Hindawi Publishing Corporation
Boundary Value Problems
Volume 2007, Article ID 64579, 12 pages
doi:10.1155/2007/64579

Research Article

Properties of Positive Solution for Nonlocal Reaction-Diffusion Equation with Nonlocal Boundary

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Received 21 January 2007; Accepted 11 April 2007

Recommended by Robert Finn

This paper considers the properties of positive solutions for a nonlocal equation with nonlocal boundary condition $u(x,t) = \int_\Omega f(x,y)u(y,t)dy$ on $\partial\Omega \times (0,T)$. The conditions on the existence and nonexistence of global positive solutions are given. Moreover, we establish the uniform blow-up estimates for the blow-up solution.

1. Introduction

In this paper, we consider the following nonlocal equation with nonlocal boundary condition:

$$\begin{gathered} u_t = \Delta u + \int_\Omega u^q(y,t)dy - ku^p, \quad x \in \Omega,\ t > 0, \\ u(x,t) = \int_\Omega f(x,y)u(y,t)dy, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{gathered} \tag{1.1}$$

where $p,q \geq 1$, $k > 0$, and $\Omega \subset R^N$ is a bounded domain with smooth boundary. The function $f(x,y) \not\equiv 0$ is nonnegative, continuous, and defined for $x \in \partial\Omega$, $y \in \overline{\Omega}$, while u_0 is a nonnegative continuous function and satisfies the compatibility condition $u_0(x) = \int_\Omega f(x,y)u_0(y)dy$ for $x \in \partial\Omega$.

Many physical phenomena were formulated into nonlocal mathematical models (see [1–3]) and studied by many authors. And in recent few years, the reaction-diffusion equation with nonlocal source has been studied extensively. In particular, M. Wang and

Y. Wang [4] studied the heat equation with nonlocal source and local damping term

$$u_t - \Delta u = \int_\Omega u^q(y,t)dy - ku^p, \tag{1.2}$$

which is subjected to homogeneous Dirichlet boundary condition. They concluded that the blowup occurs for large initial data if $q > p \geq 1$ while all solutions exist globally if $1 \leq q < p$. In case of $p = q$, the issue depends on the comparison of $|\Omega|$ and k. Using the Green's function, they also proved the blowup set is $\overline{\Omega}$. In [3], Souplet introduced a new method for investigating the rate and profile of blowup of solutions of diffusion equations with nonlocal reaction terms. He obtained the uniform blow-up rate and blow-up profile for large classes of equations. Particularly, for problem (1.2) with homogeneous Dirichlet boundary condition, Souplet [3] obtained the following blow-up estimate when $q > p \geq 1$:

$$\lim_{t\to T}(T-t)^{1/(q-1)}u(x,t) = \lim_{t\to T}(T-t)^{1/(q-1)}|u(t)|_\infty = \left[(q-1)|\Omega|\right]^{-1/(q-1)}, \tag{1.3}$$

where T is the blow-up time of $u(x,t)$. For $q = p > 1$, Souplet [5] gave the blow-up rate as

$$\lim_{t\to T}(T-t)^{1/(q-1)}u(x,t) = \lim_{t\to T}(T-t)^{1/(q-1)}|u(t)|_\infty = \left[(q-1)(|\Omega|-k)\right]^{-1/(q-1)}. \tag{1.4}$$

On the other hand, parabolic equations with nonlocal boundary conditions are also encountered in other physical applications. For example, in the study of the heat conduction within linear thermoelasticity, Day [6, 7] investigated a heat equation subject to a nonlocal boundary condition. Friedman [8] generalized Day's result to a parabolic equation

$$u_t = \Delta u + g(x,u), \quad x \in \Omega,\ t > 0, \tag{1.5}$$

which is subject to the following boundary condition:

$$u(x,t) = \int_\Omega f(x,y)u(y,t)dy. \tag{1.6}$$

He established the global existence of solution and discussed its monotonic decay property, and then proved that $\max_{\overline{\Omega}}|u(x,t)| \leq ke^{-\gamma t}$ under some hypotheses on $f(x,y)$ and $g(x,u)$. Some further results are also obtained on problem (1.5) coupled with boundary condition (1.6) (see [9–11]) later.

Nonlocal problems coupled with nonlocal boundary condition, such as (1.6), to our knowledge, has not been well studied. Recently, Lin and Liu [12] studied a parabolic equation with nonlocal source

$$u_t = \Delta u + \int_\Omega g(u)dx, \quad x \in \Omega,\ t > 0, \tag{1.7}$$

which is subject to boundary condition (1.6). The authors considered the global existence and nonexistence of solutions. Moreover, they derived the blow-up estimate for some special $g(u)$.

For other works on nonlocal problems, we refer readers to [1, 3, 13–21] and references therein.

The main purpose of this paper is to investigate problem with nonlocal source and nonlocal boundary, which is a combination of the work of [4] and that of [6–8, 12]. Precisely, we are interested in the combined effect of the nonlocal nonlinear term $\int_\Omega u^q(y,t)dy$, the damping term and the nonlocal boundary upon the behavior of the solution of problem (1.1). We will give the conditions of existence and nonexistence of global solution for (1.1), and establish the precise estimate of the blow-up rate under some suitable hypotheses. Due to the appearance of the kernel $f(x,y)$, the blow-up conditions will be some different from those of above works.

In order to state our results, we introduce some useful symbols. Throughout this paper, we let λ and ϕ be the first eigenvalue and the corresponding normalized eigenfunction of the problem

$$-\Delta\phi(x) = \lambda\phi, \quad x \in \Omega; \qquad \phi(x) = 0, \quad x \in \partial\Omega. \tag{1.8}$$

Then $\lambda > 0$, $\int_\Omega \phi(x)dx = 1$.

Our main results could be stated as followed. Firstly, for the global existence and finite time blow-up condition, we have the following theorems.

Theorem 1.1. *If $1 \le q < p$, all solutions of problem (1.1) exist globally.*

Theorem 1.2. *If $q > p \ge 1$, problem (1.1) has solutions blowing up in a finite time as well as global solutions. Precisely,*

(i) *if $\int_\Omega f(x,y) \le 1$ and $u_0(x) \le (k/|\Omega|)^{1/(q-p)}$, then the solution exists globally;*

(ii) *if $\int_\Omega f(x,y) > 1$ and $u_0(x) > (k/(|\Omega| - k))^{1/q}$, $(|\Omega| > k)$, then the solution blows up in finite time;*

(iii) *for any $f(x,y) \ge 0$, there exists $a_2 > 0$ such that the solution blows up in finite time provided that $u_0(x) > a_2\phi(x)$.*

Theorem 1.3. *Suppose $p = q > 1$. For any $f(x,y) \ge 0$, the solution blows up in finite time when $u_0(x)$ is large enough. If $\int_\Omega f(x,y)dy < 1$, the solution exists globally when $u_0(x) \le a_1\psi(x)$ for some $a_1 > 0$ (where $\psi(x)$ is defined in (3.8)).*

Remark 1.4. When $p = q = 1$, it is obvious that the problem has no blow-up solution.

For the blow-up rate estimate, we could derive the following results in the case of $\int_\Omega f(x,y)dy \le 1$.

Theorem 1.5. *Let $q > p \ge 1$ and $\int_\Omega f(x,y)dy \le 1$. If u is the solution of (1.1) which blows up at finite time T, then*

$$\lim_{t\to T}(T-t)^{1/(q-1)}u(x,t) = \lim_{t\to T}(T-t)^{1/(q-1)}|u(t)|_\infty = \left[(q-1)|\Omega|\right]^{-1/(q-1)} \tag{1.9}$$

uniformly on compact subsets of Ω.

In the case of $q = p$, the sharp blow-up rate is affected by the presence of the local damping term.

Theorem 1.6. *Let $q = p > 1$ and $\int_\Omega f(x,y)dy \le 1$. If $0 < k < |\Omega|$ and u is the solution of (1.1) which blows up at finite time T, then*

$$\lim_{t\to T}(T-t)^{1/(q-1)}u(x,t) = \lim_{t\to T}(T-t)^{1/(q-1)}|u(t)|_\infty = \left[(q-1)(|\Omega|-k)\right]^{-1/(q-1)} \tag{1.10}$$

uniformly on compact subsets of Ω.

Remark 1.7. Theorems 1.5 and 1.6 imply that the blow-up set of a blow-up solution is $\overline{\Omega}$.

Remark 1.8. Comparing the results of Theorems 1.5-1.6 with (1.3) and (1.4), we find that in the case of $\int_\Omega f(x,y)dy \le 1$, the occurrence of the kernel function $f(x,y)$ do not change the blow-up rate.

The rest of this paper is organized as follows. In Section 2, we give the comparison principle and the local existence of a positive solution. Using sub- and supersolution methods, we will give the proof of Theorems 1.1–1.3 in Section 3. Finally, we establish the uniform blow-up rate estimate and prove Theorems 1.5 and 1.6 in Section 4.

2. Comparison principal and local existence

Let $\Omega_T = \Omega \times (0,T)$ and $\Omega_T \cup \Gamma_T = \overline{\Omega} \times [0,T)$. We begin with the definition of subsolution and supersolution of (1.1).

Definition 2.1. A function $\underline{u}(x,t)$ is called a subsolution of (1.1) on Ω_T if $\underline{u} \in C^{2,1}(\Omega_T) \cap C(\Omega_T \cup \Gamma_T)$ satisfies

$$\begin{aligned}
&\underline{u}_t \le \Delta \underline{u} + \int_\Omega \underline{u}^q(y,t)dy - k\underline{u}^p, \quad x \in \Omega,\ t > 0,\\
&\underline{u}(x,t) \le \int_\Omega f(x,y)\underline{u}(y,t)dy, \quad x \in \partial\Omega,\ t > 0,\\
&\underline{u}(x,0) \le u_0(x), \quad x \in \Omega.
\end{aligned} \tag{2.1}$$

A supersolution is defined analogously with each inequality reversed.

Proposition 2.2. *Let $\underline{u}$ and $\overline{v}$ be a nonnegative subsolution and supersolution, respectively, with $\underline{u}(x,0) < \overline{v}(x,0)$ for $x \in \overline{\Omega}$. Then, $\underline{u} < \overline{v}$ in Ω_T.*

To prove this comparison principle, we need the following lemma.

Lemma 2.3. *Suppose that $w(x,t) \in C^{2,1}(\Omega_T) \cap C(\Omega_T \cup \Gamma_T)$ satisfies*

$$\begin{aligned}
&w_t - \Delta w \ge c_1(x,t)w + \int_\Omega c_2(y,t)w(y,t)dy, \quad x \in \Omega,\ t > 0,\\
&w(x,t) \ge \int_\Omega c_3(x,y)w(y,t)dy, \quad x \in \partial\Omega,\ t > 0,
\end{aligned} \tag{2.2}$$

where c_1, c_2, c_3 are bounded functions and $c_2(x,t) \ge 0$ in Ω_T, $c_3(x,y) \ge 0$ for $x \in \partial\Omega$, $y \in \Omega$ and is not identically zero. Then $w(x,0) > 0$ for $x \in \overline{\Omega}$ implies $w(x,t) > 0$ in Ω_T.

Proof. Set $\theta(x,t)=e^{\lambda t}w(x,t)$, $\lambda\geq\sup|c_1|$, then

$$\begin{gathered}\theta_t\geq\Delta\theta+(\lambda+c_1)\theta+\int_\Omega c_2(y,t)\theta(y,t)dy,\\ \theta(x,t)\big|_{\partial\Omega}\geq\int_\Omega c_3(x,y)\theta(y,t)dy,\\ \theta(x,0)>0,\quad x\in\overline{\Omega}.\end{gathered}\tag{2.3}$$

Since $\theta(x,0)>0$ for all $x\in\Omega$, by continuity, there exists a $t_0>0$ such that $\theta(x,t)>0$ for $(x,t)\in\Omega_{t_0}$. Suppose that t_1 $(t_0\leq t_1<T)$ is the first time at which θ has a zero for some $x_0\in\Omega$. Let $G(x,y;t)$ denote the Green's function for $Lu=u_t-\Delta u$ with boundary condition $u=0$, $x\in\partial\Omega$, $t>0$. Then for $y\in\partial\Omega$, $G(x,y;t)=0$ and $(\partial G/\partial n)(x,y;t)\leq 0$;

$$\begin{aligned}\theta(x,t)\geq{}&\int_\Omega G(x,y;t)\theta(y,0)dy+\int_0^t\int_\Omega G(x,y;t-\eta)[\lambda+c_1(y,\eta)]\theta(y,\eta)dy\,d\eta\\ &+\int_0^t\left(\int_\Omega c_2(x',\eta)\theta(x',t)dx'\right)\int_\Omega G(x,y;t-\eta)dy\,d\eta\\ &-\int_0^t\int_{\partial\Omega}\frac{\partial G}{\partial n}(x,\xi;t-\eta)\int_\Omega c_3(\xi,y)\theta(y,\eta)dy\,d\xi\,d\eta.\end{aligned}\tag{2.4}$$

Since $\theta(x,t)>0$ for all $x\in\Omega$, $0<t<t_1$, we find that

$$\theta(x,t_1)\geq\int_\Omega G(x,y;t)\theta(y,0)dy>0.\tag{2.5}$$

In particular, $\theta(x_0,t_1)>0$, which contradicts our assumption. □

Remark 2.4. If $\int_\Omega c_3(x,y)dy\leq 1$, $w(x,0)\geq 0$ implies that $w(x,t)\geq 0$ in Ω_T. In this case, for any $\delta>0$, $\theta(x,t)=e^{\lambda t}(w(x,t)+\delta)$ satisfies all inequalities in (2.3). Therefore, $w+\delta>0$ for any δ, and it follows that $w(x,t)\geq 0$.

Using Lemma 2.3, we could prove Proposition 2.2 easily.

Local in time existence and uniqueness of classical solutions of (1.1) could be obtained by using the representation formula and the contraction mapping principle as in [9]. We omit the standard argument here. From Proposition 2.2, we know that the classical solution is positive when $u_0(x)$ is positive. We assume that $u_0(x)>0$ in the rest of the paper.

3. Global existence and blowup in finite time

In this section, we will use super- and subsolution techniques to derive some conditions on the existence or nonexistence of global solution.

Proof of Theorem 1.1. Remember that λ and ϕ be the first eigenvalue and the corresponding normalized eigenfunction of $-\Delta$ with homogeneous Dirichlet boundary condition.

We choose l to satisfy that for some $0<\varepsilon<1$,

$$M\int_{\Omega}\frac{1}{l\phi(y)+\varepsilon}\le 1, \tag{3.1}$$

where $M=\sup_{y\in\overline{\Omega},x\in\partial\Omega}f(x,y)$. Let

$$v(x,t)=\frac{ce^{\gamma t}}{l\phi(x)+\varepsilon}, \tag{3.2}$$

where

$$\begin{gathered}c=\max\left\{\sup_{\overline{\Omega}}\left(u_0(x)+1\right)\left(l\phi(x)+\varepsilon\right),\sup_{\overline{\Omega}}\left[\frac{(l\phi+\varepsilon)^p}{k}\int_{\Omega}\frac{1}{(l\phi+\varepsilon)^q}dy\right]^{1/(p-q)}\right\},\\ \gamma\ge\lambda+\sup_{\overline{\Omega}}\frac{2l^2|\nabla\phi|^2}{(l\phi+\varepsilon)^2}.\end{gathered} \tag{3.3}$$

Then we have

$$\begin{aligned}v_t-\Delta v-\int_{\Omega}v^q\,dy+kv^p&=\gamma v-v\left[\frac{\lambda l\phi}{l\phi+\varepsilon}+\frac{2l^2|\nabla\phi|^2}{(l\phi+\varepsilon)^2}\right]\\ &\quad-c^qe^{q\gamma t}\int_{\Omega}\frac{1}{(l\phi+\varepsilon)^q}dy+kc^pe^{\gamma pt}\frac{1}{(l\phi+\varepsilon)^p}\ge 0,\\ v(x,0)&>u_0(x).\end{aligned} \tag{3.4}$$

On the other hand, for any $x\in\partial\Omega$, we have

$$v(x,t)=\frac{ce^{\gamma t}}{\varepsilon}>ce^{\gamma t}\ge\int_{\Omega}\frac{ce^{\gamma t}}{l\phi(y)+\varepsilon}f(x,y)dy=\int_{\Omega}f(x,y)v(y,t)dy. \tag{3.5}$$

Therefore, $v(x,t)$ is a supersolution of (1.1) and the solution $u(x,t)<v(x,t)$ by Proposition 2.2. Therefore, $u(x,t)$ exists globally. □

Proof of Theorem 1.2. (i) Let $v(x,t)=(k/|\Omega|)^{1/(q-p)}$. It is easy to see that $v(x,t)$ is a supersolution of (1.1) if $\int_{\Omega}f(x,y)\le 1$ and $u_0(x)\le(k/|\Omega|)^{1/(q-p)}$. By Proposition 2.2, the solution $u(x,t)$ exists globally.

(ii) Consider the following problem:

$$v'(t)=|\Omega|v^q-kv^p,\qquad v(0)=v_0. \tag{3.6}$$

As $q>p$, $v^p\le v^q+1$. From then $|\Omega|v^q-kv^p\ge(|\Omega|-k)v^q-k$.

Therefore, the solution of (3.6) is a supersolution of the following equation:

$$v'(t)=(|\Omega|-k)v^q-k,\qquad v(0)=v_0. \tag{3.7}$$

When $|\Omega|>k$ and $q>1$, it is known that the solution to this equation blows up in finite time if $v_0>(k/(|\Omega|-k))^{1/q}$.

Obviously, the solution of problem (3.6) is a subsolution of problem (1.1) when $\int_\Omega f(x,y)dy > 1$ and $u_0(x) > v_0$. By comparison principle, $u(x,t)$ is a blow-up solution.

(iii) Notice that $u(x,t) > 0$ when $u_0(x) > 0$. From [4, Theorem 3.4], we could obtain our conclusion directly. □

Proof of Theorem 1.3. Firstly, noticing that the solution to (1.2) coupled with zero boundary condition blows up in finite time if the initial data is large enough (see [4, Theorem 3.3]), we obtain our blow-up result immediately.

Now, we show there exists global solutions if $\int_\Omega f(x,y)dy < 1$.

Let $\psi(x)$ be the unique positive solution of the linear elliptic problem

$$\begin{gathered} -\Delta\psi(x) = \delta, \quad x \in \Omega; \\ \psi(x) = \int_\Omega f(x,y)dy, \quad x \in \partial\Omega. \end{gathered} \tag{3.8}$$

δ is a positive constant such that $0 \le \psi(x) \le 1$ (as $\int_\Omega f(x,y)dy < 1$, there exists such δ).

Let $v(x) = a_1\psi(x)$, where $a_1 > 0$ is chosen such that

$$-\Delta v(x) = \delta a_1 > a_1^p\left(\int_\Omega \psi^p(x)dx - k\psi^p(x)\right) = \int_\Omega v^p(x)dx - kv^p(x). \tag{3.9}$$

For $x \in \partial\Omega$, $v(x) = a_1\int_\Omega f(x,y)dy \ge \int_\Omega f(x,y)v(y)dy$.

By Proposition 2.2 it follows that $u(x,t)$ exists globally provided that $u_0(x) \le a_1\psi(x)$. □

4. Uniform blow-up estimate

In this section, we will obtain the uniform blow-up rate estimate of problem (1.1). Our method is based on the general ideas of [3]. But technically, it is quite different due to the difference of the boundary condition.

In the process of proving Theorem 1.5, we denote

$$g(t) = \int_\Omega u^q(y,t)dy, \qquad G(t) = \int_0^t g(s)ds, \qquad H(t) = \int_0^t G(s)ds. \tag{4.1}$$

Lemma 4.1. *Assume that $\int_\Omega f(x,y)dy \le 1$ for $x \in \partial\Omega$. Let $u(x,t)$ be the solution of (1.1). Then*

$$0 \le u(x,t) \le C_1 + G(t) \tag{4.2}$$

in $[T/2,T) \times \overline{\Omega}$ for some $C_1 > 0$.

Proof. Setting $v = \Delta u$ and taking the Laplacian of the first equality in (1.1) yield

$$v_t - \Delta v = -kp\left(u^{p-1}v + (p-1)u^{p-2}|\nabla u|^2\right) \le -kpu^{p-1}v \quad \text{in } (0,T)\times\Omega. \tag{4.3}$$

Therefore, by the maximum principle, v cannot achieve an interior positive maximum.

For $x \in \partial\Omega$, $y \in \Omega$, we have

$$\begin{aligned}
v(x,t) &= u_t(x,t) - \int_\Omega u^q(y,t)dy + ku^p \\
&= \int_\Omega f(x,y)u_t(y,t)dy - \int_\Omega u^q(y,t)dy + k\left(\int_\Omega f(x,y)u(y,t)dy\right)^p \\
&= \int_\Omega f(x,y)v(y,t)dy - \left(1 - \int_\Omega f(x,y)dy\right)g(t) \\
&\quad - k\left[\int_\Omega f(x,y)u^p(y,t)dy - \left(\int_\Omega f(x,y)u(y,t)dy\right)^p\right].
\end{aligned} \tag{4.4}$$

As $0 < F(x) = \int_\Omega f(x,y)dy \le 1$, we can apply Jensen's inequality to obtain

$$\begin{aligned}
&\int_\Omega f(x,y)u^p(y,t)dy - \left[\int_\Omega f(x,y)u(y,t)dy\right]^p \\
&\quad \ge F(x)\left[\int_\Omega f(x,y)u(y,t)\frac{dy}{F(x)}\right]^p - \left[\int_\Omega f(x,y)u(y,t)dy\right]^p \ge 0.
\end{aligned} \tag{4.5}$$

And this leads to $v(x,t) \le \int_\Omega f(x,y)v(y,t)dy - (1 - \int_\Omega f(x,y)dy)g(t)$ for $x \in \partial\Omega$, $y \in \Omega$.

We first consider the case $0 < \int_\Omega f(x,y)dy < 1$. If $v(x,t)$ achieves nonnegative maximum at $x_0 \in \partial\Omega$ in this case, then

$$v(x_0,t) \le -g(t) \le 0. \tag{4.6}$$

If $\int_\Omega f(x,y)dy = 1$, then $v(x,t)$ necessarily achieves nonnegative maximum at $t = 0$. In fact, if $v(x,t)$ achieves nonnegative maximum at $x_0 \in \partial\Omega$ in this case, we have $v(x_0,t) \le \int_\Omega f(x_0,y)v(y,t)dy$. If $v(x,t)$ is a constant, we obtain our result directly, or else, there exists an $\Omega_1 \subset\subset \Omega$ such that $x_0 \in \Omega_1$ and $v(x,t) < v(x_0,t)$ for arbitrary $x \neq x_0$, $x \in \Omega_1$. Then,

$$\begin{aligned}
\int_\Omega f(x,y)v(y,t)dy &= \int_{\Omega_1} f(x,y)v(y,t)dy + \int_{\Omega\setminus\Omega_1} f(x,y)v(y,t)dy \\
&< v(x_0,t)\int_{\Omega_1} f(x,y) + \int_{\Omega\setminus\Omega_1} f(x,y)v(y,t)dy \\
&\le v(x_0,t)\int_{\Omega_1} f(x,y) + v(x_0,t)\int_{\Omega\setminus\Omega_1} f(x,y)dy \\
&= v(x_0,t).
\end{aligned} \tag{4.7}$$

This is a contradiction.

So, Δu is bounded above.

Integrating the first equation in (1.1) between $T/2$ and $t \in (T/2, T)$, we obtain $0 \le u(x,t) \le C_1 + G(t)$. □

Lemma 4.2. *Assume that $q > p \geq 1$ and $\int_\Omega f(x,y)dy \leq 1$ for $x \in \partial\Omega$. Let $u(x,t)$ be the solution of (1.1). Then*

$$\sup_{x\in K_\rho}[G(t)-u(x,t)] \leq \frac{C_2}{\rho^{n+1}}(1+H(t)+M(t)) \tag{4.8}$$

in $[T/2,T)\times\overline{\Omega}$ for some $C_2>0$; where $K_\rho = \{y\in\Omega,\ \mathrm{dist}(y,\partial\Omega)\geq\rho\}$, $M(t)=o(G(t))$, as $t\to T$.

Proof. Let $\beta(t)=\int_\Omega(G(t)-u(x,t))\phi(y)dy$, then

$$\begin{aligned}\beta'(t) &= \int_\Omega (g(t)-u_t)\phi(y)dy \\ &= \lambda\int_\Omega u(y,t)\phi(y)dy + \int_{\partial\Omega} u\cdot\frac{\partial\phi}{\partial n}dS + k\int_\Omega u^p(y,t)\phi(y)dy \\ &\leq \lambda\int_\Omega u(y,t)\phi(y)dy + k\int_\Omega u^p(y,t)\phi(y)dy \\ &= -\lambda\beta(t)+\lambda G(t)+k\int_\Omega u^p(y,t)\phi(y)dy,\end{aligned} \tag{4.9}$$

which yields

$$\beta(t)\leq C\left(1+H(t)+\int_0^t\int_\Omega u^p(y,s)dy\,ds\right). \tag{4.10}$$

As $q>p\geq 1$, Hölder's inequality implies that

$$\int_0^t\int_\Omega u^p(y,s)dy\,ds \leq \left(\int_0^t\int_\Omega u^q(y,s)dy\,ds\right)^{p/q}(T|\Omega|)^{1-p/q} \equiv M(t)=o(G(t)), \tag{4.11}$$

as $t\to T$. This yields $\beta(t)\leq C(1+H(t)+M(t))$.

Similar to [3, Lemma 4.5], we can obtain $\sup_{x\in K_\rho}[G(t)-u(x,t)]\leq(C/\rho^{n+1})(1+H(t)+M(t))$, in $[T/2,T)\times\overline{\Omega}$ for some $C>0$, where $K_\rho=\{y\in\Omega,\ \mathrm{dist}(y,\partial\Omega)\geq\rho\}$.

Henceforth, we could obtain the following. □

Proposition 4.3. *Suppose that $q > p >= 1$ and $\int_\Omega f(x,y)dy \leq 1$. Then*

$$\lim_{t\to T}\sup_\Omega|u(\cdot,t)| = \infty \tag{4.12}$$

if and only if

$$\int_0^T g(s)ds = \infty. \tag{4.13}$$

Furthermore, if (4.12) or (4.13) is fulfilled, then

$$\lim_{t\to T}\frac{u(x,t)}{G(t)}=\lim_{t\to T}\frac{|u(t)|_\infty}{G(t)}=1 \tag{4.14}$$

uniformly on compact subsets of Ω.

Using Lemmas 4.1 and 4.2, the proof of Proposition 4.3 is trivial modification of [3, Lemma 4.5 and Theorem 4.1]. So we omit it here.

By Proposition 4.3 we can prove our Theorem 1.5. The proof is due to Souplet, his method in [3] works for this problem. We present it here for completeness and significance.

Proof of Theorem 1.5. From (4.14), we know

$$u^q(x,t)\sim G^q(t),\quad t\longrightarrow T. \tag{4.15}$$

By Lebesgue's dominated convergence theorem we obtain that

$$\int_\Omega u^q(y,t)dy\sim|\Omega|G^q(t)\quad t\longrightarrow T. \tag{4.16}$$

Hence

$$G'(t)=g(t)\sim|\Omega|G^q(t),\qquad (G^{1-q})'(t)\sim-(q-1)|\Omega|. \tag{4.17}$$

Therefore,

$$G(t)\sim\left[(q-1)|\Omega|(T-t)\right]^{-1/(q-1)}. \tag{4.18}$$

From (4.14), that is

$$u(x,t)\sim\left[(q-1)|\Omega|(T-t)\right]^{-1/(q-1)},\quad \text{as } t\longrightarrow T. \tag{4.19}$$

We complete our proof. □

Proof of Theorem 1.6. We denote $g_0(t)=\int_\Omega u^q(y,t)dy$, $U(t)=|u(t)|_\infty=\max_{x\in\overline{\Omega}}u(x,t)$, $g(t)=g_0(t)-kU^q(t)$, $G(t)=\int_0^t g(s)ds$, $H(t)=\int_0^t G(s)ds$.

Then, similar to Proposition 4.3, we can obtain

$$\lim_{t\to T}\frac{|u(t)|_\infty}{G(t)}=\lim_{t\to T}\frac{u(x,t)}{G(t)}=1 \tag{4.20}$$

uniformly on compact subsets of Ω. Therefore, similar to the proof of Theorem 1.5, we could conclude that

$$G'(t) = g(t) \sim \left(|\Omega| - k\right) G^{q}(t), \quad \text{as } t \longrightarrow T. \tag{4.21}$$

Then, the blow-up estimate comes from (4.21). □

Acknowledgments

This work is supported in part by NNSF of China (10571126) and in part by Program for New Century Excellent Talents in University, and by the Youth Foundation of Science and Technology of UESTC.

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