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Research Article

On the Sets of Regularity of Solutions for a Class of Degenerate Nonlinear Elliptic Fourth-Order Equations with L^1 Data

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We establish Hölder continuity of generalized solutions of the Dirichlet problem, associated to a degenerate nonlinear fourth-order equation in an open bounded set $\Omega \subset \mathbb{R}^n$, with L^1 data, on the subsets of Ω where the behavior of weights and of the data is regular enough.

1. Introduction

In this paper, we will deal with equations involving an operator $A : \mathring{W}^{1,q}_{2,p}(\nu,\mu,\Omega) \to (\mathring{W}^{1,q}_{2,p}(\nu,\mu,\Omega))^\star$ of the form

$$Au = \sum_{|\alpha|=1,2} (-1)^{|\alpha|} D^\alpha A_\alpha(x, \nabla_2 u), \tag{1.1}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $n > 4$, $2 < p < n/2$, $\max(2p, \sqrt{n}) < q < n$, ν and μ are positive functions in Ω with properties precised later, $\mathring{W}^{1,q}_{2,p}(\nu,\mu,\Omega)$ is the Banach space of all functions $u : \Omega \to \mathbb{R}$ with the properties $|u|^q, \nu|D^\alpha u|^q, \mu|D^\beta u|^p \in L^1(\Omega)$, $|\alpha| = 1$, $|\beta| = 2$, and "zero" boundary values; $\nabla_2 u = \{D^\alpha u : |\alpha| \le 2\}$.

The functions A_α satisfy growth and monotonicity conditions, and in particular, the following strengthened ellipticity condition (for a.e. $x \in \Omega$ and $\xi = \{\xi_\alpha : |\alpha| = 1,2\}$):

$$\sum_{|\alpha|=1,2} A_\alpha(x,\xi)\xi_\alpha \ge c_2 \left\{ \sum_{|\alpha|=1} \nu(x) \,|\, \xi_\alpha \,|^{\,q} + \sum_{|\alpha|=2} \mu(x) \,|\, \xi_\alpha \,|^{\,p} \right\} - g_2(x), \tag{1.2}$$

where $c_2 > 0$, $g_2(x) \in L^1(\Omega)$.

We will assume that the right-hand sides of our equations, depending on unknown function, belong to $L^1(\Omega)$.

A model representative of the given class of equations is the following:

$$-\sum_{|\alpha|=1} D^\alpha\left[\nu\left(\sum_{|\beta|=1}|D^\beta u|^2\right)^{(q-2)/2} D^\alpha u\right]+\sum_{|\alpha|=2} D^\alpha\left[\mu\left(\sum_{|\beta|=2}|D^\beta u|^2\right)^{(p-2)/2} D^\alpha u\right] = -|u|^{\sigma-1}u+f \quad \text{in } \Omega, \tag{1.3}$$

where $\sigma > 1$ and $f \in L^1(\Omega)$.

The assumed conditions and known results of the theory of monotone operators allow us to prove existence of generalized solutions of the Dirichlet problem associated to our operator (see, e.g., [1]), bounded on the sets $G \subset \Omega$ where the behavior of weights and of the data of the problem is regular enough (see [2]).

In our paper, following the approach of [3], we establish on such sets a result on Hölder continuity of generalized solutions of the same Dirichlet problem.

We note that for one high-order equation with degenerate nonlinear operator satisfying a strengthened ellipticity condition, regularity of solutions was studied in [4, 5] (nondegenerate case) and in [6, 7] (degenerate case). However, it has been made for equations with right-hand sides in L^t with $t > 1$.

2. Hypotheses

Let $n \in \mathbb{N}$, $n > 4$, and let Ω be a bounded open set of $\mathbb{R}^n$. Let p, q be two real numbers such that $2 < p < n/2$, $\max(2p, \sqrt{n}) < q < n$.

Let $\nu : \Omega \to \mathbb{R}^+$ be a measurable function such that

$$\nu \in L^1_{\text{loc}}(\Omega), \qquad \left(\frac{1}{\nu}\right)^{1/(q-1)} \in L^1_{\text{loc}}(\Omega). \tag{2.1}$$

$W^{1,q}(\nu,\Omega)$ is the space of all functions $u \in L^q(\Omega)$ such that their derivatives, in the sense of distribution, $D^\alpha u$, $|\alpha| = 1$, are functions for which the following properties hold: $\nu^{1/q} D^\alpha u \in L^q(\Omega)$ if $|\alpha| = 1$; $W^{1,q}(\nu,\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{1,q,\nu} = \left(\int_\Omega |u|^q dx + \sum_{|\alpha|=1}\int_\Omega \nu|D^\alpha u|^q dx\right)^{1/q}. \tag{2.2}$$

$\overset{\circ}{W}{}^{1,q}(\nu,\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,q}(\nu,\Omega)$.

Let $\mu(x) : \Omega \to \mathbb{R}^+$ be a measurable function such that

$$\mu \in L^1_{\text{loc}}(\Omega), \qquad \left(\frac{1}{\mu}\right)^{1/(p-1)} \in L^1_{\text{loc}}(\Omega). \tag{2.3}$$

$W^{1,q}_{2,p}(\nu,\mu,\Omega)$ is the space of all functions $u \in W^{1,q}(\nu,\Omega)$, such that their derivatives, in the sense of distribution, $D^\alpha u$, $|\alpha| = 2$, are functions with the following properties:

$\mu^{1/p}D^\alpha u \in L^p(\Omega)$, $|\alpha| = 2$; $W^{1,q}_{2,p}(\nu,\mu,\Omega)$ is a Banach space with respect to the norm

$$\|u\| = \|u\|_{1,q,\nu} + \left(\sum_{|\alpha|=2}\int_\Omega \mu\,|D^\alpha u|^p dx\right)^{1/p}. \tag{2.4}$$

$\overset{\circ}{W}{}^{1,q}_{2,p}(\nu,\mu,\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,q}_{2,p}(\nu,\mu,\Omega)$.

Hypothesis 2.1. Let $\nu(x)$ be a measurable positive function:

$$\begin{aligned} \frac{1}{\nu} &\in L^t(\Omega) \quad \text{with } t > \frac{nq}{q^2-n}, \\ \nu &\in L^{\bar t}(\Omega) \quad \text{with } \bar t > \frac{nt}{qt-n}. \end{aligned} \tag{2.5}$$

We put $\tilde q = nqt/(n(1+t)-qt)$. We can easily prove that a constant $c_0 > 0$ exists such that if $u \in W^{1,q}(\nu,\Omega)$, the following inequality holds:

$$\int_\Omega |u|^{\tilde q} dx \le c_0 \left\{\int_{\operatorname{supp} u}\left(\frac{1}{\nu}\right)^t dx\right\}^{\tilde q/qt}\left\{\sum_{|\alpha|=1}\int_\Omega \nu |D^\alpha u|^q dx\right\}^{\tilde q/q}. \tag{2.6}$$

We set $\tilde\nu = \mu^{q/(q-2p)}(1/\nu)^{2p/(q-2p)}$.

Hypothesis 2.2. $\tilde\nu \in L^1(\Omega)$.

Hypothesis 2.3. There exists a real number $r > \tilde q(q-1)/(\tilde q(q-1)(p-1)-q)$ such that

$$\frac{1}{\mu} \in L^r(\Omega). \tag{2.7}$$

For more details about weight functions, see [8, 9].

Let Ω_1 be a nonempty open set of $\mathbb{R}^n$ such that $\Omega_1 \subset \Omega$.

Definition 2.4. It is said that G closed set of $\mathbb{R}^n$ is a "*regular set*" if $\overset{\circ}{G}$ is nonempty and $G \subset \Omega_1$.

Denote by $\mathbb{R}^{n,2}$ the space of all sets $\xi = \{\xi_\alpha \in \mathbb{R} : |\alpha| = 1,2\}$ of real numbers; if a function $u \in L^1_{\text{loc}}(\Omega)$ has the weak derivatives $D^\alpha u$, $|\alpha| = 1,2$ then $\nabla_2 u = \{D^\alpha u : |\alpha| = 1,2\}$. Suppose that $A_\alpha : \Omega \times \mathbb{R}^{n,2} \to \mathbb{R}$ are Carathéodory functions.

Hypothesis 2.5. There exist $c_1, c_2 > 0$ and $g_1(x)$, $g_2(x)$ nonnegative functions such that $g_1, g_2 \in L^1(\Omega)$ and, for almost every $x \in \Omega$, for every $\xi \in \mathbb{R}^{n,2}$, the following inequalities hold:

$$\sum_{|\alpha|=1}[\nu(x)]^{-1/(q-1)}|A_\alpha(x,\xi)|^{q/(q-1)}+\sum_{|\alpha|=2}[\mu(x)]^{-1/(p-1)}|A_\alpha(x,\xi)|^{p/(p-1)}$$
$$\le c_1\left\{\sum_{|\alpha|=1}\nu(x)|\xi_\alpha|^q+\sum_{|\alpha|=2}\mu(x)|\xi_\alpha|^p\right\}+g_1(x), \tag{2.8}$$

$$\sum_{|\alpha|=1,2}A_\alpha(x,\xi)\xi_\alpha\ge c_2\left\{\sum_{|\alpha|=1}\nu(x)|\xi_\alpha|^q+\sum_{|\alpha|=2}\mu(x)|\xi_\alpha|^p\right\}-g_2(x). \tag{2.9}$$

Moreover, we will assume that for almost every $x\in\Omega$ and every $\xi,\xi'\in\mathbb{R}^{n,2}$, $\xi\neq\xi'$,

$$\sum_{|\alpha|=1,2}[A_\alpha(x,\xi)-A_\alpha(x,\xi')](\xi_\alpha-\xi'_\alpha)>0. \tag{2.10}$$

Let $F:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function such that

(a) for almost every $x\in\Omega$, the function $F(x,\cdot)$ is nonincreasing in $\mathbb{R}$;

(b) for every $x\in\Omega$, the function $F(\cdot,s)$ belongs to $L^1(\Omega)$.

Let $A:\mathring{W}^{1,q}_{2,p}(\nu,\mu,\Omega)\to(\mathring{W}^{1,q}_{2,p}(\nu,\mu,\Omega))^\star$ be the operator such that for every $u,v\in\mathring{W}^{1,q}_{2,p}(\nu,\mu,\Omega)$,

$$\langle Au,v\rangle=\int_\Omega\left\{\sum_{|\alpha|=1,2}A_\alpha(x,\nabla_2u)D^\alpha v\right\}dx. \tag{2.11}$$

We consider the following Dirichlet problem:

$$(P)=\begin{cases}Au=F(x,u) & \text{in }\Omega\\ D^\alpha u=0,\quad |\alpha|=0,1, & \text{on }\partial\Omega.\end{cases} \tag{2.12}$$

Definition 2.6. A W-solution of problem (P) is a function $u\in\mathring{W}^{2,1}(\Omega)$ such that

(i) $F(x,u)\in L^1(\Omega)$;

(ii) $A_\alpha(x,\nabla_2u)\in L^1(\Omega)$, for every $\alpha:|\alpha|=1,2$;

(iii) $\langle Au,\phi\rangle=\langle F(x,u),\phi\rangle$ in distributional sense.

It is well known that Hypotheses 2.1–2.3, 2.5, and assumptions on $F(x,s)$ imply the existence of a W-solution of problem (P) (see [1]). Moreover, a boundedness local result for such solution has been established in [2] under more restrictive hypotheses on data and weight functions.

More precisely, the following holds (see [2, Theorem 5.1]).

Theorem 2.7. *Suppose that Hypotheses 2.1–2.3 and 2.5 are satisfied. Let $q_1\in(q,\widetilde{q}(q-1)/q)$, $\tau>\widetilde{q}/(\widetilde{q}-q_1)$. Assume that restrictions of the functions $\nu^{q_1/(q_1-q)}$, $\widetilde{\nu}$, g_1, g_2, and $|F(\cdot,0)|^{q_1/(q_1-1)}$ on G belong to $L^\tau(G)$, for every "regular set" G.*

Then there exists $\overline{u}$ W-solution of problem (P) such that for every G, $\operatorname{ess}_G\sup|\overline{u}|\le M_G<+\infty$, with M_G positive constant depending only on known values.

3. Main result

In the sequel of paper, G will be a "*regular set*." In order to obtain our regularity result on G, we need the following further hypotheses.

Hypothesis 3.1. There exists a constant $c' > 0$ such that for all $y \in \overset{\circ}{G}$ and for all $\rho > 0$, with $\overline{B(y,\rho)} \subset \overset{\circ}{G}$, we have

$$\left\{\rho^{-n}\int_{B(y,\rho)}\left(\frac{1}{\nu}\right)^{t}dx\right\}^{1/t}\left\{\rho^{-n}\int_{B(y,\rho)}\nu^{\tau}dx\right\}^{1/\tau} \leq c'. \tag{3.1}$$

With regard to this assumption, see [3].

Hypothesis 3.2. There exist a real positive number σ and two real functions $h(x)(\geq 0)$, $f(x)(>0)$ defined on G, such that

$$|F(x,s)| \leq h(x)|s|^{\sigma} + f(x), \quad \text{for almost every } x \in G \text{ and every } s \in \mathbb{R}. \tag{3.2}$$

Moreover, we assume that

$$h(x), f(x) \in L^{\tau}(G), \tag{3.3}$$

with τ defined as above.

Using considerations stated in [1], following the approach of [3], we establish the following result.

Theorem 3.3. *Let all above-stated hypotheses hold and let conditions of Theorem 2.7 be satisfied. Then, the W-solution $\overline{u}$ of Dirichlet problem (P), essentially bounded on G, is also locally Hölderian on G.*

More precisely, there exist positive constant C and λ $(0 < \lambda < 1)$ such that for every open set Ω', $\overline{\Omega}' \subset \overset{\circ}{G}$, and every $x, y \in \Omega'$

$$|\overline{u}(x) - \overline{u}(y)| \leq C[d(\Omega', \partial\overset{\circ}{G})]^{-\lambda}|x-y|^{\lambda}, \tag{3.4}$$

where C and λ depend only on c_1, c_2, c_0, c', n, q, p, t, τ, σ, M_G, $\operatorname{diam} G$, $\operatorname{meas} G$, $\|f\|_{L^{\tau}(G)}$, $\|h\|_{L^{\tau}(G)}$, $\|g_1\|_{L^{\tau}(G)}$, $\|g_2\|_{L^{\tau}(G)}$, $\|\widetilde{\nu}\|_{L^{\tau}(G)}$, and $\|1/\nu\|_{L^{t}(\Omega)}$.

Proof. For every $l \in \mathbb{N}$, we define the function $F_l : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$F_l(x,s) = \begin{cases} -l & \text{if } F(x,0) - F(x,s) < -l, \\ F(x,0) - F(x,s) & \text{if } |F(x,0) - F(x,s)| \leq l, \\ l & \text{if } F(x,0) - F(x,s) > l, \end{cases} \tag{3.5}$$

and the function $f_l : \Omega \to \mathbb{R}$ by

$$f_l(x) = \begin{cases} F(x,0) & \text{if } |F(x,0)| \leq l, \\ 0 & \text{if } |F(x,0)| > l. \end{cases} \tag{3.6}$$

By Lebesgue's theorem and property (b) of $F(x,s)$, we have that $f_l(x)$ goes to $F(x,0)$ in $L^1(\Omega)$.

Next, inequalities (2.6), (2.8)–(2.10), property (a) of $F(x,s)$, and known results of the theory of monotone operators (see, e.g., [10]) imply that for any $l \in \mathbb{N}$, there exists $u_l \in \mathring{W}^{1,q}_{2,p}(\nu,\mu,\Omega)$ such that

$$\int_\Omega \left\{ \sum_{|\alpha|=1,2} A_\alpha(x,\nabla_2 u_l) D^\alpha v + F_l(x,u_l)v \right\} dx = \int_\Omega f_l v\, dx, \tag{3.7}$$

for every $v \in \mathring{W}^{1,q}_{2,p}(\nu,\mu,\Omega)$.

From considerations stated in [1, Section 3], we deduce that there exists a W-solution $\overline{u}$ of problem (P) such that

$$u_l \longrightarrow \overline{u} \quad \text{a.e. in } \Omega. \tag{3.8}$$

Moreover, see proof of Theorem 2.7,

$$\operatorname*{ess\,sup}_G |u_l| \leq M_G, \quad \text{for every } l \in \mathbb{N}. \tag{3.9}$$

We set $\overline{n} = q^2/(q-2p)$, $a = (1/\overline{n})(q - n/t - n/\tau)$.

Let us fix $y \in G$, $\rho > 0$ and $\overline{B(y,2\rho)} \subset G$. Let us put

$$\omega_{1,l} = \operatorname*{ess\,inf}_{B(y,2\rho)} u_l, \qquad \omega_{2,l} = \operatorname*{ess\,sup}_{B(y,2\rho)} u_l,$$
$$\omega_l = \omega_{2,l} - \omega_{1,l}. \tag{3.10}$$

We will show that

$$\operatorname{osc}\{u_l, B(y,\rho)\} \leq \widetilde{c}\omega_l + \rho^a, \tag{3.11}$$

with $\widetilde{c} \in]0,1[$ independent of $l \in \mathbb{N}$.

To this aim, we fix $l \in \mathbb{N}$ and we set

$$\Phi_l = \sum_{|\alpha|=1} \nu |D^\alpha u_l|^q + \sum_{|\alpha|=2} \mu |D^\alpha u_l|^p,$$
$$\psi(x) = \rho^{-a\overline{n}}\left(1 + f(x) + h(x) + g_1(x) + g_2(x) + \widetilde{\nu}(x)\right) + \rho^{-q}\nu. \tag{3.12}$$

Obviously, we will assume that

$$\omega_l \geq \rho^a \quad \text{(otherwise, it is clear that (3.11) is true).} \tag{3.13}$$

We introduce now the following functions:

$$F_{1,l}(x) = \begin{cases} \dfrac{2e\omega_l}{u_l(x) - \omega_{1,l} + \rho^a} & \text{if } x \in B(y,2\rho), \\ e & \text{if } x \in \Omega \setminus B(y,2\rho); \end{cases} \tag{3.14}$$

$\varphi \in C_0^\infty(\Omega)$: $0 \le \varphi \le 1$ in Ω, $\varphi = 0$ in $\Omega \setminus B(y, 2\rho)$ and satisfying

$$|D^\alpha \varphi| \le \bar{c}\rho^{-|\alpha|}, \quad |\alpha| = 1, 2, \tag{3.15}$$

where the positive constant $\bar{c}$ depends only on n.

Let us fix $s > q$ and $r \ge 0$ and define

$$\begin{aligned} v_l &= (\lg F_{1,l})^r F_{1,l}^{q-1} \varphi^s, \\ z_l &= -\frac{1}{2e\omega_l} [r(\lg F_{1,l})^{r-1} + (q-1)(\lg F_{1,l})^r] F_{1,l}^q \varphi^s. \end{aligned} \tag{3.16}$$

From Hypothesis 2.2 and (3.15), we have that $v_l \in \overset{\circ}{W}{}^{1,q}_{2,p}(\nu, \mu, \Omega)$ and the next inequalities are true:

$$|D^\alpha v_l - z_l D^\alpha u_l| \le \bar{c} s \varphi^{s-1} (\lg F_{1,l})^r F_{1,l}^{q-1} \rho^{-1} \quad \text{if } |\alpha| = 1 \text{ a.e. in } B(y, 2\rho), \tag{3.17}$$

$$\begin{aligned} |D^\alpha v_l - z_l D^\alpha u_l| \le{}& 5q^2 s(r+1)^2 (\lg F_{1,l})^r F_{1,l}^{q-1} \varphi^s \left\{ \sum_{|\beta|=1} \frac{|D^\beta u_l|^2}{(u_l - \omega_{1,l} + \rho^a)^2} \right\} \\ &+ 2nqs^2 \bar{c}^2 \rho^{-2} (\lg F_{1,l})^r F_{1,l}^{q-1} \varphi^{s-2} \quad \text{if } |\alpha| = 2 \text{ a.e. in } B(y, 2\rho). \end{aligned} \tag{3.18}$$

Since $u_l(x)$ satisfies (3.7), for $v = v_l$, we obtain

$$\int_\Omega \left\{ \sum_{|\alpha|=1,2} A_\alpha(x, \nabla_2 u_l) D^\alpha v_l + F_l(x, u_l) v_l \right\} dx = \int_\Omega f_l v_l dx. \tag{3.19}$$

From this, taking into account (3.9) and Hypothesis 3.2, we have

$$\int_\Omega \sum_{|\alpha|=1,2} A_\alpha(x, \nabla_2 u_l) D^\alpha v_l dx \le (3 + M_G^\sigma) \int_\Omega \{1 + f(x) + h(x)\} v_l dx. \tag{3.20}$$

Hence

$$\int_\Omega \sum_{|\alpha|=1,2} \{A_\alpha(x, \nabla_2 u_l) D^\alpha u_l\}(-z_l) dx \le (3 + M_G^\sigma) \int_\Omega \{1 + f(x) + h(x)\} v_l dx + I_1 + I_2, \tag{3.21}$$

where

$$I_i = \int_\Omega \sum_{|\alpha|=i} |A_\alpha(x, \nabla_2 u_l)| \, |D^\alpha v_l - z_l D^\alpha u_l| dx, \quad i = 1, 2. \tag{3.22}$$

Using Hypothesis 2.5 and definition of z_l, we have

$$\begin{aligned} \frac{(q-1)c_2}{2e\omega_l} \int_\Omega \Phi_l (\lg F_{1,l})^r F_{1,l}^q \varphi^s dx \le{}& (3 + M_G^\sigma) \int_\Omega \{1 + f(x) + h(x)\} (\lg F_{1,l})^r F_{1,l}^{q-1} \varphi^s dx \\ &+ \int_\Omega g_2(x)(-z_l) dx + I_1 + I_2. \end{aligned} \tag{3.23}$$

Note that

$$F_{1,l}^{q-1} \le (\operatorname{diam} G)^a (2e\omega_l)^{q-1} \rho^{-aq},$$
$$-z_l \le (q-1)(r+1)(2e\omega_l)^{q-1} \rho^{-aq} \varphi^s (\lg F_{1,l})^r \quad \text{a.e. in } B(y,2\rho), \tag{3.24}$$

consequently, from (3.23), we obtain

$$\frac{c_2}{2e\omega_l} \int_{B(y,2\rho)} \Phi_l (\lg F_{1,l})^r F_{1,l}^q \varphi^s dx$$
$$\le c_3 (r+1)(2e\omega_l)^{q-1} \int_{B(y,2\rho)} \rho^{-aq} \{1 + f(x) + h(x) + g_2(x)\} (\lg F_{1,l})^r \varphi^s dx + I_1 + I_2, \tag{3.25}$$

where $c_3 = (q-1)(3 + M_G^{\sigma})(\operatorname{diam} G + 1)$.

Let us fix $|\alpha| = 1$. Let $\epsilon > 0$, then, applying Young's inequality and using (2.8) and (3.17), we establish

$$I_1 \le \frac{c_1 \epsilon}{2e\omega_l} \int_{B(y,2\rho)} \Phi_l F_{1,l}^q (\lg F_{1,l})^r \varphi^s dx$$
$$+ c_1 \epsilon (2e\omega_l)^{q-1} \int_{B(y,2\rho)} \rho^{-aq} g_1(x) (\lg F_{1,l})^r \varphi^s dx \tag{3.26}$$
$$+ \epsilon^{1-q} (2e\omega_l)^{q-1} n (\bar{c} s)^q \int_{B(y,2\rho)} \rho^{-q} \nu (\lg F_{1,l})^r \varphi^{s-q} dx.$$

Let us fix $|\alpha| = 2$ and estimate I_2. To this aim, it will be useful to observe that the following equalities are true:

$$\frac{p-1}{p} + \frac{2}{q} + \frac{q-2p}{qp} = 1, \qquad q-1 = \frac{p-1}{p} q + \left(\frac{q}{p} - 1\right). \tag{3.27}$$

Moreover,

$$\rho^{-aq-2p} \mu \le \rho^{-a\bar{n}} \tilde{\nu} + \rho^{-q} \nu \quad \text{in } \Omega. \tag{3.28}$$

Furthermore, due to (2.8), (3.18), and Young's inequality, we have

$$I_2 \le \frac{c_4 \epsilon}{2e\omega_l} \int_{B(y,2\rho)} \Phi_l F_{1,l}^q (\lg F_{1,l})^r \varphi^s dx$$
$$+ c_5 (2e\omega_l)^{q-1} \epsilon \left(1 + \frac{1}{\epsilon}\right)^{\bar{n}} s^{\bar{n}} (r+1)^{\bar{n}} \int_{B(y,2\rho)} \{\rho^{-a\bar{n}} (g_1(x) + \tilde{\nu}(x)) + \rho^{-q} \nu\} (\lg F_{1,l})^r \varphi^{s-q} dx, \tag{3.29}$$

where c_4 depends only on c_1, n, q; and c_5 depends only on c_1, n, q, p, $\bar{c}$, and $\operatorname{diam} G$.

From (3.25), (3.26), and (3.29), we get

$$\begin{aligned}\frac{c_2}{2e\omega_l}\int_{B(y,2\rho)}&\Phi_l(\lg F_{1,l})^r F_{1,l}^q\varphi^s dx\\&\leq \frac{(c_1+c_4)\epsilon}{2e\omega_l}\int_{B(y,2\rho)}\Phi_l F_{1,l}^q(\lg F_{1,l})^r\varphi^s dx\\&\quad+(2e\omega_l)^{q-1}c_6(r+1)^{\overline{n}}s^{\overline{n}}\left(1+\epsilon+\frac{1}{\epsilon}\right)^{\overline{n}+1}\int_{B(y,2\rho)}\psi(\lg F_{1,l})^r\varphi^{s-q}dx,\end{aligned}\tag{3.30}$$

where the constant c_6 depends only on c_1, $\overline{c}$, n, q, p, M_G, σ, and $\operatorname{diam} G$.

Setting

$$\epsilon=\frac{c_2}{2(c_1+c_4)},\tag{3.31}$$

from the last inequality, we deduce

$$\int_{B(y,2\rho)}\Phi_l(\lg F_{1,l})^r F_{1,l}^q\varphi^s dx\leq c_7(2e\omega_l)^q(r+1)^{\overline{n}}s^{\overline{n}}\int_{B(y,2\rho)}\psi(\lg F_{1,l})^r\varphi^{s-q}dx,\tag{3.32}$$

where the constant c_7 depends only on c_1, c_2, $\overline{c}$, n, q, p, M_G, σ, and $\operatorname{diam} G$.

Now, if we choose φ such that $\varphi=1$ in $B(y,(4/3)\rho)$, from (3.32), with $r=0$ and $s=q+1$, we get

$$\int_{B(y,(4/3)\rho)}\left\{\sum_{|\alpha|=1}\nu|D^\alpha u_l|^q\right\}F_{1,l}^q dx\leq c_7(2e\omega_l)^q(q+1)^{\overline{n}}\int_{B(y,2\rho)}\psi dx.\tag{3.33}$$

Moreover, if we take in (3.32) instead of φ the function $\varphi_1\in C_0^\infty(\Omega)$ with the properties $0\leq\varphi_1\leq 1$ in Ω, $\varphi_1=0$ in $\Omega\setminus B(y,(4/3)\rho)$, $\varphi_1=1$ in $B(y,\rho)$, and $|D^\alpha\varphi|\leq\overline{c}\rho^{-|\alpha|}$ in Ω, $|\alpha|=1,2$, we obtain that for every $r>0$ and $s>q$,

$$\int_{B(y,2\rho)}\left\{\sum_{|\alpha|=1}\nu|D^\alpha u_l|^q\right\}(\lg F_{1,l})^r F_{1,l}^q dx\leq c_7(2e\omega_l)^q s^{\overline{n}}(r+1)^{\overline{n}}\int_{B(y,2\rho)}\psi(\lg F_{1,l})^r\varphi_1^{s-q}dx.\tag{3.34}$$

We fix arbitrary $r>0$ and $s>\widetilde{q}$, and let

$$z_l=(\lg F_{1,l})^{r/\widetilde{q}}\varphi_1^{s/\widetilde{q}}.\tag{3.35}$$

By means of Hypothesis 2.1, we establish that $z_l\in\overset{\circ}{W}{}^{1,q}(\nu,\Omega)$ and for $|\alpha|=1$,

$$\begin{aligned}\nu|D^\alpha z_l|^q&\leq 2^{q-1}\left(\frac{r}{\widetilde{q}}\right)^q(\lg F_{1,l})^{(r/\widetilde{q}-1)q}(F_{1,l})^q\frac{1}{(2e\omega_l)^q}|D^\alpha u_l|^q\nu\varphi_1^{sq/\widetilde{q}}\\&\quad+2^{q-1}\left(\frac{s}{\widetilde{q}}\right)^q(\lg F_{1,l})^{rq/\widetilde{q}}\varphi_1^{(s/\widetilde{q}-1)q}\overline{c}^q\rho^{-q}\nu.\end{aligned}\tag{3.36}$$

Now, it is convenient to observe that $\tilde{q}/(\tilde{q}-q_1) > nt/(qt-n)$, then $\tau > nt/(qt-n)$; moreover, $\psi(x) \in L^\tau(G)$. From (3.34) and (3.36), we deduce

$$\int_\Omega \nu |D^\alpha z_l|^q dx$$
$$\leq c_8 s^{\overline{n}} (r+1)^{\overline{n}+q} \left(\int_{B(y,2\rho)} \psi^\tau dx \right)^{1/\tau} \left(\int_{B(y,2\rho)} (\lg F_{1,l})^{r(q/\tilde{q})(\tau/(\tau-1))} \varphi_1^{(s/\tilde{q}-1)q(\tau/(\tau-1))} dx \right)^{(\tau-1)/\tau}, \tag{3.37}$$

where the constant c_8 depends only on c_1, c_2, $\overline{c}$, n, q, p, M_G, σ, and $\operatorname{diam} G$.

We set

$$\theta = \frac{\tilde{q}(\tau-1)}{q\tau}, \qquad m = \frac{q\tau}{\tau-1}, \tag{3.38}$$

and for every $r, s > 0$, we define

$$I(r,s) = \int_{B(y,2\rho)} (\lg F_{1,l})^r \varphi_1^s dx. \tag{3.39}$$

Consequently, last inequality can be rewritten in this manner:

$$\int_\Omega \nu |D^\alpha z_l|^q dx \leq c_8 s^{\overline{n}} (r+1)^{\overline{n}+q} \left(\int_{B(y,2\rho)} \psi^\tau dx \right)^{1/\tau} \left[I\left(\frac{r}{\theta}, \frac{s}{\theta} - m \right) \right]^{(\tau-1)/\tau}. \tag{3.40}$$

Due to Hypothesis 2.1,

$$I(r,s) = \int_{B(y,2\rho)} z_l^{\tilde{q}} dx \leq c_0 \left[\int_{B(y,2\rho)} \left(\frac{1}{\nu} \right)^t dx \right]^{\tilde{q}/qt} \left[\sum_{|\alpha|=1} \int_\Omega \nu |D^\alpha z_l|^q dx \right]^{\tilde{q}/q}. \tag{3.41}$$

Let us denote by $\prod_G$ the norm of $(1+f(x)+h(x)+g_1(x)+g_2(x)+\tilde{\nu}(x))$ in $L^\tau(G)$. By simple computation, we have

$$\left(\int_{B(y,2\rho)} \psi^\tau dx \right)^{1/\tau} \leq \rho^{-q} \left(\int_{B(y,2\rho)} \nu^\tau dx \right)^{1/\tau} + \prod\nolimits_G \rho^{-a\overline{n}}. \tag{3.42}$$

Now, it is convenient to observe that $(q - n/t - n/\tau)(\tilde{q}/q) = n(\theta - 1)$.

Then, from (3.40)–(3.42), using Hypothesis 3.1, we get

$$I(r,s) \leq M (r+s)^{\overline{m}} \rho^{n(1-\theta)} \left[I\left(\frac{r}{\theta}, \frac{s}{\theta} - m \right) \right]^\theta, \quad \text{for every } r > 0,\ s > \tilde{q}, \tag{3.43}$$

where $\overline{m} = 2(q+\overline{n})\tilde{q}$ and the positive constant M depends only on c_1, c_2, $\overline{c}$, c_0, c', n, q, p, t, $\|1/\nu\|_{L^t(\Omega)}$, M_G, σ, $\operatorname{meas} G$, $\operatorname{diam} G$, and $\prod_G$.

We set for $i = 0, 1, 2, \ldots$ that

$$r_i = \frac{tq}{t+1}\theta^i, \qquad s_i = \frac{m\theta}{\theta - 1}\left(\theta^{i+1} - 1\right). \tag{3.44}$$

Then by (3.43), it is trivial to establish the following iterative relation:

$$I(r_i, s_i) \le M c_9 \rho^{n(1-\theta)} \theta^{i\overline{m}} \left[I(r_{i-1}, s_{i-1})\right]^{\theta} \quad \text{for every } i \in \mathbb{N}, \tag{3.45}$$

where c_9 depends only on n, q, p, t, and τ.

Using this recurrent relation, we obtain that for every $i \in \mathbb{N}$,

$$I(r_i, s_i) \le \left[(M c_9 + 1)^{1/(1-\theta)} \theta^{S\overline{m}} (\operatorname{diam} G + 1)^n \rho^{-n} I(r_0, s_0)\right]^{\theta^i}, \tag{3.46}$$

where S is a positive constant depending only on n, q, t, and τ.

Now, we assume that

$$\operatorname{meas}\left\{x \in B\left(y, \frac{4}{3}\rho\right) : u_l(x) \ge \frac{\omega_{1,l} + \omega_{2,l}}{2}\right\} \ge \frac{1}{2} \operatorname{meas} B\left(y, \frac{4}{3}\rho\right). \tag{3.47}$$

We observe that if $x \in B(y, (4/3)\rho)$ satisfies $u_l(x) \ge (\omega_{1,l} + \omega_{2,l})/2$, then $F_{1,l}(x) \le 4e$, so by [11, Lemma 4], we deduce

$$\int_{B(y,(4/3)\rho)} \left(\lg F_{1,l}\right)^{r_0} dx \le c\rho^n + \frac{c\rho r_0}{2e\omega_l} \int_{B(y,(4/3)\rho)} \left\{\sum_{|\alpha|=1} \left|D^\alpha u_l\right| \left(\lg F_{1,l}\right)^{r_0 - 1} F_{1,l}\right\} dx, \tag{3.48}$$

where c depends only on n.

Then, using Young's inequality, we get

$$\int_{B(y,(4/3)\rho)} \left(\lg F_{1,l}\right)^{r_0} dx \le c r_0 \rho^n + r_0 \left(\frac{c r_0 \rho}{2e\omega_l}\right)^{r_0} \int_{B(y,(4/3)\rho)} \left\{\sum_{|\alpha|=1} \left|D^\alpha u_l\right|\right\}^{r_0} F_{1,l}^{r_0} dx. \tag{3.49}$$

Last inequality, using Hölder's inequality and (3.33), gives

$$\begin{aligned} \int_{B(y,(4/3)\rho)} \left(\lg F_{1,l}\right)^{r_0} dx \le{}& c r_0 \rho^n + r_0 \left[c r_0\right]^{r_0} 2^{r_0 - 1} \left[c_7 (q+1)^{\overline{n}}\right]^{t/(t+1)} \rho^{r_0} \\ &\times \left(\int_{B(y,2\rho)} \psi\, dx\right)^{t/(t+1)} \left(\int_{B(y,2\rho)} \left(\frac{1}{\nu}\right)^t dx\right)^{1/(t+1)}. \end{aligned} \tag{3.50}$$

Observe that due to (3.42) and Hypothesis 3.1,

$$\left(\int_{B(y,2\rho)} \psi\, dx\right)^{t/(t+1)} \left(\int_{B(y,2\rho)} \left(\frac{1}{\nu}\right)^t dx\right)^{1/(t+1)} \le c_{10} (1 + M) \rho^{n - r_0}, \tag{3.51}$$

where c_{10} depends only on measure of the unit ball in $\mathbb{R}^n$.

Consequently, from (3.50), we obtain

$$\int_{B(y,(4/3)\rho)} (\lg F_{1,l})^{r_0} dx \leq \left(c_{10}(1+M)r_0[cr_0]^{r_0} 2^{r_0-1}[c_7(q+1)^{\overline{n}}]^{t/(t+1)} + cr_0\right)\rho^n. \tag{3.52}$$

Taking into account that

$$I(r_0,s_0) \leq \int_{B(y,(4/3)\rho)} (\lg F_{1,l})^{r_0} dx, \tag{3.53}$$

from (3.46) we get

$$I(r_i,s_i) \leq [c_{11}]^{\theta^i}, \quad \text{for every } i \in \mathbb{N}. \tag{3.54}$$

Last inequality allow us to conclude that

$$\operatorname*{ess\,sup}_{B(y,\rho)} F_{1,l}(x) \leq (1+c_{11}), \tag{3.55}$$

and so

$$\operatorname{osc}\{u_l, B(y,\rho)\} \leq (1-2e^{-1-c_{11}})\omega_l + \rho^a. \tag{3.56}$$

Recall that we proved (3.11) under assumption (3.47). If (3.47) is not true, we take instead of $F_{1,l}$ the function $F_{2,l}: \Omega \to \mathbb{R}^n$ such that $F_{2,l} = 2e\omega_l(\omega_{2,l} - u_l + \rho^a)^{-1}$ in $B(y,2\rho)$, and arguing as above, we establish (3.11) again.

It is important to observe that the positive constant c_{11} depends only on c_1, c_2, c, $\overline{c}$, c_0, c', n, q, p, t, $\|1/\nu\|_{L^t(\Omega)}$, M_G, σ, $\operatorname{diam} G$, and $\prod_G$, and is independent of $l \in \mathbb{N}$.

Now from (3.11), taking into account [12, Chapter 2, Lemma 4.8], we deduce that there exist positive constant C and $\lambda(<1)$ depending on c_{11} and a but independent of $l \in \mathbb{N}$ such that

$$\operatorname{osc}\{u_l, B(y,\rho)\} \leq C[d(y,\partial\mathring{G})]^{-\lambda}\rho^{\lambda}, \quad \text{for every } \rho \in]0, d(y,\partial\mathring{G})]. \tag{3.57}$$

This and (3.8) imply that

$$\operatorname{osc}\{\overline{u}, B(y,\rho)\} \leq C[d(y,\partial\mathring{G})]^{-\lambda}\rho^{\lambda}, \quad \text{for every } \rho \in]0, d(y,\partial\mathring{G})]. \tag{3.58}$$

The proof is complete. □

4. An example

Let $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$, $0 < \gamma < \min(q - n/q, q/2)$, and let ν, μ be the restriction in $\Omega \setminus \{0\}$ of real functions

$$|x|^{\gamma}, \qquad |x|^{2p\gamma/q}. \tag{4.1}$$

According to considerations stated in [3, Section 7], we have that functions ν, μ satisfy Hypotheses 2.1 and 2.3.

Now, we will verify that $\nu(x)$ satisfies Hypothesis 3.1, for all t: $nq/(q^2 - n) < t < n/\gamma$. To this aim, let $G \subset \Omega \setminus \{0\}$ be a "*regular set*," and fix $y \in G$, $\rho > 0 : \overline{B(y,\rho)} \subset G$.

If $|y| < 2\rho$, it follows that $B(y,\rho) \subset B(0,3\rho)$. Hence, we have

$$\begin{aligned}
&\int_{B(y,\rho)} \frac{1}{|x|^{\gamma t}} dx \le \int_{B(0,3\rho)} \frac{1}{|x|^{\gamma t}} dx = n\chi_n \int_0^{3\rho} r^{n-1-\gamma t} dr = n\chi_n \frac{3^{n-\gamma t}}{n-\gamma t}\rho^{n-\gamma t}, \\
&\int_{B(y,\rho)} |x|^{\gamma\tau} dx \le \int_{B(0,3\rho)} |x|^{\gamma\tau} dx = n\chi_n \frac{3^{n+\gamma\tau}}{n+\gamma\tau}\rho^{n+\gamma\tau}.
\end{aligned} \tag{4.2}$$

From (4.2), taking into account that $\tau > nt/(qt - n)$, we get

$$\left(\rho^{-n}\int_{B(y,\rho)} \frac{1}{|x|^{\gamma t}} dx\right)^{1/t} \left(\rho^{-n}\int_{B(y,\rho)} |x|^{\gamma\tau} dx\right)^{1/\tau} \le (n\chi_n + 1)3^n\left(\frac{1}{n-\gamma t} + 1\right) \quad \text{if } |y| < 2\rho. \tag{4.3}$$

Instead if $|y| \ge 2\rho$, we denote by Ξ that

$$\Xi = \left\{k \in \mathbb{N} : \frac{|y|}{\rho} \ge k\right\}. \tag{4.4}$$

Note that $\Xi \ne \emptyset$ and is bounded from above. Consequently, if we denote $\overline{k} = \max \Xi$, we obtain

$$\overline{k}\rho \le |y| < \rho(\overline{k} + 1). \tag{4.5}$$

Last inequality implies that for every $x \in B(y,\rho)$, it results that

$$(\overline{k} - 1)\rho \le |x| \le (\overline{k} + 2)\rho. \tag{4.6}$$

From (4.6), we obtain

$$\begin{aligned}
&\int_{B(y,\rho)} \frac{1}{|x|^{\gamma t}} dx \le \frac{\chi_n}{(\overline{k} - 1)^{\gamma t}}\rho^{n-\gamma t}, \\
&\int_{B(y,\rho)} |x|^{\gamma\tau} dx \le \chi_n(\overline{k} + 2)^{\gamma\tau}\rho^{n+\gamma\tau},
\end{aligned} \tag{4.7}$$

where χ_n is the measure of the unit ball in $\mathbb{R}^n$.

Therefore, we get

$$\left(\rho^{-n}\int_{B(y,\rho)}\frac{1}{|x|^{\gamma t}}\right)^{1/t}\left(\rho^{-n}\int_{B(y,\rho)}|x|^{\gamma\tau}dx\right)^{1/\tau}\leq 4^n(\chi_n+1)\quad \text{if } |y|\geq 2\rho. \tag{4.8}$$

We can conclude that (3.1) holds with $c' = 4^n(n\chi_n+1)(1/(n-\gamma t)+1)$.

Next, let $f:\Omega\to\mathbb{R}$ be the function such that for every $x\in\Omega\setminus\{0\}$,

$$f(x)=\frac{|x|^{-n}}{(1-\lg|x|)^2}+\frac{1}{\sqrt{1-|x|}}. \tag{4.9}$$

Observe that $f(x)\in L^1(\Omega)$ but $f(x)$ does not belong to $L^\gamma(\Omega)$, for every $\gamma>1$.

Let $\sigma>1$, we consider the following Dirichlet problem:

$$\begin{gathered}
-\sum_{|\alpha|=1}D^\alpha\left[\nu\left(\sum_{|\beta|=1}|D^\beta u|^2\right)^{(q-2)/2}D^\alpha u\right]+\sum_{|\alpha|=2}D^\alpha\left[\mu\left(\sum_{|\beta|=2}|D^\beta u|^2\right)^{(p-2)/2}D^\alpha u\right]\\
=-|u|^{\sigma-1}u+f\quad \text{in } \Omega,\\
D^\alpha u=0,\quad |\alpha|=0,1,\ \text{on } \partial\Omega.
\end{gathered} \tag{4.10}$$

By Theorem 2.7, we establish that there exists a W-solution $\overline{u}$ of problem (4.10), bounded in every "*regular set*" $G\subset\Omega\setminus\{0\}$, and moreover, applying our result, Hölderian in every open set $A:\overline{A}\subset\Omega\setminus\{0\}$.

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