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Research Article

Hölder Regularity of Solutions to Second-Order Elliptic Equations in Nonsmooth Domains

Sungwon Cho and Mikhail Safonov

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We establish the global Hölder estimates for solutions to second-order elliptic equations, which vanish on the boundary, while the right-hand side is allowed to be unbounded. For nondivergence elliptic equations in domains satisfying an exterior cone condition, similar results were obtained by J. H. Michael, who in turn relied on the barrier techniques due to K. Miller. Our approach is based on special growth lemmas, and it works for both divergence and nondivergence, elliptic and parabolic equations, in domains satisfying a general "exterior measure" condition.

1. Introduction

In the theory of partial differential equations, it is important to have estimates of solutions, which do not depend on the smoothness of the given data. Such kind of estimates include different versions of the maximum principle, which are crucial for investigation of boundary value problems for second-order elliptic and parabolic equations. More delicate properties of solutions, such as Hölder estimates and Harnack inequalities, are very essential for the building of general theory of nonlinear equations (see [1–6]).

In this paper, we establish the global Hölder regularity of solutions to the *Dirichlet problem*, or the *first boundary value problem*, for second-order elliptic equations. We deal with the *Dirichlet problem*

$$Lu = f \quad \text{in } \Omega, \; u = 0 \text{ on } \partial\Omega. \tag{DP}$$

Here Ω is a bounded open set in $\mathbb{R}^n, n \geq 1$, satisfying the following "exterior measure" condition (A). This condition appeared in the books [4, 5].

Definition 1.1. An open set $\Omega \subset \mathbb{R}^n$ satisfies the condition (A) if there exists a constant $\theta_0 > 0$, such that for each $y \in \partial\Omega$ and $r > 0$, the Lebesgue measure

$$|B_r(y) \setminus \Omega| \geq \theta_0 |B_r|, \tag{A}$$

where $B_r(y)$ is the ball of radius $r > 0$, centered at y.

We deal simultaneously with the cases when the elliptic operator L in (DP) is either in the *divergence form:*

$$Lu := -(D, aDu) = -\sum_{i,j} D_i(a_{ij}D_j u), \tag{D}$$

or in the *nondivergence form:*

$$Lu := -(aD, Du) = -\sum_{i,j} a_{ij}D_{ij}u, \tag{ND}$$

where $D_j u := \partial u/\partial x_j$, $D_{ij}u := D_i D_j u$, and $a = [a_{ij}] = [a_{ij}(x)]$ is a matrix function with real entries, which satisfies the *uniform ellipticity condition*

$$(a\xi, \xi) \geq \nu|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \qquad \|a\| := \max_{|\xi|\leq 1} |a\xi| \leq \nu^{-1}, \tag{U}$$

with a constant $\nu \in (0,1]$. In (D), (ND), and throughout this paper, D is a symbolic column vector with components $D_i := \partial/\partial x_i$, which helps to write explicit expressions for Lu in a shorter form. Note that the conditions (U) are invariant with respect to rotations in $\mathbb{R}^n$, and $\nu = 1$, if and only if $-L = \Delta := \sum_i D_{ii}$—the Laplace operator. Indeed, from (U) with $\nu = 1$, it follows

$$|\xi|^2 \leq (a\xi, \xi) \leq |a\xi| \cdot |\xi| \leq |\xi|^2 \quad \forall \xi \in \mathbb{R}^n; \tag{1.1}$$

hence $|a\xi| \equiv |\xi|$, $(a\xi, \xi) \equiv |\xi|^2$, which is possible if and only if $a = I =$ the identity matrix, so that $L = -\Delta$. The notations $(\cdot,\cdot)$ and $|\cdot|$ are explained at the end of this section.

For operators L in the divergence form (D), it has been proved by Littman et al. [7] that the boundary points of Ω are regular if and only if they are regular for $L = -\Delta$. In particular, isolated points cannot be regular in the divergence case (D). On the other hand, from the results by Gilbarg and Serrin in [8, Section 7], it follows that the functions $u(x) := |x|^\gamma$ and $\gamma = \text{const} \in (0,1)$ satisfy the equation $Lu = 0$ in $\Omega := \{x \in \mathbb{R}^n : 0 < |x| < 1\}$, $n \geq 2$, with some operators L in the nondivergence form (ND). For such operators, the boundary regularity of solutions to problem (DP) is usually investigated by the standard method of barrier functions. However, this method requires certain smoothness of the boundary $\partial\Omega$. For domains Ω satisfying an exterior cone condition, such barrier functions were constructed by Miller [9], and his construction was then widely used by many authors. In particular, Michael [10, 11] used Miller's technique in his general Schauder-type existence theory, which is based on the interior estimates only. One of the key elements in his theory is the following estimate for solutions to problem (DP):

$$\sup_\Omega d^{-\gamma}|u| \leq NF, \quad \text{where } F := \sup_\Omega d^{2-\gamma}|f|, \tag{M}$$

$d = d(x) := \operatorname{dist}(x, \partial\Omega)$, and the constants $\gamma \in (0,1)$ and $N > 0$ depend only on n, ν, and the characteristics of exterior cones. Note that the function f is allowed to be unbounded near $\partial\Omega$. At about the same time, Gilbarg and Hörmander [12] also used Miller's barriers in their theory of intermediate Schauder estimates. Once again, Schauder estimates in Lipschitz domains are treated there on the grounds of estimates similar to (M) (see [12, Lemma 7.1]). All these results deal with operators L in the nondivergence form (ND).

Our method is applied to general domains satisfying the "exterior measure" condition (A), and it works for both divergence and nondivergence equations. However, the natural functional spaces for solutions in these two cases are different. We use the same notation $W(\Omega)$ for classes of solutions, which are different in the case (D) or (ND), in order to treat these cases simultaneously. The classes $W(\Omega)$ are introduced in Definition 2.1 at the beginning of Section 2. In the rest of Section 2, we discuss the three basic facts: (i) *maximum principle* (Lemma 2.2), (ii) *pointwise estimate* (Lemma 2.4), and (iii) *growth lemma* (Lemma 2.5). Growth lemmas originate from methods of Landis [13]. They were essentially used in the proof of the interior Harnack inequality for solutions to elliptic and parabolic equations in the non-divergence form (ND) (see [3, 14, 15]). One can also use growth lemmas for an alternative proof of Moser's Harnack inequality in the divergence case (D); see [16, 17].

In Section 3, we prove estimate (M) with $0 < \gamma < \gamma_1 \le 1$, where γ_1 depends only on the dimension n, the ellipticity constant ν in (U), and the constant $\theta_0 > 0$ in the condition (A). This estimate, together with the interior Hölder regularity of solutions implies the global estimates for solutions to problem (DP) in the Hölder space $C^{0,\gamma}(\overline{\Omega})$, with an appropriate $\gamma > 0$.

Remark 1.2. Estimate (M) means that from $f = O(d^{\gamma-2})$, $0 < \gamma < \gamma_1 \le 1$, it follows $u = O(d^{\gamma})$ and in particular, $u \to 0$ as $d = d(x) \to 0^+$. The assumption $0 < \gamma < 1$ is essential even in the one-dimensional case:

$$-u'' = f := d^{\gamma-2} = \left(1 - |x|\right)^{\gamma-2} \quad \text{in } \Omega = (-1,1),\ u(\pm 1) = 0. \tag{1.2}$$

Indeed, if $\gamma \le 0$, then any solution to the equation $-u'' = f$ blows to $+\infty$ near $\partial\Omega = \{1, -1\}$. If $\gamma > 1$, then this problem has a unique solution u, but estimate (M) cannot hold, because it implies the equalities $u'(\pm 1) = 0$, conflicting the properties $u(\pm 1) = 0$ and $u'' < 0$ in $(-1,1)$. Finally, in the case $\gamma = 1$, from (M) and $u(\pm 1) = 0$ it follows $|u'(\pm 1)| \le NF$, while $-u'' = d^{-1}$ implies that $u'(\pm 1)$ are unbounded. Therefore, the restrictions $0 < \gamma < 1$ are necessary for validity of estimate (M). They are also sufficient for operators L in the form (ND) and the boundary $\partial\Omega$ of class C^2 (see [10]). In Theorem 3.9, we extend this result to domains Ω satisfying an exterior sphere condition. The proof of this theorem uses elementary comparison arguments only.

Basic notations. $\mathbb{R}^n$ is the n-dimensional Euclidean space, $n \ge 1$, with points $x = (x_1, \dots, x_n)^t$, where x_i are real numbers. Here the symbol t stands for the transposition of vectors, which indicates that vectors in $\mathbb{R}^n$ are treated as column vectors. For $x = (x_1, \dots, x_n)^t$ and $y = (y_1, \dots, y_n)^t$ in $\mathbb{R}^n$, the *scalar product* $(x,y) := \sum x_i y_i$, the *length* of x is $|x| := (x,x)^{1/2}$. For $y \in \mathbb{R}^n$, $r > 0$, the ball $B_r(y) := \{x \in \mathbb{R}^n : |x - y| < r\}$. $Du := (D_1 u, \dots, D_n u)^t \in \mathbb{R}^n$, where $D_i := \partial / \partial x_i$.

Let Ω be an open set in $\mathbb{R}^n$. For $1 \le p \le \infty$ and $k = 0, 1, \ldots, W^{k,p}(\Omega)$ denotes the Sobolev space of functions, which belongs to the Lebesgue space $L^p(\Omega)$ together with all its derivatives of order $\le k$. The norm of functions $u \in W^{k,p}(\Omega)$ is defined as $\|u\|_{W^{k,p}(\Omega)} := \sum_{|l| \le k} \|D^l u\|_{p,\Omega}$, where summation is taken over all multi-indices (vectors with nonnegative integer components) $l = (l_1, \ldots, l_n)$ of order $|l| := l_1 + \cdots + l_n$. In this expression, $D^l u := D_1^{l_1} \cdots D_n^{l_n} u$, and $\|f\|_{p,\Omega}$ is the norm of f in $L^p(\Omega)$, that is,

$$\|f\|_{p,\Omega}^p := \int_\Omega |f|^p dx \quad \text{for } 1 \le p < \infty; \qquad \|f\|_{\infty,\Omega} := \operatorname*{ess\,sup}_\Omega |f|. \tag{1.3}$$

Furthermore, $W_{\text{loc}}^{k,p}(\Omega)$ denotes the class of functions which belong to $W^{k,p}(\Omega')$ for arbitrary open subsets $\Omega' \subset \overline{\Omega}' \subset \Omega$.

$\partial\Gamma$ is the *boundary* of a set Γ in $\mathbb{R}^n$, $\overline{\Gamma} := \Gamma \cup \partial\Gamma$ is the *closure* of Γ, and $\operatorname{diam}\Gamma := \sup\{|x-y| : x, y \in \Gamma\}$—the *diameter* of Γ. Moreover, $|\Gamma| := |\Gamma|_n$ is the n-dimensional Lebesgue measure of a measurable set Γ in $\mathbb{R}^n$. $c_+ := \max(c, 0)$, $c_- := \max(-c, 0)$, where c is a real number. "$A := B$" or "$B =: A$" is the definition of A by means of the expression B.

$N = N(\cdots)$ denotes a constant depending only on the prescribed quantities, such as n, ν, and so forth, which are specified in the parentheses. Constants N in different expressions may be different. For convenience of cross-references, we assign indices to some of them.

2. Auxiliary statements

Let Ω be a bounded open set in $\mathbb{R}^n$, and let L be an elliptic operator in the form (D) or (ND) with coefficients $a_{ij} = a_{ij}(x)$ satisfying the uniform ellipticity condition (U) with a constant $\nu \in (0, 1]$. Using the notation for Sobolev spaces $W^{k,p}(\Omega)$, we introduce the class of functions $W(\Omega)$, which depends on the case (D) or (ND).

Definition 2.1. (i) In the divergence case (D), $W(\Omega) := W_{\text{loc}}^{1,2}(\Omega) \cap C(\overline{\Omega})$. Functions $u \in W(\Omega)$ and $f \in L_{\text{loc}}^2(\Omega)$ satisfy $Lu := -(D, aDu) \le (\ge, =) f$ in Ω (in a weak sense) if

$$\int_\Omega (D\phi, aDu)\,dx \le (\ge, =) \int_\Omega \phi f\,dx \quad \text{for any function } \phi \in C_0^\infty(\Omega),\ \phi \ge 0. \tag{2.1}$$

If $Lu = f$, then (2.1) holds for all functions $\phi \in C_0^\infty(\Omega)$ (ϕ can change sign).

(ii) In the non-divergence case (ND), $W(\Omega) := W_{\text{loc}}^{2,n}(\Omega) \cap C(\overline{\Omega})$. For $u \in W(\Omega)$ and measurable functions f on Ω, the relations $Lu := -(aD, Du) \le (\ge, =) f$ in Ω (in a strong sense) are understood almost everywhere (a.e.) in Ω.

By approximation, the property (2.1) is easily extended to nonnegative functions $\phi \in W^{1,2}(\Omega)$ with compact support in Ω. If $u \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$, then (2.1) holds true for $\phi \in W_0^{1,2}(\Omega)$—the closure of $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega)$.

Lemma 2.2 (maximum principle). *Let u be a function in $W(\Omega)$ satisfying $Lu \le 0$ in Ω. Then*

$$\sup_\Omega u = \sup_{\partial\Omega} u. \tag{2.2}$$

This is a well-known classical result. It is contained, for example, in [2, Theorem 8.1 (case (D)) and Theorem 9.1 (case (ND))]. Since our assumptions in the case (D) are slightly different from those in [2], we give a sketch of the proof.

Proof (in the case (D)). Suppose the equality (2.2) fails, that is, the left-hand side in (2.2) is strictly larger than the right-hand side. Replacing u by $u-\text{const}$, we can assume that the set $\Omega' := \Omega \cap \{u > 0\}$ is not empty, and $u < 0$ on $\partial\Omega$. Then automatically $u = 0$ on $\partial\Omega'$. Approximating $u_+ := \max(u,0)$ in $W^{1,2}(\Omega)$ by functions $\phi \in C_0^\infty(\Omega)$, one can see that the inequality (2.1) holds with $\phi = u_+$ and $f = 0$. This yields

$$\nu \int_{\Omega'} |Du|^2 dx \le \int_{\Omega'} (Du, aDu)\, dx \le 0. \tag{2.3}$$

Hence $Du = 0$ and $u = \text{const}$ on each open connected component of Ω'. Since $u = 0$ on $\partial\Omega'$, we must have $u \equiv 0$ in Ω', in contradiction to our assumption $\Omega' := \Omega \cap \{u > 0\} \neq \varnothing$. □

Applying this lemma to the function $u - v$, we immediately get the following.

Corollary 2.3 (comparison principle). If $u, v \in W(\Omega)$ satisfy $Lu \le Lv$ in Ω, and $u \le v$ on $\partial\Omega$, then $u \le v$ in Ω.

Lemma 2.4 (pointwise estimate). (i) *For an arbitrary elliptic operator L (in the form (D) or (ND)) with coefficients a_{ij} which are defined on a ball $B_R := B_R(x_0) \subset \mathbb{R}^n$ and satisfy (U) with a constant $\nu \in (0,1]$, there exists a function $w \in W(B_R)$ such that*

$$0 \le w \le N_0 R^2, \quad Lw \ge 1 \quad \text{in } B_R; \qquad w = 0 \quad \text{on } \partial B_R, \tag{2.4}$$

where the constant $N_0 = N_0(n,\nu)$.

(ii) *Moreover, for an arbitrary open set $\Omega \subseteq B_R$ and an arbitrary function $u \in W(\Omega)$,*

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u + N_0 R^2 \cdot \sup_{\Omega} (Lu)_+. \tag{2.5}$$

Proof. (i) By rescaling $x \to R^{-1}x$, we reduce the proof to the case $R = 1$.

In the divergence case (D), consider the Dirichlet problem

$$Lw := -(D, aDw) = 1 \quad \text{in } B_1; \qquad w = 0 \quad \text{on } \partial B_1. \tag{2.6}$$

It is known (see [2, Theorems 8.3 and 8.16]) that there exists a unique solution w to this problem, which belongs to $W^{1,2}(B_1) \cap C(\overline{B}_1) \subset W(B_1)$ and satisfies $0 \le w \le N_0 = N_0(n,\nu)$ on $\overline{B}_1$. This function w satisfies all the properties (2.4) (with $R = 1$).

In the nondivergence case (ND), we take $w(x) := (2n\nu)^{-1} \cdot (1 - |x - x_0|^2)$. Since $\operatorname{tr} a := \sum_i a_{ii} \ge n\nu$, we have

$$Lw := -(aD, Dw) = (n\nu)^{-1} \cdot \operatorname{tr} a \ge 1 \quad \text{in } B_1,\ w = 0 \text{ on } \partial B_1, \tag{2.7}$$

so that (2.4) holds with $N_0 := \sup w = (2n\nu)^{-1}$.

(ii) We will compare $u = u(x)$ with the function

$$v = v(x) := \sup_{\partial\Omega} u + \sup_{\Omega}(Lu)_+ \cdot w(x). \tag{2.8}$$

We have

$$Lu \le \sup_{\Omega}(Lu)_+ \le Lv \quad \text{in } \Omega, \qquad u \le \sup_{\partial\Omega} u \le v \quad \text{on } \partial\Omega. \tag{2.9}$$

By the comparison principle, $u \le v$ in Ω. Since $w \le N_0 R^2$, the inequality (2.5) follows. □

Lemma 2.5 (growth lemma). *Let $x_0 \in \mathbb{R}^n$ and let $r > 0$ be such that the Lebesgue measure*

$$|B_r \setminus \Omega| \ge \theta |B_r|, \quad \theta > 0, \tag{2.10}$$

where $B_r := B_r(x_0)$. Then for any function $u \in W(\Omega)$, satisfying $u > 0$, $Lu \le 0$ in Ω, and $u = 0$ on $(\partial\Omega) \cap B_{4r}$,

$$\sup_{B_r} u \le \beta \cdot \sup_{B_{4r}} u = \beta \cdot \sup_{\partial B_{4r}} u, \tag{2.11}$$

where $\beta = \beta(n,\nu,\theta) \in (0,1)$. Assume that u is extended as $u \equiv 0$ on $B_{4r} \setminus \Omega$, so that both sides of (2.11) are always well defined.

The last equality in (2.11) is a consequence of the maximum principle.

In the divergence case (D), Lemma 2.5 (in equivalent formulations) is contained in [13, Chapter 2, Lemma 3.5], or in [17, formula (39)]. In the nondivergence case (ND), this follows from [15, Corollary 2.1]. In dealing with these references, or more generally, with different versions of growth lemmas, one can always impose the additional simplifying assumptions.

Assumptions 2.6. (i) The function u is defined on the whole ball B_{4r} in such a way that

$$u \in W(B_{4r}), \quad Lu \le 0 \quad \text{in } B_{4r}, \tag{2.12}$$

and $\Omega := B_{4r} \cap \{u > 0\}$ satisfies

$$|B_r \setminus \Omega| = |B_r \cap \{u \le 0\}| > \theta |B_r|, \quad \theta > 0. \tag{2.13}$$

(ii) All the functions a_{ij} and u belong to $C^\infty(\overline{B_{4r}})$.

Here we show that if the previous lemma is true under these additional assumptions, then it holds true in its original form. We proceed in two steps accordingly to parts (i), extension of u from $\Omega \cap B_{4r}$ to B_{4r}, and (ii), approximation of a_{ij} and u by smooth functions.

(i) *Extension to B_{4r}.* Fix $\varepsilon > 0$ and choose a function $G \in C^\infty(\mathbb{R}^1)$ (depending on ε) such that

$$G, G', G'' \ge 0 \quad \text{on } \mathbb{R}^1, \qquad G \equiv 0 \quad \text{on } (-\infty, \varepsilon], \qquad G' \equiv 1 \quad \text{on } [2\varepsilon, \infty). \tag{2.14}$$

Further, define

$$u_\varepsilon := G(u) \quad \text{in } \Omega \cap B_{4r}, \qquad u_\varepsilon \equiv 0 \quad \text{on } \overline{B_{4r}} \setminus \Omega. \tag{2.15}$$

From the above properties of the function G it follows

$$(u-2\varepsilon)_+ \le u_\varepsilon \le (u-\varepsilon)_+ \quad \text{in } \Omega. \tag{2.16}$$

Since $u = 0$ on the set $(\partial\Omega) \cap B_{4r}$, the function u_ε vanishes near this set. Hence in both cases (D) and (ND), we have $u_\varepsilon \in W(B_{4r})$ and $u_\varepsilon \ge 0$ in B_{4r}. Moreover, we claim that $Lu_\varepsilon \le 0$ in B_{4r}. In the non-divergence case (ND), this follows immediately from $Lu_\varepsilon \equiv 0$ on $B_{4r} \setminus \Omega$ and

$$Lu_\varepsilon = LG(u) = G'(u) \cdot Lu - G''(u) \cdot (Du, aDu) \le 0 \quad \text{in } \Omega. \tag{2.17}$$

In the divergence case (D), the inequality $Lu_\varepsilon \le 0$ is understood in a weak sense (2.1). Let ϕ be an arbitrary nonnegative function in $C_0^\infty(B_{4r})$. Then the function $\phi_0 := \phi \cdot G'(u)$ is also non-negative, belongs to $W_2^1(\Omega)$, and has compact support in $\Omega \cap B_{4r}$. By approximation, we can put ϕ_0 in place of ϕ in the inequality (2.1) corresponding to $Lu \le 0$ in Ω, that is,

$$\int (D\phi_0, aDu)\, dx \le 0. \tag{2.18}$$

Having in mind that $Du_\varepsilon = DG(u) = G'(u)Du$ (see [2, Section 7.4]), and

$$D\phi_0 = D(\phi G'(u)) = G'(u)D\phi + \phi G''(u)Du, \tag{2.19}$$

we obtain

$$\begin{aligned} \int (D\phi, aDu_\varepsilon)\, dx &= \int G'(u) \cdot (D\phi, aDu) dx \\ &= \int (D\phi_0, aDu) dx - \int \phi G''(u) \cdot (Du, aDu) dx \le 0. \end{aligned} \tag{2.20}$$

Since this is true for any $\phi \in C_0^\infty(B_{4r})$, $\phi \ge 0$, it follows $Lu_\varepsilon := -(D, aDu_\varepsilon) \le 0$ in B_{4r} (in a weak sense).

Now suppose that Lemma 2.5 is true under additional Assumptions 2.6(i). For any small $\varepsilon > 0$, we can apply this weaker formulation to the function $u_\varepsilon := G(u)$ in $\Omega_\varepsilon := \{u_\varepsilon > 0\} \cap B_{4r}$. We know that $u_\varepsilon \in W(B_{4r})$ and $Lu_\varepsilon \le 0$ in B_{4r}. Moreover, estimate (2.10) for Ω implies a bit stronger estimate (2.13) for $\Omega_\varepsilon \subset \Omega$. In addition, obviously $u_\varepsilon = 0$ on $(\partial\Omega_\varepsilon) \cap B_{4r}$. Hence the functions u_ε satisfy estimate (2.11) with the same $\beta = \beta(n,\nu,\theta) \in (0,1)$. By virtue of (2.16), $u_\varepsilon \to u$ as $\varepsilon \to 0^+$, uniformly on Ω, and we get estimate (2.11) under the original assumptions in Lemma 2.5.

(ii) *Approximation by smooth functions.* The additional Assumptions 2.6(i) help in approximation of a_{ij} and u by smooth functions. Note that since both sides of (2.13) are continuous with respect to r, we also have

$$|\{u \le 0\} \cap B_\rho| > \theta |B_\rho| \tag{2.21}$$

for all $\rho < r$ which are close enough to r. Fix such $\rho < r$ and approximate a_{ij} by convolutions $a_{ij}^{(\varepsilon)}$, $0 < \varepsilon < \varepsilon_0 := r - \rho > 0$, which are defined in a standard way:

$$f^{(\varepsilon)}(x) := (\eta_\varepsilon * f)(x) := \int \eta_\varepsilon(x-y) f(y)\,dy = \int f(x-y)\eta_\varepsilon(y)\,dy. \tag{2.22}$$

Here η_ε are fixed functions satisfying the properties

$$\eta_\varepsilon \in C^\infty(\mathbb{R}^n), \quad \eta_\varepsilon \geq 0 \quad \text{in } \mathbb{R}^n, \quad \eta_\varepsilon(x) \equiv 0 \quad \text{for } |x| \geq \varepsilon, \quad \int \eta_\varepsilon\,dx = 1. \tag{2.23}$$

Then $a_{ij}^{(\varepsilon)} \in C^\infty(\overline{B_{4\rho}})$ and the matrices $a^{(\varepsilon)} := [a_{ij}^{(\varepsilon)}]$ satisfy the uniform ellipticity condition (U) with the same constant ν. Further, we consider the cases (D) and (ND) separately.

Divergence case (D). Denote $r_0 := 4\rho + \varepsilon_0 < 4r$. From $u \in W(B_{4r}) := W^{1,2}_{\text{loc}}(B_{4r}) \cap C(\overline{B_{4r}})$ it follows $u \in W^{1,2}(B_{r_0}) \cap C(\overline{B_{r_0}})$ and $aDu \in L^2(B_{r_0})$. Hence the functions

$$f_\varepsilon := -\left(D, (aDu)^{(\varepsilon)}\right) \in C^\infty\left(\overline{B_{4\rho}}\right), \quad 0 < \varepsilon < \varepsilon_0. \tag{2.24}$$

Without loss of generality, assume $x_0 = 0$. Then for fixed $x \in B_{4\rho} = B_{4\rho}(0)$ and $0 < \varepsilon < \varepsilon_0$, the function $\phi(y) := \eta_\varepsilon(x-y)$ is non-negative, belongs to C^∞, and has compact support in B_{r_0}. Since $Lu := -(D, aDu) \leq 0$ in B_{r_0}, and $D\phi(y) = -D\eta_\varepsilon(x-y)$, we have

$$f_\varepsilon(x) := -\left(D, \int \left(\eta_\varepsilon(x-y), aDu(y)\right) dy\right) = \int \left(D\phi(y), aDu(y)\right) dy \leq 0 \tag{2.25}$$

for $x \in B_{4\rho}$ and $0 < \varepsilon < \varepsilon_0$. In terms of Schwartz distributions, this property simply means that $Lu \leq 0$ implies $f_\varepsilon = (Lu)^{(\varepsilon)} \leq 0$.

Next, consider the Dirichlet problem

$$L_\varepsilon u_\varepsilon := -\left(D, a^{(\varepsilon)} D u_\varepsilon\right) = f_\varepsilon \quad \text{in } B_{4\rho}, \qquad u_\varepsilon = u^{(\varepsilon)} \quad \text{on } \partial B_{4\rho}, \tag{2.26}$$

where $0 < \varepsilon < \varepsilon_0$. Here $a^{(\varepsilon)}$, f_ε, and $u^{(\varepsilon)}$ belong to $C^\infty(\overline{B_{4\rho}})$, so that this problem has a unique classical solution u_ε, which belongs to $C^\infty(\overline{B_{4\rho}})$ (see, e.g., [2, Theorem 6.19]). Then the functions

$$v_\varepsilon := u_\varepsilon - u^{(\varepsilon)}, \qquad g_\varepsilon := (aDu)^{(\varepsilon)} - a^{(\varepsilon)} D u^{(\varepsilon)} \in C^\infty\left(\overline{B_{4\rho}}\right), \tag{2.27}$$

and $v_\varepsilon = 0$ on $\partial B_{4\rho}$. Integrating by parts over the ball $B_{4\rho}$, and then applying the Cauchy-Schwartz inequality, we derive

$$\begin{aligned}
\int \left(Dv_\varepsilon, a^{(\varepsilon)} D u_\varepsilon\right) dx &= \int v_\varepsilon L_\varepsilon u_\varepsilon\,dx = \int v_\varepsilon f_\varepsilon\,dx = \int \left(Dv_\varepsilon, (aDu)^{(\varepsilon)}\right) dx, \\
\nu \int |Dv_\varepsilon|^2 dx &\leq \int \left(Dv_\varepsilon, a^{(\varepsilon)} D v_\varepsilon\right) dx = \int \left(Dv_\varepsilon, a^{(\varepsilon)} D u_\varepsilon - a^{(\varepsilon)} D u^{(\varepsilon)}\right) dx \\
&= \int \left(Dv_\varepsilon, g_\varepsilon\right) dx \leq \frac{\nu}{2} \int |Dv_\varepsilon|^2 dx + \frac{1}{2\nu} \int g_\varepsilon^2\,dx.
\end{aligned} \tag{2.28}$$

It follows $\int |Dv_\varepsilon|^2 dx \le \nu^{-2} \int |g_\varepsilon|^2 dx$, and then by the Poincaré inequality,

$$\int |v_\varepsilon|^2 dx \le N(n,\nu,\rho) \cdot \int g_\varepsilon^2 dx, \quad 0 < \varepsilon < \varepsilon_0. \tag{2.29}$$

Further, we will use the property of convolution: for any open set $\Omega \subset \mathbb{R}^n$, and any bounded open subset $\Omega' \subset \overline{\Omega'} \subset \Omega$, we have

$$\begin{gathered} h^{(\varepsilon)} \longrightarrow h \quad \text{in } L^p(\Omega') \text{ as } \varepsilon \longrightarrow 0^+, \text{ if } h \in L^p(\Omega),\ 1 \le p < \infty; \\ h^{(\varepsilon)} \longrightarrow h \quad \text{a.e. in } \Omega' \text{ as } \varepsilon \longrightarrow 0^+, \text{ if } h \in L^p(\Omega),\ 1 \le p \le \infty; \\ h^{(\varepsilon)} \longrightarrow h \quad \text{in } L^\infty(\Omega') \text{ as } \varepsilon \longrightarrow 0^+, \text{ if } h \in C(\Omega). \end{gathered} \tag{2.30}$$

In our case $\Omega' := B_{4\rho} \subset \Omega := B_{r_0}$. We write $g_\varepsilon = g_{1,\varepsilon} + g_{2,\varepsilon} + g_{3,\varepsilon}$, where

$$g_{1,\varepsilon} := (aDu)^{(\varepsilon)} - aDu, \qquad g_{2,\varepsilon} := aDu - a^{(\varepsilon)}Du, \qquad g_{3,\varepsilon} := a^{(\varepsilon)}Du - a^{(\varepsilon)}Du^{(\varepsilon)}. \tag{2.31}$$

From $a \in L^\infty(B_{r_0})$ and $Du, aDu \in L^2(B_{r_0})$, it follows $g_{1,\varepsilon} \to 0$ in $L^2(B_{4\rho})$. We also have $a^{(\varepsilon)} \to a$ a.e. in $B_{4\rho}$, and by the dominated convergence theorem, $g_{2,\varepsilon} \to 0$ in $L^2(B_{4\rho})$. Finally, since all the matrices $a^{(\varepsilon)}$ satisfy (U) with same constant ν, the norm of $g_{3,\varepsilon}$ in $L^2(B_{4\rho})$,

$$||g_{3,\varepsilon}||_2 \le ||a^{(\varepsilon)}|| \cdot ||Du - Du^{(\varepsilon)}||_2 \le \nu^{-1} \cdot ||Du - (Du)^{(\varepsilon)}||_2 \longrightarrow 0; \tag{2.32}$$

therefore,

$$||g_\varepsilon||_2 \le ||g_{1,\varepsilon}||_2 + ||g_{2,\varepsilon}||_2 + ||g_{3,\varepsilon}||_2 \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0^+. \tag{2.33}$$

By virtue of (2.29), $\|v_\varepsilon\|_2 \to 0$ as $\varepsilon \to 0^+$. Furthermore, since $u \in C(B_{r_0})$, the convolutions $u^{(\varepsilon)} \to u$ uniformly on $B_{4\rho}$, which implies convergence in $L^2(B_{4\rho})$. Summarizing the above arguments, we obtain

$$u_\varepsilon = v_\varepsilon + u^{(\varepsilon)} \longrightarrow u \quad \text{in } L^2(B_{4\rho}) \text{ as } \varepsilon \longrightarrow 0^+. \tag{2.34}$$

Fix a small constant $h > 0$, and note that

$$u_\varepsilon - u > h \quad \text{on } S_{\varepsilon,h,\rho} := \{u_\varepsilon > h,\ u \le 0\} \cap B_\rho. \tag{2.35}$$

By virtue of (2.34), the measure

$$|S_{\varepsilon,h,\rho}| \le h^{-2} \int_{S_{\varepsilon,h,\rho}} (u_\varepsilon - u)^2 dx \le h^{-2} \int_{B_\rho} (u_\varepsilon - u)^2 dx \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0^+. \tag{2.36}$$

Now from $\{u_\varepsilon \le h\} \cap B_\rho \supseteq (\{u \le 0\} \cap B_\rho) \setminus S_{\varepsilon,h,\rho}$ and (2.21), it follows

$$|\{u_\varepsilon \le h\} \cap B_\rho| \ge |\{u \le 0\} \cap B_\rho| - |S_{\varepsilon,h,\rho}| > \theta |B_\rho|, \tag{2.37}$$

provided $\varepsilon > 0$ is small enough.

Now suppose that Lemma 2.5 is true for smooth a_{ij} and u. We can apply it to the function $u_\varepsilon - h$ which satisfies $L_\varepsilon(u_\varepsilon - h) = f_\varepsilon \le 0$ in $B_{4\rho}$. By Lemma 2.2, the maximum of u_ε on $\overline{B_{4\rho}}$ is attained on the boundary $\partial B_{4\rho}$, so that for small $\varepsilon > 0$,

$$\sup_{B_\rho}(u_\varepsilon - h) \le \beta \cdot \sup_{B_{4\rho}}(u_\varepsilon - h) < \beta \cdot \sup_{B_{4\rho}} u_\varepsilon = \beta \cdot \sup_{\partial B_{4\rho}} u^{(\varepsilon)} \le \beta \cdot \sup_{B_{4\rho}} u^{(\varepsilon)}. \tag{2.38}$$

Further, since u is continuous on $\overline{B_\rho}$, we have

$$\sup_{B_\rho} u < u + h \quad \text{on an open nonempty set } O \subseteq B_\rho. \tag{2.39}$$

From the convergence $u_\varepsilon \to u$ in $L^2(B_{4\rho})$, it follows the convergence in $L^1(O)$. Using also (2.38) and the uniform convergence $u^{(\varepsilon)} \to u$ on $B_{4\rho}$, we obtain

$$\begin{aligned}\sup_{B_\rho} u &< \frac{1}{|O|}\int_O (u+h)dx = \lim_{\varepsilon\to 0^+}\frac{1}{|O|}\int_O (u_\varepsilon + h)\,dx \\ &= 2h + \lim_{\varepsilon\to 0^+}\frac{1}{|O|}\int_O (u_\varepsilon - h)\,dx \le 2h + \beta \cdot \sup_{B_{4\rho}} u.\end{aligned} \tag{2.40}$$

Letting $h \to 0^+$ and then $\rho \to r^-$, we arrive at the estimate

$$\sup_{B_r} u \le \beta \cdot \sup_{B_{4r}} u \tag{2.41}$$

which is equivalent to (2.11) (under additional Assumptions 2.6(i)). Thus we have reduced Lemma 2.5 for divergence operators (D) to the smooth case.

Nondivergence case (ND). We will partially follow the previous arguments, with obvious simplification. Now from $u \in W(B_r) := W^{2,n}_{\text{loc}}(B_r) \cap C(\overline{B_r})$, it follows $u \in W^{2,n}(B_{r_0}) \cap C(\overline{B_{r_0}})$, where $r_0 := 4\rho + \varepsilon_0 < r$. Then $f := Lu := (aD, Du) \in L^n(B_{4\rho})$, and from $f \le 0$ in B_{r_0}, it follows $f^{(\varepsilon)} \le 0$ in $B_{4\rho}$, for $0 < \varepsilon \le \varepsilon_0$. For such ε, the Dirichlet problem

$$L_\varepsilon u_\varepsilon := -(a^{(\varepsilon)}D, Du_\varepsilon) = f^{(\varepsilon)} \quad \text{in } B_{4\rho}, \qquad u_\varepsilon = u^{(\varepsilon)} \quad \text{on } \partial B_{4\rho}, \tag{2.42}$$

has a unique classical solution u_ε which belongs to $C^\infty(\overline{B_{4\rho}})$. Then $u_\varepsilon - u \in W^{2,n}(B_{4\rho}) \cap C(\overline{B_{4\rho}})$, and

$$g_\varepsilon := L_\varepsilon(u_\varepsilon - u) \longrightarrow 0 \quad \text{in } L^n(B_{4\rho}) \text{ as } \varepsilon \longrightarrow 0^+, \tag{2.43}$$

because $g_\varepsilon = g_{1,\varepsilon} + g_{2,\varepsilon}$, where

$$g_{1,\varepsilon} = L_\varepsilon u_\varepsilon - Lu = f^{(\varepsilon)} - f \longrightarrow 0, \qquad g_{2,\varepsilon} = Lu - L_\varepsilon u = \big((a^{(\varepsilon)} - a)D, Du\big) \longrightarrow 0. \tag{2.44}$$

In addition, $u_\varepsilon - u = u^{(\varepsilon)} - u \to 0$ uniformly on $\partial B_{4\rho}$. By the Aleksandrov-type estimate (see, [2, Theorem 9.1]),

$$\sup_{B_{4\rho}} |u_\varepsilon - u| \le \sup_{\partial B_{4\rho}} |u_\varepsilon - u| + N(n,\nu,\rho) \cdot \|g_\varepsilon\|_{n,B_{4\rho}} \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0^+. \tag{2.45}$$

Now for fixed $h > 0$, we have $|u_\varepsilon - u| \le h$ on $B_{4\rho}$ for small $\varepsilon > 0$, and from estimate (2.21), it follows (2.37), which in turn yields (2.38). The desired estimate is obtained from (2.38) by taking $\varepsilon \to 0^+$, then $\delta \to 0^+$, and finally, $\rho \to r^-$.

Therefore, in Lemma 2.5 (and other similar statements) we can always impose the additional Assumptions 2.6.

Remark 2.7. In a simple case $L := -\Delta$, Lemma 2.5 follows immediately from the *mean value theorem* for subharmonic functions. Indeed, in this case u is positive subharmonic function in Ω, which vanishes on $(\partial\Omega) \cap B_{4r}$. By defining $u \equiv 0$ on $B_{4r} \setminus \Omega$, we get a nonnegative subharmonic function u in B_{4r}. For arbitrary $y \in B_r = B_r(x_0)$, we have $B_r \subset B_{2r}(y) \subset B_{3r}$, and by the mean value theorem,

$$u(y) \le \frac{1}{|B_{2r}|} \int_{B_{2r}(y)} u\,dx = \frac{1}{|B_{2r}|} \int_{\Omega \cap B_{2r}(y)} u\,dx \le \frac{|\Omega \cap B_{2r}(y)|}{|B_{2r}|} \sup_{B_{3r}(x_0)} u. \tag{2.46}$$

Condition (2.10) implies

$$\begin{aligned} |B_{2r}(y) \setminus \Omega| &\ge |B_r \setminus \Omega| \ge \theta |B_r| = 2^{-n}\theta \cdot |B_{2r}|, \\ |\Omega \cap B_{2r}(y)| &= |B_{2r}| - |B_{2r}(y) \setminus \Omega| \le (1 - 2^{-n}\theta) \cdot |B_{2r}|, \end{aligned} \tag{2.47}$$

and estimate (2.11) holds true with $\beta := 1 - 2^{-n}\theta \in (0,1)$.

Remark 2.8. Consider another special case, when the operator L is in the nondivergence form (ND), and instead of (2.10), we have a stronger assumption

$$B_r(x_0) \setminus \Omega \text{ contains a ball } B_{\theta_1 r}(z), \quad \theta_1 = \text{const} \in (0,1). \tag{2.48}$$

In this case, Lemma 2.5 can be proved by the elementary comparison argument. For the proof of this weaker version of this lemma, we can assume $r = 1$ and $z = 0$. The general case is obtained from here by a linear transformation. Note that

$$\begin{aligned} L(|x|^{-m}) := -(aD, D(|x|^{-m})) &= m\Big[\operatorname{tr} a - (m+2)\frac{(ax,x)}{|x|^2}\Big] \cdot |x|^{-m-2} \\ &\le m[n\nu^{-1} - (m+2)\nu] \cdot |x|^{-m-2} \le 0 \quad \text{for } x \ne 0, \end{aligned} \tag{2.49}$$

provided the constant $m = m(n,\nu) > 0$ is large enough; for example, one can take $m := n\nu^{-2}$. Fix such a constant m and compare u with

$$v(x) := \frac{\theta_1^{-m} - |x|^{-m}}{\theta_1^{-m} - 3^{-m}} M, \quad \text{where } M := \sup_{\Omega'} u > 0, \tag{2.50}$$

on the set $\Omega' := \Omega \cap B_3(0)$. We have $L(u - v) \le 0$ in Ω', and

$$u = 0 \le v \quad \text{on } (\partial\Omega) \cap B_3(0), \qquad u \le M = v \quad \text{on } (\partial B_3(0)) \cap \Omega, \tag{2.51}$$

that is, $u \le v$ on $\partial\Omega'$. By the comparison principle, $u \le v$ in Ω'. Having in mind that

$$\Omega \cap B_1(x_0) \subseteq \Omega \cap B_2(0) \subseteq \Omega' := \Omega \cap B_3(0) \subseteq \Omega \cap B_4(x_0), \tag{2.52}$$

we finally derive estimate (2.11):

$$\sup_{\Omega\cap B_1(x_0)} u \le \sup_{\Omega\cap B_2(0)} v \le \beta\cdot M \le \beta\cdot \sup_{\Omega\cap B_4(x_0)} u, \quad \text{where } \beta := \frac{\theta_1^{-m}-2^{-m}}{\theta_1^{-m}-3^{-m}} \in (0,1). \tag{2.53}$$

3. Main results

Throughout this section, Ω is a bounded domain in $\mathbb{R}^n$ satisfying the "exterior measure" condition (A) in Definition 1.1 with a constant $\theta_0>0$, and L is a second-order elliptic operator in the divergence (D) or nondivergence (ND) form with coefficients a_{ij} satisfying the uniform ellipticity condition (U) with a constant $\nu\in(0,1]$. We apply L to functions in the classes $W(\Omega)$ described in Definition 2.1. For $x\in\Omega$, we set $d=d(x):=\operatorname{dist}(x,\partial\Omega)$.

Here we prove estimate (M) for solutions to the Dirichlet problem (DP) in Ω. This statement is contained in Theorem 3.5 and Corollary 3.6, which are preceded with a few more technical results. Estimate (M) is true with $\gamma\in(0,\gamma_0)$, where the constant $\gamma_0\in(0,1]$ depends only on n, ν, and $\theta_0>0$. Theorem 3.9 is devoted to a special case, when the operator L is in the nondivergence form (ND), and Ω satisfies the exterior sphere condition in Definition 3.8; in this case this estimate (M) holds true with $\gamma_0=1$. Finally, this estimate together with Lemmas 2.4 and 2.5 imply the global Hölder regularity of solutions to problem (DP), which is contained in Theorem 3.10.

Lemma 3.1. *Let $\omega(\rho)$ be a nonnegative, nondecreasing function on an interval $(0,\rho_0]$, such that*

$$\omega(\rho)\le q^{-\alpha}\omega(q\rho) \quad \forall\rho\in(0,q^{-1}\rho_0], \tag{3.1}$$

with some constants $q>1$ and $\alpha>0$. Then

$$\omega(\rho)\le\left(\frac{q\rho}{\rho_0}\right)^{\alpha}\omega(\rho_0) \quad \forall\rho\in(0,\rho_0]. \tag{3.2}$$

Proof. For an arbitrary $\rho\in(0,\rho_0]$, we have $q^{-j-1}\rho_0<\rho\le q^{-j}\rho_0$ for some integer $j\ge0$. From these inequalities it follows $q^{-j}<q\rho/\rho_0$ and $q^j\rho\le\rho_0$. Iterating (3.1) and using monotonicity of ω, we obtain

$$\omega(\rho)\le q^{-\alpha}\omega(q\rho)\le q^{-2\alpha}\omega(q^2\rho)\le\cdots\le q^{-j\alpha}\omega(q^j\rho)\le\left(\frac{q\rho}{\rho_0}\right)^{\alpha}\omega(\rho_0). \tag{3.3}$$

□

Lemma 3.2. *Let $y\in\partial\Omega$, $r_0=\text{const}>0$, an open subset $\Omega'\subseteq\Omega$, and let $u\in W(\Omega')$ be such that*

$$u>0,\quad Lu\le 0 \quad \text{in } \Omega'\cap B_{r_0}(y); \qquad u=0 \quad \text{on } (\partial\Omega')\cap B_{r_0}(y). \tag{3.4}$$

Set $u\equiv0$ on $\Omega\setminus\Omega'$. Then

$$\omega_y(\rho):=\sup_{\Omega\cap B_\rho(y)} u\le\left(\frac{4\rho}{r_0}\right)^{\gamma_1}\omega_y(r_0) \quad \forall\rho\in(0,r_0], \tag{3.5}$$

where the constant $\gamma_1 = \gamma_1(n,\nu,\theta_0) := -\log_4 \beta \in (0,1]$, $\beta = \beta(n,\nu,\theta_0) \in (0,1)$, *is the constant in Lemma 2.5 corresponding to* $\theta = \theta_0$ *in condition (A).*

Proof. Condition (A) implies

$$|B_\rho(y) \setminus \Omega'| \geq |B_\rho(y) \setminus \Omega| \geq \theta_0 |B_\rho| \quad \forall \rho > 0. \tag{3.6}$$

Applying Lemma 2.5 to the function u in Ω', we obtain $\omega_y(\rho) \leq \beta \cdot \omega_y(4\rho)$ for all $\rho \in (0, 4^{-1} r_0]$, with a constant $\beta = \beta(n,\nu,\theta_0) \in (0,1)$. We can write $\beta = 4^{-\gamma_1}$, where $\gamma_1 = \gamma_1(n,\nu,\theta_0) := -\log_4 \beta > 0$, then (3.5) follows by the previous lemma.

It remains to show that $\gamma_1 \leq 1$, or equivalently, the above properties cannot hold *uniformly* for $\gamma_1 > 1$, $y \in \partial\Omega$, and operators L under consideration. Indeed, consider the case $\Omega' = \Omega$ and $L = -\Delta$. One can always find a ball $B := B_\rho(z) \subseteq \Omega$ which touches the boundary $\partial\Omega$ at some point y, that is, $y \in (\partial B) \cap (\partial\Omega)$. In the assumptions of this lemma, consider $\Delta u = 0$ as a special case of $Lu \leq 0$. Then estimate (3.5) with $\gamma_1 > 1$ would imply that the normal derivative to ∂B of u at the point y is zero. But this is impossible by boundary Hopf's lemma (see, [2, Lemma 3.4]). Therefore, we must have $\gamma_1 \leq 1$. □

Corollary 3.3. Let $r_0 = \text{const} > 0$, an open subset $\Omega' \subseteq \Omega$, and let $u \in W(\Omega')$ be such that

$$u > 0, \quad Lu \leq 0 \quad \text{in } \Omega' \cap \{d < r_0\}; \qquad u = 0 \quad \text{on } (\partial\Omega') \cap \{d < r_0\}, \tag{3.7}$$

where $d = d(x) := \text{dist}(x, \partial\Omega)$. Set $u \equiv 0$ on $\Omega \setminus \Omega'$. Then

$$\omega(\rho) := \sup_{\Omega \cap \{d < \rho\}} u \leq \left(\frac{4\rho}{r_0}\right)^{\gamma_1} \omega(r_0) \quad \forall \rho \in (0, r_0], \tag{3.8}$$

where $\gamma_1 = \gamma_1(n,\nu,\theta_0) \in (0,1]$ is the constant in Lemma 3.2.

Proof. The proof follows immediately from (3.5), because $\omega(\rho) = \sup_{y \in \partial\Omega} \omega_y(\rho)$. □

Lemma 3.4. *Let* $r_1 = \text{const} > 0$, *and a nonempty open subset* $\Omega' \subseteq \Omega$. *Let* $u \in W(\Omega')$ *be such that* $u > 0$ *in* Ω', $u = 0$ *on* $\partial\Omega'$, *and*

$$Lu \leq 0 \quad \text{in } \Omega_0' := \Omega' \cap \{d > r_1\}, \qquad Lu \leq 1 \quad \text{in } \Omega_1' := \Omega' \cap \{d < r_1\}. \tag{3.9}$$

Then

$$\sup_{\Omega'} u \leq N_1 r_1^2, \tag{3.10}$$

where the constant $N_1 = N_1(n,\nu,\theta_0) > 0$.

Proof. The set Ω_1' is nonempty, because otherwise we would have $\Omega_0' = \Omega'$, and from $Lu \leq 0$ in Ω' and $u = 0$ on $\partial\Omega'$, it follows $u \leq 0$ in Ω', in contradiction to our assumption $u > 0$ in Ω'.

Since u is continuous on $\overline{\Omega}_1'$, we can choose a point $x_0 \in \overline{\Omega}_1'$ at which

$$u(x_0) = M := \sup_{\Omega_1'} u > 0. \tag{3.11}$$

Then $u \le M$ on the set $\Omega' \cap \{d = r_1\} \subseteq (\partial\Omega_0') \cap (\partial\Omega_1')$. On the remaining part of $\partial\Omega_0'$, which is contained in $\partial\Omega'$, we have $u = 0 < M$. By the maximum principle, $u \le M$ on $\overline{\Omega}_0'$. Therefore, $u \le M$ on $\overline{\Omega}' = \overline{\Omega}_0' \cup \overline{\Omega}_1'$, and $M = \sup_{\Omega'} u$ by virtue of (3.11).

Further, since $x_0 \in \overline{\Omega}_1'$, we have $d(x_0) := \operatorname{dist}(x_0, \partial\Omega) \le r_1$. Choose a point $y_0 \in \partial\Omega$ for which $d(x_0) = |x_0 - y_0|$. Set $R := 4r_1$, and $B_\rho := B_\rho(y_0)$ for $\rho > 0$. By Lemma 2.4(i), there exists a function $w \in W(B_R)$ satisfying the properties (2.4) with a constant $N_0 = N_0(n, \nu)$. If $M \le N_0R^2 = 16N_0r_1^2$, then estimate (3.10) holds with $N_1 = 16N_0$. In the remaining case $M > N_0R^2$, we will apply Lemma 2.5 to the function

$$v(x) := u(x) - w(x) \quad \text{in } V := \Omega' \cap B_R \cap \{u > w\} \subseteq \Omega' \subseteq \Omega. \tag{3.12}$$

Condition (A) implies $|B_{r_1} \setminus V| \ge |B_{r_1} \setminus \Omega| \ge \theta_0 |B_{r_1}|$. From $0 \le w \le N_0R^2$ and $u = 0$ on $\partial\Omega'$, it follows $v > 0$ in V, $v = 0$ on $(\partial V) \cap B_R$, and $x_0 \in V \cap \overline{B}_{r_1}$. Moreover, since $Lu \le 1 \le Lw$, we also have $Lv \le 0$ in V. Now we can use Lemma 2.5 with $x_0 = y_0$, $r = r_1$, and $\theta = \theta_0$, which yields

$$0 < M - N_0R^2 \le v(x_0) \le \sup_{V \cap B_{r_1}} v \le \beta \cdot \sup_{V \cap \partial(B_{4r_1})} v, \tag{3.13}$$

where $\beta = \beta(n, \nu, \theta_0) \in (0, 1)$. Having in mind that $w = 0$ on $\partial B_{4r_1} = \partial B_R$, we finally get

$$M - N_0R^2 \le \beta \cdot \sup_{\overline{V} \cap (\partial B_R)} u \le \beta \cdot \sup_{\Omega'} u = \beta \cdot M. \tag{3.14}$$

This gives us the desired estimate $M \le (1 - \beta)^{-1} N_0R^2 = N_1 r_1^2$, where $N_1 = N_1(n, \nu, \theta_0) = 16N_0(1 - \beta)^{-1} > 0$. ☐

Now we can proceed to the equations $Lu = f$ with $f \in L^\infty_{\mathrm{loc}}$.

Theorem 3.5. *Let Ω be a bounded open set in $\mathbb{R}^n$ satisfying condition (A) with a constant $\theta_0 > 0$, and let $\gamma_1 = \gamma_1(n, \nu, \theta_0) \in (0, 1]$ be the constant in Lemma 3.2. Let an open subset $\Omega' \subseteq \Omega$, a uniformly elliptic operator L (in the form (D) or (ND)), and functions $u \in W(\Omega')$, $f \in L^\infty_{\mathrm{loc}}(\Omega')$ be such that*

$$Lu = f \quad \text{in } \Omega', \qquad u = 0 \quad \text{on } \partial\Omega'. \tag{3.15}$$

Then for any constant $\gamma \in (0, \gamma_1)$,

$$\sup_{\Omega'} d^{-\gamma} u \le NF, \quad \text{where } F := \sup_{\Omega'} d^{2-\gamma} f_+, \tag{3.16}$$

$d = d(x) := \operatorname{dist}(x, \partial\Omega)$, and $N = N(n, \nu, \theta_0, \gamma) > 0$.

Applying this theorem to the functions u in $\Omega' = \Omega_+ := \Omega \cap \{u > 0\}$ and $-u$ in $\Omega' = \Omega_- := \Omega \cap \{u < 0\}$, we derive the following.

Corollary 3.6. The solutions to the Dirichlet problem (DP) satisfy estimate (M) with a constant $N = N(n, \nu, \theta_0, \gamma) > 0$, provided $0 < \gamma < \gamma_1 \le 1$.

We first expose an idea of the proof of (3.16), which is based on the previous auxiliary results. For simplicity, we assume that all problems (3.15) under consideration (with different f) have solutions in $W(\Omega')$. Multiplying u and f by a constant, we can always assume that $F = 1$, that is, $f \le \overline{f} := d^{\gamma-2}$. Let $\overline{u} \in W(\Omega')$ be a solution to problem (3.15) with $f = \overline{f}$. Then by the maximum principle $u \le \overline{u}$, so that it suffices to derive estimate (3.16) for the functions $\overline{u}$ and $\overline{f}$.

Thus we can restrict ourselves to the case $f = \overline{f} := d^{\gamma-2}$. In this case, using substitution $s = r^{\gamma-2}$, we can write

$$f = \int_f^{2f} ds = c_1 \int_{d/c_2}^{d} r^{\gamma-3} dr, \quad \text{where } c_1 := 2-\gamma > 1,\ c_2 := 2^{1/(2-\gamma)} > 1. \tag{3.17}$$

Note that $d/c_2 < r < d$ if and only if $r < d < c_2 r$. Therefore, introducing the indicator functions,

$$f_r(x) = \begin{cases} 1 & \text{for } x \in \Omega' \cap \{r < d < c_2 r\}, \\ 0 & \text{otherwise,} \end{cases} \tag{3.18}$$

we rewrite $f(x)$ in the form

$$f(x) = c_1 \int_0^{\infty} r^{\gamma-3} f_r(x) dr. \tag{3.19}$$

By linearity, we can expect that the solution u to problem (3.15) is represented in the form

$$u(x) = c_1 \int_0^{\infty} r^{\gamma-3} u_r(x) dr, \tag{3.20}$$

where u_r is a solution to the same problem corresponding to the function f_r, that is, $Lu_r = f_r$ in Ω'. The functions $f_r \equiv 0$ for $d < r$; hence we can rely on Corollary 3.3 with $r_0 = r$, which implies

$$u_r(x) \le \min\left\{1, \left(\frac{4d(x)}{r}\right)^{\gamma_1}\right\} \sup_{\Omega'} u_r. \tag{3.21}$$

We also have $f_r \equiv 0$ for $d > c_2 r$. Lemma 3.4 with $r_1 = c_2 r$ yields

$$\sup_{\Omega'} u_r \le N_1 r_1^2 = N_1 c_2^2 r^2. \tag{3.22}$$

Combining these two estimates together, we obtain

$$u_r \le N_2 r^2 \min\left\{1, \left(\frac{d}{r}\right)^{\gamma_1}\right\}, \quad \text{where } N_2 := 4^{\gamma_1} N_1 c_2^2. \tag{3.23}$$

Finally, we insert this estimate into (3.20):

$$\begin{aligned} u &\le c_1 N_2 \int_0^\infty r^{\gamma-1} \min\left\{1, \left(\frac{d}{r}\right)^{\gamma_1}\right\} dr \\ &= c_1 N_2 \left(\int_0^d r^{\gamma-1} dr + d^{\gamma_1} \int_d^\infty r^{\gamma-1-\gamma_1} dr \right) = N d^\gamma, \end{aligned} \tag{3.24}$$

where $N := c_1 N_2 \cdot [\gamma^{-1} + (\gamma_1 - \gamma)^{-1}]$, and estimate (3.16) follows.

This approach uses some technical assumptions, such as the existence of solutions $u_r \in W(\Omega')$ to problems (3.15) with $f = f_r$, and also the possibility of interchanging L with integration with respect to r in (3.20), which implies

$$Lu = c_1 \int_0^\infty r^{\gamma-3} L u_r dr = c_1 \int_0^\infty r^{\gamma-3} f_r dr = f. \tag{3.25}$$

The validation of these assumptions requires some standard work. Instead, we present another proof which implicitly uses the same idea, only without involving auxiliary existence results, except for the existence of a function w in Lemma 2.4(i) in the divergence case (D).

Proof of Theorem 3.5. Without loss of generality, we can always assume that $\operatorname{dist}(\Omega', \partial\Omega) > 0$. Indeed, if estimate (3.16) is true under this additional assumption, we can apply it to the functions $u - c$ in $\Omega'_c := \Omega' \cap \{u > c\}$ with small $c = \text{const} > 0$. Then we have

$$\sup_{\Omega'_c} d^{-\gamma}(u - c) \le N \sup_{\Omega'_c} d^{2-\gamma} f_+ \le N \sup_{\Omega'} d^{2-\gamma} f_+ = NF, \tag{3.26}$$

and (3.16) follows by the limit passage as $c \to 0^+$. We also assume that the set $\Omega' \cap \{u > 0\}$ is nonempty; otherwise there is nothing to prove. The assumption $\operatorname{dist}(\Omega', \partial\Omega) > 0$ allows us to claim that $d^{-\gamma} u \in C(\overline{\Omega}')$, and hence there is a point $x_0 \in \Omega'$ at which

$$d^{-\gamma} u(x_0) = M := \sup_{\Omega'} d^{-\gamma} u > 0. \tag{3.27}$$

Fix such a point x_0, set $r := d(x_0) := \operatorname{dist}(x_0, \partial\Omega) > 0$, and choose a point $y_0 \in \partial\Omega$ for which $d(x_0) = |x_0 - y_0|$. By Lemma 2.4(i), there exists a function $w \in W(B_{4r}(y_0))$ satisfying the properties (2.4) in the ball $B_R := B_{4r}(y_0)$. For a small constant $\varepsilon \in (0,1)$ to be specified later, consider the function

$$v(x) := u(x) - (\varepsilon r)^\gamma M - (\varepsilon r)^{\gamma-2} F w, \quad \text{where } F := \sup_{\Omega'} d^{2-\gamma} f_+, \tag{3.28}$$

on the set

$$V := \Omega' \cap B_{4r}(y_0) \cap \{d > \varepsilon r\} \cap \{v > 0\}. \tag{3.29}$$

We first assume $v(x_0) > 0$. In this case, $x_0 \in V \cap \overline{B}_r(y_0)$. By the condition (A) we have that $|B_r(y_0) \setminus V| \ge |B_r(y_0) \setminus \Omega| \ge \theta_0 |B_r|$. From $u = 0$ on $\partial\Omega'$ and $u \le M d^\gamma \le (\varepsilon r)^\gamma M$ on

$\{d \le \varepsilon r\}$, it follows $v = 0$ on $(\partial V) \cap B_{4r}(y_0)$. Moreover,

$$Lv = f - (\varepsilon r)^{\gamma-2} F \cdot Lw \le d^{\gamma-2} F - (\varepsilon r)^{\gamma-2} F \le 0 \quad \text{in } V. \tag{3.30}$$

Applying Lemma 2.5, we get

$$v(x_0) \le \sup_{V \cap B_r(y_0)} v \le \beta \cdot \sup_{V \cap B_{4r}(y_0)} v \le \beta \cdot \sup_{V \cap B_{4r}(y_0)} u, \tag{3.31}$$

where $\beta = \beta(n, \nu, \theta_0) \in (0,1)$. Note that $u \le d^\gamma M \le (4r)^\gamma M$ on $V \cap B_{4r}(y_0)$, which implies $v(x_0) \le (4r)^\gamma \beta \cdot M$. Of course, the last estimate also holds in case $v(x_0) \le 0$. From this estimate, together with (3.27) and (3.28), it follows

$$r^\gamma M = u(x_0) \le r^\gamma (\varepsilon^\gamma + 4^\gamma \beta) M + (\varepsilon r)^{\gamma-2} F w(x_0). \tag{3.32}$$

By the properties (2.4) of the function w on $B_R = B_{4r}(y_0)$, we have

$$w(x_0) \le N_0 R^2 = 16 N_0 r^2, \quad \text{where } N_0 = N_0(n, \nu) > 0. \tag{3.33}$$

Therefore,

$$M \le (\varepsilon^\gamma + 4^\gamma \beta) M + 16 N_0 \varepsilon^{\gamma-2} F. \tag{3.34}$$

Finally, note that our assumption $\gamma < \gamma_1 := -\log_4 \beta$ implies $4^\gamma \beta = 4^{\gamma-\gamma_1} < 1$. This allows us to choose $\varepsilon = \varepsilon(n, \nu, \theta_0, \gamma) \in (0,1)$ such that $\lambda := \varepsilon^\gamma + 4^\gamma \beta < 1$. Now from (3.34) it follows

$$M \le NF, \quad \text{where } N = N(n, \nu, \theta_0, \gamma) := 16 N_0 \varepsilon^{\gamma-2} (1-\lambda)^{-1} > 0. \tag{3.35}$$

The theorem is proved. □

Remark 3.7. Roughly speaking, Lemma 3.2 and Corollary 3.3 guarantee that if $Lu = 0$ near $\partial\Omega$, and $u = 0$ on $\partial\Omega$, then $u = O(d^{\gamma_1})$ with a constant $\gamma_1 = \gamma_1(n, \nu, \theta_0) \in (0,1]$. It is well known (see, [2, Problem 3.6]) that the "optimal" value $\gamma_1 = 1$ is attained for operators L in the non-divergence form (ND), and domains Ω satisfying the *exterior sphere condition* which is specified in Definition 3.8 below. The argument after Corollary 3.6 shows that under these assumptions, estimate (3.16) in Theorem 3.5 should be true for any constant $\gamma \in (0,1)$. We give a direct proof of this fact in Theorem 3.9 below. For domains of class C^2, one can prove it using special barrier functions depending only on $d = d(x) := \operatorname{dist}(x, \partial\Omega)$ (see [10]). This approach looks especially simple when Ω is a ball B_R. For certainty, take $\Omega = B_1 := B_1(0)$ and fix $\gamma \in (0,1)$. Obviously,

$$d(x) = \operatorname{dist}(x, \partial\Omega) = 1 - |x| \le d_1(x) := 1 - |x|^2 \le 2d(x) \quad \text{in } B_1. \tag{3.36}$$

Following [2, the proof of Lemma 6.21], consider the function $w_1 := d_1^\gamma$ which satisfies

$$\begin{aligned} Lw_1 = -(aD, Dw_1) &= \gamma d_1^{\gamma-1} L d_1 + \gamma(1-\gamma) d_1^{\gamma-2} (aDd_1, Dd_1) \\ &= 2\gamma d_1^{\gamma-2} [d_1 \operatorname{tr} a + 2(1-\gamma)(ax, x)] \\ &\ge 2\gamma\nu d_1^{\gamma-2} [n(1-|x|^2) + 2(1-\gamma)|x|^2] \ge c_1 d_1^{\gamma-2}, \end{aligned} \tag{3.37}$$

where $c_1 := 2\gamma\nu \cdot \min\{n, 2(1-\gamma)\} > 0$. On the other hand, for any solution to problem (3.15) in an open set $\Omega' \subseteq B_1$, we have

$$Lu = f \le f_+ \le F d^{\gamma-2} \le 2^{2-\gamma} F d_1^{\gamma-2} \le 2^{2-\gamma} c_1^{-1} F \cdot L w_1 \quad \text{in } \Omega', \tag{3.38}$$

where $F := \sup d^{2-\gamma} f_+$. By the comparison principle,

$$u \le 2^{2-\gamma} c_1^{-1} F \cdot w_1 = 2^{2-\gamma} c_1^{-1} F \cdot d_1^{\gamma} \le 4 c_1^{-1} F \cdot d^{\gamma} \quad \text{in } \Omega', \tag{3.39}$$

so that estimate (3.16) holds with $N := 4c_1^{-1}$.

Definition 3.8. An open set $\Omega \subset \mathbb{R}^n$ satisfies the *exterior sphere condition* with a constant $r_0 > 0$, if for each $y \in \partial\Omega$, there exists a ball $B_{r_0}(z) \subset \mathbb{R}^n \setminus \Omega$ such that $\overline{\Omega} \cap \overline{B}_{r_0}(z) = \{y\}$.

THEOREM 3.9. *Let Ω be a bounded open set in $\mathbb{R}^n$ satisfying the exterior sphere condition with a constant $r_0 > 0$. Let a uniformly elliptic operator L in the non-divergence form (ND) be defined in an open subset $\Omega' \subseteq \Omega$, and let functions $u \in W(\Omega')$, $f \in L^\infty_{\mathrm{loc}}(\Omega')$ satisfy (3.15). Then for any constant $\gamma \in (0,1)$, estimate (3.16) holds true with a constant N depending only on n, ν, γ, and R/r_0, where $R := \operatorname{diam}\Omega$.*

Proof. By rescaling $x \to \text{const} \cdot x$, we reduce the proof to the case $r_0 = 1$. Instead of the distance function $d(x) := \operatorname{dist}(x, \partial\Omega)$, it is convenient to use another "equivalent" function:

$$d_1(x) := h(d(x)) \quad \text{in } \Omega, \quad \text{where } h(\rho) := 1 - (1+\rho)^{-m} \quad \text{for } \rho \ge 0, \tag{3.40}$$

and $m := n\nu^{-2}$. By the concavity of $h(\rho)$, we have $c_1 d \le d_1 \le c_2 d$ in Ω, with the constants $c_1 := R^{-1} h(R)$ and $c_2 := d_1'(0) = m$. Therefore, for any fixed constant $\gamma \in (0,1)$, estimate (3.16) is equivalent to the following one:

$$M_1 := \sup_{\Omega'} d_1^{-\gamma} u \le N_1 F_1, \quad \text{where } F_1 := \sup_{\Omega'} d_1^{2-\gamma} f_+, \tag{3.41}$$

and $N_1 = N_1(n, \nu, \gamma, R) > 0$.

As in the proof of Theorem 3.5, we assume that $\operatorname{dist}(\Omega', \partial\Omega) > 0$, and fix a point $x_0 \in \Omega'$ such that $M_1 = d_1^{-\gamma} u(x_0)$, and $y_0 \in \partial\Omega$ for which $|x_0 - y_0| = d(x_0)$. By our assumptions, Ω satisfies the exterior sphere condition with $r_0 = 1$, so that $\overline{\Omega} \cap \overline{B}_1(z_0) = \{y_0\}$ for some $z_0 \in \mathbb{R}^n$. Set $r := d(x_0)$. From the properties

$$B_r(x_0) \subset \Omega, \qquad B_1(z_0) \subset \mathbb{R}^n \setminus \Omega, \qquad y_0 \in (\partial B_r(x_0)) \cap (\partial B_1(z_0)), \tag{3.42}$$

it follows

$$d(x) \le |x - z_0| - 1 \quad \text{in } \Omega, \qquad r = d(x_0) = |x_0 - y_0| = |x_0 - z_0| - 1. \tag{3.43}$$

Therefore,

$$d_1(x) := h(d(x)) \le d_0(x) := h(|x - z_0| - 1) = 1 - |x - z_0|^{-m} \quad \text{in } \Omega, \tag{3.44}$$

and $d_1(x_0) = d_0(x_0)$. By definition of M_1 in (3.41) and the choice of x_0, we get

$$u \le M_1 d_1^\gamma \le M_1 d_0^\gamma \quad \text{in } \Omega, \qquad u(x_0) = M_1 d_1^\gamma(x_0) = M_1 d_0^\gamma(x_0). \tag{3.45}$$

In Remark 2.8, we pointed out that the choice $m := n\nu^{-2}$ guarantees the inequality $L(|x|^{-m}) \le 0$ for $x \ne 0$ and any operator L in the form (ND) with the ellipticity constant $\nu \in (0,1]$. Therefore, for $x \ne z_0$ we have $Ld_0(x) = -L(|x - z_0|^{-m}) \ge 0$ and

$$L(d_0^\gamma) = \gamma d_0^{\gamma-1} L d_0 + \gamma(1-\gamma) d_0^{\gamma-2}(aDd_0, Dd_0) \ge \nu\gamma(1-\gamma) d_0^{\gamma-2} |Dd_0|^2. \tag{3.46}$$

Here, $|Dd_0(x)| = m \cdot |x - z_0|^{-m-1} \ge m \cdot (1+R)^{-m-1}$ in Ω, so that

$$L(d_0^\gamma) \ge c_3 d_0^{\gamma-2} \quad \text{in } \Omega, \text{ where } c_3 = c_3(n,\nu,\gamma,R) > 0. \tag{3.47}$$

On the other hand, for $0 < \varepsilon < r$,

$$r - \varepsilon \le d(x) \le |x - z_0| - 1 \le |x - y_0| \le r + \varepsilon \quad \text{in } B_\varepsilon(x_0). \tag{3.48}$$

Since $h(\rho)$ is a concave, increasing function with $h(0) = 0$, we derive

$$\begin{aligned} d_0(x) = h(|x - z_0| - 1) &\le h(r+\varepsilon) \le \frac{r+\varepsilon}{r-\varepsilon} h(r-\varepsilon) \\ &\le \frac{r+\varepsilon}{r-\varepsilon} h(d(x)) = \frac{r+\varepsilon}{r-\varepsilon} d_1(x) \quad \text{in } B_\varepsilon(x_0). \end{aligned} \tag{3.49}$$

This inequality, together with $\gamma - 2 < 0$, yields

$$Lu = f \le f_+ \le F_1 d_1^{\gamma-2} \le \left(\frac{r+\varepsilon}{r-\varepsilon}\right)^{2-\gamma} F_1 d_0^{\gamma-2} \quad \text{in } \Omega' \cap B_\varepsilon(x_0). \tag{3.50}$$

We claim that the desired estimate (3.41) holds with $N_1 = N_1(n,\nu,\gamma,R) := c_3^{-1} > 0$. Suppose that this is not the case, that is, $M_1 > N_1 F_1$. Then we can choose $\varepsilon \in (0,1)$ so small that $B_\varepsilon(x_0) \subset \Omega'$ and $M_1 > ((r+\varepsilon)/(r-\varepsilon))^{2-\gamma} N_1 F_1$. By (3.47) and (3.50), the function $v := u - M_1 d_0^\gamma$ satisfies $Lv \le -c = \text{const} < 0$ in $B_\varepsilon(x_0)$, and by (3.45), v has a local maximum at x_0. Then the desired contradiction follows by the strong maximum, which can be proved for functions $v \in W^{2,n}$ in the same way as for $v \in C^2$. Alternatively, one can compare the functions v and $w_\varepsilon := \varepsilon^2 \cdot |x - x_0|^2 - \varepsilon^4$ in $B_\varepsilon(x_0)$. We can assume that the constant $\varepsilon > 0$ is so small that $Lw_\varepsilon = -2\varepsilon^2 \cdot \operatorname{tr} a \ge -c \ge Lv$ in $B_\varepsilon(x_0)$. Since also $v \le 0 = w_\varepsilon$ on $\partial B_\varepsilon(x_0)$, by the comparison principle we then get $v(x_0) \le w_\varepsilon(x_0) = -\varepsilon^4 < 0$, in contradiction to $v(x_0) = 0$. The theorem is proved. □

Estimate (3.16) for solutions u to the Dirichlet problem (DP), together with the "interior" estimates in Hölder spaces $C^{0,\gamma}$, results in the "global" Hölder regularity of solutions: $u \in C^{0,\gamma}(\overline{\Omega})$, that is, u has finite norm

$$|u|_{0,\gamma;\Omega} := \sup_\Omega |u| + [u]_{0,\gamma;\Omega}, \quad \text{where } [u]_{0,\gamma;\Omega} := \sup_{x \ne y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\gamma}. \tag{3.51}$$

This property of solutions is established in the following theorem. Its proof combines the methods of de Giorgi [18], Landis [13], Moser [19], Serrin [20], and other mathematicians.

Theorem 3.10. *Let Ω be a bounded open set in $\mathbb{R}^n$ satisfying the condition (A) with a constant $\theta_0 > 0$. There exists a constant $\overline{\gamma} = \overline{\gamma}(n,\nu,\theta_0) \in (0,1]$, such that for any constant $\gamma \in (0,\overline{\gamma})$, any uniformly elliptic operator L (in the form (D) or (ND)) and $f \in L^\infty_{\text{loc}}(\Omega)$, the solutions $u \in W(\Omega)$ to the Dirichlet problem (DP) satisfy the estimate*

$$|u|_{0,\gamma;\Omega} \le NF, \quad \text{where } F := \sup_\Omega d^{2-\gamma}|f|, \tag{3.52}$$

$d = d(x) := \operatorname{dist}(x,\partial\Omega)$, and the constant $N > 0$ depends only on n, ν, θ_0, γ, and $R := \operatorname{diam}\Omega$.

Proof. Throughout the proof, different constants N, including $N', \overline{N}, N_1, \dots$, depend only on n, ν, θ_0, γ, and R. By Corollary 3.6, the solutions to (DP) satisfy estimate (M) with a constant $N = N(n,\nu,\theta_0,\gamma) > 0$, provided $0 < \gamma < \gamma_1 = \gamma_1(n,\nu,\theta_0) \le 1$. This estimate implies

$$\sup_\Omega |u| \le R^\gamma \sup_\Omega d^{-\gamma}|u| \le N_1 \cdot F. \tag{3.53}$$

Now it remains to prove the estimate $[u]_{0,\gamma;\Omega} \le NF$ with $N = N(n,\nu,\theta_0,\gamma,R) > 0$. In turn, for this purpose it suffices to show that

$$\omega(\rho,x_0) := \sup_{\Omega\cap B_\rho(x_0)} u - \inf_{\Omega\cap B_\rho(x_0)} u \le NF\rho^\gamma \tag{3.54}$$

for all $\rho > 0$ and $x_0 \in \overline{\Omega}$, because $|u(x) - u(y)| \le \omega(\rho,x)$ for $\rho := |x-y|$. For the proof of (3.54), we fix $x_0 \in \overline{\Omega}$, and consider separately the cases (i) and (ii).

(i) $\rho \ge \rho_0 := \min\{1,(1/2)d(x_0)\}$. If $\rho \ge (1/2)d(x_0)$, then

$$d(x) \le d(x_0) + |x - x_0| \le 3\rho \quad \forall x \in \Omega \cap B_\rho(x_0), \tag{3.55}$$

which gives us

$$\omega(\rho,x_0) \le 2 \sup_{\Omega\cap B_\rho(x_0)} |u| \le 2 \sup_{\Omega\cap B_\rho(x_0)} \left(\frac{3\rho}{d}\right)^\gamma |u| \le 2\cdot(3\rho)^\gamma \sup_\Omega d^{-\gamma}|u|, \tag{3.56}$$

and (3.54) follows from (M). In the remaining subcase of (i), we must have $\rho \ge 1$, and then (3.54) follows immediately from (3.53) with $N := 2N_1$.

(ii) $0 < \rho < \rho_0$. For brevity, we write $B_\rho := B_\rho(x_0)$ and $\omega(\rho) := \omega(\rho,x_0)$. Note that $d \ge \rho_0$ on B_{ρ_0}, which yields

$$|Lu| = |f| \le Fd^{\gamma-2} \le F\rho_0^{\gamma-2} \quad \text{in } B_{\rho_0}. \tag{3.57}$$

For $0 < \rho < \rho_0$, the balls B_ρ are contained in Ω, so that

$$\omega(\rho) = M(\rho) - m(\rho), \quad \text{where } M(\rho) := \sup_{B_\rho} u,\ m(\rho) := \inf_{B_\rho} u. \tag{3.58}$$

Fix $\rho \in (0,(1/4)\rho_0]$ and set $\lambda := (1/2)[M(\rho)+m(\rho)]$. Suppose that the measure

$$|B_\rho \cap \{u \le \lambda\}| \ge \frac{1}{2}|B_\rho|. \tag{3.59}$$

By Lemma 2.4 applied to the ball $B_R := B_{4\rho} \subseteq B_{\rho_0}$, there exists a function $w \in W(B_{4\rho})$ such that

$$0 \le w \le N_0(4\rho)^2, \quad Lw \ge 1 \text{ in } B_{4\rho};\ w = 0 \text{ on } \partial B_{4\rho}, \tag{3.60}$$

where $N_0 = N_0(n,\nu) > 0$ is the constant in (2.4). Consider the function

$$v(x) := u(x) - \lambda - F\rho_0^{\gamma-2}w(x) \quad \text{in } V := B_{4\rho} \cap \{v > 0\}. \tag{3.61}$$

Note that

$$\frac{1}{2}\omega(\rho) = M(\rho) - \lambda = \sup_{B_\rho}(u-\lambda) = \sup_{B_\rho}\left(v + F\rho_0^{\gamma-2}w\right). \tag{3.62}$$

Since $0 < \rho \le (1/4)\rho_0$ and $0 < \gamma \le 1$, we have

$$\begin{gathered} F\rho_0^{\gamma-2}w \le 16N_0F\rho_0^{\gamma-2}\rho^2 \le N_0F\rho^\gamma, \\ \omega(\rho) \le 2\sup_{B_\rho} v + 2N_0F\rho^\gamma. \end{gathered} \tag{3.63}$$

We first consider a more interesting subcase, when v attains positive values in B_ρ. In this case, the set $V \cap B_\rho$ is nonempty. By virtue of (3.57), (3.59), and the properties of the function w, we have

$$\begin{gathered} Lv = f - F\rho_0^{\gamma-2}Lw \le f - F\rho_0^{\gamma-2} \le 0 \quad \text{in } V, \\ |B_\rho \setminus V| = |B_\rho \cap \{v \le 0\}| \ge |B_\rho \cap \{u \le \lambda\}| \ge \frac{1}{2}|B_\rho|. \end{gathered} \tag{3.64}$$

Obviously, $v > 0$ in V, and $v = 0$ on $(\partial V) \cap B_{4\rho}$. Applying Lemma 2.5 to the function v in V, we get

$$\sup_{B_\rho} v = \sup_{V \cap B_\rho} v \le \beta \cdot \sup_V v \le \beta \cdot \sup_{B_{4\rho}}[u(x) - \lambda] = \beta \cdot [M(4\rho) - \lambda], \tag{3.65}$$

where $\beta = \beta(n,\nu) \in (0,1)$ is the constant in that lemma corresponding to $\theta = 1/2$. The previous estimate, together with (3.63), yields

$$\omega(\rho) \le 2\beta \cdot [M(4\rho) - \lambda] + 2N_0F\rho^\gamma. \tag{3.66}$$

Here,

$$M(4\rho) - \lambda = M(4\rho) - m(\rho) - \frac{1}{2}[M(\rho) - m(\rho)] \le \omega(4\rho) - \frac{1}{2}\omega(\rho). \tag{3.67}$$

From these relations it follows

$$\omega(\rho) \le 2\beta \cdot \left[\omega(4\rho) - \frac{1}{2}\omega(\rho)\right] + 2N_0 F\rho^{\gamma}. \tag{3.68}$$

By virtue of (3.63), the last inequality also holds when $\nu \le 0$ in B_ρ. After simplification, it is reduced to the following one:

$$\omega(\rho) \le \beta' \cdot \omega(4\rho) + N' F\rho^{\gamma}, \tag{3.69}$$

where the constants $\beta' := 2\beta/(1+\beta) \in (0,1)$ and $N' := (2/(1+\beta))N_0$ depend only on n and ν. This estimate was proved under assumption (3.59). If this assumption fails, that is, $|B_\rho \cap \{u \le \lambda\}| < (1/2)|B_\rho|$, then

$$|B_\rho \cap \{-u \le -\lambda\}| = |B_\rho| - |B_\rho \cap \{u < \lambda\}| > \frac{1}{2}|B_\rho|. \tag{3.70}$$

Therefore, we can apply the previous argument to $-u$ and $-\lambda$ in place of u and λ correspondingly. Since $\omega(\rho)$ and $\omega(4\rho)$ remain the same after such a substitution, estimate (3.69) holds true in any case.

Finally, we set $\overline{\gamma} = \overline{\gamma}(n,\nu,\theta_0) := \min\{\gamma_1, -\log_4\beta'\} \in (0,1]$, and let $0 < \gamma < \overline{\gamma}$, or equivalently, $0 < \gamma < \gamma_1$ and $4^{\gamma}\beta' < 1$. The restriction $0 < \gamma < \gamma_1$ was needed in the case (i), which was based on Corollary 3.6. The inequality $\tau := 4^{\gamma}\beta' < 1$ is essentially used in the following argument. Set

$$\overline{\omega}(\rho) := \max\{\omega(\rho), \overline{N}F\rho^{\gamma}\}, \quad \text{where } \overline{N} = \overline{N}(n,\nu,\gamma) := (1-\tau)^{-1}N' > 0. \tag{3.71}$$

From (3.69) it follows

$$\begin{aligned} \omega(\rho) &\le 4^{-\gamma} \cdot [\tau\omega(4\rho) + (1-\tau)\overline{N}F(4\rho)^{\gamma}] \\ &\le 4^{-\gamma} \cdot \max\{\omega(4\rho), \overline{N}F(4\rho)^{\gamma}\} = 4^{-\gamma}\overline{\omega}(4\rho). \end{aligned} \tag{3.72}$$

In addition,

$$\overline{N}F\rho^{\gamma} = 4^{-\gamma} \cdot \overline{N}F(4\rho)^{\gamma} \le 4^{-\gamma}\overline{\omega}(4\rho). \tag{3.73}$$

Therefore, we have

$$\overline{\omega}(\rho) = \max\{\omega(\rho), \overline{N}F\rho^{\gamma}\} \le 4^{-\gamma}\overline{\omega}(4\rho). \tag{3.74}$$

This estimate is true for all $\rho \in (0,(1/4)\rho_0]$, so applying Lemma 3.1 with $\alpha = \gamma$ and $q = 4$, we get

$$\omega(\rho) \le \overline{\omega}(\rho) \le \left(\frac{4\rho}{\rho_0}\right)^{\gamma}\overline{\omega}(\rho_0) \quad \forall \rho \in (0,\rho_0]. \tag{3.75}$$

Note that $\rho = \rho_0 := \min\{1,(1/2)d(x_0)\}$ belongs to the case (i) in which we already have estimate (3.54), that is, $\omega(\rho_0) \le NF\rho_0^{\gamma}$. Then $\overline{\omega}(\rho_0) \le (N+\overline{N})F\rho_0^{\gamma} =: N_2F\rho_0^{\gamma}$ and by (3.75), $\omega(\rho) \le N_3\rho^{\gamma}$ with $N_3 := 4^{\gamma}N_2$. Thus estimate (3.54) holds true for all $\rho > 0$ and $x_0 \in \overline{\Omega}$.

The theorem is proved. □

Remark 3.11. The results of this paper can be generalized, with minor natural modifications, to elliptic operators *with lower-order terms* in the *divergence form:*

$$Lu := -(D, aDu) + (b, Du) + (D, cu) + c_0 u, \tag{D1}$$

or in the *nondivergence form:*

$$Lu := -(aD, Du) + (b, Du) + c_0 u, \tag{ND1}$$

where $a = [a_{ij}] = [a_{ij}(x)]$ satisfies the condition (U), the vector functions $b := (b_1(x), \dots, b_n(x))^t$ and $c := (c_1(x), \dots, c_n(x))^t$, and the scalar function $c_0 := c_0(x)$ are locally bounded in Ω. We can allow a "moderate" growth near $\partial\Omega$:

$$|b| \le K d^{\alpha-1}, \quad |c| \le K d^{\alpha-1}, \quad |c_0| \le K d^{\alpha-2} \quad \text{in } \Omega, \tag{3.76}$$

with some constants $K > 0$ and $\alpha > 0$, where $d = d(x) := \operatorname{dist}(x, \partial\Omega)$. For applicability of the maximum principle, we also need the additional condition $L1 \ge 0$, which means $c_0 \ge 0$ for operators L in (ND1). In the divergence case (D1), the inequality $L1 = (D, c) + c_0 \ge 0$ is understood in a weak sense (see, [2, Section 8.1]).

Remark 3.12. The main results can also be extended to the parabolic equations, in both divergence and non-divergence forms. For this purpose, one can use the "parabolic" versions of growth lemmas in [3, 13, 14], or [16].

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Sungwon Cho: Department of Mathematics, Michigan State University, East Lansing,
MI 48824, USA
Email address: cho@math.msu.edu

Mikhail Safonov: School of Mathematics, University of Minnesota, 127 Vincent Hall,
Minneapolis, MN 55455, USA
Email address: safonov@math.umn.edu