middle-point theorem we have  $|f(u_1) - f(u_2)| = |f'(\xi)||u_1 - u_2|$ ,  $|\xi| \leq \max\{|u_1|, |u_2|\}$ ,  $|f'(\xi)| \leq b|\xi| \leq b/B$ ,

$$\begin{aligned} & \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} |u_1 - u_2| + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} |u_1 - u_2| \right) d\tau ds \\ &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) |u_1 - u_2| d\tau ds \\ &\leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \max_{t \in [0,1]} \frac{v''(t)}{v(t)} + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) |u_1 - u_2| d\tau ds \\ &\leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) |u_1 - u_2| d\tau ds, \end{aligned} \tag{3.21}$$

that is, for every fixed  $t \in [0, 1]$  and for every  $r \in [0, \infty)$  we have

$$|L(u_1) - L(u_2)| \leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) |u_1 - u_2| d\tau ds. \tag{3.22}$$

From here

$$\begin{aligned} & \|L(u_1) - L(u_2)\|_{L^2([0, \infty))}^2 \\ &\leq \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) |u_1 - u_2| d\tau ds \right)^2 dr. \end{aligned} \tag{3.23}$$

Now we will use the Hölder inequality

$$\begin{aligned} & \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) d\tau \right)^{1/2} \right. \\ & \quad \left. \times \left( \int_s^\infty |u_1 - u_2|^2 d\tau \right)^{1/2} ds \right)^2 dr \\ &\leq \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) d\tau \right)^2 \right)^{1/2} \\ & \quad \times \left( \int_0^\infty |u_1 - u_2|^2 d\tau \right)^{1/2} ds \Big)^2 dr \\ &= \|u_1 - u_2\|_{L^2([0, \infty))}^2 \\ &\times \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) d\tau \right)^2 \right)^{1/2} ds \Big)^2 dr, \end{aligned} \tag{3.24}$$

that is, for every fixed  $t \in [0, 1]$  we have

$$\begin{aligned} & \|L(u_1) - L(u_2)\|_{L^2([0, \infty))}^2 \\ & \leq \|u_1 - u_2\|_{L^2([0, \infty))}^2 \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) d\tau \right)^{1/2} ds \right)^2 dr, \end{aligned} \quad (3.25)$$

from (H4) we have

$$\int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) d\tau \right)^{1/2} ds \right)^2 dr < 1. \quad (3.26)$$

From here and from (3.25) we get

$$\|L(u_1) - L(u_2)\|_{L^2([0, \infty))}^2 < \|u_1 - u_2\|_{L^2([0, \infty))}^2. \quad (3.27)$$

Consequently, the operator  $L : M \rightarrow M$  is contractive operator. We note that the set  $M$  is closed subset of the space  $\mathcal{C}((0, 1]L^2([0, \infty)))$  (for the proof see Lemma A.1 in the appendix of this paper). Therefore, (1\*) has unique nontrivial solution in the set  $M$ .  $\square$

Let  $\tilde{u}$  be the solution from Theorem 3.1, that is,  $\tilde{u}$  is a solution to the integral equation (1\*). From Proposition 2.2, we have that  $\tilde{u}$  satisfies (1). Consequently,  $\tilde{u}$  is solution to the Cauchy problem (1), (2) with initial data

$$\begin{aligned} u_0 &= v(1)\omega(r) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} v''(1)\omega(\tau) - \sqrt{h_1(\tau)h_2(\tau)} f(v(1)\omega(\tau)) \right) d\tau ds, \\ u_1 &\equiv 0. \end{aligned} \quad (3.28)$$

We have  $\tilde{u} \in \mathcal{C}((0, 1]L^2([0, \infty)))$ ,  $u_0 \in L^2([0, \infty))$ ,  $u_1 \in \dot{H}^{-1}([0, \infty))$ .

**3.2. Blow up of the solutions of homogeneous Cauchy problem (1), (2).** Let  $v(t)$  be the same function as in Theorem 3.1.

**THEOREM 3.2.** *Let  $n \geq 2$  be fixed, let  $h_1(r)$ ,  $h_2(r)$  fixed, which satisfy the conditions (i1), be the positive constants  $a$ ,  $b$  be fixed, let the positive constants  $c$ ,  $d$ ,  $A$ ,  $B$ ,  $A_1$ ,  $A_2$  be fixed which satisfy the conditions (H1), ..., (H4) and  $f \in \mathcal{C}^1((-\infty, \infty))$ ,  $f(0) = 0$ ,  $a|u| \leq f'(u) \leq b|u|$ . Then for the solution  $\tilde{u}$  of the Cauchy problem (1), (2) one has*

$$\lim_{t \rightarrow 0} \|\tilde{u}\|_{L^2([0, \infty))} = \infty. \quad (3.29)$$

*Proof.* For every fixed  $t \in (0, 1]$  and for every  $r \in [0, \infty)$  we have

$$\tilde{u}(t, r) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \tilde{u}(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} f(\tilde{u}) \right) d\tau ds. \quad (3.30)$$

Since for every fixed  $t \in (0, 1]$  and for every  $r \in [c, d]$  we have that  $\tilde{u} \geq 1/A$  follows that there exists subinterval  $\Delta$  in  $[0, \infty)$  such that

$$\tilde{u} \geq \frac{1}{A} \quad \text{for } r \in \Delta. \quad (3.31)$$

Let us fix the subinterval  $\Delta$ . From here

$$\begin{aligned} & \|\tilde{u}\|_{L^2([0, \infty))}^2 \\ &= \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \tilde{u}(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} f(\tilde{u}) \right) d\tau ds \right)^2 dr \\ &= \int_{[0, \infty) \setminus \Delta} \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \tilde{u}(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} f(\tilde{u}) \right) d\tau ds \right)^2 dr \\ &\quad + \int_\Delta \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \tilde{u}(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} f(\tilde{u}) \right) d\tau ds \right)^2 dr. \end{aligned} \quad (3.32)$$

Let

$$\begin{aligned} I_1 &:= \int_{[0, \infty) \setminus \Delta} \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \tilde{u}(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} f(\tilde{u}) \right) d\tau ds \right)^2 dr, \\ I_2 &:= \int_\Delta \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \tilde{u}(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} f(\tilde{u}) \right) d\tau ds \right)^2 dr. \end{aligned} \quad (3.33)$$

Then

$$\|\tilde{u}\|_{L^2([0, \infty))}^2 = I_1 + I_2. \quad (3.34)$$

For  $I_1$  we have the following estimate:

$$\begin{aligned} I_1 &\leq \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \tilde{u}(t, \tau) + \sqrt{h_1(\tau)h_2(\tau)} |f(\tilde{u})| \right) d\tau ds \right)^2 dr \\ &\leq \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \max_{t \in [0, 1]} \frac{v''(t)}{v(t)} \tilde{u}(t, \tau) + \sqrt{h_1(\tau)h_2(\tau)} |f(\tilde{u})| \right) d\tau ds \right)^2 dr \\ &\leq \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 \tilde{u}(t, \tau) + \sqrt{h_1(\tau)h_2(\tau)} |f(\tilde{u})| \right) d\tau ds \right)^2 dr. \end{aligned} \quad (3.35)$$

Now we suppose that  $f'(\tilde{u}) \leq b\tilde{u}$ ,  $f(0) = 0$ ,  $\tilde{u} \leq 1/B$  for every fixed  $t \in [0, 1]$  and for every  $r \in [0, \infty)$ . Therefore  $|f(\tilde{u})| \leq (b/2)\tilde{u}^2 \leq (b/2B)\tilde{u}$  for every fixed  $t \in [0, 1]$  and for every  $r \in [0, \infty)$ .

$$\begin{aligned} & \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 \tilde{u}(t, \tau) + \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \tilde{u} \right) d\tau ds \right)^2 dr \\ &= \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right) \tilde{u} d\tau ds \right)^2 dr. \end{aligned} \quad (3.36)$$

Now we apply the Hölder inequality

$$\begin{aligned} & \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right) d\tau \right)^{1/2} \left( \int_s^\infty |\tilde{u}|^2 d\tau \right)^{1/2} ds \right)^2 dr \\ & \leq \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right) d\tau \right)^{1/2} \left( \int_0^\infty |\tilde{u}|^2 d\tau \right)^{1/2} ds \right)^2 dr \\ & = \|\tilde{u}\|_{L^2([0, \infty))}^2 \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right) d\tau \right)^{1/2} ds \right)^2 dr. \end{aligned} \quad (3.37)$$

Let

$$Q := \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right) d\tau \right)^{1/2} ds \right)^2 dr. \quad (3.38)$$

Then

$$I_1 \leq Q \|\tilde{u}\|_{L^2([0, \infty))}^2. \quad (3.39)$$

Now we consider  $I_2$ . For it we have

$$\begin{aligned} I_2 &= \int_\Delta \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \tilde{u}(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} f(\tilde{u}) \right) d\tau ds \right)^2 dr \\ &\leq \int_\Delta \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_0^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \tilde{u}(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} f(\tilde{u}) \right) d\tau ds \right)^2 dr \\ &= \int_\Delta \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_\Delta \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \tilde{u}(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} f(\tilde{u}) \right) d\tau ds \right. \\ &\quad \left. + \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_{[0, \infty) \setminus \Delta} \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \tilde{u}(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} f(\tilde{u}) \right) d\tau ds \right)^2 dr. \end{aligned} \quad (3.40)$$

Let

$$\begin{aligned}
 I_{21} &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_\Delta \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \tilde{u}(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} f(\tilde{u}) \right) d\tau ds, \\
 I_{22} &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_{[0, \infty) \setminus \Delta} \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \tilde{u}(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} f(\tilde{u}) \right) d\tau ds.
 \end{aligned}
 \tag{3.41}$$

Consequently,

$$I_2 = \int_\Delta (I_{21} + I_{22})^2 dr \leq 2 \int_\Delta I_{21}^2 dr + 2 \int_\Delta I_{22}^2 dr.
 \tag{3.42}$$

Also we have

$$\begin{aligned}
 &\int_\Delta I_{21}^2 dr \\
 &= \int_\Delta \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_\Delta \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \tilde{u}(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} f(\tilde{u}) \right) d\tau ds \right)^2 dr \\
 &= \int_\Delta \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_\Delta \sqrt{h_1(\tau)h_2(\tau)} \left( \frac{1}{h_1(\tau)} \frac{v''(t)}{v(t)} \tilde{u}(t, \tau) - f(\tilde{u}) \right) d\tau ds \right)^2 dr \\
 &\leq \int_\Delta \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_\Delta \max_{\tau \in [0, \infty)} \sqrt{h_1(\tau)h_2(\tau)} \left( \max_{\tau \in [0, \infty)} \frac{1}{h_1(\tau)} \frac{v''(t)}{v(t)} \tilde{u}(t, \tau) - f(\tilde{u}) \right) d\tau ds \right)^2 dr.
 \end{aligned}
 \tag{3.43}$$

Case 1. Let

$$\max_{r \in [0, \infty)} \frac{1}{h_1(r)} \leq \frac{1}{A}.
 \tag{3.44}$$

Then, after we suppose that for every fixed  $t \in [0, 1]$  and for every  $r \in \Delta$  we have  $f(\tilde{u}) \geq (a/2)\tilde{u}^2 \geq (a/2A)\tilde{u}$ , we get

$$\begin{aligned}
 &\int_\Delta \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_\Delta \max_{\tau \in [0, \infty)} \sqrt{h_1(\tau)h_2(\tau)} \left( \max_{\tau \in [0, \infty)} \frac{1}{h_1(\tau)} \frac{v''(t)}{v(t)} \tilde{u}(t, \tau) - \frac{a}{2A} \tilde{u} \right) d\tau ds \right)^2 dr \\
 &\leq \int_\Delta \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_\Delta \max_{\tau \in [0, \infty)} \sqrt{h_1(\tau)h_2(\tau)} \frac{1}{A} \left( \frac{v''(t)}{v(t)} - \frac{a}{2} \right) \tilde{u} d\tau ds \right)^2 dr.
 \end{aligned}
 \tag{3.45}$$

Now we suppose that for every fixed  $t \in [0, 1]$  and for every  $r \in \Delta$  we have  $\tilde{u} \geq 1/A$ ,  $A\tilde{u} \geq 1$ ,

$$\begin{aligned}
& \left( \max_{\tau \in [0, \infty)} \sqrt{h_1(\tau)h_2(\tau)} \frac{1}{A} \left( \frac{v''(t)}{v(t)} - \frac{a}{2} \right) \right)^2 \int_{\Delta} \left( \int_r^{\infty} \sqrt{\frac{h_2(s)}{h_1(s)}} \int_{\Delta} A\tilde{u} \frac{1}{A} d\tau ds \right)^2 dr \\
& \leq \left( \max_{\tau \in [0, \infty)} \sqrt{h_1(\tau)h_2(\tau)} \frac{1}{A} \left( \frac{v''(t)}{v(t)} - \frac{a}{2} \right) \right)^2 \int_{\Delta} \left( \int_r^{\infty} \sqrt{\frac{h_2(s)}{h_1(s)}} \int_{\Delta} A^2 \tilde{u}^2 \frac{1}{A} d\tau ds \right)^2 dr \\
& \leq A^2 \left( \max_{\tau \in [0, \infty)} \sqrt{h_1(\tau)h_2(\tau)} \frac{1}{A} \left( \frac{v''(t)}{v(t)} - \frac{a}{2} \right) \right)^2 \int_0^{\infty} \left( \int_r^{\infty} \sqrt{\frac{h_2(s)}{h_1(s)}} \int_0^{\infty} \tilde{u}^2 d\tau ds \right)^2 dr \\
& = \|\tilde{u}\|_{L^2([0, \infty))}^4 A^2 \left( \max_{\tau \in [0, \infty)} \sqrt{h_1(\tau)h_2(\tau)} \frac{1}{A} \left( \frac{v''(t)}{v(t)} - \frac{a}{2} \right) \right)^2 \int_0^{\infty} \left( \int_r^{\infty} \sqrt{\frac{h_2(s)}{h_1(s)}} ds \right)^2 dr,
\end{aligned} \tag{3.46}$$

that is,

$$\int_0^{\infty} I_{21}^2 dr \leq \|\tilde{u}\|_{L^2([0, \infty))}^4 A^2 \left( \max_{\tau \in [0, \infty)} \sqrt{h_1(\tau)h_2(\tau)} \frac{1}{A} \left( \frac{v''(t)}{v(t)} - \frac{a}{2} \right) \right)^2 \int_0^{\infty} \left( \int_r^{\infty} \sqrt{\frac{h_2(s)}{h_1(s)}} ds \right)^2 dr. \tag{3.47}$$

Let

$$F = A^2 \left( \max_{\tau \in [0, \infty)} \sqrt{h_1(\tau)h_2(\tau)} \frac{1}{A} \left( \frac{v''(t)}{v(t)} - \frac{a}{2} \right) \right)^2 \int_0^{\infty} \left( \int_r^{\infty} \sqrt{\frac{h_2(s)}{h_1(s)}} ds \right)^2 dr. \tag{3.48}$$

Then

$$\int_{\Delta} I_{21}^2 dr \leq F \|\tilde{u}\|_{L^2[0, \infty)}^4. \tag{3.49}$$

Case 2. Let

$$\max_{r \in [0, \infty)} \frac{1}{h_1(r)} \geq \frac{1}{A}. \tag{3.50}$$

Let

$$\max_{r \in [0, \infty)} \frac{1}{h_1(r)} = G. \tag{3.51}$$

Then

$$\begin{aligned}
\int_{\Delta} I_{21}^2 dr & \leq 2 \int_{\Delta} \left( \int_r^{\infty} \sqrt{\frac{h_2(s)}{h_1(s)}} \int_{\Delta} \max_{\tau \in [0, \infty)} \sqrt{h_1(\tau)h_2(\tau)} \left( \left( G - \frac{1}{A} \right) \frac{v''(t)}{v(t)} \tilde{u}(t, \tau) - f(\tilde{u}) \right) d\tau ds \right)^2 dr \\
& \quad + 2 \int_{\Delta} \left( \int_r^{\infty} \sqrt{\frac{h_2(s)}{h_1(s)}} \int_{\Delta} \max_{\tau \in [0, \infty)} \sqrt{h_1(\tau)h_2(\tau)} \frac{1}{A} \left( \frac{v''(t)}{v(t)} \tilde{u}(t, \tau) - \frac{a}{2} \tilde{u} \right) d\tau ds \right)^2 dr.
\end{aligned} \tag{3.52}$$

As mentioned above

$$2Q\|\tilde{u}\|_{L^2([0,\infty))}^2 + 2F\|\tilde{u}\|_{L^2([0,\infty))}^4, \tag{3.53}$$

that is,

$$\int_{\Delta} I_{21}^2 dr \leq 2Q\|\tilde{u}\|_{L^2([0,\infty))}^2 + 2F\|\tilde{u}\|_{L^2([0,\infty))}^4. \tag{3.54}$$

From (3.49) and (3.54) we have

$$\int_{\Delta} I_{21}^2 dr \leq 2Q\|\tilde{u}\|_{L^2([0,\infty))}^2 + 2F\|\tilde{u}\|_{L^2([0,\infty))}^4. \tag{3.55}$$

As in the estimate for  $I_1$  we have

$$\int_0^{\infty} I_{22}^2 dr \leq Q\|\tilde{u}\|_{L^2[0,\infty)}^2. \tag{3.56}$$

From (3.42), (3.55), and (3.56) we get

$$I_2 \leq 4F\|\tilde{u}\|_{L^2([0,\infty))}^4 + 6Q\|\tilde{u}\|_{L^2[0,\infty)}^2. \tag{3.57}$$

From the last inequality and from (3.34), (3.39) we have

$$\begin{aligned} \|\tilde{u}\|_{L^2([0,\infty))}^2 &\leq 4F\|\tilde{u}\|_{L^2([0,\infty))}^4 + 7Q\|\tilde{u}\|_{L^2([0,\infty))}^2, \\ (1 - 7Q)\|\tilde{u}\|_{L^2([0,\infty))}^2 &\leq 4F\|\tilde{u}\|_{L^2([0,\infty))}^4. \end{aligned} \tag{3.58}$$

From (H3) we have

$$Q < \frac{1}{7}, \tag{3.59}$$

from here

$$\|\tilde{u}\|_{L^2([0,\infty))}^2 \geq \frac{1 - 7Q}{4F}. \tag{3.60}$$

From (H7) we have

$$\lim_{t \rightarrow 0} F = +0. \tag{3.61}$$

Therefore

$$\lim_{t \rightarrow 0} \|\tilde{u}\|_{L^2([0,\infty))} = \infty. \tag{3.62}$$

□

#### 4. Proof of Theorem 1.2

**4.1. Local existence of nontrivial solutions of nonhomogeneous Cauchy problem (1), (2).** In this section we will prove that the nonhomogeneous Cauchy problem (1), (2) has nontrivial solution in the form  $u(t, r) = v(t)\omega(r)$ .

Let us consider the integral equation

$$u(t, r) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} (f(u) + g(\tau)) \right) d\tau ds. \quad (1^{**})$$

**THEOREM 4.1.** *Let  $n \geq 2$  be fixed, let  $h_1(r), h_2(r)$  be fixed, which satisfy the conditions (i1), the positive constants  $a, b, a \leq b$  are fixed, and let the positive constants  $c, d, A, B, A_1, A_2$  be fixed which satisfy the conditions (H1), ..., (H4) and  $f \in \mathcal{C}^1((-\infty, \infty))$ ,  $f(0) = 0$ ,  $a|u| \leq f'(u) \leq b|u|$ ,  $b/2 \geq f(1) \geq a/2$ ,  $g \in \mathcal{C}([0, \infty))$ ,  $g(r) \geq 0$  for every  $r \in [0, \infty)$ ,  $g(r) \leq b/2 - f(1)$  for every  $r \in [0, \infty)$ . Let also  $v(t)$  be fixed function which satisfies the conditions (H5), ..., (H7). Then (1<sup>\*\*</sup>) has unique nontrivial solution  $u(t, r) = v(t)\omega(r)$  for which  $u(t, r) \in \mathcal{C}([0, 1] \times [0, \infty))$ ,  $u(t, r) \leq 1/B$  for every  $t \in [0, 1]$  and for every  $r \in [0, \infty)$ , for every  $t \in [0, 1]$  and for every  $r \in [c, d]$ ,  $u(t, r) \geq 1/A$ ,  $u(t, r) \geq 0$  for every  $t \in [0, 1]$  and for every  $r \in [0, \infty)$ ,  $u(t, \infty) = u_r(t, \infty) = 0$  for every  $t \in [0, 1]$ ,  $u(t, r) \in C((0, 1]L^2([0, \infty)))$ .*

*Proof.* Let  $M$  be the set of the proof of Theorem 3.1. Let  $t \in [0, 1]$  be fixed. We define the operator  $R$  as follows:

$$R(u)(t, r) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} (f(u) + g(\tau)) \right) d\tau ds \quad (4.1)$$

for  $u \in M$ . First we will see that  $R: M \rightarrow M$ . Let  $u \in M$ . Then the following holds.

(1) Since  $v(t) \in \mathcal{C}^4([0, 1])$ ,  $u(t, r) \in \mathcal{C}([0, 1] \times [0, \infty))$  and  $f \in \mathcal{C}(-\infty, \infty)$  and from (i1) we have that  $R(u) \in \mathcal{C}([0, 1] \times [0, \infty))$ . Also we have

$$\begin{aligned} R(u)|_{r=\infty} &= 0, \\ \frac{\partial}{\partial r} R(u) &= -\sqrt{\frac{h_2(r)}{h_1(r)}} \int_r^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u(t, \tau) - \sqrt{h_1(\tau)h_2(\tau)} (f(u) + g(\tau)) \right) d\tau, \\ \frac{\partial}{\partial r} R(u)|_{r=\infty} &= 0. \end{aligned} \quad (4.2)$$

(2) Now we will prove that for every fixed  $t \in [0, 1]$  and for every  $r \in [0, \infty)$  we have that  $R(u) \geq 0$ . Really,

$$\begin{aligned} R(u) &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u - \sqrt{h_1(\tau)h_2(\tau)} (f(u) + g(\tau)) \right) d\tau ds \\ &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \min_{t \in [0,1]} \frac{v''(t)}{v(t)} u - \sqrt{h_1(\tau)h_2(\tau)} (f(u) + g(\tau)) \right) d\tau ds \\ &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 u - \sqrt{h_1(\tau)h_2(\tau)} (f(u) + g(\tau)) \right) d\tau ds. \end{aligned} \tag{4.3}$$

Now we suppose that for every fixed  $t \in [0, 1]$  and for every  $r \in [0, \infty)$  we have  $u(t, r) \geq 0$ , from here  $f'(u) \leq bu$  after integration from 1 to  $u$  then we get that

$$\begin{aligned} f(u) - f(1) &\leq \frac{b}{2}u^2 - \frac{b}{2}, \\ f(u) &\leq \frac{b}{2}u^2 - \frac{b}{2} + f(1), \\ f(u) + g(r) &\leq \frac{b}{2}u^2 - \frac{b}{2} + f(1) + g(r). \end{aligned} \tag{4.4}$$

From the conditions of Theorem 4.1 we have that

$$g(r) \leq \frac{b}{2} - f(1). \tag{4.5}$$

Therefore

$$f(u) + g(r) \leq \frac{b}{2}u^2. \tag{4.6}$$

From here and from  $u(t, r) \leq 1/B$  for every fixed  $t \in [0, 1]$  and for every  $r \in [0, \infty)$  we get

$$f(u) + g(r) \leq \frac{b}{2B}u. \tag{4.7}$$

Consequently, from (4.3) we get

$$\begin{aligned} R(u) &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 u - \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} u \right) d\tau ds \\ &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 - \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right) u d\tau ds \\ &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \min_{\tau \in [0, \infty)} \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 - \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right) u d\tau ds. \end{aligned} \tag{4.8}$$

From (H1) we have that

$$\min_{\tau \in [0, \infty)} \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 - \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right) \geq 0 \quad (4.9)$$

and since  $u(t, r) \geq 0$  for every fixed  $t \in [0, 1]$  and for every  $r \in [0, \infty)$  from (4.8) we get

$$\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \min_{\tau \in [0, \infty)} \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 - \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right) u \, d\tau \, ds \geq 0, \quad (4.10)$$

that is, for every fixed  $t \in [0, 1]$  and for every  $r \in [0, \infty)$  we have

$$R(u) \geq 0. \quad (4.11)$$

(3) Now we will see that for every fixed  $t \in [0, 1]$  and for every  $r \in [c, d]$  we have  $R(u) \geq 1/A$ . Really, after we use (4.7),

$$R(u) \geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 - \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right) u \, d\tau \, ds. \quad (4.12)$$

Let

$$R_1(u) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 - \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right) u \, d\tau \, ds. \quad (4.13)$$

From here

$$R'_1(u) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 - \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right) d\tau \, ds \geq 0. \quad (4.14)$$

Since for every  $r \in [c, d]$  we have  $u(t, r) \geq 1/A$ , we get

$$\begin{aligned} R_1(u) &\geq R_1\left(\frac{1}{A}\right) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 - \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right) \frac{1}{A} d\tau \, ds \\ &= \frac{1}{A} \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_1 - \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right) d\tau \, ds \geq \frac{1}{A} \end{aligned} \quad (4.15)$$

(in the last inequality we use (H1)). Consequently, for every fixed  $t \in [0, 1]$  and for every  $r \in [c, d]$  we have that

$$R(u) \geq \frac{1}{A}. \tag{4.16}$$

(4) Now we will prove that for every fixed  $t \in [0, 1]$  and for every  $r \in [0, \infty)$  we have that

$$R(u) \leq \frac{1}{B}. \tag{4.17}$$

Really, for every fixed  $t \in [0, 1]$  and for every  $r \in [0, \infty)$  we have

$$\begin{aligned} R(u) &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u - \sqrt{h_1(\tau)h_2(\tau)}(f(u) + g(\tau)) \right) d\tau ds \\ &\leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} u + \sqrt{h_1(\tau)h_2(\tau)}(f(u) + g(\tau)) \right) d\tau ds \\ &\leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \max_{t \in [0,1]} \frac{v''(t)}{v(t)} |u| + \sqrt{h_1(\tau)h_2(\tau)}(f(u) + g(\tau)) \right) d\tau ds. \end{aligned} \tag{4.18}$$

Now we use (4.7)

$$\begin{aligned} &\int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 u + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} u \right) d\tau ds \\ &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} \right) u d\tau ds. \end{aligned} \tag{4.19}$$

here we use

$$\begin{aligned} u &\leq \frac{1}{B} \leq \frac{1}{B} \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} \right) d\tau ds \\ &\leq \frac{1}{B} \max_{r \in [0, \infty)} \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} \right) d\tau ds \leq \frac{1}{B}. \end{aligned} \tag{4.20}$$

In the last inequality, we use (H2). Consequently, for every fixed  $t \in [0, 1]$  and for every  $r \in [0, \infty)$  we have

$$R(u) \leq \frac{1}{B}. \quad (4.21)$$

(5) Now we will prove that for every fixed  $t \in [0, 1]$  we have that  $R(u) \in L^2([0, \infty))$ . Really, for every fixed  $t \in [0, 1]$  after we use the inequality (4.18) we get

$$\begin{aligned} & \int_0^\infty |R(u)|^2 dr \\ & \leq \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} \right) |u| d\tau ds \right)^2 dr. \end{aligned} \quad (4.22)$$

Now we use the Hölder inequality

$$\begin{aligned} & \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} \right) d\tau \right)^{1/2} \right. \\ & \quad \left. \times \left( \int_s^\infty |u|^2 d\tau \right)^{1/2} ds \right)^2 dr \\ & \leq \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} \right)^2 d\tau \right)^{1/2} \right. \\ & \quad \left. \times \left( \int_0^\infty |u|^2 d\tau \right)^{1/2} ds \right)^2 dr \\ & = \|u\|_{L^2([0, \infty))}^2 \\ & \quad \times \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{2B} \right)^2 d\tau \right)^{1/2} ds \right)^2 dr \\ & \leq \|u\|_{L^2([0, \infty))}^2 \frac{1}{7}. \end{aligned} \quad (4.23)$$

(In the last inequality, we use the condition (H3).) Consequently, for every fixed  $t \in [0, 1]$  we have  $R(u) \in L^2([0, \infty))$ .

From (1), (2), (3), (4), and (5) we get that for every fixed  $t \in [0, 1]$  we have

$$R: M \longrightarrow M. \quad (4.24)$$

Now we will prove that the operator  $R: M \rightarrow M$  is contractive operator. Let  $u_1$  and  $u_2$  be two elements of the set  $M$ . Then for every fixed  $t \in [0, 1]$  we have

$$\begin{aligned} & |R(u_1) - R(u_2)| \\ &= \left| \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} (u_1 - u_2) - \sqrt{h_1(\tau)h_2(\tau)} (f(u_1) - f(u_2)) \right) d\tau ds \right| \\ &\leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} |u_1 - u_2| + \sqrt{h_1(\tau)h_2(\tau)} |f(u_1) - f(u_2)| \right) d\tau ds, \end{aligned} \quad (4.25)$$

then from the middle-point theorem we have  $|f(u_1) - f(u_2)| = |f'(\xi)||u_1 - u_2|$ ,  $|\xi| \leq \max\{|u_1|, |u_2|\}$ ,  $|f'(\xi)| \leq b|\xi| \leq b/B$ ,

$$\begin{aligned} & \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} |u_1 - u_2| + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} |u_1 - u_2| \right) d\tau ds \\ &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) |u_1 - u_2| d\tau ds \\ &\leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \max_{t \in [0,1]} \frac{v''(t)}{v(t)} + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) |u_1 - u_2| d\tau ds \\ &\leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) |u_1 - u_2| d\tau ds, \end{aligned} \quad (4.26)$$

that is, for every fixed  $t \in [0, 1]$  and for every  $r \in [0, \infty)$  we have

$$|R(u_1) - R(u_2)| \leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) |u_1 - u_2| d\tau ds. \quad (4.27)$$

From here

$$\begin{aligned}
& \|R(u_1) - R(u_2)\|_{L^2([0, \infty))}^2 \\
& \leq \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau) h_2(\tau)} \frac{b}{B} \right) |u_1 - u_2| d\tau ds \right)^2 dr.
\end{aligned} \tag{4.28}$$

Now we will use the Hölder inequality

$$\begin{aligned}
& \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau) h_2(\tau)} \frac{b}{B} \right)^2 d\tau \right)^{1/2} \right. \\
& \quad \left. \times \left( \int_s^\infty |u_1 - u_2|^2 d\tau \right)^{1/2} ds \right)^2 dr \\
& \leq \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau) h_2(\tau)} \frac{b}{B} \right)^2 d\tau \right)^{1/2} \right. \\
& \quad \left. \times \left( \int_0^\infty |u_1 - u_2|^2 d\tau \right)^{1/2} ds \right)^2 dr \\
& = \|u_1 - u_2\|_{L^2([0, \infty))}^2 \\
& \quad \times \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau) h_2(\tau)} \frac{b}{B} \right)^2 d\tau \right)^{1/2} ds \right)^2 dr,
\end{aligned} \tag{4.29}$$

that is, for every fixed  $t \in [0, 1]$  we have

$$\begin{aligned}
& \|R(u_1) - R(u_2)\|_{L^2([0, \infty))}^2 \\
& \leq \|u_1 - u_2\|_{L^2([0, \infty))}^2 \\
& \quad \times \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau) h_2(\tau)} \frac{b}{B} \right)^2 d\tau \right)^{1/2} ds \right)^2 dr,
\end{aligned} \tag{4.30}$$

from (H4) we have

$$\int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau) h_2(\tau)} \frac{b}{B} \right)^2 d\tau \right)^{1/2} ds \right)^2 dr < 1. \tag{4.31}$$

From here and from (4.30) we get

$$\|R(u_1) - R(u_2)\|_{L^2([0,\infty))}^2 < \|u_1 - u_2\|_{L^2([0,\infty))}. \tag{4.32}$$

Consequently, the operator  $R : M \rightarrow M$  is contractive operator. We note that the set  $M$  is closed subset of the space  $\mathcal{C}((0,1]L^2([0,\infty)))$  (for the proof see Lemma A.1 in the appendix of this paper). Therefore  $(1^{**})$  has unique nontrivial solution in the set  $M$ .  $\square$

Let  $\bar{u}$  be the solution from Theorem 4.1, that is,  $\bar{u}$  is a solution to the integral equation  $(1^{**})$ . From Proposition 2.4, we have that  $\tilde{u}$  satisfies (1). Consequently,  $\bar{u}$  is solution to the nonhomogeneous Cauchy problem (1), (2) with initial data

$$\begin{aligned} u_o &= v(1)\omega(r) \\ &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} v''(1)\omega(\tau) - \sqrt{h_1(\tau)h_2(\tau)}(f(v(1)\omega(\tau)) + g(\tau)) \right) d\tau ds, \\ u_1 &\equiv 0. \end{aligned} \tag{4.33}$$

We have  $\bar{u} \in \mathcal{C}((0,1]L^2([0,\infty)))$ ,  $u_o \in L^2([0,\infty))$ ,  $u_1 \in \dot{H}^{-1}([0,\infty))$ .

**4.2. Blow up of the solutions of nonhomogeneous Cauchy problem (1), (2).** Let  $v(t)$  be the same function as in Theorem 4.1.

**THEOREM 4.2.** *Let  $n \geq 2$  be fixed, let  $h_1(r), h_2(r)$  be fixed, which satisfy the conditions (i1), let the positive constants  $a, b, a \leq b$  be fixed, and let the positive constants  $c, d, A, B, A_1, A_2$  be fixed which satisfy the conditions (H1), ..., (H4) and  $f \in \mathcal{C}^1((-\infty, \infty))$ ,  $f(0) = 0$ ,  $a|u| \leq f'(u) \leq b|u|$ ,  $b/2 \geq f(1) \geq a/2$ ,  $g(r) \in \mathcal{C}([0, \infty))$ ,  $g(r) \geq 0$  for every  $r \geq 0$ ,  $g(r) \leq b/2 - f(1)$  for every  $r \geq 0$ . Then for the solution  $\bar{u}$  of the Cauchy problem (1), (2) one has*

$$\lim_{t \rightarrow 0} \|\bar{u}\|_{L^2([0,\infty))} = \infty. \tag{4.34}$$

*Proof.* For every fixed  $t \in (0, 1]$  and for every  $r \in [0, \infty)$  we have

$$\bar{u}(t,r) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \bar{u}(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)}(f(\bar{u}) + g(\tau)) \right) d\tau ds. \tag{4.35}$$

Since for every fixed  $t \in (0, 1]$  and for every  $r \in [c, d]$  we have that  $\bar{u} \geq 1/A$  follows that there exists subinterval  $\Delta$  in  $[0, \infty)$  such that

$$\bar{u} \geq \frac{1}{A} \quad \text{for } r \in \Delta. \quad (4.36)$$

Let us fix the subinterval  $\Delta$ . From here

$$\begin{aligned} & \|\bar{u}\|_{L^2([0, \infty))}^2 \\ &= \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \bar{u}(t, \tau) \right. \right. \\ & \quad \left. \left. - \sqrt{h_1(\tau)h_2(\tau)}(f(\bar{u}) + g(\tau)) \right) d\tau ds \right)^2 dr \\ &= \int_{[0, \infty) \setminus \Delta} \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \bar{u}(t, \tau) \right. \right. \\ & \quad \left. \left. - \sqrt{h_1(\tau)h_2(\tau)}(f(\bar{u}) + g(\tau)) \right) d\tau ds \right)^2 dr \\ &+ \int_\Delta \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \bar{u}(t, \tau) \right. \right. \\ & \quad \left. \left. - \sqrt{h_1(\tau)h_2(\tau)}(f(\bar{u}) + g(\tau)) \right) d\tau ds \right)^2 dr. \end{aligned} \quad (4.37)$$

Let

$$\begin{aligned} I_1 &:= \int_{[0, \infty) \setminus \Delta} \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \bar{u}(t, \tau) \right. \right. \\ & \quad \left. \left. - \sqrt{h_1(\tau)h_2(\tau)}(f(\bar{u}) + g(\tau)) \right) d\tau ds \right)^2 dr, \\ I_2 &:= \int_\Delta \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \bar{u}(t, \tau) \right. \right. \\ & \quad \left. \left. - \sqrt{h_1(\tau)h_2(\tau)}(f(\bar{u}) + g(\tau)) \right) d\tau ds \right)^2 dr. \end{aligned} \quad (4.38)$$

Then

$$\|\bar{u}\|_{L^2([0, \infty))}^2 = I_1 + I_2. \quad (4.39)$$

For  $I_1$  we have the following estimate:

$$\begin{aligned}
 I_1 &\leq \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \bar{u}(t, \tau) \right. \right. \\
 &\quad \left. \left. + \sqrt{h_1(\tau)h_2(\tau)}(f(\bar{u}) + g(\tau)) \right) d\tau ds \right)^2 dr \\
 &\leq \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \max_{t \in [0,1]} \frac{v''(t)}{v(t)} \bar{u}(t, \tau) \right. \right. \\
 &\quad \left. \left. + \sqrt{h_1(\tau)h_2(\tau)}(f(\bar{u}) + g(\tau)) \right) d\tau ds \right)^2 dr \tag{4.40} \\
 &\leq \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 \bar{u}(t, \tau) \right. \right. \\
 &\quad \left. \left. + \sqrt{h_1(\tau)h_2(\tau)}(f(\bar{u}) + g(\tau)) \right) d\tau ds \right)^2 dr.
 \end{aligned}$$

Now we use (4.7)

$$\begin{aligned}
 &\int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 \bar{u}(t, \tau) + \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \bar{u} \right) d\tau ds \right)^2 dr \\
 &= \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right) \bar{u} d\tau ds \right)^2 dr. \tag{4.41}
 \end{aligned}$$

Now we apply the Hölder inequality

$$\begin{aligned}
 &\int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right)^2 d\tau \right)^{1/2} \left( \int_s^\infty |\bar{u}|^2 d\tau \right)^{1/2} ds \right)^2 dr \\
 &\leq \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right)^2 d\tau \right)^{1/2} \left( \int_0^\infty |\bar{u}|^2 d\tau \right)^{1/2} ds \right)^2 dr \\
 &= \|\bar{u}\|_{L^2[0,\infty)}^2 \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right)^2 d\tau \right)^{1/2} ds \right)^2 dr. \tag{4.42}
 \end{aligned}$$

Let

$$Q := \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \frac{b}{2B} \sqrt{h_1(\tau)h_2(\tau)} \right) d\tau \right)^{1/2} ds \right)^2 dr. \quad (4.43)$$

Then

$$I_1 \leq Q \|\bar{u}\|_{L^2([0,\infty))}^2. \quad (4.44)$$

Now we consider  $I_2$ . For it we have

$$\begin{aligned} I_2 &= \int_\Delta \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \bar{u}(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} (f(\bar{u}) + g(\tau)) \right) d\tau ds \right)^2 dr \\ &\leq \int_\Delta \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_0^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \bar{u}(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} (f(\bar{u}) + g(\tau)) \right) d\tau ds \right)^2 dr \\ &= \int_\Delta \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_\Delta \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \bar{u}(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} (f(\bar{u}) + g(\tau)) \right) d\tau ds \right. \\ &\quad \left. + \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_{[0,\infty) \setminus \Delta} \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \bar{u}(t,\tau) \right. \right. \\ &\quad \left. \left. - \sqrt{h_1(\tau)h_2(\tau)} (f(\bar{u}) + g(\tau)) \right) d\tau ds \right)^2 dr. \end{aligned} \quad (4.45)$$

Let

$$\begin{aligned} I_{21} &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_\Delta \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \bar{u}(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} (f(\bar{u}) + g(\tau)) \right) d\tau ds, \\ I_{22} &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_{[0,\infty) \setminus \Delta} \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \bar{u}(t,\tau) - \sqrt{h_1(\tau)h_2(\tau)} (f(\bar{u}) + g(\tau)) \right) d\tau ds. \end{aligned} \quad (4.46)$$

Consequently,

$$I_2 \leq \int_\Delta \left( I_{21} + I_{22} \right)^2 dr \leq 2 \int_\Delta I_{21}^2 dr + 2 \int_\Delta I_{22}^2 dr. \quad (4.47)$$

Also we have

$$\begin{aligned}
 \int_{\Delta} I_{21}^2 dr &= \int_{\Delta} \left( \int_r^{\infty} \sqrt{\frac{h_2(s)}{h_1(s)}} \int_{\Delta} \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} \bar{u}(t, \tau) \right. \right. \\
 &\quad \left. \left. - \sqrt{h_1(\tau)h_2(\tau)} (f(\bar{u}) + g(\tau)) \right) d\tau ds \right)^2 dr \\
 &= \int_{\Delta} \left( \int_r^{\infty} \sqrt{\frac{h_2(s)}{h_1(s)}} \int_{\Delta} \sqrt{h_1(\tau)h_2(\tau)} \left( \frac{1}{h_1(\tau)} \frac{v''(t)}{v(t)} \bar{u}(t, \tau) \right. \right. \\
 &\quad \left. \left. - (f(\bar{u}) + g(\tau)) \right) d\tau ds \right)^2 dr \tag{4.48} \\
 &\leq \int_{\Delta} \left( \int_r^{\infty} \sqrt{\frac{h_2(s)}{h_1(s)}} \int_{\Delta} \max_{\tau \in [0, \infty)} \sqrt{h_1(\tau)h_2(\tau)} \right. \\
 &\quad \left. \times \left( \max_{\tau \in [0, \infty)} \frac{1}{h_1(\tau)} \frac{v''(t)}{v(t)} \bar{u}(t, \tau) - (f(\bar{u}) + g(\tau)) \right) d\tau ds \right)^2 dr.
 \end{aligned}$$

Case 1. Let

$$\max_{r \in [0, \infty)} \frac{1}{h_1(r)} \leq \frac{1}{A}. \tag{4.49}$$

Then after we use that for every fixed  $t \in [0, 1]$  and for every  $r \in \Delta$  we have  $f'(\bar{u}) \geq a\bar{u}$  and integrate the last inequality from 1 to  $\bar{u}$ , we get

$$\begin{aligned}
 f(\bar{u}) &\geq \frac{a}{2} \bar{u}^2 + f(1) - \frac{a}{2}, \\
 f(\bar{u}) + g(r) &\geq \frac{a}{2} \bar{u}^2 - \frac{a}{2} + f(1) + g(r) \\
 &\geq \frac{a}{2} \bar{u}^2 - \frac{a}{2} + f(1) - f(1) + \frac{a}{2} \geq \frac{a}{2} \bar{u}^2 \geq \frac{a}{2A} \bar{u},
 \end{aligned} \tag{4.50}$$

for every  $r \in \Delta$ ,

$$\begin{aligned}
 &\int_{\Delta} \left( \int_r^{\infty} \sqrt{\frac{h_2(s)}{h_1(s)}} \int_{\Delta} \max_{\tau \in [0, \infty)} \sqrt{h_1(\tau)h_2(\tau)} \left( \max_{\tau \in [0, \infty)} \frac{1}{h_1(\tau)} \frac{v''(t)}{v(t)} \bar{u}(t, \tau) - \frac{a}{2A} \bar{u} \right) d\tau ds \right)^2 dr \\
 &\leq \int_{\Delta} \left( \int_r^{\infty} \sqrt{\frac{h_2(s)}{h_1(s)}} \int_{\Delta} \max_{\tau \in [0, \infty)} \sqrt{h_1(\tau)h_2(\tau)} \frac{1}{A} \left( \frac{v''(t)}{v(t)} - \frac{a}{2} \right) \bar{u} d\tau ds \right)^2 dr.
 \end{aligned} \tag{4.51}$$

Now we use that for every fixed  $t \in [0, 1]$  and for every  $r \in \Delta$  we have  $\bar{u} \geq 1/A$ ,  $A\bar{u} \geq 1$ ,

$$\begin{aligned}
& \left( \max_{\tau \in [0, \infty)} \sqrt{h_1(\tau)h_2(\tau)} \frac{1}{A} \left( \frac{v''(t)}{v(t)} - \frac{a}{2} \right) \right)^2 \int_{\Delta} \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_{\Delta} A\bar{u} \frac{1}{A} d\tau ds \right)^2 dr \\
& \leq \left( \max_{\tau \in [0, \infty)} \sqrt{h_1(\tau)h_2(\tau)} \frac{1}{A} \left( \frac{v''(t)}{v(t)} - \frac{a}{2} \right) \right)^2 \int_{\Delta} \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_{\Delta} A^2 \bar{u}^2 \frac{1}{A} d\tau ds \right)^2 dr \\
& \leq A^2 \left( \max_{\tau \in [0, \infty)} \sqrt{h_1(\tau)h_2(\tau)} \frac{1}{A} \left( \frac{v''(t)}{v(t)} - \frac{a}{2} \right) \right)^2 \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_0^\infty \bar{u}^2 d\tau ds \right)^2 dr \\
& \leq A^2 \left( \max_{\tau \in [0, \infty)} \sqrt{h_1(\tau)h_2(\tau)} \frac{1}{A} \left( \frac{v''(t)}{v(t)} - \frac{a}{2} \right) \right)^2 \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_0^\infty \bar{u}^2 d\tau ds \right)^2 dr \\
& = \|\bar{u}\|_{L^2([0, \infty))}^4 A^2 \left( \max_{\tau \in [0, \infty)} \sqrt{h_1(\tau)h_2(\tau)} \frac{1}{A} \left( \frac{v''(t)}{v(t)} - \frac{a}{2} \right) \right)^2 \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} ds \right)^2 dr,
\end{aligned} \tag{4.52}$$

that is,

$$\int_{\Delta} I_{21}^2 dr \leq \|\bar{u}\|_{L^2([0, \infty))}^4 A^2 \left( \max_{\tau \in [0, \infty)} \sqrt{h_1(\tau)h_2(\tau)} \frac{1}{A} \left( \frac{v''(t)}{v(t)} - \frac{a}{2} \right) \right)^2 \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} ds \right)^2 dr. \tag{4.53}$$

Let

$$F = A^2 \left( \max_{\tau \in [0, \infty)} \sqrt{h_1(\tau)h_2(\tau)} \frac{1}{A} \left( \frac{v''(t)}{v(t)} - \frac{a}{2} \right) \right)^2 \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} ds \right)^2 dr. \tag{4.54}$$

Then

$$\int_{\Delta} I_{21}^2 dr \leq F \|\bar{u}\|_{L^2[0, \infty)}^4. \tag{4.55}$$

Case 2. Let

$$\max_{r \in [0, \infty)} \frac{1}{h_1(r)} \geq \frac{1}{A}. \tag{4.56}$$

Let

$$\max_{r \in [0, \infty)} \frac{1}{h_1(r)} = G. \tag{4.57}$$

Then

$$\begin{aligned} \int_{\Delta} I_{21}^2 dr &\leq 2 \int_{\Delta} \left( \int_r^{\infty} \sqrt{\frac{h_2(s)}{h_1(s)}} \int_{\Delta} \max_{\tau \in [0, \infty)} \sqrt{h_1(\tau)h_2(\tau)} \right. \\ &\quad \times \left. \left( \left( G - \frac{1}{A} \right) \frac{v''(t)}{v(t)} \bar{u}(t, \tau) - (f(\bar{u} + g(\tau))) \right) d\tau ds \right)^2 dr \\ &\quad + 2 \int_{\Delta} \left( \int_r^{\infty} \sqrt{\frac{h_2(s)}{h_1(s)}} \int_{\Delta} \max_{\tau \in [0, \infty)} \sqrt{h_1(\tau)h_2(\tau)} \frac{1}{A} \left( \frac{v''(t)}{v(t)} \bar{u}(t, \tau) - \frac{a}{2} \bar{u} \right) d\tau ds \right)^2 dr. \end{aligned} \quad (4.58)$$

As mentioned above, we get

$$2Q \|\bar{u}\|_{L^2([0, \infty))}^2 + 2F \|\bar{u}\|_{L^2([0, \infty))}^4, \quad (4.59)$$

that is,

$$\int_{\Delta} I_{21}^2 dr \leq 2Q \|\bar{u}\|_{L^2([0, \infty))}^2 + 2F \|\bar{u}\|_{L^2([0, \infty))}^4. \quad (4.60)$$

From (4.55) and (4.60) we have

$$\int_{\Delta} I_{21}^2 dr \leq 2Q \|\bar{u}\|_{L^2([0, \infty))}^2 + 2F \|\bar{u}\|_{L^2([0, \infty))}^4. \quad (4.61)$$

As in the estimate for  $I_1$  we have

$$\int_{\Delta} I_{22}^2 dr \leq \int_0^{\infty} I_{22}^2 dr \leq Q \|\bar{u}\|_{L^2([0, \infty))}^2. \quad (4.62)$$

From (4.47), (4.61), and (4.62) we get

$$I_2 \leq 4F \|\bar{u}\|_{L^2([0, \infty))}^4 + 6Q \|\bar{u}\|_{L^2([0, \infty))}^2. \quad (4.63)$$

From the last inequality and from (4.39), (4.44) we have

$$\begin{aligned} \|\bar{u}\|_{L^2([0, \infty))}^2 &\leq 4F \|\bar{u}\|_{L^2([0, \infty))}^4 + 7Q \|\bar{u}\|_{L^2([0, \infty))}^2, \\ (1 - 7Q) \|\bar{u}\|_{L^2([0, \infty))}^2 &\leq 4F \|\bar{u}\|_{L^2([0, \infty))}^4. \end{aligned} \quad (4.64)$$

From (H3) we have

$$Q < \frac{1}{7}, \quad (4.65)$$

from here

$$\|\bar{u}\|_{L^2([0, \infty))}^2 \geq \frac{1 - 7Q}{4F}. \quad (4.66)$$

From (H7) we have

$$\lim_{t \rightarrow 0} F = +0. \quad (4.67)$$

Therefore

$$\lim_{t \rightarrow 0} \|\bar{u}\|_{L^2([0, \infty))} = \infty. \quad (4.68)$$

□

## Appendix

LEMMA A.1. *The set  $M$  is closed subset of the space  $\mathcal{C}((0, 1]L^2([0, \infty)))$ .*

*Proof.* Let  $t \in (0, 1]$  be fixed. Let  $\{u_n\}$  be a sequence of elements of the set  $M$  for which

$$\lim_{n \rightarrow \infty} \|u_n - \tilde{u}\|_{L^2([0, \infty))} = 0, \quad (A.1)$$

where  $\tilde{u} \in L^2([0, \infty))$ .

First, we note that for  $u \in M$  we have that  $L(u)$  is continuous function because  $f(u) \in \mathcal{C}^1(-\infty, \infty)$ . Also for  $u \in M$  we have that  $L'(u)$  is continuous function and

$$L'(u) = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} - \sqrt{h_1(\tau)h_2(\tau)} f'(u) \right) d\tau ds. \quad (A.2)$$

Now, we will prove that  $|L'(u)| > 0$  for  $u \in M$ . Really,

$$\begin{aligned} L'(u) &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} - \sqrt{h_1(\tau)h_2(\tau)} f'(u) \right) d\tau ds \\ &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} - b\sqrt{h_1(\tau)h_2(\tau)} |u| \right) d\tau ds \\ &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \min_{t \in [0, 1]} \frac{v''(t)}{v(t)} - b\sqrt{h_1(\tau)h_2(\tau)} |u| \right) d\tau ds \\ &\geq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( A_1 \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} - \frac{b}{B} \sqrt{h_1(\tau)h_2(\tau)} \right) d\tau ds. \end{aligned} \quad (A.3)$$

From (H1) we have

$$A_1 \sqrt{\frac{h_2(r)}{h_1(r)}} - \frac{b}{B} \sqrt{h_1(r)h_2(r)} \geq 0 \quad (A.4)$$

for every  $r \geq 0$ . Therefore

$$L'(u) \geq 0 \tag{A.5}$$

for every fixed  $t \in [0, 1]$  and for every  $r \geq 0$ . Also for every  $r \in [0, 1]$  we have that  $L'(u)(r)$  is decrease function of  $r$  and for every  $r \in [0, 1]$  we have

$$L'(u)(r) \geq \int_1^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( A_1 \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} - \frac{b}{B} \sqrt{h_1(\tau)h_2(\tau)} \right) d\tau ds \geq \frac{A_1}{10^{10}}. \tag{A.6}$$

(In the last inequality, we use (H1).) From here it follows that for every  $u \in M$  there exists

$$N := \min_{r \in [0,1]} |L'(u)(r)| > 0. \tag{A.7}$$

Let

$$M_1 = \max_{r \in [0,\infty)} \left| \frac{\partial}{\partial r} L'(u)(r) \right|. \tag{A.8}$$

Now, we will prove that for every  $\epsilon > 0$  there exists  $\delta = \delta(\epsilon) > 0$  such that from  $|x - y| < \delta$  we have

$$|u_m(x) - u_m(y)| < \epsilon \quad \forall m \in \mathcal{N}. \tag{A.9}$$

We suppose that there exists  $\tilde{\epsilon} > 0$  such that for every  $\delta > 0$  there exist natural  $m$  and  $x, y \in [0, \infty)$ ,  $|x - y| < \delta$  for which  $|u_m(x) - u_m(y)| \geq \tilde{\epsilon}$ . We choose  $\tilde{\epsilon} > 0$  such that  $\tilde{\epsilon} < N\tilde{\epsilon}$ . We note that  $L(u_m)(x)$  is uniformly continuous function for  $x \in [0, \infty)$  (for  $u \in M$   $L(u)(r)$  is uniformly continuous function for  $r \in [0, \infty)$  because  $L(u)(r) \in \mathcal{C}([0, \infty))$ ) and as in the proof of Theorem 3.1 (after we use the second condition of (H2)) we have that  $|\partial/\partial r L(u)(r)| \leq 1/B$ . Then there exists  $\delta_1 = \delta_1(\tilde{\epsilon}) > 0$  such that for every natural  $n$  we have

$$|L(u)(x) - L(u)(y)| < \tilde{\epsilon} \quad \forall x, y \in [0, \infty) : |x - y| < \delta_1. \tag{A.10}$$

Consequently, we can choose

$$0 < \delta < \min \left\{ 1, \delta_1, \frac{(N\tilde{\epsilon} - \tilde{\epsilon})B}{M_1} \right\}, \tag{A.11}$$

such that there exist natural  $m$  and  $x_1, x_2 \in [0, \infty)$  for which

$$|x_1 - x_2| < \delta, \quad |u_m(x_1 - x_2) - u_m(0)| \geq \tilde{\epsilon}. \quad (\text{A.12})$$

In particular,

$$|L(u_m)(x_1 - x_2) - L(u_m)(0)| < \tilde{\epsilon}. \quad (\text{A.13})$$

Let us suppose for convenience that  $x_1 - x_2 > 0$ . Then,  $x_1 - x_2 < 1$  and for every  $u \in M$  we have  $L'(u)(x_1 - x_2) \geq N$ . Then, from the middle point theorem we have

$$\begin{aligned} L(0) &= 0, & L(u_m)(x_1 - x_2) &= L'(\xi)(x_1 - x_2)u_m(x_1 - x_2), \\ L(u_m)(0) &= L'(\xi)(0)u_m(0), \\ |L(u_m)(x_1 - x_2) - L(u_m)(0)| & \\ &= |L'(\xi)(x_1 - x_2)u_m(x_1 - x_2) - L'(\xi)(0)u_m(0)| \\ &= |L'(\xi)(x_1 - x_2)u_m(x_1 - x_2) - L'(\xi)(x_1 - x_2)u_m(0) \\ &\quad + L'(\xi)(x_1 - x_2)u_m(0) - L'(\xi)(0)u_m(0)| \\ &\geq |L'(\xi)(x_1 - x_2)u_m(x_1 - x_2) - L'(\xi)(x_1 - x_2)u_m(0)| \\ &\quad - |L'(\xi)(x_1 - x_2)u_m(0) - L'(\xi)(0)u_m(0)| \\ &= |L'(\xi)(x_1 - x_2)| |u_m(x_1 - x_2) - u_m(0)| - |L'(\xi)(x_1 - x_2) - L'(\xi)(0)| |u_m(0)| \\ &= |L'(\xi)(x_1 - x_2)| |u_m(x_1 - x_2) - u_m(0)| - \left| \frac{\partial}{\partial r} (L'(\xi)) \right| |x_1 - x_2| |u_m(0)| \\ &\geq N\tilde{\epsilon} - M_1\delta \frac{1}{B} \geq \tilde{\epsilon}, \end{aligned} \quad (\text{A.14})$$

which is contradiction with (A.13). Therefore,

$$\begin{aligned} \forall \epsilon > 0 \quad \exists \delta = \delta(\epsilon) > 0: \text{ from } |x - y| < \delta \\ \implies |u_m(x) - u_m(y)| < \epsilon \quad \forall m. \end{aligned} \quad (\text{A.15})$$

On the other hand, from the definition of the set  $M$  we have

$$|u_m| \leq \frac{1}{B} \quad \forall m \in \mathcal{N}. \quad (\text{A.16})$$

From (A.15) and (A.16) it follows that the set  $\{u_n\}$  is compact subset of the space  $\mathcal{C}([0, \infty))$ . Therefore, there exists a subsequence  $\{u_{n_k}\}$  and function  $u \in \mathcal{C}([0, \infty))$  for which

$$|u_{n_k}(x) - u(x)| < \epsilon \quad (\text{A.17})$$

for every  $x \in [0, \infty)$ . From here and from

$$\|u_{n_k} - \tilde{u}\|_{L^2([0, \infty))} = 0 \quad (\text{A.18})$$

we have that for every  $\epsilon > 0$  there exists  $F = F(\epsilon) > 0$  such that for every  $n_k > F$  we have

$$\max_{r \in [0, \infty)} |u_{n_k} - u| < \delta, \quad \|u_{n_k} - \tilde{u}\|_{L^2([0, \infty))} < \epsilon. \quad (\text{A.19})$$

Then, for every  $n_k > F$  we have

$$|u - \tilde{u}| = |u - u_{n_k} + u_{n_k} - \tilde{u}| \leq |u - u_{n_k}| + |u_{n_k} - \tilde{u}| < \epsilon + |u_{n_k} - \tilde{u}|. \quad (\text{A.20})$$

Also

$$\begin{aligned} & |L(u) - L(\tilde{u})| \\ &= \left| \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} (u - \tilde{u}) - \sqrt{h_1(\tau)h_2(\tau)} (f(u) - f(\tilde{u})) \right) d\tau ds \right| \\ &\leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} |u - \tilde{u}| + \sqrt{h_1(\tau)h_2(\tau)} |f(u) - f(\tilde{u})| \right) d\tau ds. \end{aligned} \quad (\text{A.21})$$

Now we apply the middle point theorem

$$\begin{aligned}
& \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \frac{v''(t)}{v(t)} |u - \tilde{u}| + \sqrt{h_1(\tau)h_2(\tau)} |f'(\xi)| |u - \tilde{u}| \right) d\tau ds \\
& \leq |\xi| \leq \max\{|u|, |\tilde{u}|\} \\
& \leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \max_{t \in [0,1]} \frac{v''(t)}{v(t)} |u - \tilde{u}| + \sqrt{h_1(\tau)h_2(\tau)} |f'(\xi)| |u - \tilde{u}| \right) d\tau ds \\
& \leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 |u - \tilde{u}| + \sqrt{h_1(\tau)h_2(\tau)} |f'(\xi)| |u - \tilde{u}| \right) d\tau ds \\
& \leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 |u - \tilde{u}| + \sqrt{h_1(\tau)h_2(\tau)} b |\xi| |u - \tilde{u}| \right) d\tau ds,
\end{aligned} \tag{A.22}$$

(1)  $|\xi| \leq |u|$ . Then,  $|\xi| \leq 1/B$ . From here

$$\begin{aligned}
|L(u) - L(\tilde{u})| & \leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 |u - \tilde{u}| + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} |u - \tilde{u}| \right) d\tau ds \\
& = \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) |u - \tilde{u}| d\tau ds \\
& \leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) (\epsilon + |u_{n_k} - \tilde{u}|) d\tau ds \\
& = \epsilon \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) d\tau ds \\
& \quad + \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) |u_{n_k} - \tilde{u}| d\tau ds, \\
|L(u) - L(\tilde{u})|^2 & \leq \left( \epsilon \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) d\tau ds \right. \\
& \quad \left. + \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) |u_{n_k} - \tilde{u}| d\tau ds \right)^2 \\
& \leq 2\epsilon^2 \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) d\tau ds \right)^2 \\
& \quad + 2 \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) |u_{n_k} - \tilde{u}| d\tau ds \right)^2.
\end{aligned} \tag{A.23}$$

Now we apply the Hölder inequality

$$\begin{aligned}
& 2\epsilon^2 \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) d\tau ds \right)^2 \\
& \quad + 2 \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) d\tau \right)^2 ds \right)^{1/2} \\
& \quad \times \left( \int_s^\infty |u_{n_k} - \tilde{u}|^2 d\tau \right)^{1/2} ds \Big)^2 \\
& \leq 2\epsilon^2 \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) d\tau ds \right)^2 \\
& \quad + 2 \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) d\tau \right)^2 ds \right)^{1/2} \\
& \quad \times \left( \int_0^\infty |u_{n_k} - \tilde{u}|^2 d\tau \right)^{1/2} ds \Big)^2 \\
& = 2\epsilon^2 \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) d\tau ds \right)^2 \\
& \quad + 2 \|u_{n_k} - \tilde{u}\|_{L^2([0, \infty))}^2 \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) d\tau \right)^2 ds \right)^{1/2} \Big)^2 \\
& \leq 2\epsilon^2 \left[ \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) d\tau ds \right)^2 \right. \\
& \quad \left. + \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) d\tau \right)^2 ds \right)^{1/2} \right]^2.
\end{aligned} \tag{A.24}$$

From here

$$\begin{aligned}
& \int_0^\infty |L(u) - L(\tilde{u})|^2 dr \\
& \leq 2\epsilon^2 \left[ \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) d\tau ds \right)^2 dr \right. \\
& \quad \left. + \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) d\tau \right)^2 ds \right)^{1/2} dr \right]^2.
\end{aligned} \tag{A.25}$$

From (i1) we have that there exist constants  $C_1$  and  $C_2$  such that

$$\int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) d\tau ds \right)^2 dr \leq C_1, \quad (\text{A.26})$$

$$\int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + \sqrt{h_1(\tau)h_2(\tau)} \frac{b}{B} \right) d\tau \right)^{1/2} ds \right)^2 dr \leq C_2.$$

Consequently,

$$\int_0^\infty |L(u) - L(\tilde{u})|^2 dr \leq 2\epsilon^2(C_1 + C_2). \quad (\text{A.27})$$

(2)  $|\xi| \leq |\tilde{u}|$ . Then,

$$\begin{aligned} |L(u) - L(\tilde{u})| &\leq \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} A_2 + b\sqrt{h_1(\tau)h_2(\tau)} |\tilde{u}| \right) (\epsilon + |u_{n_k} - \tilde{u}|) d\tau ds \\ &= \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \left( \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} \epsilon A_2 + \epsilon b\sqrt{h_1(\tau)h_2(\tau)} |\tilde{u}| + A_2 \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} |u_{n_k} - \tilde{u}| \right. \\ &\quad \left. + b\sqrt{h_1(\tau)h_2(\tau)} |\tilde{u}| |u_{n_k} - \tilde{u}| \right) d\tau ds \\ &= \epsilon A_2 \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} d\tau ds \\ &\quad + \epsilon b \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \sqrt{h_1(\tau)h_2(\tau)} |\tilde{u}| d\tau ds \\ &\quad + A_2 \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} |u_{n_k} - \tilde{u}| d\tau ds \\ &\quad + b \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \sqrt{h_1(\tau)h_2(\tau)} |\tilde{u}| |u_{n_k} - \tilde{u}| d\tau ds \end{aligned}$$

$$\begin{aligned}
&\leq \epsilon A_2 \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} d\tau ds \\
&\quad + \epsilon b \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty h_1(\tau) h_2(\tau) d\tau \right)^{1/2} \left( \int_s^\infty |\tilde{u}|^2 d\tau \right)^{1/2} ds \\
&\quad + A_2 \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \frac{h_2(\tau)}{h_1(\tau)} d\tau \right)^{1/2} \left( \int_s^\infty |u_{n_k} - \tilde{u}|^2 d\tau \right)^{1/2} ds \\
&\quad + b \max_{r \in [0, \infty)} \sqrt{h_1(\tau) h_2(\tau)} \\
&\quad \times \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty |\tilde{u}|^2 d\tau \right)^{1/2} \left( \int_s^\infty |u_{n_k} - \tilde{u}|^2 d\tau \right)^{1/2} ds,
\end{aligned} \tag{A.28}$$

from (i1) we have that there exists constant  $C_3 > 0$  such that

$$\begin{aligned}
&\max_{r \in [0, \infty)} \sqrt{h_1(r) h_2(r)} \leq C_3, \\
&\epsilon A_2 \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} d\tau ds \\
&\quad + \epsilon b \|\tilde{u}\|_{L^2[0, \infty)} \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty h_1(\tau) h_2(\tau) d\tau \right)^{1/2} ds \\
&\quad + A_2 \|u_{n_k} - \tilde{u}\|_{L^2([0, \infty))} \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \frac{h_2(\tau)}{h_1(\tau)} d\tau \right)^{1/2} ds \\
&\quad + b C_3 \|\tilde{u}\|_{L^2([0, \infty))} \|u_{n_k} - \tilde{u}\|_{L^2([0, \infty))} \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} ds \\
&\leq \epsilon A_2 \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} d\tau ds \\
&\quad + \epsilon b \|\tilde{u}\|_{L^2[0, \infty)} \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty h_1(\tau) h_2(\tau) d\tau \right)^{1/2} ds \\
&\quad + A_2 \epsilon \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \frac{h_2(\tau)}{h_1(\tau)} d\tau \right)^{1/2} ds + \epsilon b C_3 \|\tilde{u}\|_{L^2([0, \infty))} \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} ds,
\end{aligned} \tag{A.29}$$

from here

$$\begin{aligned}
|L(u) - L(\tilde{u})|^2 &\leq 2\epsilon^2 \left[ A_2^2 \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} d\tau ds \right)^2 \right. \\
&\quad + b^2 \|\tilde{u}\|_{L^2[0,\infty)}^2 \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty h_1(\tau)h_2(\tau) d\tau \right) \frac{1}{2} ds \right)^2 \\
&\quad + A_2^2 \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \frac{h_2(\tau)}{h_1(\tau)} d\tau \right)^{1/2} ds \right)^2 \\
&\quad \left. + b^2 C_3^2 \|\tilde{u}\|_{L^2([0,\infty))}^2 \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} ds \right)^2 \right],
\end{aligned} \tag{A.30}$$

from here

$$\begin{aligned}
&\int_0^\infty |L(u) - L(\tilde{u})|^2 dr \\
&\leq 2\epsilon^2 \left[ A_2^2 \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} d\tau ds \right)^2 dr \right. \\
&\quad + b^2 \|\tilde{u}\|_{L^2[0,\infty)}^2 \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty h_1(\tau)h_2(\tau) d\tau \right) \frac{1}{2} ds \right)^2 dr \\
&\quad + A_2^2 \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \frac{h_2(\tau)}{h_1(\tau)} d\tau \right)^{1/2} ds \right)^2 dr \\
&\quad \left. + b^2 C_3^2 \|\tilde{u}\|_{L^2([0,\infty))}^2 \int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} ds \right)^2 dr \right].
\end{aligned} \tag{A.31}$$

From (i1) we have that there exist constants  $C_4$ ,  $C_5$ ,  $C_6$ , and  $C_7$  such that

$$\begin{aligned}
&\int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \int_s^\infty \sqrt{\frac{h_2(\tau)}{h_1(\tau)}} d\tau ds \right)^2 dr \leq C_4, \\
&\int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty h_1(\tau)h_2(\tau) d\tau \right)^{1/2} ds \right)^2 dr \leq C_5, \\
&\int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} \left( \int_s^\infty \frac{h_2(\tau)}{h_1(\tau)} d\tau \right)^{1/2} ds \right)^2 dr \leq C_6, \\
&\int_0^\infty \left( \int_r^\infty \sqrt{\frac{h_2(s)}{h_1(s)}} ds \right)^2 dr \leq C_7.
\end{aligned} \tag{A.32}$$

Consequently,

$$\int_0^\infty |L(u) - L(\tilde{u})|^2 dr \leq 2\epsilon^2 (A_2^2 C_4 + b^2 \|\tilde{u}\|_{L^2([0, \infty))}^2 C_5 + C_6 A_2^2 + \|\tilde{u}\|_{L^2([0, \infty))}^2 b^2 C_3^2 C_7). \tag{A.33}$$

Since  $\tilde{u} \in L^2([0, \infty))$  and from the inequalities (A.27) and (A.33) it follows that there exists constant  $C_8$  such that

$$\int_0^\infty |L(u) - L(\tilde{u})|^2 dr \leq C_8 \epsilon^2. \tag{A.34}$$

From here it follows that  $L(u) = L(\tilde{u})$  a.e. in  $[0, \infty)$ . Now we suppose that it is not true  $u = \tilde{u}$  a.e. in  $[0, \infty)$ . Then there exist  $\epsilon_1 > 0$  and subinterval  $\Delta \subset [0, \infty)$  such that

$$|u - \tilde{u}| > \epsilon_1 \quad \text{for } r \in \Delta. \tag{A.35}$$

From the middle point theorem we have

$$|L(u) - L(\tilde{u})| = |L'(\xi)| |u - \tilde{u}|, \quad |\xi| \leq \{|u|, |\tilde{u}|\}. \tag{A.36}$$

Also there exists constant  $M_1 > 0$  such that

$$\min_{r \in \Delta} |L'(\xi)(r)| \geq M_1. \tag{A.37}$$

Then for

$$\epsilon < M_1^2 \mu(\Delta) \epsilon_1^2 \tag{A.38}$$

we have

$$\begin{aligned} \epsilon &> \int_0^\infty |L(u) - L(\tilde{u})|^2 dr \geq \int_\Delta |L(u) - L(\tilde{u})|^2 dr \\ &\geq \int_\Delta |L'(\xi)|^2 |u - \tilde{u}|^2 dr \geq M_1^2 \mu(\Delta) \epsilon_1^2 \end{aligned} \tag{A.39}$$

which is contradiction with (A.38). From here it follows that  $u = \tilde{u}$  a.e. in  $[0, \infty)$ ,  $|u_n - u|^2 = |u_n - \tilde{u}|^2$  a.e. in  $[0, \infty)$ ,

$$\|u_n - u\|_{L^2([0, \infty))} = \|u_n - \tilde{u}\|_{L^2([0, \infty))}. \tag{A.40}$$

Consequently, for every sequence  $\{u_n\}$  from elements of the set  $M$  which is convergent in  $L^2([0, \infty))$  there exists function  $u \in \mathcal{C}([0, \infty))$ ,  $u \in L^2([0, \infty))$  for which

$$\lim_{n \rightarrow \infty} \|u_n - u\|_{L^2([0, \infty))} = 0. \tag{A.41}$$

Below we suppose that the sequence  $\{u_n\}$  is a sequence from elements of the set  $M$  which is convergent in  $L^2([0, \infty))$ . Then, there exists function  $u \in \mathcal{C}([0, \infty))$ ,  $u \in L^2([0, \infty))$  for which  $\lim_{n \rightarrow \infty} \|u_n - u\|_{L^2([0, \infty))} = 0$ .

Now we suppose that  $u(t, \infty) \neq 0$ . Since  $u(t, r) \in \mathcal{C}([0, \infty))$ ,  $u_n(t, r) \in \mathcal{C}([0, \infty))$ ,  $u_n(t, \infty) = 0$  for every natural  $n$  we have that there exists  $\epsilon_2 > 0$  and constant  $Q > 0$  such that for every  $r > Q$

$$|u_n| < \frac{\epsilon_2}{2}, \quad |u| > \epsilon_2. \quad (\text{A.42})$$

Then, for every natural  $n$  and for every  $r > Q$  we have

$$|u_n - u| \geq |u| - |u_n| > \frac{\epsilon_2}{2}. \quad (\text{A.43})$$

Also there exist constants  $M > 0$  and  $M_1 > 0$  such that

$$\min_{r \in [Q, Q+1]} |L'(\xi)(r)| \geq M, \quad |L'(\xi)(r)| \leq \frac{1}{M_1} \quad \text{for } r \in [0, \infty). \quad (\text{A.44})$$

Let  $\epsilon_3 > 0$  be such that

$$\epsilon_3 < \frac{M\epsilon_2}{2(1/M_1)}. \quad (\text{A.45})$$

Therefore, there exists natural  $n$  and for every  $r > Q$  we have

$$|u_n(r) - u(r)| > \frac{\epsilon_2}{2}, \quad \|u_n - u\|_{L^2([0, \infty))} < \epsilon_3. \quad (\text{A.46})$$

From here

$$\begin{aligned} M^2 \frac{\epsilon_2^2}{4} &\leq M^2 \int_Q^{Q+1} |u - u_n|^2 dr \leq \int_Q^{Q+1} |L(u) - L(u_n)|^2 dr \\ &= \int_Q^{Q+1} |L'(\xi)|^2 |u - u_n|^2 dr \leq \frac{1}{M_1^2} \int_0^\infty |u - u_n|^2 dr \\ &\leq \frac{1}{M_1^2} \|u_n - u\|_{L^2([0, \infty))}^2 \leq \frac{1}{M_1^2} \epsilon_3^2 \end{aligned} \quad (\text{A.47})$$

or

$$M^2 \frac{\epsilon_2^2}{4} \leq \frac{1}{M_1^2} \epsilon_3^2 \quad (\text{A.48})$$

or

$$\epsilon_3 \geq \frac{M\epsilon_2}{2(1/M_1)}, \quad (\text{A.49})$$

which is contradiction with (A.45). Therefore,  $u(t, \infty) = 0$  and for every enough large  $Q > 0$  we have  $u(t, r) = 0$  for every  $r > Q$ . From here  $u_r(t, \infty) = 0$ .

Now we suppose that the inequality

$$u(t, r) \leq \frac{1}{B} \quad (\text{A.50})$$

is not true for every  $r \in [0, \infty)$ . Since  $u \in \mathcal{C}([0, \infty))$ , we can take  $\epsilon_4 > 0$  and  $\Delta_1 \subset [0, \infty)$  such that

$$u \geq \frac{1}{B} + \epsilon_4 \quad \text{for } r \in \Delta_1. \quad (\text{A.51})$$

Then for every natural  $n$  and for every  $r \in \Delta_1$  we have

$$|u_n - u| \geq |u| - |u_n| \geq \frac{1}{B} + \epsilon_4 - \frac{1}{B} = \epsilon_4. \quad (\text{A.52})$$

Let  $\epsilon_5 > 0$  be such that

$$\epsilon_5 \leq \epsilon_4 (\mu(\Delta_1))^{1/2}. \quad (\text{A.53})$$

Then, there exists constant  $F > 0$  such that for every  $n > F$  we have

$$\|u_n - u\|_{L^2([0, \infty))} < \epsilon_5. \quad (\text{A.54})$$

Therefore, for every  $n > F$  and for every  $r \in \Delta_1$  we have

$$|u_n(r) - u(r)| \geq \epsilon_4, \quad \|u_n - u\|_{L^2([0, \infty))} < \epsilon_5. \quad (\text{A.55})$$

Also

$$\begin{aligned} \epsilon_4 \mu(\Delta_1) &\leq \int_{\Delta_1} |u_n - u| dx \leq (\mu(\Delta_1))^{1/2} \left( \int_{\Delta_1} |u_n - u|^2 dx \right)^{1/2} \\ &\leq (\mu(\Delta_1))^{1/2} \left( \int_0^\infty |u_n - u|^2 dx \right)^{1/2} \\ &\leq (\mu(\Delta_1))^{1/2} \|u_n - u\|_{L^2([0, \infty))} < \epsilon_5 (\mu(\Delta_1))^{1/2} \end{aligned} \quad (\text{A.56})$$

or

$$\epsilon_4 \mu(\Delta_1) < \epsilon_5 (\mu(\Delta_1))^{1/2}. \quad (\text{A.57})$$

From here

$$\epsilon_4 (\mu(\Delta_1))^{1/2} < \epsilon_5, \quad (\text{A.58})$$

which is contradiction with (A.53). Consequently, for every  $r \in [0, \infty)$  we have

$$u(t, r) \leq \frac{1}{B}. \quad (\text{A.59})$$

Now we suppose that the inequality

$$u(t, r) \geq \frac{1}{A} \quad (\text{A.60})$$

is not true for every  $r \in [c, d]$ . Since  $u \in \mathcal{C}([0, \infty))$ , we can take  $\epsilon_6 > 0$  and  $\Delta_2 \subset [c, d]$  such that

$$|u| \leq \frac{1}{A} - \epsilon_6 \quad \text{for } r \in \Delta_2. \quad (\text{A.61})$$

Then, for every natural  $n$  and for every  $r \in \Delta_2$  we have

$$|u_n - u| \geq |u_n| - |u| \geq \frac{1}{A} + \epsilon_6 - \frac{1}{A} = \epsilon_6. \quad (\text{A.62})$$

Let  $\epsilon_7 > 0$  be such that

$$\epsilon_7 \leq \epsilon_6 (\mu(\Delta_2))^{1/2}, \quad (\text{A.63})$$

then there exists natural  $n$  such that

$$\begin{aligned} \|u_n - u\|_{L^2([0, \infty))} &< \epsilon_7, \\ |u_n(r) - u(r)| &\geq \epsilon_6, \quad r \in \Delta_2. \end{aligned} \quad (\text{A.64})$$

Also

$$\begin{aligned} \epsilon_6 \mu(\Delta_2) &\leq \int_{\Delta_2} |u_n - u| dx \leq (\mu(\Delta_2))^{1/2} \left( \int_{\Delta_2} |u_n - u|^2 dx \right)^{1/2} \\ &\leq (\mu(\Delta_2))^{1/2} \left( \int_0^\infty |u_n - u|^2 dx \right)^{1/2} \\ &\leq (\mu(\Delta_2))^{1/2} \|u_n - u\|_{L^2([0, \infty))} < \epsilon_7 (\mu(\Delta_2))^{1/2} \end{aligned} \quad (\text{A.65})$$

or

$$\epsilon_6 \mu(\Delta_2) < \epsilon_7 (\mu(\Delta_2))^{1/2}. \quad (\text{A.66})$$

From here

$$\epsilon_6 (\mu(\Delta_2))^{1/2} < \epsilon_7, \quad (\text{A.67})$$

which is contradiction with (A.63). Consequently, for every  $r \in [c, d]$  we have

$$u(t, r) \geq \frac{1}{A}. \quad (\text{A.68})$$

Now we suppose that the inequality

$$u(t, r) \geq 0 \quad (\text{A.69})$$

is not true for every  $r \in [0, \infty)$ . Since  $u \in \mathcal{C}([0, \infty))$ , we can take  $\epsilon_8 > 0$  and  $\Delta_3 \subset [0, \infty)$  such that

$$u < 0 \quad \text{for } r \in \Delta_3, \quad |u_n - u| \geq \epsilon_8. \quad (\text{A.70})$$

Let  $\epsilon_9 > 0$  be such that

$$\epsilon_9 \leq \epsilon_8 (\mu(\Delta_3))^{1/2}, \quad (\text{A.71})$$

then there exists natural  $n$  such that

$$\begin{aligned} \|u_n - u\|_{L^2([0, \infty))} &< \epsilon_9, \\ |u_n(r) - u(r)| &\geq \epsilon_8, \quad r \in \Delta_3. \end{aligned} \quad (\text{A.72})$$

Also

$$\begin{aligned} \epsilon_8 \mu(\Delta_3) &\leq \int_{\Delta_3} |u_n - u| dx \leq (\mu(\Delta_3))^{1/2} \left( \int_{\Delta_3} |u_n - u|^2 dx \right)^{1/2} \\ &\leq (\mu(\Delta_3))^{1/2} \left( \int_0^\infty |u_n - u|^2 dx \right)^{1/2} \\ &\leq (\mu(\Delta_3))^{1/2} \|u_n - u\|_{L^2([0, \infty))} < \epsilon_9 (\mu(\Delta_3))^{1/2} \end{aligned} \quad (\text{A.73})$$

or

$$\epsilon_8 \mu(\Delta_3) < \epsilon_9 (\mu(\Delta_3))^{1/2}. \quad (\text{A.74})$$

From here

$$\epsilon_8 (\mu(\Delta_3))^{1/2} < \epsilon_9, \quad (\text{A.75})$$

which is contradiction with (A.71). Consequently, for every  $r \in [0, \infty)$  we have

$$u(t, r) \geq 0. \quad (\text{A.76})$$

Consequently, for every sequence  $\{u_n\} \subset M$  which is convergent in  $L^2([0, \infty))$  there exists  $u \in M$  such that

$$\lim_{n \rightarrow \infty} \|u_n - u\|_{L^2([0, \infty))} = 0. \quad (\text{A.77})$$

□

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Svetlin Georgiev Georgiev: Department of Differential Equations, University of Sofia,  
1164 Sofia, Bulgaria  
*Email address:* sgg2000bg@yahoo.com