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Research Article

Existence and Multiplicity Results for Degenerate Elliptic Equations with Dependence on the Gradient

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We study the existence of positive solutions for a class of degenerate nonlinear elliptic equations with gradient dependence. For this purpose, we combine a blowup argument, the strong maximum principle, and Liouville-type theorems to obtain a priori estimates.

1. Introduction

We consider the following nonvariational problem:

$$-\Delta_m u = f(x,u,\nabla u) - a(x)g(u,\nabla u) + \tau \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \qquad (P)_\tau$$

where Ω is a bounded domain with smooth boundary of $\mathbb{R}^N$, $N \geq 3$. Δ_m denotes the usual m-Laplacian operators, $1 < m < N$ and $\tau \geq 0$. We will obtain a priori estimate to positive solutions of problem $(P)_\tau$ under certain conditions on the functions f, g, a. This result implies nonexistence of positive solutions to τ large enough.

Also we are interested in the existence of a positive solutions to problem $(P)_0$, which does not have a clear variational structure. To avoid this difficulty, we make use of the blow-up method over the solutions to problem $(P)_\tau$, which have been employed very often to obtain a priori estimates (see, e.g., [1, 2]). This analysis allows us to apply a result due to [3], which is a variant of a Rabinowitz bifurcation result. Using this result, we obtain the existence of positive solutions.

Throughout our work, we will assume that the nonlinearities f and g satisfy the following conditions.

(H_1) $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous function.

(H_2) $g : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous function.

(H_3) There exist $L > 0$ and $c_0 \geq 1$ such that $u^p - L|\eta|^\alpha \leq f(x,u,\eta) \leq c_0 u^p + L|\eta|^\alpha$ for all $(x,u,\eta) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, where $p \in (m-1, m_* - 1)$ and $\alpha \in (m-1, mp/(p+1))$. Here, we denote $m_* = m(N-1)/(N-m)$.

(H_4) There exist $M > 0$, $c_1 \geq 1$, $q > p$, and $\beta \in (m-1, mp/(p+1))$ such that $|u|^q - M|\eta|^\beta \leq g(u,\eta) \leq c_1|u|^q + M|\eta|^\beta$ for all $(u,\eta) \in \mathbb{R} \times \mathbb{R}^N$.

We also assume the following hypotheses on the function a.

(A_1) $a : \overline{\Omega} \to \mathbb{R}$ is a nonnegative continuous function.

(A_2) There is a subdomain Ω_0 with C^2-boundary so that $\overline{\Omega_0} \subset \Omega$, $a \equiv 0$ in $\overline{\Omega_0}$, and $a(x) > 0$ for $x \in \Omega \setminus \overline{\Omega_0}$.

(A_3) We assume that the function a has the following behavior near to $\partial\Omega_0$:

$$a(x) = b(x)d(x, \partial\Omega_0)^\gamma, \tag{1.1}$$

$x \in \Omega \setminus \overline{\Omega_0}$, where γ is positive constant and $b(x)$ is a positive continuous function defined in a small neighborhood of $\partial\Omega_0$.

Observe that particular situations on the nonlinearities have been considered by many authors. For instance, when $a \equiv 0$ and f verifies (H_3), Ruiz has proved that the problem $(P)_0$ has a bounded positive solution (see [2] and reference therein). On the other hand, when $f(x,u,\eta) = u^p$ and $g(x,u,\eta) = u^q$, $q > p$ and $m < p$, and $a \equiv 1$, a multiplicity of results was obtained by Takeuchi [4] under the restriction $m > 2$. Later, Dong and Chen [5] improve the result because they established the result for all $m > 1$. We notice that the Laplacian case was studied by Rabinowitz by combining the critical point theory with the Leray-Schauder degree [6]. Then, when $m \geq p$, since $(f(x,u) - g(x,u))/u^{m-1}$ becomes monotone decreasing for $0 < u$, we know that the solution to $(P)_0$ is unique (as far as it exists) from the Díaz and Saá's uniqueness result (see [7]). For more information about this type of logistic problems, see [1, 8–13] and references cited therein.

Our main results are the following.

Theorem 1.1. *Let $u \in C^1(\Omega)$ be a positive solution of problem $(P)_\tau$. Suppose that the conditions (H_1)–(H_4) and the hypotheses (A_1)–(A_3) are satisfied with $\gamma \neq m(q-p)/(1-m+p)$. Then, there is a positive constant C, depending only on the function a and Ω, such that*

$$0 \leq u(x) + \tau \leq C \tag{1.2}$$

for any $x \in \Omega$.

Moreover, if $\gamma = m(q-p)/(1-m+p)$, then there exists a positive constant $c_1 = c_1(p,\alpha,\beta,N,c_0)$ such that the conclusion of the theorem is true, provided that $\inf_{\partial\Omega_0} b(x) > c_1$.

Observe that this result implies in particular that there is no solution for $0 < \tau$ large enough. By using a variant of a Rabinowitz bifurcation result, we obtain an existence result for positive solutions.

Theorem 1.2. *Under the hypotheses of Theorem 1.1, the problem $(P)_0$ has at least one positive solution.*

2. A priori estimates and proof of Theorem 1.1

We will use the following lemma which is an improvement of Lemma 2.4 by Serrin and Zou [14] and was proved in Ruiz [2].

Lemma 2.1. *Let u be a nonnegative weak solution to the inequality*

$$-\Delta_m u \geq u^p - M|\nabla u|^\alpha, \tag{2.1}$$

in a domain $\Omega \subset \mathbb{R}^N$, where $p > m-1$ and $m-1 \leq \alpha < mp/(p+1)$. Take $\lambda \in (0,p)$ and let $B(\cdot,R_0)$ be a ball of radius R_0 such that $B(\cdot,2R_0)$ is included in Ω.

Then, there exists a positive constant $C = C(N,m,q,\alpha,\lambda,R_0)$ such that

$$\int_{B(\cdot,R)} u^\lambda \leq CR^{(N-m\lambda)/(p+1-m)}, \tag{2.2}$$

for all $R \in (0,R_0]$.

We will also make use of the following weak Harnack inequality, which was proved by Trudinger [15].

Lemma 2.2. *Let $u \geq 0$ be a weak solution to the inequality $\Delta_m u \leq 0$ in Ω. Take $\lambda \in [1, m_* - 1)$ and $R > 0$ such that $B(\cdot,2R) \subset \Omega$. Then there exists $C = C(N,m,\lambda)$ (independent of R) such that*

$$\inf_{B(\cdot,R)} u \geq CR^{-N/\lambda}\left(\int_{B(\cdot,2R)} u^\lambda\right)^{1/\lambda}. \tag{2.3}$$

The following lemma allows us to control the parameter τ in the Blow-Up analysis. (See Section 2.1.)

Lemma 2.3. *Let u be a solution to the problem $(P)_\tau$. Then there is a positive constant k_0 which depends only on Ω_0 such that*

$$\tau \leq k_0\left(\max_{x\in\overline{\Omega}} u\right)^{m-1}. \tag{2.4}$$

Proof. Since u is a positive solution, the inequality holds if $\tau = 0$. Now if $\tau > 0$, then from (H_1) and (A_2) we get

$$-\Delta_m u = f(x,u,\nabla u) - a(x)g(u,\nabla u) + \tau \geq \tau \quad \forall x \in \Omega_0. \tag{2.5}$$

Let v be the positive solution to

$$\begin{aligned} -\Delta_m v &= 1 \quad \text{in } \Omega_0, \\ v &= 0 \quad \text{on } \partial\Omega_0 \end{aligned} \tag{2.6}$$

and $w = (\tau/2)^{1/(m-1)}v$ in Ω_0, then it follows that $-\Delta_m w = \tau/2 < -\Delta_m u$ in Ω_0 and $u > w$ on $\partial\Omega_0$. Thus, using the comparison lemma (see [16]), we obtain $u \geq w$ in Ω_0. Therefore, there is a positive constant k_0 such that

$$\tau \leq k_0 u^{m-1} \tag{2.7}$$

at the maximum point of v and the conclusion follows. □

2.1. A priori estimates. We suppose that there is a sequence $\{(u_n,\tau_n)\}_{n\in\mathbb{N}}$ with u_n being a C^1-solution of $(P)_{\tau_n}$ such that $\|u_n\|+\tau_n \xrightarrow[n\to\infty]{} \infty$. By Lemma 2.3, we can assume that there exists $x_n \in \Omega$ such that $u_n(x_n) = \|u_n\| =: S_n \xrightarrow[n\to\infty]{} \infty$. Let $d_n := d(x_n,\partial\Omega)$, we define $w_n(y) = S_n^{-1}u_n(x)$, where $x = S_n^{-\theta}y + x_n$ for some positive θ that will be defined later. The functions w_n are well defined at least $B(0,d_nS_n^{\theta})$, and $w_n(0) = \|w_n\| = 1$. Easy computations show that

$$\begin{aligned} -\Delta_m w_n(y) = S_n^{1-(\theta+1)m}\big[&f(S_n^{-\theta}y+x_n,S_nw_n(y),S_n^{1-\theta}\nabla w_n(y)) \\ &- a(S_n^{-\theta}y+x_n)g(S_nw_n(y),S_n^{1-\theta}\nabla w_n(y)) + \tau_n\big]. \end{aligned} \tag{2.8}$$

From our conditions on the functions f and g, the right-hand side of (2.8) reads as

$$\begin{aligned} S_n^{1-(\theta+1)m}&\big[f(S_n^{-\theta}y+x_n,S_nw_n(y),S_n^{1-\theta}\nabla w_n(y)) \\ &\quad - a(S_n^{-\theta}y+x_n)g(S_nw_n(y),S_n^{1-\theta}\nabla w_n(y)) + \tau_n\big] \\ &\leq S_n^{1-(\theta+1)m+q}\Big[c_0S_n^{p-q}w_n(y)^p + MS_n^{(1-\theta)\alpha-q}\,|\nabla w_n(y)|^{\alpha} \\ &\qquad - a(S_n^{-\theta}y+x_n)\Big(w_n(y)^q - g_0S_n^{\beta(1-\theta)-q}\,|\nabla w_n(y)|^{\beta}\Big)\Big] + S_n^{1-(\theta+1)m}\tau_n. \end{aligned} \tag{2.9}$$

We note that from Lemma 2.3 we have $S_n^{1-(\theta+1)m}\tau_n \leq c_0S_n^{1-(\theta+1)m}S_n^{m-1} \xrightarrow[n\to\infty]{} 0$.

We split this section into the following three steps according to location of the limit point x_0 of the sequence $\{x_n\}_n$.

(1) $x_0 \in \overline{\Omega}\setminus\overline{\Omega_0}$. Here, up to subsequence, we may assume that $\{x_n\}_n \subset \Omega\setminus\overline{\Omega_0}$. We define $\delta_n' = \min\{\operatorname{dist}(x_n,\partial\Omega),\operatorname{dist}(x_n,\partial\Omega_0)\}$ and $B = B(0,\delta_n'S_n^{\theta})$ if $\operatorname{dist}(x_0,\partial\Omega) > 0$, or $\delta_n' = \operatorname{dist}(x_n,\partial\Omega_0)$ and $B = B(0,\delta_n'S_n^{\theta})\cap\Omega$ if $\operatorname{dist}(x_0,\partial\Omega) = 0$. Then, w_n is well defined in B and satisfies

$$\sup_{y\in B} w_n(y) = w_n(0) = 1. \tag{2.10}$$

Now, taking $\theta = (q+1-m)/m$ in (2.9) and applying regularity theorems for the m-Laplacian operator, we can obtain estimates for w_n such that for a subsequence $w_n \to w$, locally uniformly, with w be a C^1-function defined in $\mathbb{R}^N$ or in a halfspace, if $\operatorname{dist}(x_0,\partial\Omega)$ is positive or zero, satisfying

$$-\Delta_m w \leq -a(x_0)w^q, \quad w \geq 0,\ w(0) = \max w = 1, \tag{2.11}$$

which is a contradiction with the strong maximum principle (see [17]).

(2) $x_0 \in \Omega_0$. In this case, up to subsequence we may assume that $\{x_n\}_n \subset \Omega_0$. Let $d_n = \text{dist}(x_n, \partial\Omega_0)$ and $\theta = (1+p-m)/m$. Then, w_n is well defined in $B(0, d_n S_n^{\theta})$ and satisfies

$$\sup_{y \in B(0, d_n S_n^{\theta})} w_n(y) = w_n(0) = 1. \tag{2.12}$$

On the other hand, for any $n \in \mathbb{N}$, we have $a(S_n^{-\theta} y + x_n) = 0$ and

$$-\Delta_m w_n(y) = S_n^{1-(\theta+1)m}\left[f\left(S_n^{-\theta} y + x_n, S_n w_n(y), S_n^{1-\theta} \nabla w_n(y)\right) + \tau_n\right]. \tag{2.13}$$

From the hypothesis (H_4),

$$\begin{aligned} -\Delta_m w_n(y) &= S_n^{1-(\theta+1)m}\left[f\left(S_n^{-\theta} y + x_n, S_n w_n(y), S_n^{1-\theta} \nabla w_n(y)\right) + \tau_n\right] \\ &\geq w_n(y)^p - M S_n^{\alpha(1-\theta)+1-(\theta+1)m} \left| \nabla w_n(y) \right|^{\alpha} + \tau_n S_n^{1-(\theta+1)m}. \end{aligned} \tag{2.14}$$

From our choice of the constants α and θ, we have $\alpha(1-\theta)+1-(\theta+1)m = \alpha(2m-(1+p))/m - p < 0$, that is, $S_n^{\alpha(1-\theta)+1-(\theta+1)m} |\nabla w_n(y)|^{\alpha}$ and $\tau_n S_n^{1-(\theta+1)m}$ tend to 0 as n goes to ∞. This implies that for a subsequence w_n converges to a solution of $-\Delta_m v \geq v^p$, $v \geq 0$ in $\mathbb{R}^N$, $v(0) = \max v = 1$. This is a contradiction with [14, Theorem III].

(3) $x_0 \in \partial\Omega_0$. Let $\delta_n = d(x_n, z_n)$, where $z_n \in \partial\Omega_0$. Denote by ν_n the unit normal of $\partial\Omega_0$ at z_n pointing to $\Omega \setminus \Omega_0$.

Up to subsequences, We may distinguish two cases: $x_n \in \partial\Omega_0$ for all n or $x_n \in \Omega \backslash \partial\Omega_0$ for all n.

Case 1 ($x_n \in \partial\Omega_0$ for all n). In this case, $x_n = z_n$. For ε sufficiently small but fixed take $\tilde{x}_n = z_n - \varepsilon\nu_n$. Then we have the following.

Claim 1. For any large n we have

$$u_n(\tilde{x}_n) < \frac{S_n}{4}. \tag{2.15}$$

Proof of Claim 1. In other cases, define for all n sufficiently large, passing to a subsequence if necessary, the following functions

$$\tilde{w}_n(y) = S_n^{-1} u_n\left(\tilde{x}_n + S_n^{-(p+1-m)/m} y\right), \tag{2.16}$$

which are well defined at least in $B(0, \varepsilon S_n^{(p+1-m)/m})$, $w_n(0) \geq 1/4$ and $\sup_{B(0, \varepsilon S_n^{(p+1-m)/m})} \tilde{w}_n \leq 1$.

Arguing as in the previous case $x_0 \in \Omega_0$, we arrive to a contradiction. □

Now, by continuity, for any large n there exist two points in $\Omega_0 x_n^* = x_n - t_n^* \nu_n$ and $x_n^{**} = x_n - t_n^{**} \nu_n$, $0 < t_n^* < t_n^{**} < \varepsilon$ such that

$$u_n(x_n^*) = \frac{S_n}{2}, \qquad u_n(x_n^{**}) = \frac{S_n}{4}. \tag{2.17}$$

Claim 2. There exists a number $\tilde{\delta}_n \in (0, \min\{d(x_n, x_n^*), d(x_n^*, x_n^{**})\})$ such that $S_n/4 < u_n(x) < S_n$ for all $x \in B(x_n^*, \tilde{\delta}_n)$. Moreover, there exists y_n satisfying $d(x_n^*, y_n) = \tilde{\delta}_n$ and either $u_n(y_n) = S_n/4$ or else $u_n(y_n) = S_n$.

Proof of Claim 2. Define $\widetilde{\delta}_n = \sup\{\delta > 0 : S_n/4 < u_n(x) < S_n \text{ for all } x \in B(x_n^*,\delta)\}$. It is easy to prove that $\widetilde{\delta}_n$ is well defined. Thus, the continuity of u_n ensures the existence of y_n. □

Now we will obtain an estimate from below of $\widetilde{\delta}_n S_n^{(p+1-m)/m}$.

Claim 3. There exists a positive constant $c = c(p,\alpha,\beta,N,c_0)$ such that

$$\widetilde{\delta}_n S_n^{(p+1-m)/m} \geq c, \tag{2.18}$$

for any n sufficiently large.

Proof of Claim 3. Assume, passing to a subsequence if necessary, that $\widetilde{\delta}_n S_n^{(p+1-m)/m} < 1$ for any n. We have that the functions $\widetilde{w}_n(y) = S_n^{-1} u_n(x_n^* + S_n^{-(p+1-m)/m} y)$ are well defined in $B(0,1)$ for n sufficiently large and satisfy

$$-\Delta_m \widetilde{w}_n \leq c_0 \widetilde{w}_n^p + |\nabla \widetilde{w}_n|^\alpha + |\nabla \widetilde{w}_n|^\beta. \tag{2.19}$$

Applying Lieberman's regularity (see [18]), we obtain that there exists a positive constant $k = k(p,\alpha,\beta,N,c_0)$ such that $|\nabla \widetilde{w}_n| \leq k$ in $B(0,1)$. Assume for example that $u_n(y_n) = S_n/4$. By the generalized mean value theorem, we have

$$\frac{1}{4} = \frac{1}{2} - \frac{1}{4} = \widetilde{w}_n(0) - \widetilde{w}_n(S_n^\theta(y_n - x_n^*)) \leq |\nabla \widetilde{w}_n(\xi)| \widetilde{\delta}_n S_n^\theta. \tag{2.20}$$

□

Claim 4. For any n sufficiently large, we have $B(x_n^*,\widetilde{\delta}_n) \subset B(\widetilde{x}_n,\varepsilon)$.

Proof of Claim 4. Take $x \in B(x_n^*,\widetilde{\delta}_n)$, by Claim 2 we get

$$\begin{aligned} d(x,\widetilde{x}_n) &\leq d(x,x_n^*) + d(x_n^*,\widetilde{x}_n) < \widetilde{\delta}_n + d(x_n^*,\widetilde{x}_n) \\ &\leq d(x_n,x_n^*) + d(x_n^*,\widetilde{x}_n) = d(x_n,\widetilde{x}_n) \leq \varepsilon. \end{aligned} \tag{2.21}$$

So, $x \in B(\widetilde{x}_n,\varepsilon)$.

Let λ be a number such that $N(p+1-m)/m < \lambda < p$ (this is possible because $p < m_* - 1$). By Claims 3 and 4, and by Lemma 2.2, we get

$$\begin{aligned} \left(\inf_{B(\widetilde{x}_n,\varepsilon/2)} u_n\right)^\lambda &\geq c\varepsilon^{-N} \int_{B(\widetilde{x}_n,\varepsilon)} u_n^\lambda \geq \int_{B(x_n^*,\widetilde{\delta}_n)} u_n^\lambda \\ &\geq C\widetilde{\delta}_n^N S_n^\lambda/4 \geq C_1 S_n^{N(m-1-p)/m+\lambda} \xrightarrow[n\to\infty]{} \infty. \end{aligned} \tag{2.22}$$

Therefore, the last inequality tells us that

$$\int_{B(\widetilde{x}_n,\varepsilon/2)} u_n^\lambda \xrightarrow[n\to\infty]{} \infty, \tag{2.23}$$

which contradicts Lemma 2.1. □

Now, we will analyze the other case.

Case 2 ($x_n \in \Omega\backslash\partial\Omega_0$ for all n). Define $2d = \mathrm{dist}(x_0,\partial\Omega) > 0$. Since Ω_0 has C^2-boundary as in [19], we have

$$d(x_n + S_n^{-\theta}y, \partial\Omega_0) = |\delta_n + S_n^{-\theta}\nu_n \cdot y + o(S_n^{-\theta})|,$$

$$a(x_n + S_n^{-\theta}y) = \begin{cases} b(x_n + S_n^{-\theta}y)S_n^{-\gamma\theta}|\delta_n S_n^{\theta} + \nu_n \cdot y + o(1)|^{\gamma}, & \text{if } x_n + S_n^{-\theta}y \in \Omega\setminus\Omega_0, \\ 0, & \text{if } x_n + S_n^{-\theta}y \in \Omega_0. \end{cases} \tag{2.24}$$

We define $b_n(x_n + S_n^{-\theta}y) = S_n^{\gamma\theta}a(x_n + S_n^{-\theta}y)$.

For n large enough, w_n is well defined in $B(0, dS_n^{\theta})$ and we get

$$\sup_{y\in B(0,dS_n^{\theta})} w_n(y) = w_n(0) = 1. \tag{2.25}$$

By (2.9), we obtain

$$\begin{aligned} -\Delta_m w_n(y) \le S_n^{1-(\theta+1)m+q}\Big[&c_0 S_n^{p-q}w_n(y)^p + MS_n^{(1-\theta)\alpha-q}|\nabla w_n(y)|^{\alpha} \\ &- b_n(x_n + S_n^{-\theta}y)S_n^{-\gamma\theta}\left(w_n(y)^q - g_0 S_n^{\beta(1-\theta)-q}|\nabla w_n(y)|^{\beta}\right)\Big] \\ &+ S_n^{1-(\theta+1)m}\tau_n. \end{aligned} \tag{2.26}$$

Now we need to consider the following cases.

If $0 < \gamma < m(q-p)/(1-m+p)$, we choose $\theta = (1-m+q)/(\gamma+m)$.

We first assume that $\{\delta_n S_n^{\theta}\}_{n\in\mathbb{N}}$ is bounded. Up to subsequence, we may assume that $\delta_n S_n^{\theta} \xrightarrow[n\to\infty]{} d_0 \ge 0$, from (2.26) we get

$$\begin{aligned} -\Delta_m w_n(y) &\le S_n^{\gamma\theta}\Big[c_0 S_n^{p-q}w_n(y)^p + MS_n^{(1-\theta)\alpha-q}|\nabla w_n(y)|^{\alpha} \\ &\qquad - b_n(x_n + S_n^{-\theta}y)S_n^{-\gamma\theta}\left(w_n(y)^q - g_0 S_n^{\beta(1-\theta)-q}|\nabla w_n(y)|^{\beta}\right)\Big] + S_n^{1-(\theta+1)m}\tau_n \\ &= c_0 S_n^{p-q+\gamma\theta}w_n(y)^p + MS_n^{\gamma\theta+(1-\theta)\alpha-q}|\nabla w_n(y)|^{\alpha} \\ &\quad - b_n(x_n + S_n^{-\theta}y)\left(w_n(y)^q - g_0 S_n^{\beta(1-\theta)-q}|\nabla w_n(y)|^{\beta}\right) + S_n^{1-(\theta+1)m}\tau_n. \end{aligned} \tag{2.27}$$

Thus, up to a subsequence, we may assume that w_n converges to a C^1 function w defined in $\mathbb{R}^N$ and satisfying $w \ge 0$, $w(0) = \max w = 1$ in $\mathbb{R}^N$, and

$$-\Delta_m w(y) \le \begin{cases} -b(x_0)|d_0 + \nu_0 \cdot y|^{\gamma}w^q(y), & \text{if } \nu_0 \cdot y > \sigma, \\ 0, & \text{if } \nu_0 \cdot y < \sigma, \end{cases} \tag{2.28}$$

where $\sigma = -d_0$ if $x_n \in \Omega\setminus\overline{\Omega}_0$ or $\sigma = d_0$ if $x_n \in \overline{\Omega}_0$ and ν_0 is a unitary vector in $\mathbb{R}^N$. This is impossible by the strong maximum principles.

Suppose now that $\{\delta_n S_n^{\theta}\}$ is unbounded, we may assume that $\beta_n = (\delta_n^{-1} S_n^{-\theta})^{\gamma/m} \xrightarrow[n\to\infty]{} 0$ for any $r > 0$. Let us introduce $z = y/\beta_n$ and $v_n(z) = w_n(\beta_n z)$, using (2.26) we see that v_n satisfies

$$\begin{aligned}
-\Delta_m v_n(z) \le \beta_n^m S_n^{\gamma\theta}\Big[&c_0 S_n^{p-q} v_n(z)^p + M S_n^{(1-\theta)\alpha-q}\beta_n^{-\alpha}\,|\,\nabla v_n(z)\,|^{\alpha} \\
&- b_n(x_n + S_n^{-\theta}\beta_n z) S_n^{-\gamma\theta}\Big(v_n(z)^q - g_0 S_n^{\beta(1-\theta)-q}\beta_n^{-\beta}\,|\,\nabla v_n(z)\,|^{\beta}\Big)\Big] \\
&+ S_n^{1-(\theta+1)m}\tau_n \\
= {} & c_0\beta_n^m S_n^{\gamma\theta+p-q} v_n(z)^p + M S_n^{\gamma\theta+(1-\theta)\alpha-q}\beta_n^{m-\alpha}\,|\,\nabla v_n(z)\,|^{\alpha} \\
&- \beta_n^m b_n(x_n + S_n^{-\theta}\beta_n z)\Big(v_n(z)^q - g_0 S_n^{\beta(1-\theta)-q}\beta_n^{m-\beta}\,|\,\nabla v_n(z)\,|^{\beta}\Big) + S_n^{1-(\theta+1)m}\tau_n.
\end{aligned} \tag{2.29}$$

On the other hand,

$$\beta_n^m b_n(x_n + S_n^{-\theta}\beta_n z) = b(x_n + S_n^{-\theta}\beta_n z)\left[1 + \beta_n^{(m+\gamma)/\gamma} \nu_n \cdot z + o(\beta_n^{m/\gamma})\right]^{\gamma} \xrightarrow[n\to\infty]{} b(x_0). \tag{2.30}$$

Thus, since $\gamma < m(q-p)/(1-m+p)$ and our choice of θ and β_n, it is easy to see that $S_n^{\gamma\theta+p-q}$, $S_n^{\gamma\theta+(1-\theta)\alpha-q}\beta_n^{m-\alpha}$ and $S_n^{\beta(1-\theta)-q}\beta_n^{m-\beta}$ tend to 0 as n goes to $+\infty$. Therefore, we obtain a limit function v that satisfies $-\Delta_m v \le -b(x_0)v^q$, $v \ge 0$, $v(0) = \max v = 1$ in $\mathbb{R}^N$ which is again impossible.

If $\gamma = m(q-p)/(1-m+p)$, in this case, by our assumptions on the function b, we obtain for $\theta = (1-m+p)/m$

$$\begin{aligned}
-\Delta_m w_n(y) \le {} & c_0 w_n(y)^p + M S_n^{(1-\theta)\alpha-p}\,|\,\nabla w_n(y)\,|^{\alpha} \\
&- b_n(x_n + S_n^{-\theta} y)\Big(w_n(y)^q - g_0 S_n^{\beta(1-\theta)-q}\,|\,\nabla w_n(y)\,|^{\beta}\Big) + S_n^{1-(\theta+1)m}\tau_n.
\end{aligned} \tag{2.31}$$

Arguing as in the proof of Claim 3 in the above case $x_n \in \partial\Omega_0$ for all n, we may assume that $\delta_n S_n \theta \ge d_0 = d_0(p,\alpha,\beta,N,c_0) > 0$. Therefore, the limit w of the sequence w_n satisfies

$$-\Delta_m w(y) \le c_0 w(y)^p - b(x_0)\,|\,d_0 - |\,\nu_0 \cdot y + o(1)\,|\,|^{\gamma} w(y)^q. \tag{2.32}$$

Now, evaluating in $x = 0$, the last inequality reads as

$$-\Delta_m w(0) \le c_0 - b(x_0) d_0^{\gamma} < 0, \tag{2.33}$$

provided that $b(x_0) > c_0/d_0^{\gamma}$. This contradicts the strong maximum principle.

If $\gamma > m(q-p)/(1-m+p)$, we choose $\theta = (p-m+1)/m$, then we get

$$\begin{aligned}
-\Delta_m w_n(y) \ge {} & w_n(y)^p - M S_n^{(1-\theta)\alpha-p}\,|\,\nabla w_n(y)\,|^{\alpha} \\
&- S_n^{q-p-\gamma\theta} b_n(x_n + S_n^{-\theta} y)\Big(g_1 w_n(y)^q + g_2 S_n^{\beta(1-\theta)-q}\,|\,\nabla w_n(y)\,|^{\beta}\Big) + S_n^{1-(\theta+1)m}\tau_n.
\end{aligned} \tag{2.34}$$

Arguing as seen before, that is, $\{\delta_n S_n^{-\theta}\}$ is whether bounded or unbounded, we obtain that the limit equation of the last inequality becomes

$$-\Delta_m v \geq v^p, \quad v \geq 0 \text{ in } \mathbb{R}^N,\ v(0) = \max v = 1, \tag{2.35}$$

which is a contradiction with [14, Theorem III].

3. Proof of Theorem 1.2

The following result is due to Azizieh and Clément (see [3]).

Lemma 3.1. *Let* $\mathbb{R}^+ := [0,+\infty)$ *and let* $(E, \|\cdot\|)$ *be a real Banach space. Let* $G : \mathbb{R}^+ \times E \to E$ *be continuous and map bounded subsets on relatively compact subsets. Suppose moreover that* G *satisfies the following:*

(a) $G(0,0) = 0$,

(b) *there exists* $R > 0$ *such that*

  (i) $u \in E$, $\|u\| \leq R$, *and* $u = G(0,u)$ *imply that* $u = 0$,

  (ii) $\deg(\mathrm{Id} - G(0,\cdot), B(0,R), 0) = 1$.

Let J *denote the set of the solutions to the problem*

$$u = G(t,u) \tag{$\mathfrak{P}$}$$

in $\mathbb{R}^+ \times E$. *Let* $\mathfrak{C}$ *denote the component (closed connected maximal subset with respect to the inclusion) of* J *to which* $(0,0)$ *belongs. Then if*

$$\mathfrak{C} \cap (\{0\} \times E) = \{(0,0)\}, \tag{3.1}$$

then $\mathfrak{C}$ *is unbounded in* $\mathbb{R}^+ \times E$.

Proof of Theorem 1.2. First, we consider the following problem:

$$\begin{gathered} -\Delta_m u = f(x,u^+,\nabla u^+) - a(x) g(u^+,\nabla u^+) + \tau \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{$(P)_\tau^+$}$$

and let u be a nontrivial solution to the problem above, then u is nonnegative and so is solution for the problem $(P)_\tau$. In fact, suppose that $U = \{x \in \Omega : u(x) < 0\}$ is nonempty. Then u is a weak solution to

$$\begin{gathered} -\Delta_m u = \tau \geq 0 \quad \text{in } U, \\ u = 0 \quad \text{on } \partial U. \end{gathered} \tag{3.2}$$

Using Lemma 2.3, we obtain that $u(x) \geq 0$, which is a contradiction with the definition of U.

Consider $T : L^\infty(\Omega) \to C^1(\overline{\Omega})$ as the unique weak solution $T(v)$ to the problem

$$\begin{gathered} -\Delta_m T(v) = v \quad \text{in } \Omega, \\ T(v) = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.3}$$

It is well known that the function T is continuous and compact (e.g., see [3, Lemma 1.1]).

Next, denote by $G(\tau,u):=T(f(x,u^+,\nabla u^+)-a(x)g(u^+,\nabla u^+)+\tau)$, then $G:\mathbb{R}^+\times C^1(\overline{\Omega})\to C^1(\overline{\Omega})$ is continuous and compact. Now, we will verify the hypotheses of Lemma 3.1. It is clear that $G(0,0)=0$. On the other hand, consider the compact homotopy $H(\lambda,u):[0,1]\times C^1(\overline{\Omega})\to C^1(\overline{\Omega})$ given by $H(\lambda,u)=u-\lambda G(0,u)$. We will show that

$$\text{if } u \text{ is a nontrivial solution to } H(\lambda,u)=0, \text{ then } \|u\|>R>0. \tag{3.4}$$

This fact implies that condition (i) of (b) holds. Moreover, (3.4) also implies that $\deg(H(\lambda,\cdot)B(0,R),0)$ is well defined since there is not solution on $\partial B(0,R)$. By the invariance property of the degree, we have

$$\deg\left(\mathrm{Id}-\lambda G(0,\cdot),B(0,R),0\right)=\deg\left(\mathrm{Id},B(0,R),0\right)=1,\quad \forall\lambda\in(0,1] \tag{3.5}$$

and (ii) of (b) holds.

In order to prove (3.4), note that $H(\lambda,u)=0$ implies that u is a solution to the problem

$$\begin{gathered}-\Delta_m u=\lambda\left(f(x,u^+,\nabla u^+)-a(x)g(u^+,\nabla u^+)\right)\quad \text{in } \Omega,\\ u=0\quad \text{on } \partial\Omega.\end{gathered} \tag{3.6}$$

Multiplying (3.6) by u, integrating over Ω the equation obtained, and applying Hölder's and Poincare's inequalities, we have that

$$\begin{aligned}\int_\Omega|\nabla u|^m &\le c_0\int_\Omega u^{p+1}+M_1\left[\int_\Omega|\nabla u|^\alpha u+\int_\Omega|\nabla u|^\beta u\right]\\ &\le C\left(\int_\Omega|\nabla u|^m\right)^{(p+1)/m}+M_1\left(\int_\Omega|\nabla u|^m\right)^{\alpha/m}\left(\int_\Omega u^{m/(m-\alpha)}\right)^{(m-\alpha)/m}\\ &\quad+M_1\left(\int_\Omega|\nabla u|^m\right)^{\beta/m}\left(\int_\Omega u^{m/(m-\beta)}\right)^{(m-\beta)/m}\\ &\le C\left(\int_\Omega|\nabla u|^m\right)^{(p+1)/m}+C_1\left(\int_\Omega|\nabla u|^m\right)^{(\alpha+1)/m}+C_1\left(\int_\Omega|\nabla u|^m\right)^{(\beta+1)/m}.\end{aligned} \tag{3.7}$$

This inequality implies that $\int_\Omega|\nabla u|^m>c>0$. Hence, we have $\|u\|>R>0$.

Now, we note that Theorem 1.1 and $C^{1,\rho}$ estimates imply that the component $\mathfrak{C}$ which contains $(0,0)$ is bounded. So, applying Lemma 3.1, we obtain that $\mathfrak{C}\cap(\{0\}\times C^1(\overline{\Omega}))\neq(0,0)$. Therefore, we have a positive solution u to the problem $(P)_0$. □

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References

[1] W. Dong, "A priori estimates and existence of positive solutions for a quasilinear elliptic equation," *Journal of the London Mathematical Society*, vol. 72, no. 3, pp. 645–662, 2005.

[2] D. Ruiz, "A priori estimates and existence of positive solutions for strongly nonlinear problems," *Journal of Differential Equations*, vol. 199, no. 1, pp. 96–114, 2004.

[3] C. Azizieh and P. Clément, "A priori estimates and continuation methods for positive solutions of p-Laplace equations," *Journal of Differential Equations*, vol. 179, no. 1, pp. 213–245, 2002.

[4] S. Takeuchi, "Positive solutions of a degenerate elliptic equation with logistic reaction," *Proceedings of the American Mathematical Society*, vol. 129, no. 2, pp. 433–441, 2001.

[5] W. Dong and J. T. Chen, "Existence and multiplicity results for a degenerate elliptic equation," *Acta Mathematica Sinica*, vol. 22, no. 3, pp. 665–670, 2006.

[6] P. H. Rabinowitz, "Pairs of positive solutions of nonlinear elliptic partial differential equations," *Indiana University Mathematics Journal*, vol. 23, pp. 173–186, 1973/1974.

[7] J. I. Díaz and J. E. Saá, "Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires. [Existence and uniqueness of positive solutions of some quasilinear elliptic equations]," *Comptes Rendus des Séances de l'Académie des Sciences. Série I. Mathématique*, vol. 305, no. 12, pp. 521–524, 1987.

[8] J. García Melián and J. S. de Lis, "Uniqueness to quasilinear problems for the p-Laplacian in radially symmetric domains," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 43, no. 7, pp. 803–835, 2001.

[9] Z. Guo and H. Zhang, "On the global structure of the set of positive solutions for some quasilinear elliptic boundary value problems," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 46, no. 7, pp. 1021–1037, 2001.

[10] S. Takeuchi and Y. Yamada, "Asymptotic properties of a reaction-diffusion equation with degenerate p-Laplacian," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 42, no. 1, pp. 41–61, 2000.

[11] S. Takeuchi, "Multiplicity result for a degenerate elliptic equation with logistic reaction," *Journal of Differential Equations*, vol. 173, no. 1, pp. 138–144, 2001.

[12] S. Takeuchi, "Stationary profiles of degenerate problems with inhomogeneous saturation values," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 63, no. 5–7, pp. e1009–e1016, 2005.

[13] S. Kamin and L. Véron, "Flat core properties associated to the p-Laplace operator," *Proceedings of the American Mathematical Society*, vol. 118, no. 4, pp. 1079–1085, 1993.

[14] J. Serrin and H. Zou, "Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities," *Acta Mathematica*, vol. 189, no. 1, pp. 79–142, 2002.

[15] N. S. Trudinger, "On Harnack type inequalities and their application to quasilinear elliptic equations," *Communications on Pure and Applied Mathematics*, vol. 20, pp. 721–747, 1967.

[16] L. Damascelli, "Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results," *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, vol. 15, no. 4, pp. 493–516, 1998.

[17] J. L. Vázquez, "A strong maximum principle for some quasilinear elliptic equations," *Applied Mathematics and Optimization*, vol. 12, no. 3, pp. 191–202, 1984.

[18] G. M. Lieberman, "Boundary regularity for solutions of degenerate elliptic equations," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 12, no. 11, pp. 1203–1219, 1988.

[19] H. Amann and J. López-Gómez, "A priori bounds and multiple solutions for superlinear indefinite elliptic problems," *Journal of Differential Equations*, vol. 146, no. 2, pp. 336–374, 1998.

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